ANNALES MATHÉMATIQUES BLAISE PASCAL

CHRISTIAN LICHT
GÉRARD MICHAILLE

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Annales mathématiques Blaise Pascal, tome 9, nº 1 (2002), p. 21-62
<http://www.numdam.org/item?id=AMBP_2002__9_1_21_0>

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Christian Licht
Gérard Michaille

Abstract

We establish a global-local ergodic theorem about subadditive processes which seems to be a flexible tool to identify some limit problems in homogenization involving several small parameters. When the subadditive process is parametrized in a separable space, we show that the convergence takes place in the variational sense of the epiconvergence (or Γ-convergence). Some applications are given in the setting of nonlinear elasticity.

1 Introduction

The Ackoglu-Krengel subadditive ergodic theorem asserts, for a subadditive process $A \mapsto \mathcal{S}_A$, the existence of a pointwise limit for the sequence $\mathcal{S}_{A_n}/\text{meas}\ (A_n)$ where $(A_n)_n$ is a family of cubes in $\mathbf{R}^d$ whose size tends to infinity. This result seems to be firstly used in the setting of the calculus of variation by G. Dal Maso-L. Modica [10]. In this context, we would like to generalize this theorem to sequences indexed by convex sets. Indeed, homogenization of nonconvex integral functionals with linear growth seems to require this generalisation (see Y. Abddaimi-C. Licht-G. Michaille [2]). In these applications, the limit density (or its regular part in a nonreflexive case) appears to be the limit of a suitable subadditive process and it is of interest to study, from a variational point of view, the "stability" of the limit with respect to perturbations. This is the reason why we study the variational property of the previous convergence when the process depends on a parameter in a metric space. On the other hand many mathematical modelings in homogenization involve several small parameters and the limit problem, in the sense of *epiconvergence*, depends on their relative behavior. The previous (global) subadditive theorem or the local and more generally the global-local version, according to the various relative behaviors, seems to be an efficient mathematical tool to identify the limit problem. Consequently, we study the pointwise limit of $\mathcal{S}_{A_n \times Q_r}/\text{meas}\,(A_n)r^q$ when the "size" $\rho(A_n)$ tends to infinity and that of the cube Q_r tends to zero where $\mathcal{S}$ is defined on the product $\mathcal{B}_b(\mathbf{R}^d) \times \mathcal{B}_b(\mathbf{R}^q)$ of bounded Borel sets of $\mathbf{R}^d$ and $\mathbf{R}^q$.

The paper is organized as follows. In section 2, we investigate the invariant case : the subadditive set function is invariant when the set index is translated in $\mathbf{Z}^d$ in the global version, when the set index is translated in $\mathbf{R}^q$ in the local version. The result obtained in the global version is well known when the indices are $[0, n[^d$. We give a complete proof of the generalization to a suitable family of convex indices $(A_n)_n$ through some arguments of Nguyen Xuhan Xanh-H. Zessin [18] and various ideas explained in M.A. Ackoglu-U. Krengel [3] and U. Krengel [11]. After giving the local theorem, we mix the two versions to obtain a global-local subadditive theorem and a complete description of the limit.

In view of some applications (see G. Bouchitté-I. Fonseca-L. Mascarenhas [7]), we generalize, in section 3, the previous global result to the quasiperiodic case.

Section 4 is devoted to the random case. The subadditive set function takes its values in $L^1(\Omega, \mathcal{T}, P)$ where $(\Omega, \mathcal{T}, P)$ is a probability space and the translation of the index in $\mathbf{Z}^d$ modifies the function through a group of P-preserving transformations in the global version. When the family $(A_n)_n$ is constituted of suitable intervals of $\mathbf{R}^d$, we recover the Ackoglu-Krengel ergodic theorem. Our generalisation is perhaps known (see for instance various remarks in U. Krengel [11], chapter 7) but we give an exhaustive proof which is a natural extension of the proof of the invariant case and a complete description of the limit in the nonergodic case. We recall without proof the local version due to M.A. Ackoglu-U. Krengel [3] and we give a global-local subadditive theorem.

In section 5, when the subbaditive process depends on a parameter varying through a separable metric space and when the set valued maps $\omega \mapsto$ epi $\mathcal{S}_A(\omega, .)$ are random sets, where epi $\mathcal{S}_A(\omega, .)$ denotes the epigraph of $\mathcal{S}_A(\omega, .)$, we establish, in the global case, a variational almost sure convergence of previous sequences with respect to the parameter : the limit is obtained in the sense of epiconvergence (also called Γ-convergence). The method consists in applying the previous results to the Baire approximate of $-\mathcal{S}_{A_n}/\text{meas}\,(A_n)$ which is a superadditive process. The conclusion then follows thanks to a characterization of epiconvergence by the pointwise convergence of the Baire approximate. We do not give the local or global-local version which are easy adaptations of the previous method.

In the last section, we first recall some results about stochastic homogenization of nonconvex integral functionals and particularly those with linear growth, and give three applications. In the two first one, using Theorem 5.2 about almost sure epiconvergence of parametrized subadditive processes, we establish the continuity of homogenized energy or homogenized density energy with respect to some parameters. The last application concerns a modeling of elastic adhesive bonded joints. At least three parameters appears : the stiffness of the adhesive, the thickness ε of the layer filled by the adhesive and the size λ of heterogenities. Using the global or the local subadditive ergodic theorem, we give the limit problems corresponding to the cases $\lambda << \varepsilon$ or $\varepsilon << \lambda$.

2 The invariant case

2.1 The global theorem

In the sequel, $\mathcal{B}_b(\mathbf{R}^d)$ will denote the family of all the bounded Borel sets of $\mathbf{R}^d$, δ the euclidean distance in $\mathbf{R}^d$. For every A in $\mathcal{B}_b(\mathbf{R}^d)$, $|A|$ will denote its Lebesgue measure and we define the positive number $\rho(A) := \sup\{r \geq 0 : \exists \bar{B}_r(x) \subset A\}$ where $\bar{B}_r(x) = \{y \in \mathbf{R}^d : \delta(x,y) \leq r\}$.
A sequence $(B_n)_{n\in\mathbf{N}}$ of sets of $\mathcal{B}_b(\mathbf{R}^d)$ is said to be regular, if there exists an increasing sequence of intervals I_n in $\mathbf{Z}^d$ and a positive constant C independent of n such that $B_n \subset I_n$ and $|I_n| \leq C|B_n|$, $\forall n \in \mathbf{N}$. This last inequality will be used only in section 3.

A subadditive $\mathbf{Z}^d$-invariant set function is a map, $\mathcal{S} : \mathcal{B}_b(\mathbf{R}^d) \longrightarrow \mathbf{R}$, $A \mapsto \mathcal{S}_A$, such that

(i) $\forall A,\ B \in \mathcal{B}_b(\mathbf{R}^d)$ with $A \cap B = \emptyset$, $\mathcal{S}_{A\cup B} \leq \mathcal{S}_A + \mathcal{S}_B$,

(ii) $\forall A \in \mathcal{B}_b(\mathbf{R}^d)$, $\forall z \in \mathbf{Z}^d$, $\mathcal{S}_{z+A} = \mathcal{S}_A$.

Theorem 2.1: *Let $\mathcal{S}$ be a subadditive $\mathbf{Z}^d$-invariant set function such that*

$$\gamma(\mathcal{S}) := \inf\{\frac{\mathcal{S}_I}{|I|} : I = [a,b[,\ a,\ b \in \mathbf{Z}^d,\ \forall i = 1,\dots,d,\ a_i < b_i\ \} > -\infty,$$

and which satisfies the following domination property : there exists a positive constant $C(\mathcal{S}) < +\infty$ *such that* $|\mathcal{S}_A| \leq C(\mathcal{S})$ *for all Borel sets* A *included in* $[0,1[^d$*. Let* $(A_n)_{n\in\mathbf{N}}$ *be a regular sequence of Borel convex sets of* $\mathcal{B}_b(\mathbf{R}^d)$ *satisfying* $\lim_{n\to+\infty} \rho(A_n) = +\infty$*. Then*

$$\lim_{n\to+\infty} \frac{\mathcal{S}_{A_n}}{|A_n|} = \inf_{m\in\mathbf{N}^*} \{\frac{\mathcal{S}_{[0,m[^d}}{m^d}\} = \gamma(\mathcal{S}).$$

PROOF: The proof is divided in four steps. In what follows, $[t]$ denotes the integer part of the real t.

First step. We establish $\lim_{n\to+\infty} \frac{\mathcal{S}_{[0,n[^d}}{n^d} = \inf_{m\in\mathbf{N}^*} \{\frac{\mathcal{S}_{[0,m[^d}}{m^d}\}$. This is a well known result but, for the sake of completeness, we give its proof.

Let $m < n$ be in $\mathbf{N}^*$ and consider the following partition

$$[0,n[^d= \cup_{z\in m\mathbf{Z}^d\cap[0,n-m]^d}(z + [0,m[^d) \cup R_{n,m}$$

where $card(m\mathbf{Z}^d \cap [0,n-m]^d) = [\frac{n}{m}]^d$ and $R_{n,m}$ is a finite union of $\mathbf{Z}^d$ translated of $[0,1[^d$ with $card(\mathbf{Z}^d \cap R_{n,m}) = n^d - [\frac{n}{m}]^d m^d$. Thus, by subadditivity and invariance

$$\frac{\mathcal{S}_{[0,n[^d}}{n^d} \leq (\frac{m}{n})^d [\frac{n}{m}]^d \frac{\mathcal{S}_{[0,m[^d}}{m^d} + (1 - (\frac{m}{n})^d [\frac{n}{m}]^d) \mathcal{S}_{[0,1[^d}.$$

Letting $n \to +\infty$, we obtain, for every $m \in \mathbf{N}^*$

$$\limsup_{n\to+\infty} \frac{\mathcal{S}_{[0,n[^d}}{n^d} \leq \frac{\mathcal{S}_{[0,m[^d}}{m^d},$$

thus

$$\limsup_{n\to+\infty} \frac{\mathcal{S}_{[0,n[^d}}{n^d} = \liminf_{m\to+\infty} \frac{\mathcal{S}_{[0,m[^d}}{m^d} = \inf_{m\in\mathbf{N}^*} \{\frac{\mathcal{S}_{[0,m[^d}}{m^d}\}.$$

Second step. We establish $\lim_{n\to+\infty} \frac{\mathcal{S}_{[0,n[^d}}{n^d} = \gamma(\mathcal{S})$.

Fix $I = [a,b[$, a, $b \in \mathbf{Z}^d$, $a_i < b_i$, $i = 1,\ldots,n$. By invariance, we may assume $a = 0$. For m large enough, let us consider the partition

$$[0,m[^d= A_{I,m} \cup R_{I,m}$$

where $A_{I,m}$ is the subset of $[0,m[^d$ constituted of all the disjoint $\mathbf{Z}^d$-translated of I included in $[0,m[^d$. Considering each component of $b \in \mathbf{Z}^d$, it is straightforward to check that

$$\frac{|A_{I,m}|}{m^d} \sim \frac{1}{|I|} \text{ and } \frac{|R_{I,m}|}{m^d} \sim 0$$

when m tends to $+\infty$. Therefore, by the arguments of the first step $\lim_{m\to+\infty} \frac{\mathcal{S}_{[0,m[^d}}{m^d} \leq \frac{\mathcal{S}_I}{|I|}$ so that $\lim_{n\to+\infty} \frac{\mathcal{S}_{[0,n[^d}}{n^d} \leq \gamma(\mathcal{S})$, which concludes this second step, the converse inequality being obvious.

Third step. We adopt the following notations. Let $(A_n)_{n\in\mathbf{N}}$ be a sequence of convex sets of $\mathcal{B}_b(\mathbf{R}^d)$ such that $\lim_{n\to+\infty} \rho(A_n) = +\infty$. For $m < n$, $m \in \mathbf{N}^*$, we set

$$\underline{A}_{n,m} = \cup_{\{z\in m\mathbf{Z}^d \,:\, (z+[0,m[^d \subset A_n\}}(z + [0,m[^d)$$
$$\overline{A}_{n,m} = \cup_{\{z\in m\mathbf{Z}^d \,:\, (z+[0,m[^d)\cap A_n \neq \emptyset\}}(z + [0,m[^d).$$

We will need the following lemma (see Nguyen-Zessin [18])

Lemma 2.2: *If $(A_n)_{n\in\mathbf{N}}$ is a sequence of convex sets of $\mathcal{B}_b(\mathbf{R}^d)$ with $\lim_{n\to+\infty} \rho(A_n) = +\infty$, then*

$$\frac{|\overline{A}_{n,m} \setminus \underline{A}_{n,m}|}{|A_n|} \sim 0$$

when n tends to $+\infty$.

Finally, let

$$\bar{l}_m := \limsup_{n\to+\infty} \frac{\mathcal{S}_{\underline{A}_{n,m}}}{|\underline{A}_{n,m}|},\ \underline{l}_m = \liminf_{n\to+\infty} \frac{\mathcal{S}_{\overline{A}_{n,m}}}{|\overline{A}_{n,m}|}$$
$$\bar{l} := \limsup_{n\to+\infty} \frac{\mathcal{S}_{A_n}}{|A_n|},\ \underline{l} = \liminf_{n\to+\infty} \frac{\mathcal{S}_{A_n}}{|A_n|}.$$

Our aim, in this step, is to establish $\underline{l} = \bar{l} := l$.

The finitness of $\bar{l}_m$ follows from

$$\begin{aligned}\frac{\mathcal{S}_{\underline{A}_{n,m}}}{|\underline{A}_{n,m}|} &\leq \frac{card\{z \in m\mathbf{Z}^d \,:\, (z+[0,m[^d) \subset A_n\}}{|\underline{A}_{n,m}|}\mathcal{S}_{[0,m[^d} \\ &= \frac{\mathcal{S}_{[0,m[^d}}{m^d}.\end{aligned}$$

The inclusion $\underline{A}_{n,m} \subset A_n$ implies, by subadditivity, invariance and domination

$$\begin{aligned}\frac{\mathcal{S}_{A_n}}{|A_n|} &\leq \frac{\mathcal{S}_{\underline{A}_{n,m}}}{|\underline{A}_{n,m}|}\frac{|\underline{A}_{n,m}|}{|A_n|} + \frac{\mathcal{S}_{A_n \setminus \underline{A}_{n,m}}}{|A_n|} \\ &\leq \frac{\mathcal{S}_{\underline{A}_{n,m}}}{|\underline{A}_{n,m}|}\frac{|\underline{A}_{n,m}|}{|A_n|} + C(\mathcal{S})\frac{|\overline{A}_{n,m} \setminus \underline{A}_{n,m}|}{|A_n|}\end{aligned}$$

thus, by Lemma 2.1

$$l \leq \bar{l}_m. \tag{2.1}$$

With the same computation, the second inclusion $A_n \subset \overline{A}_{n,m}$ implies

$$\underline{l}_m \leq \underline{l}. \tag{2.2}$$

Let $\varepsilon > 0$ and $m(\varepsilon) \in \mathbf{N}$ be such that, for every $m \geq m(\varepsilon)$

$$\frac{\mathcal{S}_{[0,m[^d}}{m^d} - \gamma(\mathcal{S}) \leq \varepsilon.$$

For a fixed $m \geq m(\varepsilon)$ and for every $I \in \mathcal{B}_b(\mathbf{R}^d)$, finite union of $m\mathbf{Z}^d$-translated of $[0, m[^d$, by subadditivity and invariance

$$\mathcal{S}_I^m := \mathcal{S}_I - card\{I \cap m\mathbf{Z}^d\}\mathcal{S}_{[0,m[^d} \leq 0. \tag{2.3}$$

(Note that $\mathcal{S}^m$ is a non increasing subadditive $m\mathbf{Z}^d$-invariant set function.) For every subadditive set function Ψ defined on finite unions of $m\mathbf{Z}^d$-translated of $[0, m[^d$ we define

$$\gamma^m(\Psi) := \inf\{\frac{\Psi_I}{|I|} : I = [a, b[,\ a,\ b \in m\mathbf{Z}^d, \forall i = 1, \dots, d,\ a_i < b_i\}.$$

With this definition

$$\gamma^m(\mathcal{S}^m) \geq \gamma(\mathcal{S}) - \frac{\mathcal{S}_{[0,m[^d}}{m^d} \geq -\varepsilon. \tag{2.4}$$

According to the regularity of the sequence $(A_n)_{n\in\mathbf{N}}$, there exists a sequence of non decreasing intervals $(\overline{I}_{n,m})_{n\in\mathbf{N}}$ where $\overline{I}_{n,m} := \cup_{\{z \in m\mathbf{Z}^d\ :\ (z+[0,m[^d) \cap I_n \neq \emptyset\}}(z + [0, m[^d)$. Taking $I = \overline{A}_{n,m}$ in (2.3), we obtain

$$\begin{aligned}\frac{\mathcal{S}_{\overline{A}_{n,m}}}{|\overline{A}_{n,m}|} - \frac{\mathcal{S}_{[0,m[^d}}{m^d} = \frac{\mathcal{S}^m_{\overline{A}_{n,m}}}{|\overline{A}_{n,m}|} &\geq \frac{\mathcal{S}^m_{\overline{I}_{n,m}}}{|\overline{I}_{n,m}|}\frac{|\overline{I}_{n,m}|}{|\overline{A}_{n,m}|} \\ &\geq \gamma^m(\mathcal{S}^m)\frac{|\overline{I}_{n,m}|}{|\overline{A}_{n,m}|},\end{aligned}$$

thus

$$\underline{l}_m - \frac{\mathcal{S}_{[0,m[^d}}{m^d} \geq \gamma^m(\mathcal{S}^m) \tag{2.5}$$

(note that $\liminf_{n \to +\infty} \frac{|\overline{I}_{n,m}|}{|\overline{A}_{n,m}|} \geq 1$) so that $\underline{l}_m > -\infty$.
Taking $I = \underline{A}_{n,m}$ in (2.3) we obtain

$$\overline{l}_m - \frac{\mathcal{S}_{[0,m[^d}}{m^d} \leq 0. \tag{2.6}$$

Consequently, from (2.6), (2.5) and (2.4)

$$\overline{l}_m - \underline{l}_m \leq -\gamma^m(\mathcal{S}^m) \leq \varepsilon.$$

Since (2.1), (2.2) give

$$\overline{l} - \underline{l} \leq \overline{l}_m - \underline{l}_m \leq \varepsilon. \tag{2.7}$$

we obtain the desired result letting $\varepsilon \to 0$.

Last step. We identify l. By (2.1), (2.6) $l - \frac{\mathcal{S}_{[0,m[^d}}{m^d} \leq 0$ so that, according to the second step $l \leq \gamma(\mathcal{S})$. On the other hand, from (2.5) and (2.4)

$$\begin{aligned} l - \frac{\mathcal{S}_{[0,m[^d}}{m^d} &\geq \gamma^m(\mathcal{S}^m) \\ &\geq \gamma(\mathcal{S}) - \frac{\mathcal{S}_{[0,m[^d}}{m^d} \end{aligned}$$

and we complete the proof after letting $m \to +\infty$. □

Remark: In the definition of subadditivity, assertion (i) can be replaced by : $\forall A,\ B \in \mathcal{B}_b(\mathbf{R}^d)$ with $A \cap B = \emptyset$ and $|\partial A| = |\partial B| = 0$, $\mathcal{S}_{A \cup B} \leq \mathcal{S}_A + \mathcal{S}_B$. Indeed all the sets considered in the proof have a Lebesgue negligeable boundary. This remark will be applyed in section 6 for the various processes defined from infimum of integral functionals.

2.2 The local and global-local theorems

We denote by $\mathcal{P}(\mathbf{R}^q)$ the set of intervals of the form $[a, b[$ in $\mathbf{R}^q$ and we consider a subadditive $\mathbf{R}^q$-invariant set function defined in $\mathcal{P}(\mathbf{R}^q)$, that is a map $\mathcal{S} :\ \mathcal{P}(\mathbf{R}^q) \longrightarrow \mathbf{R}$ which satisfies

i) $\forall I_1, \dots, I_s$ disjoints sets in $\mathcal{P}(\mathbf{R}^q)$ such that $I = \bigcup_{i=1}^{s} I_i$ is in $\mathcal{P}(\mathbf{R}^q)$, $\mathcal{S}_I \leq \sum_{i=1}^{s} \mathcal{S}_{I_i}$,

ii) $\forall A \in \mathcal{P}(\mathbf{R}^q)$, $\forall x \in \mathbf{R}^q$, $\mathcal{S}_{x+A} = \mathcal{S}_A$.

For every x_0 in $\mathbf{R}^q$, let $Q_r(x_0)$ be the cube in $\mathcal{P}(\mathbf{R}^q)$ of size r centered at x_0. We have the following elementary local result (cf M.A. Ackoglu-U. Krengel [3])

Theorem 2.3: *Let $\mathcal{S}$ be a subadditive and $\mathbf{R}^q$-invariant set function defined as above and satisfying*

$$\delta := \sup\{\frac{|\mathcal{S}_I|}{|I|} : I \in \mathcal{P}(\mathbf{R}^q),\ |I| \neq 0\} < +\infty.$$

then

$$\lim_{r\to 0} \frac{\mathcal{S}_{Q_r(x_0)}}{r^q} = S := \sup\{\frac{\mathcal{S}_I}{|I|} : I \in \mathcal{P}(\mathbf{R}^q),\ |I| \neq 0\}.$$

PROOF: For every $\varepsilon > 0$ let I_ε be such that

$$\frac{\mathcal{S}_{I_\varepsilon}}{|I_\varepsilon|} > S - \varepsilon. \tag{2.8}$$

On the other hand, there exists $r(\varepsilon) > 0$ such that for $0 < r < r(\varepsilon)$ there exists I'_ε in $\mathcal{P}(\mathbf{R}^q)$ included in I_ε, a union of disjoint translates of $Q_r(x_0)$ with

$$I'_\varepsilon = \bigcup_{i=1}^{s} x_i + Q_r(x_0), \quad |I_\varepsilon \setminus I'_\varepsilon| < \varepsilon |I'_\varepsilon|.$$

Subadditivity and invariance yield

$$\frac{\mathcal{S}_{I'_\varepsilon}}{|I'_\varepsilon|} \leq \frac{\mathcal{S}_{Q_r(x_0)}}{r^q}. \tag{2.9}$$

But

$$\begin{aligned} \frac{\mathcal{S}_{I_\varepsilon}}{|I_\varepsilon|} &\leq \frac{\mathcal{S}_{I'_\varepsilon}}{|I'_\varepsilon|} + \delta \frac{|I_\varepsilon \setminus I'_\varepsilon|}{|I'_\varepsilon|} \\ &\leq \frac{\mathcal{S}_{I'_\varepsilon}}{|I'_\varepsilon|} + \delta\varepsilon. \end{aligned} \tag{2.10}$$

Estimates (2.8), (2.9) and (2.10) implies $\frac{\mathcal{S}_{Q_r(x_0)}}{r^q} \geq S-(\delta+1)\varepsilon$. Going to the limit in r and ε we obtain $\liminf_{r\to 0}\frac{\mathcal{S}_{Q_r(x_0)}}{r^q} \geq S$. Obviously $\limsup_{r\to 0}\frac{\mathcal{S}_{Q_r(x_0)}}{r^q} \leq S$ and the proof is complete. □

We now consider a subadditive set function $\mathcal{S}$ indexed by a product of Borel sets. More precisely the map $\mathcal{S} : \mathcal{B}_b(\mathbf{R}^d)\times\mathcal{P}(\mathbf{R}^q) \longrightarrow \mathbf{R}$, $A\times I \mapsto \mathcal{S}_{A\times I}$ is such that

i) for every I in $\mathcal{P}(\mathbf{R}^q)$, $I \subset [0,1[^q$, $A \mapsto \mathcal{S}_{A\times I}$ is a subadditive $\mathbf{Z}^d$-invariant set function satisfying all the hypothesis of section 2.1 and Theorem 2.1 and where the constant in the domination property does not depend on I.

ii) for every A in $\mathcal{B}_b(\mathbf{R}^d)$, $A \subset [0,1[^d$, $I \mapsto \mathcal{S}_{A\times I}$ is a subadditive $\mathbf{R}^q$-invariant set function satisfying all the hypothesis of previous Theorem 2.2 and where the constant δ does not depend on A.

Then as a corollary of Theorems 2.1 and 2.2 we obtain the following global-local and local-global subadditive ergodic theorems in the deterministic case

Theorem 2.4: *Let $\mathcal{S}$ be a subadditive set function satisfying hypothesis i) and ii) and $(A_n)_{n\in\mathbf{N}}$ be a regular sequence of Borel convex sets of $\mathcal{B}_b(\mathbf{R}^d)$ satisfying $\lim_{n\to+\infty}\rho(A_n) = +\infty$. Then*

$$\begin{aligned}\lim_{n\to+\infty}\lim_{r\to 0}\frac{\mathcal{S}_{A_n\times Q_r(x_0)}}{|A_n|r^q} &= \inf_{A\in\mathcal{P}(\mathbf{Z}^d)}\sup_{I\in\mathcal{P}(\mathbf{R}^q)}\{\frac{\mathcal{S}_{A\times I}}{|A|\ |I|} : |A|\neq 0,\ |I|\neq O\}\\ &= \inf_{m\in\mathbf{N}^*}\sup_{n\in\mathbf{N}^*}\frac{n^q}{m^d}\mathcal{S}_{[0,m[^d\times[0,\frac{1}{n}[^q}\end{aligned}$$

and

$$\begin{aligned}\lim_{r\to 0}\lim_{n\to+\infty}\frac{\mathcal{S}_{A_n\times Q_r(x_0)}}{|A_n|r^q} &= \sup_{I\in\mathcal{P}(\mathbf{R}^q)}\inf_{A\in\mathcal{P}(\mathbf{Z}^d)}\{\frac{\mathcal{S}_{A\times I}}{|A|\ |I|} : |A|\neq 0,\ |I|\neq O\}\\ &= \sup_{n\in\mathbf{N}^*}\inf_{m\in\mathbf{N}^*}\frac{n^q}{m^d}\mathcal{S}_{[0,m[^d\times]0,\frac{1}{n}[^q}.\end{aligned}$$

Moreover if

$$\inf_{A\in\mathcal{P}(\mathbf{Z}^d)}\sup_{I\in\mathcal{P}(\mathbf{R}^q)}\{\frac{\mathcal{S}_{A\times I}}{|A|\ |I|} : |A|\neq 0,\ |I|\neq O\} = \sup_{I\in\mathcal{P}(\mathbf{R}^q)}\inf_{A\in\mathcal{P}(\mathbf{Z}^d)}\{\frac{\mathcal{S}_{A\times I}}{|A|\ |I|} : |A|\neq 0,\ |I|\neq O\}$$

then, for every sequence $(r_n)_{n\in\mathbf{N}}$ of positive reals tending to zero, $\lim_{n\to+\infty} \frac{\mathcal{S}_{A_n\times Q_{r_n}(x_0)}}{|A_n| r_n^q}$ *exists and is equal to this common value.*

PROOF: We have

$$\frac{\mathcal{S}_{A_n\times Q_{r_n}(x_0)}}{|A_n| r_n^q} \le \sup_I \frac{\mathcal{S}_{A_n\times I}}{|A_n|\,|I|}.$$

Going to the limit on n and according to Theorem 2.1

$$\limsup_{n\to+\infty} \frac{\mathcal{S}_{A_n\times Q_{r_n}(x_0)}}{|A_n| r_n^q} \le \inf_A \sup_I \frac{\mathcal{S}_{A\times I}}{|A|\,|I|}.$$

On the other hand

$$\lim_{m\to+\infty} \frac{\mathcal{S}_{[0,m[^d\times Q_{r_n}(x_0)}}{m^d r_n^q} = \inf_A \frac{\mathcal{S}_{A\times Q_{r_n}(x_0)}}{|A| r_n^q} \le \frac{\mathcal{S}_{A_n\times Q_{r_n}(x_0)}}{|A_n| r_n^q}.$$

Going to the limit on n and according to Theorem 2.2

$$\sup_I \inf_A \frac{\mathcal{S}_{A\times I}}{|A|\,|I|} \le \liminf_{n\to+\infty} \frac{\mathcal{S}_{A_n\times Q_{r_n}(x_0)}}{|A_n| r_n^q}.$$

and the proof is complete. □

3 The almost periodic case.

With the notations of the previous section, we now consider an almost periodic subadditive set function, that is a subadditive map $\mathcal{S} : \mathcal{B}_b(\mathbf{R}^d) \longrightarrow \mathbf{R}$ satisfying : $\forall \eta > 0$, $\exists T_\eta \subset \mathbf{R}^d$, $\exists L_\eta > 0$ such that

(i) $\mathbf{R}^d = T_\eta + [0, L_\eta]^d$

(ii) $|\mathcal{S}_{t+A} - \mathcal{S}_A| \le \eta |A|$ for all t in T_η.

If moreover $\mathcal{S}$ satisfies the growth condition

(iii) $\exists C > 0$ such that $\mathcal{S}_A \le C|A|$,

we have the following global theorem.

Theorem 3.1: *For every cube A of the form $[a, b[^d$,* $\lim_{s\to+\infty} \frac{\mathcal{S}_{sA}}{|sA|}$ *exists and is equal to* $\lim_{s\to+\infty} \frac{\mathcal{S}_{[0,s[^d}}{s^d}$.

PROOF: The proof is divided in two steps.

First step. The limit $l := \lim_{s \to +\infty} \frac{\mathcal{S}_{[0,s[^d}}{s^d}$ exists.

Let s be a fixed positive real, $t > s$ intended to tend to $+\infty$, and consider a net made of the $(sz)_{z \in \mathbf{Z}^d}$-translated of $[0, s[^d$. Before to perturb this net by a family $(t_z)_{z \in \mathbf{Z}^d}$, $t_z \in T_\eta$, we begin to disconnect its elements by a distance of order L_η. More precisely, we consider the family $((s + L_\eta)z + [0, s[^d)_{z \in \mathbf{Z}^d}$. We now perturb the corresponding net by $[0, L_\eta]^d$ to obtain a family of disjoint translated of $[0, s[^d$ by suitable elements of T_η as follows : for every $(s + L_\eta)z$, there exists $t_z \in T_\eta$ such that $(s + L_\eta)z \in t_z + [0, L_\eta]^d$ and we consider the family $(t_z + [0, s[^d)_{z \in \mathbf{Z}^d}$, $t_z \in T_\eta$, $t_z \in (s + L_\eta)z - [0, L_\eta]^d$. We finally have

$$[0, t[^d = \bigcup_{z \in I_{s,t}} (t_z + [0, s[^d) \bigcup N_{s,t}$$

where $I_{s,t} = \{z \in \mathbf{Z}^d : t_z + [0, s[^d \subset [0, t[^d\}$. An easy calculation gives $[\frac{t}{s + 2L_\eta}]^d \leq \mathrm{card}(I_{s,t}) \leq [\frac{t}{s}]^d$ and the left bound gives $|N_{s,t}| \leq t^d - [\frac{t}{s + 2L_\eta}]^d s^d$. By subadditivity and hypothesis (iii), we obtain

$$\mathcal{S}_{[0,t[^d} \leq \sum_{z \in I_{s,t}} \mathcal{S}_{t_z + [0,s[^d} + C(t^d - [\frac{t}{s + 2L_\eta}]^d s^d)$$

and by (ii) and the right bound of $\mathrm{card}(I_{s,t})$, we infer that

$$\begin{aligned} \mathcal{S}_{[0,t[^d} &\leq \mathrm{card}(I_{s,t})(\mathcal{S}_{[0,s[^d} + \eta s^d) + C(t^d - [\frac{t}{s + 2L_\eta}]^d s^d) \\ &\leq [\frac{t}{s}]^d (\mathcal{S}_{[0,s[^d} + \eta s^d) + C(t^d - [\frac{t}{s + 2L_\eta}]^d s^d). \end{aligned}$$

Dividing by t^d, we obtain

$$\frac{\mathcal{S}_{[0,t[^d}}{t^d} \leq [\frac{t}{s}]^d (\frac{s}{t})^d (\frac{\mathcal{S}_{[0,s[^d}}{s^d} + \eta) + C(1 - [\frac{t}{s + 2L_\eta}]^d (\frac{s}{t})^d). \tag{3.11}$$

Letting $t \to +\infty$, we deduce that

$$\limsup_{t \to +\infty} \frac{\mathcal{S}_{[0,t[^d}}{t^d} \leq \frac{\mathcal{S}_{[0,s[^d}}{s^d} + \eta + C(1 - (\frac{s}{s + 2L_\eta})^d),$$

and going to the limit on s we finally find that

$$\limsup_{t\to+\infty}\frac{\mathcal{S}_{[0,t[^d}}{t^d}\le\liminf_{s\to+\infty}\frac{\mathcal{S}_{[0,s[^d}}{s^d}+\eta$$

and we end the proof of this step after letting $\eta\to 0$.

Second step. Let A be a cube of the form $b+[0,a[^d$. By (i), there exists $\tau_t\in T_\eta$ such that $tb\in\tau_t+[0,L_\eta]^d$. Consequentely $tA=\tau_t+l_{\eta,t}+[0,ta[^d$ where $l_{\eta,t}\in[0,L_\eta]^d$ and by (ii), we obtain

$$\frac{\mathcal{S}_{l_{\eta,t}+[0,ta[^d}}{(ta)^d}-\eta\le\frac{\mathcal{S}_{tA}}{|tA|}\le\frac{\mathcal{S}_{l_{\eta,t}+[0,ta[^d}}{(ta)^d}+\eta.$$

The conclusion will follow if we prove

$$\liminf_{\eta\to 0}\liminf_{t\to+\infty}\frac{\mathcal{S}_{l_{\eta,t}+[0,ta[^d}}{(ta)^d}\ge l,\tag{3.12}$$

$$\limsup_{\eta\to 0}\limsup_{t\to+\infty}\frac{\mathcal{S}_{l_{\eta,t}+[0,ta[^d}}{(ta)^d}\le l.\tag{3.13}$$

Proof of (3.13). The cube $l_{\eta,t}+[0,ta[^d$ is a perturbation of $[0,ta[^d$ by an element of $[0,L_\eta]^d$, so that, using again the perturbed net made of the family $(t_z+[0,s[^d)_{z\in\mathbf{Z}^d}$ considered in the first step, we infer as (3.11) that

$$\frac{\mathcal{S}_{l_{\eta,t}+[0,ta[^d}}{(ta)^d}\le[\frac{ta+L_\eta}{s}]^d(\frac{s}{ta})^d(\frac{\mathcal{S}_{[0,s[^d}}{s^d}+\eta)+C(1-[\frac{ta}{s+2L_\eta}]^d(\frac{s}{ta})^d),$$

where we have used $[\frac{ta}{s+2L_\eta}]^d\le\operatorname{card}(I_{s,t})\le[\frac{ta+L_\eta}{s}]^d$. Then (3.13) is easily obtained after letting $t\to+\infty$, $s\to+\infty$ and $\eta\to 0$.

Proof of (3.12). We have $l_{\eta,t}+[0,ta[^d\subset[0,ta+L_\eta[^d$ so that $[0,ta+L_\eta[^d=l_{\eta,t}+[0,ta[^d\cup N_{\eta,t}$ and by subadditivity $\mathcal{S}_{[0,ta+L_\eta[^d}\le\mathcal{S}_{l_{\eta,t}+[0,ta[^d}+C|N_{\eta,t}|$ with $|N_{\eta,t}|=(ta+L_\eta)^d-(ta)^d$. Therefore

$$\frac{\mathcal{S}_{[0,ta+L_\eta[^d}}{(ta+L_\eta)^d}\le\frac{\mathcal{S}_{l_{\eta,t}+[0,ta[^d}}{(ta)^d}+C(1-(\frac{ta}{ta+L_\eta})^d).$$

Letting $t\to+\infty$ and by the first step, we obtain, $\forall\eta>0$, $l\le\liminf_{t\to+\infty}\frac{\mathcal{S}_{l_{\eta,t}+[0,ta[^d}}{(ta)^d}$ which is (3.12). □

For an application of this result, consult G. Bouchitté-I Fonseca- L. Mascarenhas [7]. Similar results have already been obtained by the same arguments in the framework of the homogenization of almost-periodic integral functionals (see A. Braides [8]).

4 The Stochastic case

4.1 The global Theorem

Let $(\Omega,\ \mathcal{T},\ P)$ be a probability space and $(\tau_z)_{z\in\mathbf{Z}^d}$ be a group of P-preserving transformations on $(\Omega,\mathcal{T})$, that is

(i) τ_z is $\mathcal{T}$-measurable,

(ii) $P\circ\tau_z(E)=P(E)$, for every E in $\mathcal{T}$ and every z in $\mathbf{Z}^d$,

(iii) $\tau_z\circ\tau_t=\tau_{z+t},\quad \tau_{-z}=\tau_z^{-1}$, for every z and t in $\mathbf{Z}^d$.

In addition, if every set E in $\mathcal{T}$ such that $\tau_z(E)=E$ for every $z\in\mathbf{Z}^d$ has a probability equal to 0 or 1, $(\tau_z)_{z\in\mathbf{Z}^d}$ is said to be *ergodic*. A sufficient condition to ensure ergodicity of $(\tau_z)_{z\in\mathbf{Z}^d}$ is the following *mixing* condition : for every E and F in $\mathcal{T}$

$$\lim_{|z|\to+\infty} P(\tau_z E\cap F)=P(E)P(F)$$

which expresses an asymptotic independance. In the sequel, $\mathcal{F}$ (resp. $\mathcal{F}_m$, $m\in\mathbf{N}^*$) will denote the σ-algebra of invariant sets of $\mathcal{T}$ for $(\tau_z)_{z\in\mathbf{Z}^d}$) (resp. for $(\tau_z)_{z\in m\mathbf{Z}^d}$, and $E^{\mathcal{F}}$ (resp. $E^{\mathcal{F}_m}$) will denote the conditional expectation operator with respect to $\mathcal{F}$ (resp. to $\mathcal{F}^m$).

A subadditive process for $(\tau_z)_{z\in\mathbf{Z}^d}$ is a set function $\mathcal{S}:\ \mathcal{B}_b(\mathbf{R}^d)\longrightarrow L^1(\Omega,\ \mathcal{T},\ P)$ such that

i) $\forall A,\ B\in\mathcal{B}_b(\mathbf{R}^d)$ with $A\cap B=\emptyset$, $\mathcal{S}_{A\cup B}\le\mathcal{S}_A+\mathcal{S}_B$

ii) $\forall A\in\mathcal{B}_b(\mathbf{R}^d)$, $\forall z\in\mathbf{Z}^d$, $\mathcal{S}_{z+A}=\mathcal{S}_A\circ\tau_z$ (covariance).

The following result generalizes Theorem 2.1 in a stochastic framework and gives an explicit formula for the limit in the non ergodic case. For the study of the speed of convergence in the ergodic case (more precisely in the independent case), we refer the reader to G. Michaille-J. Michel-L. Piccinini [15]

Theorem 4.1: *Let $\mathcal{S}$ be a subadditive process for $(\tau_z)_{z\in\mathbf{Z}^d}$ such that*

$$\gamma(\mathcal{S}):=\inf\{\int_\Omega\frac{\mathcal{S}_I}{|I|}dP : I=[a,b[,\ a,\ b\in\mathbf{Z}^d,\ \forall i=1,\dots,d,\ \ a_i<b_i\ \}>-\infty,$$

and which satisfies the following domination property : there exists f in $L^1(\Omega,\ \mathcal{T},\ P)$ such that, for all Borel sets A included in $[0,1[^d$, $|\mathcal{S}_A| \leq f$. Let $(A_n)_{n\in\mathbf{N}}$ be a regular sequence of convex Borel sets of $\mathcal{B}_b(\mathbf{R}^d)$ satisfying $\lim_{n\to+\infty} \rho(A_n) = +\infty$. Then almost surely

$$\lim_{n\to+\infty} \frac{\mathcal{S}_{A_n}(\omega)}{|A_n|} = \inf_{m\in\mathbf{N}^*} E^{\mathcal{F}} \frac{\mathcal{S}_{[0,m[^d}}{m^d}(\omega).$$

Moreover, if $(\tau_z)_{z\in\mathbf{Z}^d}$ is ergodic

$$\lim_{n\to+\infty} \frac{\mathcal{S}_{A_n}(\omega)}{|A_n|} = \inf_{m\in\mathbf{N}^*} \{\int_\Omega \frac{\mathcal{S}_{[0,m[^d}}{m^d} dP\} = \gamma(\mathcal{S}).$$

PROOF: We acknowledge the result in the additive case (see for instance Nguyen Xuan Xanh-H. Zessin [18] or U. Krengel [11]. In this case, almost surely

$$\lim_{n\to+\infty} \frac{\mathcal{S}_{A_n}(\omega)}{|A_n|} = E^{\mathcal{F}} \mathcal{S}_{[0,1[^d}(\omega)$$

and more generally, if the process is associated to a $(\tau_z)_{z\in m\mathbf{Z}^d}$ group where $m \in \mathbf{N}^*$, then, almost surely

$$\lim_{n\to+\infty} \frac{\mathcal{S}_{A_n}(\omega)}{|A_n|} = E^{\mathcal{F}_m} \frac{\mathcal{S}_{[0,m[^d}}{m^d}(\omega).$$

For every subadditive process Ψ for $(\tau_z)_{z\in m\mathbf{Z}^d}$, $m \in \mathbf{N}^*$, defined on finite unions of $m\mathbf{Z}^d$-translated of $[0,m[^d$ we set

$$\gamma^m(\Psi) := \inf\{\int_\Omega \frac{\Psi_I}{|I|} dP : I = [a,b[,\ a,\ b \in m\mathbf{Z}^d, \forall i = 1,\dots,d,\ a_i < b_i\}.$$

The main ingredient of the proof is the following *maximal inequality* (this is an easy adaptation of U. Krengel [11], Theorem 2.6 and Corollary 2.7, p.205) which allows us to estimate the probability of the event $\{\omega : \overline{l}_m(\omega) - \underline{l}_m(\omega) \geq \alpha\}$ corresponding to inequality (2.7) of section 2 :

Lemma 4.2:*[maximal inequality] let $(I_n)_{n\in\mathbf{N}}$ be a regular sequence of intervals with vertices in $m\mathbf{Z}^d$, with constant of regularity C, and $\mathcal{S}^m$ be a non* *positive subadditive process for a group* $(\tau_z)_{z\in m\mathbf{Z}^d}$, $m \in \mathbf{N}^*$. *Then, for every* $\alpha > 0$ *the probability of the set*

$$E_\alpha := \{\omega \in \Omega : \inf_n \frac{\mathcal{S}^m_{I_n}(\omega)}{|I_n|} \leq -\alpha\}$$

satisfies

$$P(E_\alpha) \leq -2^d C \frac{\gamma^m(\mathcal{S}^m)}{\alpha}.$$

First step. Let $\bar{l}_m$, $\bar{l}$, $\underline{l}_m$ and $\underline{l}$ be defined as in the proof of Lemma 2.1. We establish that $\underline{l} = \bar{l}$ almost surely. We will denote by l this common value. Like in section 2, the inclusion $\underline{A}_{n,m} \subset A_n$ implies, by subadditivity and domination

$$\begin{aligned}
\frac{\mathcal{S}_{A_n}}{|A_n|} &\leq \frac{\mathcal{S}_{\underline{A}_{n,m}}}{|\underline{A}_{n,m}|}\frac{|\underline{A}_{n,m}|}{|A_n|} + \frac{\mathcal{S}_{A_n\setminus\underline{A}_{n,m}}}{|A_n|} \\
&\leq \frac{\mathcal{S}_{\underline{A}_{n,m}}}{|\underline{A}_{n,m}|}\frac{|\underline{A}_{n,m}|}{|A_n|} + \frac{1}{|A_n|}\sum_{z\in\mathbf{Z}^d\cap(\overline{A}_{n,m}\setminus\underline{A}_{n,m})} f\circ\tau_z
\end{aligned}$$

where almost surely

$$\lim_{n\to+\infty}\frac{1}{|A_n|}\sum_{z\in\mathbf{Z}^d\cap(\overline{A}_{n,m}\setminus\underline{A}_{n,m})} f\circ\tau_z = 0$$

(see Nguyen Xuan Xanh-H.Zessin [18], Corollary 4.10). Hence, almost surely

$$\bar{l} \leq \bar{l}_m. \tag{4.14}$$

Similarly, $A_n \subset \overline{A}_{n,m}$ implies, almost surely

$$\underline{l}_m \leq \underline{l}. \tag{4.15}$$

Let $\alpha > 0$ be fixed. As $\{\omega : \bar{l}(\omega) - \underline{l}(\omega) \geq \alpha\} \subset E_{m,\alpha} := \{\omega : \bar{l}_m(\omega) - \underline{l}_m(\omega) \geq \alpha\}$, it suffices to show that $\forall\varepsilon > 0$ and for m large enough

$$P(E_{m,\alpha}) \leq \frac{2^d\varepsilon}{\alpha}$$

provided that we have established for m large enough $-\infty < \underline{l}_m(\omega) \leq \overline{l}_m(\omega) < +\infty$ a.s., hence $-\infty < \underline{l}(\omega) \leq \overline{l}(\omega) < +\infty$ a.s..
Let $\varepsilon > 0$ and $m(\varepsilon) \in \mathbf{N}^*$ be such that, for $m \geq m(\varepsilon)$

$$\int_\Omega \frac{\mathcal{S}_{[0,m[^d}}{m^d} dP - \gamma(\mathcal{S}) \leq \varepsilon$$

(apply Theorem 2.1 to the $\mathbf{Z}^d$-invariant and subadditive set function $A \mapsto \int_\Omega \mathcal{S}_A dP$).
On the other hand, let $\mathcal{I}_m$ denote the family of finite unions of $m\mathbf{Z}^d$-translated of $[0,m[^d$ and consider the additive process $\mathcal{A}^m$ for the group $(\tau_z)_{z\in m\mathbf{Z}^d}$ defined in $\mathcal{I}_m$ by:

$$\mathcal{A}_I^m := \sum_{z\in I\cap m\mathbf{Z}^d} \mathcal{S}_{[0,m[^d} \circ \tau_z.$$

Substracting this process from the restriction of $\mathcal{S}$ to $\mathcal{I}_m$, we get a non positive and non increasing process $\mathcal{S}^m$ for the group $(\tau_z)_{z\in m\mathbf{Z}^d}$ defined on $\mathcal{I}_m$:

$$\mathcal{S}^m := \mathcal{S} - \mathcal{A}^m \leq O. \tag{4.16}$$

By additivity and covariance

$$\gamma^m(\mathcal{A}^m) = \int_\Omega \frac{\mathcal{S}_{[0,m[^d}}{m^d} dP$$

so that, for $m \geq m(\varepsilon)$

$$\gamma^m(\mathcal{S}^m) \geq -\varepsilon. \tag{4.17}$$

Moreover, according to the well known results related to additive processes recalled at the begining of the proof, ω almost surely

$$\begin{aligned} L_m(\omega) &:= \lim_{n\to+\infty} \frac{\mathcal{A}^m_{\overline{A}_{n,m}}(\omega)}{|\overline{A}_{n,m}|} \\ &= \lim_{n\to+\infty} \frac{\mathcal{A}^m_{\underline{A}_{n,m}}(\omega)}{|\underline{A}_{n,m}|}. \\ &= E^{\mathcal{F}_m} \frac{\mathcal{S}_{[0,m[^d}}{m^d}. \end{aligned}$$

Taking successively $(\overline{A}_{n,m})_{n\in\mathbf{N}}$ and $(\underline{A}_{n,m})_{n\in\mathbf{N}}$ in (4.16) and going to the limit on n, we obtain, as in (2.5), (2.6), ω almost surely,

$$\underline{l}_m(\omega) - L_m(\omega) \geq \inf_n \frac{\mathcal{S}^m_{\overline{I}_{n,m}}(\omega)}{|\overline{I}_{n,m}|} \tag{4.18}$$

$$\overline{l}_m(\omega) - L_m(\omega) \leq 0. \tag{4.19}$$

Inequality (4.18) implies

$$\{\omega : \underline{l}_m - L_m \leq -\alpha\} \subset E_\alpha.$$

By (4.17) and the *maximal inequality* (Lemma 4.1) applyed to the process $\mathcal{S}^m$ for the group $(\tau_z)_{z\in m\mathbf{Z}^d}$, for $m \geq m(\varepsilon)$ (note that $(\overline{I}_{n,m})_{n\in\mathbf{N}}$ is non decreasing and therefore is a regular sequence of intervals with constant 1) we get

$$P(\{\omega : \underline{l}_m - L_m \leq -\alpha\}) \leq \frac{2^d\varepsilon}{\alpha}. \tag{4.20}$$

The almost sure inequality $-\infty < \underline{l}_m$ follows after letting α tend to $+\infty$ and $\overline{l}_m < +\infty$ follows from (4.19). On the other hand (4.18) and (4.19) imply

$$\overline{l}_m(\omega) - \underline{l}_m(\omega) \leq -\inf_n \frac{\mathcal{S}^m_{\overline{I}_{n,m}}(\omega)}{|\overline{I}_{n,m}|}$$

so that $E_{m,\alpha} \subset E_\alpha$. Therefore, by the *maximal inequality* and (4.17), for $m \geq m(\varepsilon)$

$$\begin{aligned} P(\{\omega\in\Omega:\ \overline{l}(\omega) - \underline{l}(\omega) \geq \alpha\}) &\leq P(\{\omega\in\Omega:\ \overline{l}_m(\omega) - \underline{l}_m(\omega) \geq \alpha\}) \\ &\leq -\frac{2^d\gamma^m(\mathcal{S}^m)}{\alpha} \leq \frac{2^d\varepsilon}{\alpha}. \end{aligned}$$

As ε and α are arbitrary, the proof of this step is complete. Note that we have also proved : $\overline{l}_m = \underline{l}_m = l$ a.s..

Second step. We prove that l is almost surely invariant, that is $\forall z \in \mathbf{Z}^d$, $l(\omega) = l(\tau_z\omega)$ a.s..

From (4.19) and the invariance of L_m for $(\tau_z)_{z\in m\mathbf{Z}^d}$, we have

$$\begin{aligned} \{\omega:\ l(\tau_{mz}\omega) - l(\omega) > \alpha\} &= \{\omega:\ l(\tau_{mz}\omega) - L_m(\tau_{mz}\omega) + L_m(\omega) - l(\omega) > \alpha\} \\ &\subset \{\omega:\ L_m(\omega) - l(\omega) > \alpha\} \end{aligned}$$

thus, for $m \geq m(\varepsilon)$, by (4.20),

$$\begin{aligned} P(\{\omega :\ l(\tau_{mz}\omega) - l(\omega) > \alpha\}) &\leq P(\{\omega :\ L_m(\omega) - l(\omega) > \alpha\} \\ &\leq \frac{2^d \varepsilon}{\alpha} \end{aligned}$$

which yields

$$l(\tau_{mz}\omega) \leq l(\omega) \ a.s.. \tag{4.21}$$

¿From (4.21),

$$\begin{aligned} P(\{\omega :\ l(\tau_z\omega) - l(\omega) > \alpha\}) &= P(\{\omega :\ l(\tau_{mz}\omega) - l(\tau_{(m-1)z}\omega) > \alpha\} \\ &\leq P(\{\omega :\ l(\tau_{mz}\omega) - l(\omega) > \alpha\} \\ &\leq \tfrac{2^d \varepsilon}{\alpha} \end{aligned}$$

so that, going to the limit on ε

$$l(\tau_z\omega) \leq l(\omega) \ a.s..$$

On the other hand, noticing that $\underline{A}_{n,m} + z \subset \underline{A}_{n,m+|z|_1}$ where $|z|_1 = \max_{i=1,\dots,d} |z_i|$, we obtain

$$\frac{\mathcal{S}_{\underline{A}_{n,m}}(\tau_z\omega)}{|\underline{A}_{n,m}|} \geq \frac{\mathcal{S}_{\underline{A}_{n,m+|z|_1}}(\omega)}{|\underline{A}_{n,m+|z|_1}|} \frac{|\underline{A}_{n,m+|z|_1}|}{|\underline{A}_{n,m}|},$$

and finally, going to the limit on n,

$$l(\tau_z\omega \geq l(\omega) \ a.s. \tag{4.22}$$

Collecting (4.21) and (4.22), the proof of the second step is complete.

Last step. We identify l. Let us set for all $m \in \mathbf{N}^*$, $f_m(\omega) := E^{\mathcal{F}}(\frac{\mathcal{S}_{[0,m[^d}}{m^d})$. We first prove $l \leq \inf_{m \in \mathbf{N}^*} f_m$. Indeed, by (4.1), (4.6), for every $m \in \mathbf{N}^*$, $l \leq L_m = E^{\mathcal{F}_m} \frac{\mathcal{S}_{[0,m[^d}}{m^d}$ and, by invariance of l and with $\mathcal{F} \subset \mathcal{F}_m$,

$$\begin{aligned} l = E^{\mathcal{F}} l &\leq E^{\mathcal{F}} L_m \\ &= E^{\mathcal{F}}(E^{\mathcal{F}_m} \frac{\mathcal{S}_{[0,m[^d}}{m^d}) \\ &= E^{\mathcal{F}} \frac{\mathcal{S}_{[0,m[^d}}{m^d}. \end{aligned}$$

On the other hand, by (4.15), Fatou 's Lemma, (4.17), for every $E \in \mathcal{F}$ and $m \geq m(\varepsilon)$, up to a subsequence with respect to n

$$\begin{aligned}
\int_E (l - L_m) dP &\geq \int_E \lim_{n\to+\infty} \frac{\mathcal{S}^m_{\overline{A}_{n,m}}}{|\overline{A}_{n,m}|} dP \\
&\geq \int_E \lim_{n\to+\infty} \frac{\mathcal{S}^m_{\overline{I}_{n,m}}}{|\overline{I}_{n,m}|} dP \\
&\geq \limsup_{n\to+\infty} \int_E \frac{\mathcal{S}^m_{\overline{I}_{n,m}}}{|\overline{I}_{n,m}|} dP \\
&\geq \gamma^m(\mathcal{S}^m) \geq -\varepsilon.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_E l \; dP &\geq \int_E E^{\mathcal{F}_m}\left(\frac{\mathcal{S}_{[0,m[^d}}{m^d}\right) \, dP - C\varepsilon \\
&= \int_E E^{\mathcal{F}}\left(\frac{\mathcal{S}_{[0,m[^d}}{m^d}\right) \, dP - C\varepsilon \\
&\geq \int_E \inf_{m\in\mathbf{N}^*} f_m \; dP - C\varepsilon.
\end{aligned}$$

According to the previous inequality, letting $\varepsilon \to 0$

$$\forall E \in \mathcal{F}, \int_E l \; dP = \int_E \inf_{m\in\mathbf{N}^*} f_m \; dP.$$

As $\inf_{m\in\mathbf{N}^*} f_m$ is $\mathcal{F}$-measurable, we may conclude $l = E^{\mathcal{F}}(\inf_{m\in\mathbf{N}^*} f_m) = \inf_{m\in\mathbf{N}^*} f_m$ a.s. which completes the proof. □

Remark: The remark of section 2 about subadditivity reamins valid in this stochastic case.

4.2 The local and global-local theorems

With the notations of subsection 2.2, we consider a subadditive process for a group of P-preserving transformations $(T_x)_{x\in\mathbf{R}^q}$ on $(\Omega, \mathcal{T})$ defined in $\mathcal{P}(\mathbf{R}^q)$. More precisely, we consider a map $\mathcal{S} : \; \mathcal{P}(\mathbf{R}^q) \longrightarrow L^1(\Omega, \; \mathcal{T}, \; P)$ satisfying

i) $\forall I_1, \ldots, I_s$ disjoints sets in $\mathcal{P}(\mathbf{R}^q)$ such that $I = \bigcup_{i=1}^{s} I_i$ is in $\mathcal{P}(\mathbf{R}^q)$, $\mathcal{S}_I \leq \sum_{i=1}^{s} \mathcal{S}_{I_i}$,

ii) $\forall A \in \mathcal{P}(\mathbf{R}^q)$, $\forall x \in \mathbf{R}^q$, $\mathcal{S}_{x+A} = \mathcal{S}_A \circ T_x$.

M.A. Ackoglu and U. Krengel have proved in [3] the following local theorem which generalizes Theorem 2.2

Theorem 4.3: *Let $\mathcal{S}$ be a subadditive process for a group of P-preserving transformations $(T_x)_{x \in \mathbf{R}^q}$ and satisfying*

$$\delta := \sup\{\int_\Omega \frac{|\mathcal{S}_I|}{|I|}\, dP : I \in \mathcal{P}(\mathbf{R}^q),\ mes(I) \neq 0\} < +\infty.$$

then $\lim_{r \to +\infty} \frac{\mathcal{S}_{Q_r(x_0)}}{r^q}$ *exists almost surely.*

We now consider a subadditive process $\mathcal{S}$ indexed by a product of Borel sets. More precisely

$$\mathcal{S} :\ \mathcal{B}_b(\mathbf{R}^d) \times \mathcal{P}(\mathbf{R}^q) \longrightarrow L^1(\Omega, \mathcal{T}, P)$$

with

i) for every I in $\mathcal{P}(\mathbf{R}^q)$, $I \subset [0,1[^q$, $A \mapsto \mathcal{S}_{A \times I}$ is a subadditive process for $(\tau_z)_{z \in \mathbf{Z}^d}$ satisfying all the hypothesis of section 4.1 and Theorem 4.1 and where the function f in the domination property does not depend on I.

ii) for every A in $\mathcal{B}_b(\mathbf{R}^d)$, $A \subset [0,1[^d$, $I \mapsto \mathcal{S}_{A \times I}$ is a subadditive process for $(T_x)_{x \in \mathbf{R}^q}$ satisfying all the hypothesis of previous Theorem 4.3 and where the constant δ does not depend on A.

Then as a corollary of Theorems 4.1 and 4.2 we obtain the following global-local and local-global subadditive ergodic theorems, the proof of which being an easy extension of the proof of Theorem 2.2.

Theorem 4.4: *Let $\mathcal{S}$ be a subadditive process satisfying hypothesis i) and ii), $(A_n)_{n \in \mathbf{N}}$ be a regular sequence of Borel convex sets of $\mathcal{B}_b(\mathbf{R}^d)$ satisfying $\lim_{n \to +\infty} \rho(A_n) = +\infty$ and $(r_m)_{m \in \mathbf{N}}$ be a sequence of positive numbers* *tending to zero. Then almost surely the two following limits exist :*

$$\lim_{n\to+\infty}\lim_{m\to+\infty}\frac{\mathcal{S}_{A_n\times Q_{r_m}(x_0)}}{|A_n|r_m^q} \text{ and } \lim_{m\to+\infty}\lim_{n\to+\infty}\frac{\mathcal{S}_{A_n\times Q_{r_m}(x_0)}}{|A_n|r_m^q}.$$

If these two limits are equal, $\lim_{n\to+\infty}\frac{\mathcal{S}_{A_n\times Q_{r_n}(x_0)}}{|A_n|r_n^q}$ *exists and is equal to this common value.*

5 Parametric subadditive processes

In what follows, we assume that $\mathcal{T}$ is *P-complete* and that $(\tau_z)_{z\in\mathbf{Z}^d}$ is *ergodic.* This section is concerned with the variational property of the almost sure convergence studied in section 4, when the subadditive process depends on a parameter which belongs to a separable metric space. For convenience and in view to use some usual concepts of the calculus of variations, the process $\mathcal{S}$ will be assumed to be superadditive that is $-\mathcal{S}$ is subadditive.
More precisely, (X,d) being a given separable metric space, we consider the map

$$\mathcal{S}:\ \mathcal{B}_b(\mathbf{R}^d)\times X\longrightarrow L^1(\Omega,\ \mathcal{T},\ P),\ (A,x)\mapsto\mathcal{S}_A(x,.)$$

satisfying hypotheses :

(i) for every $x\in X$, $A\mapsto\mathcal{S}_A(x,.)$ is a superadditive process;

(ii) $\forall A\in\mathcal{B}_b(\mathbf{R}^d)$, $(x,\omega)\mapsto\mathcal{S}_A(x,\omega)$ is $\mathcal{B}(X)\otimes\mathcal{T}$ measurable;

(iii) $\forall A\in\mathcal{B}_b(\mathbf{R}^d)$, $\forall\omega\in\Omega$, $x\mapsto\mathcal{S}_A(x,\omega)$ is lower semicontinuous (lsc).

(iv) $\exists\alpha>0$, $\exists\beta>0$, $\exists x_0\in X$ such that $\forall A\in\mathcal{B}_b(\mathbf{R}^d)$, $\forall x\in X$,

$$\mathcal{S}_A(x,\omega)+(\alpha d(x,x_0)+\beta)|A|\geq 0.$$

In this context, under (i), (ii) and (iii) every set valued map $\omega\mapsto$ epi $\mathcal{S}_A(.,\omega)$, where epi $\mathcal{S}_A(.,\omega)$ denotes the epigraph of $x\mapsto\mathcal{S}_A(x,\omega)$, is a random set and with the terminology of R.T. Rockafellar [17] or H. Attouch-R J.B. Wets [5], every map $(x,\omega)\mapsto\mathcal{S}_A(x,\omega)$ is a *random lsc function.*

We recall that for $f,\ f_n: X\longrightarrow\overline{\mathbf{R}}$,

$$f=epilim f_n \iff epilimsup f_n\leq f\leq epiliminf f_n$$

where

$$epilimsup f_n(x) := \sup_{\varepsilon>0} \limsup_{n\to+\infty} \inf_{y\in B(x,\varepsilon)} f_n(y)$$

$$epiliminf f_n(x) := \sup_{\varepsilon>0} \liminf_{n\to+\infty} \inf_{y\in B(x,\varepsilon)} f_n(y),$$

$B(x,\varepsilon)$ denoting the open ball of X with radius ε and centered at x. We then say that f_n epiconverges to f. Let us also recall the following "variational property" of epiconvergence (cf Attouch [4]):

Proposition 5.1: *Assume that (f_n) epiconverges to f and let $x_n \in X$ be such that*

$$f_n(x_n) < \inf\{ f_n(y) : y \in X \} + \varepsilon_n,$$

assume furthermore that the set $\{x_n, : n \in \mathbf{N}\}$ is relatively compact, then any cluster point x of x_n is a minimizer of f and $\lim_{n\to+\infty} \inf\{ f_n(y) : y \in X \} = f(x)$.

For any $g : X \longrightarrow \overline{\mathbf{R}}$, and $k \in \mathbf{N}^*$, we define the *Baire approximate* of g by

$$g^k(x) := \inf_{y\in X}\{g(y) + kd(x,y)\}.$$

If g is lsc in X, non identically equal to $+\infty$ and satisfies :

$$\exists\alpha > 0,\ \exists\beta > 0\ \exists x_0 \in X \text{ such that } \forall x \in X,\ \ g(x) + \alpha d(x,x_0) + \beta \geq 0$$

then g^k is lipschitzian with Lipschitz constant k and $g = \sup_{k\in\mathbf{N}^*} g^k$.
Moreover, if the sequence $(f_n)_{n\in N}$ satisfies the above properties where the constants α, β and x_0 do not depend on n, we have :

$$epiliminf f_n = \sup_{k\in\mathbf{N}^*} \liminf_{n\to+\infty} f_n^k$$

$$epilimsup f_n = \sup_{k\in\mathbf{N}^*} \limsup_{n\to+\infty} f_n^k.$$

For more details see C. Hess [10]. For another approximation process see H. Attouch-R J.B. Wets [5] and for a complete study of epiconvergence see H. Attouch [4].

In these conditions we state in the theorem below that the almost sure convergence in Theorem 4.1 is variational in the sense of epiconvergence. When $\mathcal{S}$ is additive, we recover the law of large numbers for random lsc functions firstly established by H. Attouch-R.J.B. Wets [5]. For more details and an upper bound of the tail probabilities of the law, we refer the reader to G. Michaille-J. Michel-L. Piccinini [15].

Theorem 5.2: *If $\mathcal{S}$ satisfies $(i)-(iv)$ and $-\mathcal{S}$ satisfies the hypotheses of Theorem 4.1 including ergodicity for each fixed $x \in X$, we have ω-almost surely*

$$\begin{aligned} epilim_{n\to+\infty}\frac{\mathcal{S}_{A_n}}{|A_n|}(.,\omega) &= \sup_{m\in\mathbf{N}^*}\{\int_\Omega \frac{\mathcal{S}_{[0,m[^d}}{m^d}(.,\omega)dP(\omega)\} \\ &= \gamma(\mathcal{S}(.,.)). \end{aligned}$$

PROOF: $x \mapsto \alpha d(x,x_0)+\beta$ being a continuous perturbation of $x \mapsto \frac{\mathcal{S}_{A_n}}{|A_n|}(.,\omega)$ it suffices to prove our result for the non negative process $A \mapsto \mathcal{S}_A(x,.) + (\alpha d(x,x_0)+\beta)|A|$ (see H. Attouch [4] for stability properties of epiconvergence). We adopt the same notation for this new process.
First step. There exists $\Omega_1 \in \mathcal{T}$, $P(\Omega_1)=1$ such that $\forall\omega\in\Omega_1$

$$epiliminf\frac{\mathcal{S}_{A_n}}{|A_n|}(.,\omega) \geq \sup_{m\in\mathbf{N}^*}\{\int_\Omega \frac{\mathcal{S}_{[0,m[^d}}{m^d}(.,\omega)dP(\omega)\}.$$

It is easily seen that, for every fixed x, $A \mapsto -\inf_{y\in X}\{\mathcal{S}_A(y,.)+kd(x,y)|A|\}$ is a subadditive process satisfying all the hypothesis of Theorem 2.1 (the measurability comes from the measurability of $\omega \mapsto$ epi $\mathcal{S}_A(.,\omega)$, see H. Attouch-R J.B Wets [5] or C. Hess [11]). Therefore, D denoting a dense countable subset of X, there exists $\Omega_1 \in \mathcal{T}$, $P(\Omega_1)=1$ such that $\forall\omega\in\Omega_1$ and $\forall x\in D$

$$\begin{aligned} \lim_{n\to+\infty}\Big(\frac{\mathcal{S}_{A_n}}{|A_n|}(.,\omega)\Big)^k(x) &= \sup_{m\in\mathbf{N}^*}\int_\Omega \Big(\frac{\mathcal{S}_{[0,m[^d}}{m^d}(.,\omega)\Big)^k(x)dP(\omega) \\ &\geq \int_\Omega \Big(\frac{\mathcal{S}_{[0,m[^d}}{m^d}(.,\omega)\Big)^k(x)dP(\omega)\ \forall m\in\mathbf{N}^*. \end{aligned}$$

By equi-lipschitz property of the Baire approximations, the above inequality is satisfyed for every (ω,x) in $\Omega_1\times X$. Going to the limit on k, we obtain finally

$$epiliminf\frac{\mathcal{S}_{A_n}}{|A_n|}(.,\omega) \geq \sup_{m\in\mathbf{N}^*}\{\int_\Omega \frac{\mathcal{S}_{[0,m[^d}}{m^d}(.,\omega)dP(\omega)\}\ \forall\omega\in\Omega_1.$$

Second step. There exists $\Omega_2 \in \mathcal{T}$, $P(\Omega_2) = 1$ such that $\forall \omega \in \Omega_2$

$$epilimsup \frac{\mathcal{S}_{A_n}}{|A_n|}(.,\omega) \leq \sup_{m\in\mathbf{N}^*} \{ \int_\Omega \frac{\mathcal{S}_{[0,m[^d}}{m^d}(.,\omega)dP(\omega)\}.$$

For every $\varepsilon > 0,\ x \in X$,

$$\inf_{y\in B(x,\varepsilon)} \frac{\mathcal{S}_{A_n}}{|A_n|}(y,\omega) \leq \frac{\mathcal{S}_{A_n}}{|A_n|}(x,\omega).$$

According to Theorem 2.1, there exists $\Omega_x \in \mathcal{T},\ P(\Omega_x) = 1$ such that $\forall \omega \in \Omega_x$

$$\limsup_{n\to+\infty} \inf_{y\in B(x,\varepsilon)} \frac{\mathcal{S}_{A_n}}{|A_n|}(y,\omega) \leq \sup_{m\in\mathbf{N}^*} \{ \int_\Omega \frac{\mathcal{S}_{[0,m[^d}}{m^d}(x,\omega)dP(\omega)\}.$$

We conclude by an argument used in H. Attouch-R J.B. Wets [5], lemma 2.5. Indeed, let $\mathcal{D}$ be a dense countable subset of the epigraph of

$$\Phi : x \mapsto \sup_{m\in\mathbf{N}^*} \{ \int_\Omega \frac{\mathcal{S}_{[0,m[^d}}{m^d}(x,\omega)dP(\omega)\},$$

let $\Pi_X D$ its projection on X and set $\Omega_2 := \cap_{x\in\Pi_X D}\Omega_x$. Taking the supremum on ε in above inequality, we deduce that $\forall\omega \in \Omega_2$, $\{(x,r) \in \mathcal{D}\ :\ \Phi(x) \leq r\}$ is a subset of the epigraph of $x \mapsto epilimsup\frac{\mathcal{S}_{A_n}}{|A_n|}(x,\omega)$ which is closed. Taking closures of both sides yields the desired result. □

6 Some applications to Homogenization

Let us first recall the probabilistic setting related to general stochastic homogenization. Let $M^{m\times N}$ be the space of $m\times N$ matrices. We now consider a probability space $(\Omega,\ \mathcal{T},\ P)$ and the set $\mathcal{G}$ of all functions g from $\mathbf{R}^d \times M^{m\times N}$ into $\mathbf{R}$, measurable with respect to the first variable, and such that there exists three positive constant α, β and L with, for every a, b in $M^{m\times N}$ and x *a.e.* in $\mathbf{R}^d$

$$\alpha(|a|^p - 1) \leq g(x,a) \leq \beta(1 + |a|^p)$$

$$|g(x,a) - g(x,b)| \leq L(1 + |a|^{p-1} + |b|^{p-1})|a - b| \tag{6.23}$$

where $1 \leq p < +\infty$. We equip $\mathcal{G}$ with the trace σ-field $\sigma(\mathcal{G})$ of the product σ-field of $\mathbf{R}^{\mathbf{R}^d \times M^{m\times N}}$ and define the group of transformation $(\tau_z)_{z\in\mathbf{Z}^d}$ in $\mathcal{G}$, by $\tau_z g(x,a) = g(x+z,a)$.

We finally consider a map f from $\Omega \times \mathbf{R}^d \times M^{m\times N}$ into $\mathbf{R}$, which is $\mathcal{T} \otimes \mathcal{B}(\mathbf{R}^d) \otimes \mathcal{B}(M^{m\times d})$ measurable and such that, for every ω in Ω, $f(\omega,.,.)$ belongs to $\mathcal{G}$. In the sequel, to shorten the notations, f will also denote the partial map $\omega \mapsto f(\omega,.,.)$ from Ω into $\mathcal{G}$.

It is clear that the maps $\tau_z f$ from Ω into $\mathcal{G}$ are $(\mathcal{T}, \sigma(\mathcal{G}))$ measurable. The process f is said to be *stationary* if, for every z in $\mathbf{Z}^d$, $P \circ f^{-1} = P \circ (\tau_z f)^{-1}$, and is said to be *ergodic* if $P \circ f^{-1}(E) \in \{0,1\}$ for every E in $\sigma(\mathcal{G})$ such that, $\tau_z(E) = E$ for every z in $\mathbf{Z}^d$.
The two following sufficient conditions ensure the *stationarity and ergodicity* of f (see G. Dal Maso-L.Modica [14])

(ST) If, for all finite families $(x_i,a_i)_{i\in I}$ of $\mathbf{R}^d \times M^{m\times N}$, the random vectors $(f(.,x_i,a_i))_{i\in I}$ and $(f(.,x_i+z,a_i))_{i\in I}$ have the same law for every z in $\mathbf{Z}^d$, then f is stationary.

(ER) If, for all finite families $(x_i,a_i,r_i)_{i\in I}$ and $(y_j,b_j,s_j)_{j\in J}$ of $\mathbf{R}^d \times M^{m\times N} \times \mathbf{R}$

$$\begin{aligned}\lim_{|z|\to+\infty:\, z\in\mathbf{Z}^d} & P([f(.,x_i+z,a_i) > r_i] \cap [f(.,y_j,b_j) > s_j]) \\ &= P([f(.,x_i,a_i) > r_i])P([f(.,y_j,b_j) > s_j])\end{aligned}$$

then f is ergodic.

In the context of integral functionals and when $d = N$, applying Theorem 4.1 to the following subadditive process

$$\mathcal{S}_A(g,a) := \inf\{\int_{\overset{0}{A}} g(x, a+\nabla u)dx : \ u \in W_0^{1,p}(\overset{0}{A}, \mathbf{R}^N)\}$$

defined in the probability space $(\mathcal{G},\ \Omega(\mathcal{G})P \circ f^{-1})$ image of $(\Omega,\ \mathcal{T},\ P)$ by a given stationary process f, we obtain

$$\lim_{n\to+\infty} \frac{\mathcal{S}_{\frac{1}{\varepsilon_n}A}(f(\omega),a)}{|\frac{1}{\varepsilon_n}A|} = \inf_{m\in\mathbf{N}^*} E^{\mathcal{F}} \frac{\mathcal{S}_{[0,m[^d}(f(.),a)}{m^d}(\omega)$$

where $E^{\mathcal{F}}$ denotes the conditional expectation operator with respect to the σ-field of all the events $E \in \mathcal{T}$ satisfying $\tau_z f(E) = f(E)$, $\forall z \in \mathbf{Z}^d$. This limit is the density (or its regular part when $p = 1$) of the almost sure limit in the sense of epiconvergence for the strong topology of $L^p(\mathcal{O}, \mathbf{R}^m)$ of the following sequence indexed by ε_n and defined in $L^p(\mathcal{O}, \mathbf{R}^m)$

$$F_{\varepsilon_n}(\omega, u) = \begin{cases} \displaystyle\int_{\mathcal{O}} f(\omega, \frac{x}{\varepsilon_n}, \nabla u)\, dx \text{ if } u \in W^{1,p}(\mathcal{O}, \mathbf{R}^m) \\ +\infty \text{ otherwise} \end{cases}$$

when ε_n tends to zero.

More details can be found in G. Dal Maso-L. Modica [14] when $p > 1$, f ergodic, $f(\omega, x, .)$ convex and in Y. Abddaimi-C. Licht-G. Michaille [2] when $p = 1$ and f is assumed to be stationary only. It is precisely about this last extension that some encountered technical difficulties ([2], pp. 195-199) motivate us to generalize subadditive theorems to sequences indexed by convex sets. We now give two new applications respectively using Theorem 5, 4.1 and 2.2.

6.1 Application to optimization of integral functionals in stochastic homogenization

Let (X, d) be a separable metric space. According to the probabilistic setting stated above, we consider a map f from $X \times \Omega \times \mathbf{R}^d \times M^{m\times d}$ into $\mathbf{R}$ which is $\mathcal{B}(X) \otimes \mathcal{T} \otimes \mathcal{B}(\mathbf{R}^d) \otimes \mathcal{B}(M^{m\times d})$ measurable and which fullfils, for every fixed θ in X conditions (i) and (ii) below.

(i) $\omega \mapsto f(\theta, \omega, ., .)$ is a stationary and ergodic process.

Let now $1 \leq p < +\infty$, a be a fixed matrix in $M^{m\times d}$ and b a fixed element of $\mathbf{R}^m$. For every $\theta \in X$, $\omega \in \Omega$ and every open bounded subset A of $\mathbf{R}^d$, we define the functional $F^\theta(., \omega, A)$ in $L^p(A, \mathbf{R}^m)$ equipped with its strong topology, by

$$F^\theta(u, \omega, A) = \begin{cases} \displaystyle\int_A f(\theta, \omega, x, \nabla u)\, dx \text{ if } u \in l_a + W_0^{1,p}(A, \mathbf{R}^m) \\ +\infty \text{ otherwise} \end{cases}$$

where l_a is the function defined by $l_a(x) = a.x + b$. We assume

(ii) if $\theta_n \to \theta$ in (X, d), the sequence $(F^{\theta_n}(., \omega, A))_{n\in\mathbf{N}}$ epiconverges ω-a.s. to the lower semicontinuous envelope $\overline{F^\theta}(., \omega, A)$ of $F^\theta(., \omega, A)$.

Condition (ii) is satisfyed for instance if $(F^{\theta_n}(.,\omega,A))_{n\in\mathbf{N}}$ is non increasing. Note also, that when $p>1$, $\overline{F^\theta}(.,\omega,A)=F^\theta(.,\omega,A)$.

We now define the random infimum

$$I_\varepsilon(\omega,\theta)=\inf\{\int_{\mathcal{O}} f(\theta,\omega,\frac{x}{\varepsilon},\nabla u)\;dx : u\in l_a+W^{1,p}(\mathcal{O},\mathbf{R}^m)\}$$

where $\mathcal{O}$ is a given open bounded subset of $\mathbf{R}^d$. Note that

$$I_\varepsilon(\omega,\theta)=\text{meas}(\mathcal{O})\frac{\inf_{u\in L^p(\mathcal{O},\mathbf{R}^m)} F^\theta(u,\omega,\frac{1}{\varepsilon}\mathcal{O})}{\text{meas}(\frac{1}{\varepsilon}\mathcal{O})}.$$

We finally define the process

$$\mathcal{S}:\mathcal{B}_b(\mathbf{R}^d)\times X\to L^1(\Omega,\mathcal{T},P)$$

by

$$\mathcal{S}_A(\theta,\omega)=\inf_{u\in L^p(\overset{0}{A},\mathbf{R}^m)} F^\theta(u,\omega,\overset{0}{A}).$$

Then, according to conditions fulfiled by the process $\omega\mapsto f(\theta,\omega,.,.)$, and to hypotheses (i), (ii), $\mathcal{S}$ is a parametrized subadditive process. Thanks to (ii), we actually have continuity of $\theta\mapsto\mathcal{S}_A(\theta,\omega)$. Applying Theorem 5.2, we deduce that ω a.s., $-I_\varepsilon(\omega,.)$ epiconverges to $-I=-\text{meas}(\mathcal{O})\inf_{n\in\mathbf{N}^*}E\frac{\mathcal{S}_{[0,n[^d}(.,.)}{n^d}$. Therefore, if the set $\{\theta_\varepsilon(\omega):\varepsilon>0\}$ of ε-minimizers of $I_\varepsilon(\omega,.)$ is relatively compact in X, we have

$$\lim_{\varepsilon\to 0} I_\varepsilon(\omega,\theta)=\sup_{\theta\in X} I(\theta).$$

So, roughly speaking, for maximizing the random energy $I_\varepsilon(\omega,.)$ with respect to a (physical) parameter θ, it suffices, for the small values of ε, to maximize the deterministic homogenized energy I.

6.2 Application to the continuity of an homogenized density with respect to a geometrical parameter

Let us onsider $D_i\subset\subset]0,1[^2$, $i=1,2$ and $\Lambda=\{D_1,D_2\}$ equipped with the probability presence p_1 and p_2 of D_1 and D_2. We set $\Omega=\Lambda^{\mathbf{Z}^2}$, define the classical Bernoulli product probability space $(\Omega, \mathcal{T}, P)$ and the random chessboard $D(\omega) = \cup_{z\in\mathbf{Z}^2}(\omega_z + z\}$ in $\mathbf{R}^2$.

Let now $f : \mathbf{R}^2 \to \mathbf{R}$ be a given function satisfying the growth conditions 6.23 and a, b two numbers in $\mathbf{R}$. We denote by $\mathcal{I}(\mathbf{R}^2)$ the set of all the intervals of $\mathbf{R}^2$ of the form $[a, b[$, a, $b \in \mathbf{Z}^2$ and define a parametrized subadditive process

$$\mathcal{S} : \mathcal{I}(\mathbf{R}^2) \times [0, \delta_0] \to L^1(\Omega, \mathcal{T}, P)$$

by

$$\mathcal{S}_A(\delta, \omega) = \inf\{\int_A f(Dv)\ dx : \frac{1}{\text{meas}(A)} \int_A v\ dx = a,\ v = b \text{ in } A \cap D_\delta(\omega)\}$$

where $D_\delta(\omega) = \cup_{z\in\mathbf{Z}^2}(h_\delta\omega_z + z)$, $h_\delta\omega_z = \{x \in \omega_z : d(x, \mathbf{R}^2 \setminus \omega_z) > \delta\}$. It is easily seen that this process satisfies all conditions of Theorem 5.2. We would like to establish the continuity at $\delta = 0$ of the almost sure limit

$$L(\delta) = \lim_{n\to+\infty} \frac{\mathcal{S}_{[0,n[^2}}{n^2}(\delta, \omega).$$

We proceed as follows : ω a.s.

$$\begin{aligned}
\lim_{\delta\to 0}\lim_{n\to+\infty} \frac{\mathcal{S}_{[0,n[^2}}{n^2}(\delta, \omega) &= \sup_{\delta\in[0,\delta_0]}\lim_{n\to+\infty} \frac{\mathcal{S}_{[0,n[^2}}{n^2}(\delta, \omega) \\
&= \lim_{n\to+\infty}\sup_{\delta\in[0,\delta_0]} \frac{\mathcal{S}_{[0,n[^2}}{n^2}(\delta, \omega) \\
&= L(0)
\end{aligned}$$

where we have used Theorem 5, for processes restricted to $\mathcal{I}(\mathbf{R}^2)$, in the second equality. In the deterministic case, the limit $L(0)$ forms part of the definition of a non local homogenized problem studyed in M. Bellieud-G. Bouchitté [6]. In our case, above result is an essential tool for describing this problem in a probabilistic setting.

6.3 Application to a modeling of elastic adhesive bonded joints

Here, we extend or give more direct proofs of some results of [12], [13] to where we refer for a detailled presentation of the problem (see also [1]). This problems devoted to the modelling of elastic adhesive bonded joints.

Let $\mathcal{O}$ be a domain with lipschitz boundary in $\mathbf{R}^3$ whose intersection S with the plane $x_3 = 0$ is assumed to have a positive two dimensional Hausdorff measure $H_2(S)$. In the sequel $x = (\tilde{x}, x_3)$ denotes a current point of $\mathbf{R}^3$. If ε is a small positive parameter intended to tend to zero, $B_\varepsilon := \{x \in \mathcal{O} : \pm x_3 \leq \varepsilon\}$ (respectively $\mathcal{O}_\varepsilon := \mathcal{O} \setminus \overline{B}_\varepsilon$) denotes the interior of the part of the reference configuration filled by the adhesive (respectively by the adherents). The adhesive and the adherents are assumed to be perfectly stuck together along $S_\varepsilon^\pm := \{x \in \mathcal{O} : \pm x_3 = \varepsilon\}$. They are modeled as hyperelastic. The small positive parameters μ and λ are associated respectively with the low stiffness and the size of heterogenities of the adhesive. We will denote by s the 3-uplet $(\mu, \varepsilon, \lambda)$, and s tends to zero means that there exists a sequence $((\mu_n, \varepsilon_n, \lambda_n))_n$ going to $(0,0,0)$. Moreover, we assume that $\lim_{s\to 0} \frac{\mu}{2\varepsilon} = l$ with $l \in [0, +\infty[$. The stored strain energy associated with a displacement field v is then given by the following functional where ω denotes a random parameter

$$F_s(\omega)(v) := \int_{\mathcal{O}_\varepsilon} h(x, \nabla v(x))dx + \mu \int_{B_\varepsilon} b(\omega)(\frac{\tilde{x}}{\lambda}, \nabla v(x))dx.$$

The structure made of the elastic bodies and the adhesive is clamped on a part Γ_0 of $\partial\mathcal{O}$ with $H_2(\Gamma_0) > 0$, and is subjected to applied body forces f and applied surface forces g on $\Gamma_1 := \partial\mathcal{O} \setminus \Gamma_0$. We shall make precisely the following assumptions on the exterior loading and B_ε :

(H_1) $(f, g) \in L^2(\mathcal{O}, \mathbf{R}^3) \times L^2(\Gamma_1, \mathbf{R}^3)$ and there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$, $B_\varepsilon = S \times (-\varepsilon, +\varepsilon)$ and $(\mathrm{supp}(f) \cup \Gamma_1) \cap B_\varepsilon = \emptyset$.

If we define L by

$$L(v) := \int_{\mathcal{O}} f(x).v(x)\ dx + \int_{\Gamma_1} g(x).v(x)da,$$

equilibrium configurations of the structure are given by the displacement fields $\overline{u}_s$, solutions of the problem

$$\min\{F_s(v) - L(v)\}$$

where the minimum is taken over the space

$$V = \{v \in W^{1,2}(\mathcal{O}, \mathbf{R}^3) : v = 0 \text{ on } \Gamma_0\}.$$

We study the behavior of $\overline{u}_s$ when s tends to zero. Due to the small stiffness in the layer B_ε, the limit displacement field $\overline{u}_s$ can at the limit develop discontinuities along S to which B_ε shrinks, and converges in $L^2(\mathcal{O}, \mathbf{R}^3)$ to a solution of the limit problem :

$$\min\{\int_{\mathcal{O}} \mathcal{Q}h(x, \nabla v(x))dx + l \int_S (b^{\infty,2})^{hom}([v](x) \otimes e_3)\; dH_2 - L(v)\}$$

$\mathcal{Q}h$ is the quasiconvex envelope of h, $(b^{\infty,2})^{hom}$ is the density of the surface energy defined below and $[v]$ is the jump of the displacement field v through S. Actually, arguing as in [13], it suffices to exhibit the almost sure epilimit of F_s.

The limit problem describes the equilibrium of deformable bodies filling the closure of $\mathcal{O}^{\pm} = \mathcal{O} \cap \{\pm x_3 > 0\}$ as reference configurations, made of hyperelastic materials with energy density $\mathcal{Q}h$, subjected to the loading (f, g), clamped on Γ_0 and constrained along S to which B_ε shrinks.

The density b is assumed to be a stationary and ergodic process, that is satisfies (ST) and (ER) with $d = 2$ and $m = N = 3$, with, more precisely, value in the class $\mathcal{F}$ of bulk energy densities satisfying the two uniform conditions

$$(H_2) \begin{cases} \exists\ \alpha, \beta,\ C \in \mathbf{R}^+ \text{ such that } \tilde{x} \text{ a.e. in } \mathbf{R}^2 \text{and } \forall (Q, Q') \in M^{3\times 3} \times M^{3\times 3} \\ \alpha |Q|^2 \le b(\tilde{x}, Q) \le \beta(1 + |Q|^2) \\ |b(\tilde{x}, Q) - b(\tilde{x}, Q')| \le C|Q - Q'|(1 + |Q| + |Q'|) \end{cases}$$

and the following behavior at infinity

(H_3) There exist $b^{\infty,2}$, C', $0 < m < 2$ such that $Q \mapsto b^{\infty,2}(\tilde{x}, Q)$ is positively homogeneous of degree 2 and

$$|b^{\infty,2}(\tilde{x}, Q) - b(\tilde{x}, Q)| \le C'(1 + |Q|^{2-m})\ \forall (\tilde{x}, Q) \in \mathbf{R}^2 \times M^{3\times 3}.$$

It is easily seen that the process $\omega \mapsto b^{\infty,2}$ also satisfies (ST) and (ER). Moreover we assume that the deterministic density h satisfies (H_2). In the sequel, to shorten notations, we omit the random variable ω. In order to work in a fixed space, we extend F_s by $+\infty$ in $L^2(\mathcal{O}, \mathbf{R}^3) \setminus V$ and we define the limit energy by

$$F(v) := \begin{cases} \int_{\mathcal{O}} \mathcal{Q}h(x, \nabla v(x))dx + l \int_S (b^{\infty,2})^{hom}([v](x) \otimes e_3)\; dH_2 \text{ if } v \in V_{[]} \\ +\infty \text{ if not,} \end{cases}$$

where

$$V_{[\,]} := \{v \in L^2(\mathcal{O}, \mathbf{R}^3) \ : \ v \in W^{1,2}(\mathcal{O} \setminus S, \mathbf{R}^3), v = 0 \ on \ \Gamma_0\}$$

and where $(b^{\infty,2})^{hom}$ is defined in Theorems 6.1, 6.4 below and depends on the relative behavior of λ and ε.

The limit problem is defined in term of epiconvergence in the space $L^2(\mathcal{O}, \mathbf{R}^3)$ equipped with its strong topology. More precisely we want to prove that, almost surely, $F = \text{epi}\lim_{s\to 0} F_s$, that is, the sequence of random functions $(F_s)_s$ fulfils the two following conditions for every ω in a set Ω' of full probability and every u in $L^2(\mathcal{O}, \mathbf{R}^3)$:

(E_1) for every u_s converging to u $F(u) \leq \liminf_{s\to} F_s(u_s)$,

(E_2) there exists v_s in $L^2(\mathcal{O}, \mathbf{R}^3)$ converging to u in $L^2(\mathcal{O}, \mathbf{R}^3)$ such that $F(u) \geq \limsup_{s\to 0} F_s(u_s)$.

In [12], [13], the cases $\lim\limits_{s\to 0} \dfrac{\varepsilon}{\lambda} \in]0, +\infty]$ were studied. We give a new and more direct proof for the case $\lim\limits_{s\to 0} \dfrac{\varepsilon}{\lambda} = +\infty$ and we complete the study to the case $\lim\limits_{s\to 0} \dfrac{\varepsilon}{\lambda} = 0$.

6.1.1 Case $\lambda << \varepsilon$ ($\lim\limits_{s\to 0} \dfrac{\varepsilon}{\lambda} = +\infty$).

Theorem 6.1: *Almost surely F_s epi-converges to F where for every Q in $M^{3\times 3}$*

$$(b^{\infty,2})^{hom}(a) := \inf_{k\in\mathbf{N}^*} \frac{1}{k^3} \int_\Omega \inf\{\int_{kY} b^{\infty,2}(\tilde{y}, Q+\nabla\varphi(y))dy : \varphi \in W_0^{1,2}(kY, \mathbf{R}^3)\}\ dP.$$

PROOF: The proof is divided in three steps.

First step. We prove the lower bound (E_1) for regular elements u of $V_{[\,]}$, that is for every element of the space $\tilde{V}_{[\,]}$ of all the functions u whose restrictions $u^\pm$ to $\mathcal{O}^\pm$ are the restrictions to $\mathcal{O}^\pm$ of $\mathcal{C}^\infty(\overline{\mathcal{O}}^\pm, \mathbf{R}^3)$ – functions.

It suffices to assume $\liminf\limits_{s\to} F_s(u_s) < +\infty$. Therefore, for a subsequence not relabelled, the bounded Borel measure

$$\nu_s := \chi_{\mathcal{O}_\varepsilon} h(., \nabla u_s)\ dx + \mu\chi_{B_\varepsilon} b(\frac{\tilde{x}}{\lambda}, \nabla u_s)\ dx$$

tends weakly to a bounded Borel measure ν. Our method consists in analyzing the limit measure ν. More precisely, if $\nu = \nu^a + \nu^{sing}$ where ν^a is absolutely continuous with respect to the Lebesgue measure on $\mathcal{O}$ and ν^{sing} is the singular part of ν, we prove

$$\begin{aligned}
&\mu^a \geq Qh(.,\nabla u)\ dx \\
&\mu^{sing} \geq l\ (b^{\infty,2})^{hom\ \infty}([u] \otimes e_3) H_2 \lfloor S.
\end{aligned}$$

For proving the first inequality, by the differentiation of measures, it suffices to establish for almost all x_0 in $\mathcal{O}$

$$\lim_{\rho \to 0} \frac{\nu(B_\rho(x_0))}{\text{meas}(B_\rho(x_0))} \geq Qh(x_0, \nabla u(x_0))$$

where $B_\rho(x_0)$ denotes the open ball of $\mathbf{R}^3$ with radius ρ and centered at x_0. Let x_0 be fixed in $\mathcal{O} \setminus S$ and $\rho < d(x_0, S)$. According to the Alexandrov theorem, for $\rho \in]0, d(x_0, S)[\setminus N$ where N is a countable set

$$\begin{aligned}
\lim_{\rho \to 0} \frac{\nu(B_\rho(x_0))}{\text{meas}(B_\rho(x_0))} &= \lim_{\rho \to 0} \lim_{s \to 0} \frac{\nu_s(B_\rho(x_0))}{\text{meas}(B_\rho(x_0))} \\
&= \lim_{\rho \to 0} \lim_{s \to 0} \frac{1}{meas(B_\rho(x_0))} \int_{B_\rho(x_0)} h(x, \nabla u_s)\ dx.
\end{aligned}$$

But by coercivity of the quasiconvexification Qh and by weak lower semicontinuity of the integral functional $v \mapsto \displaystyle\int_{B_\rho(x_0)} Qh(x, \nabla v)\ dx$ in $W^{1,2}(B_\rho(x_0), \mathbf{R}^3)$, we have

$$\lim_{s \to 0} \int_{B_\rho(x_0)} h(x, \nabla u_s)\ dx \geq \int_{B_\rho(x_0)} Qh(x, \nabla u)\ dx.$$

Therefore

$$\begin{aligned}
\lim_{\rho \to 0} \frac{\nu(B_\rho(x_0))}{\text{meas}(B_\rho(x_0))} &\geq \lim_{\rho \to 0} \frac{1}{meas(B_\rho(x_0))} \int_{B_\rho(x_0)} Qh(x, \nabla u)\ dx \\
&= Qh(x_0, \nabla u(x_0))
\end{aligned}$$

for almost all x_0 in $\mathcal{O}$.

We now prove the lower bound for μ^{sing}. Denoting by $C_\rho(x_0)$ the cylinder $S_\rho(x_0) \times]-\rho, \rho[$ where $S_\rho(x_0)$ is the open ball of $\mathbf{R}^2$ with radius ρ and centered at x_0 on S, it suffices to establish for H_2 almost all x_0 in S

$$\lim_{\rho \to 0} \frac{\nu(C_\rho(x_0))}{H_2(S_\rho(x_0))} \geq (b^{\infty,2})^{hom\ \infty}([u](x_0) \otimes e_3)$$

As in the proof of the first bound

$$\begin{aligned} \frac{\nu(C_\rho(x_0))}{H_2(S_\rho(x_0))} &= \lim_{s \to 0} \frac{\nu_s(C_{\rho(x_0)})}{H_2(S_\rho(x_0))} \\ &= \lim_{s \to 0} \frac{\mu}{H_2(S_\rho(x_0))} \int_{S_{\rho(x_0)} \times]-\varepsilon,\varepsilon[} b(\frac{\tilde{x}}{\lambda}, \nabla u_s)\, dx. \qquad (6.24) \end{aligned}$$

Thanks to (H_1), the elements u of V or $V_{[\ ]}$ can be extended by zero in $\mathbf{R}^2 \times]-\varepsilon, +\varepsilon[$. We will use the same notation u for such an extension. With regard to the strain energy of the adhesive, the smoothing operator $u \in V_{[\ ]} \mapsto R_\varepsilon u \in V$ defined by

$$R_\varepsilon u(x) := \begin{cases} \dfrac{u(\tilde{x}, |x_3|) - u(\tilde{x}, -|x_3|)}{2} \Psi_\varepsilon(x) + \dfrac{u(\tilde{x}, |x_3|) + u(\tilde{x}, -|x_3|)}{2} & \text{if } l < +\infty \\ u(x) \text{ if not} \end{cases}$$

where $\Psi_\varepsilon(x) := \text{sign}(x_3)\ min(\frac{|x_3|}{\varepsilon}, 1)$, allows us to replace ∇u_s by

$$\frac{1}{2\varepsilon}(u(\tilde{x}, |x_3|) - u(\tilde{x}, -|x_3|)) \otimes e_3 + \nabla(u_s - R_\varepsilon u)$$

and finally by

$$\frac{1}{2\varepsilon}[u](x_0) \otimes e_3 + \nabla(u_s - R_\varepsilon u)$$

Indeed by the lipschitz property of b

$$\begin{aligned} &\lim_{\rho \to 0} \lim_{s \to 0} \frac{\mu}{H_2(S_\rho(x_0))} \int_{S_\rho(x_0) \times]-\varepsilon,\varepsilon[} b(\frac{\tilde{x}}{\lambda}, \nabla u_s)\, dx \qquad (6.25) \\ = &\lim_{\rho \to 0} \lim_{s \to 0} \frac{\mu}{H_2(S_\rho(x_0))} \int_{S_\rho(x_0) \times]-\varepsilon,\varepsilon[} b(\frac{\tilde{x}}{\lambda}, \nabla R_\varepsilon u + \nabla(u_s - R_\varepsilon u))\, dx \\ = &\lim_{\rho \to 0} \lim_{s \to 0} \frac{\mu}{H_2(S_\rho(x_0))} \int_{S_\rho(x_0) \times]-\varepsilon,\varepsilon[} b(\frac{\tilde{x}}{\lambda}, \frac{1}{2\varepsilon}[u](x_0) \otimes e_3 + \nabla(u_s - R_\varepsilon u))\, dx. \end{aligned}$$

But by a De Giorgi trick (see L. Modica-G. Dal Maso [14] and C. Licht-G. Michaille [12], [13]), one can modify $v_s := u_s - R_\varepsilon u$ in the boundary of $S_\rho(x_0)\times]-\varepsilon,\varepsilon[$ by a function $w_s \in W_0^{1,2}(S_{\rho(x_0)}\times]-t(\varepsilon),t(\varepsilon)[,\mathbf{R}^3)$ where $\lim \frac{t(\varepsilon)}{\varepsilon} = 1$ so that

$$\lim_{\rho\to 0}\lim_{s\to 0} \frac{\mu}{H_2(S_\rho(x_0))} \int_{S_\rho(x_0)\times]-t(\varepsilon),t(\varepsilon)[} b(\frac{\tilde{x}}{\lambda}, \frac{1}{2\varepsilon}[u](x_0)\otimes e_3 + \nabla v_s)\, dx \quad (6.26)$$

$$\geq \limsup_{\rho\to 0}\limsup_{s\to 0} \frac{\mu}{H_2(S_\rho(x_0))} \int_{S_{\rho(x_0)}\times]-t(\varepsilon),t(\varepsilon)[} b(\frac{\tilde{x}}{\lambda}, \frac{1}{2\varepsilon}[u](x_0)\otimes e_3 + \nabla w_s)\, dx.$$

Recalling (6.24), (6.25), (6.26), according to (H_3), and after a change of scale, we obtain

$$\lim_{\rho\to 0} \frac{\nu(C_\rho(x_0))}{H_2(S_\rho(x_0))}$$

$$\geq \limsup_{\rho\to 0}\limsup_{s\to 0} \frac{\mu}{H_2(S_\rho(x_0))} \int_{S_\rho(x_0)\times]-t(\varepsilon),t(\varepsilon)[} b(\frac{\tilde{x}}{\lambda}, \frac{1}{2\varepsilon}[u](x_0)\otimes e_3 + \nabla w_s)\, dx$$

$$\geq l \limsup_{\rho\to 0}\limsup_{s\to 0} \frac{1}{\mathrm{meas}(A_s)} \inf\{\int_{A_s} b^{\infty,2}(\tilde{x}, [u](x_0)\otimes e_3 + \nabla\varphi)\, dx : \varphi \in W_0^{1,2}(A_s,\mathbf{R}^3)\}$$

where $A_s := \frac{1}{\lambda} S_{\delta\rho}(x_0)\times]-\frac{t(\varepsilon)}{\lambda}, \frac{t(\varepsilon)}{\lambda}[$. By (H_2), (H_3), (ST) and (ER), the subadditive process

$$A \mapsto \mathcal{S}_A := \inf\{\int_{\overset{0}{A}} b^{\infty,2}(\tilde{x}, [u](x_0)\otimes e_3 + \nabla\varphi)\, dx : \varphi \in W_0^{1,2}(\overset{0}{A},\mathbf{R}^3)\}.$$

satisfies all the conditions of the global Theorem 4.1. Thus we finally obtain

$$\lim_{\rho\to 0} \frac{\nu(C_\rho(x_0))}{H_2(S_\rho(x_0))} \geq l(b^{\infty,2})^{hom}([u](x_0)\otimes e_3)$$

for every $\omega \in \Omega'$ with $P(\Omega') = 1$.

Second step. We prove (E_2) for every $u \in \tilde{V}_{[\,]}$, that is : there exists v_s converging to u in $L^2(\mathcal{O},\mathbf{R}^3)$ such that $F(u) \geq \limsup_{s\to 0} F_s(u_s)$.

Let $(S_I)_{i\in I(\eta)}$ be a family of disjoint cubes in $\mathbf{R}^2$ with size η such that $H_2(S \setminus \bigcup_{i\in I(\eta)} S_i) = 0$. We have

$$l\int_S (b^{\infty,2})^{hom}([u](x)\otimes e_3)\, dH_2$$

$$= \lim_{\eta\to 0} \sum_{i\in I(\eta)} l H_2(S_i)(b^{\infty,2})^{hom}([u](a_i)\otimes e_3) \quad (6.27)$$

where $a_i \in S_i$. Let us consider $u_{s,i}$ an ε-minimizer of

$$\mathcal{S}_{\frac{1}{\lambda}S_i\times]-\frac{\varepsilon}{\lambda},+\frac{\varepsilon}{\lambda}[} := \inf\Big\{\int_{\frac{1}{\lambda}S_i\times]-\frac{\varepsilon}{\lambda},+\frac{\varepsilon}{\lambda}[} b^{\infty,2}(\tilde{x},[u](a_i)\otimes e_3+\nabla\phi)dx : \phi\in W_0^{1,2}(\frac{1}{\lambda}S_i\times]-\frac{\varepsilon}{\lambda},+\frac{\varepsilon}{\lambda}[,\mathbf{R}^3)\Big\}$$

and set

$$u_{s,\eta}(x) := R_\varepsilon u(x) + \sum_{i\in I(\eta)}\frac{\lambda}{2\varepsilon}u_{s,i}(\frac{x}{\lambda})$$

which defines an element of $W^{1,2}(\mathcal{O},\mathbf{R}^3)$ if we extend $u_{s,i}$ by zero in $\mathbf{R}^3\setminus\frac{1}{\lambda}S_i\times]-\frac{\varepsilon}{\lambda},+\frac{\varepsilon}{\lambda}[$. According to the global subadditive Theorem 4.1, for every $i\in I(\eta)$:

$$\begin{aligned}
&lH_2(S_i)(b^{\infty,2})^{hom}([u](a_i)\otimes e_3)\\
&= \lim_{s\to 0}\frac{\mu}{2\varepsilon}\frac{H_2(S_i)}{H_2(\frac{1}{\lambda}S_i)}\frac{\lambda}{2\varepsilon}\int_{\frac{1}{\lambda}S_i\times]-\frac{\varepsilon}{\lambda},+\frac{\varepsilon}{\lambda}[} b^{\infty,2}(\tilde{x},[u](a_i)\otimes e_3+\nabla u_{s,i})\,dx\\
&= \lim_{s\to 0}\mu(\frac{1}{2\varepsilon})^2\int_{S_i\times]-\varepsilon,+\varepsilon[} b^{\infty,2}(\frac{\tilde{x}}{\lambda},[u](a_i)\otimes e_3+(\nabla u_{s,i})(\frac{x}{\lambda}))\,dx\\
&= \lim_{s\to 0}\mu\int_{S_i\times]-\varepsilon,+\varepsilon[} b(\frac{\tilde{x}}{\lambda},\frac{1}{2\varepsilon}[u](a_i)\otimes e_3+\frac{1}{2\varepsilon}(\nabla u_{s,i})(\frac{x}{\lambda}))\,dx\\
&= \lim_{s\to 0}\mu\int_{S_i\times]-\varepsilon,+\varepsilon[} b(\frac{\tilde{x}}{\lambda},\nabla u_{s,\eta})\,dx + o(\eta).
\end{aligned}$$

Summing over i and going to the limit on η, we obtain by (6.27)

$$l\int_S (b^{\infty,2})^{hom}([u](x)\otimes e_3)\,dH_2 = \lim_{\eta\to 0}\lim_{s\to 0}\mu\int_{S_i\times]-\varepsilon,+\varepsilon[} b(\frac{\tilde{x}}{\lambda},\nabla u_{s,\eta})\,dx.$$

By a diagonalization argument, there exists a map $s\to\eta(s)$ such that

$$l\int_S (b^{\infty,2})^{hom}([u](x)\otimes e_3)\,dH_2 = \lim_{s\to 0}\mu\int_{S_i\times]-\varepsilon,+\varepsilon[} b(\frac{\tilde{x}}{\lambda},\nabla u_s)\,dx$$

where $u_s := u_{s,\eta(s)}$. Moreover using the Poincaré inequality, it can be easily proved (see [1]) that u_s strongly tends to u in $L^2(\mathcal{O},\mathbf{R}^3)$ when s tend to zero.

Thus

$$\begin{aligned} G(u) &:= \inf\{\limsup_{s\to 0} F_s(v_s) : v_s \to u \text{ in } L^2(\mathcal{O},\mathbf{R}^3)\ \} \\ &\leq l\int_S (b^{\infty,2})^{hom}([u](x)\otimes e_3)\ dH_2 + \int_O h(x,\nabla u)\ dx \end{aligned}$$

(note that $u_s = u$ in $\mathcal{O}_\varepsilon$). Taking the lower semicontinuous envelope denoted by $cl_{w-W^{1,2}(\mathcal{O}\setminus S,\mathbf{R}^3)}$ of the two members with respect to the weak topology of $W^{1,2}(\mathcal{O}\setminus S,\mathbf{R}^3)$, we obtain

$$cl_{w-W^{1,2}(\mathcal{O}\setminus S,\mathbf{R}^3)}G(u) \leq l\int_S (b^{\infty,2})^{hom}([u]\otimes e_3)\ dH_2 + \int_O Qh(x,\nabla u)\ dx$$

where we have used the compact embedding of $W^{1,2}(\mathcal{O}\setminus S,\mathbf{R}^3)$ into $L^2(S,\mathbf{R}^3)$ for the first term of the right hand side and the integral representation (see for instance Dacorogna [9]) of the quasiconvex envelope of the second term. But (see Attouch [4] for the first equality)

$$G = cl_{L^2(\mathcal{O},\mathbf{R}^3)}G \leq cl_{w-W^{1,2}(\mathcal{O}\setminus S,\mathbf{R}^3)}G$$

so that

$$G(u) \leq l\int_S (b^{\infty,2})^{hom}([u]\otimes e_3)\ dH_2 + \int_O Qh(x,\nabla u)\ dx$$

and we conclude the proof after noticing that the infimum in the definition of G is attained.

Third step. If u is not smooth, for (E_1) we approximate u by u_δ strongly in $W^{1,2}(\mathcal{O},\mathbf{R}^3)$ and consider $u_{\delta,s} = u_s - R_\varepsilon u + R_\varepsilon u_\delta$ and conclude as in [13]. For (E_2), we reason by density and a diagonalization argument. □

Remark: It is straightforward to establish (cf [13]) $(b^{\infty,2})^{hom}(Q) = (b^{hom})^{\infty,2}(Q)$ where, for every $Q \in M^{3\times 3}$,

$$b^{hom}(Q) := \lim_{k\to+\infty}\frac{1}{k^N}\inf\{\int_{kY} b(\tilde{x},\nabla\varphi(x))\ dx : \varphi \in Q.x + W_0^{1,p}(kY,\mathbf{R}^3)\}.$$

This new expression of $(b^{\infty,2})^{hom}$ is conform to physical intuition : since λ is lower than ε, we begin to homogenize the layer, then we let the thickness of the layer tends to zero.

Remark: In the non ergodic case, according to Theorem 4.1, we obtain the following expression of the density of the surface energy :

$$(b^{\infty,2})^{hom}(a) := \inf_{k\in\mathbf{N}^*} \frac{1}{k^3} E^{\mathcal{F}} \inf\{\int_{kY} b^{\infty,2}(\tilde{y}, Q+\nabla\varphi(y))dy : \varphi \in W_0^{1,2}(kY, \mathbf{R}^3)\}.$$

6.1.2 Case $\varepsilon << \lambda$ ($\lim_{s\to+\infty} \frac{\varepsilon}{\lambda} = 0$). We assume that $b(.,Q)$ is $\mathbf{Z}^2$-periodic and that $b^{\infty,2}(\tilde{x},.)$ is convex. Let $\tilde{Y} =]0,1[^2$, we denote by $W^{1,2}_{\text{per},0}(\tilde{Y}\times]-r,r[,\mathbf{R}^3)$ the space of the elements of the Sobolev space $W^{1,2}(\tilde{Y}\times]-r,r[,\mathbf{R}^3)$ with null trace on the faces $\tilde{Y}\times\{-r\}$, $\tilde{Y}\times\{r\}$ and with equal traces on the opposite faces of $\tilde{Y}\times]-r,r[$.

Theorem 6.2: *Under above hypothesis, F_s epi-converges to F where, for every Q in $M^{3\times3}$,*

$$\begin{aligned} &(b^{\infty,2})^{hom}(Q) \\ = \ & \lim_{r\to O} \inf\{\frac{1}{2r}\int_{\tilde{Y}\times]-r,r[} b^{\infty,2}(\tilde{x}, Q+\nabla\varphi)\ dx : \varphi \in W^{1,2}_{per,0}(\tilde{Y}\times]-r,r[,\mathbf{R}^3)\} \\ = \ & \int_{\tilde{Y}} b^{\infty,2}(\tilde{x},Q)\ d\tilde{x}. \end{aligned}$$

PROOF: *First step.* We prove (E_1) for every $u \in \tilde{V}_{[\,]}$ by the strategy of the previous case. With the same notations, the bounded Borel measure

$$\nu_s := \chi_{O_\varepsilon} h(.,\nabla u_s)\ dx + \mu\chi_{B_\varepsilon} b(\frac{x}{\lambda}, \nabla u_s)\ dx$$

tends weakly to a bounded Borel measure ν and we will prove

$$\begin{aligned} &\nu^a \geq Qh(.,\nabla u)\ dx \\ &\nu^{sing} \geq l\ (b^{\infty,2})^{hom\ \infty}([u]\otimes e_3) H_2\lfloor S. \end{aligned}$$

The first inequality is already proved in Theorem 6.2. For the second, we have also

$$\lim_{\rho\to 0} \frac{\nu(C_\rho(x_0))}{H_2(S_\rho(x_0))} \tag{6.28}$$

$$\geq \lim_{\rho\to 0}\limsup_{s\to 0}\frac{\mu}{H_2(S_\rho(x_0))}\int_{S_\rho(x_0)\times]-t(\varepsilon),t(\varepsilon)[} b(\frac{\tilde{x}}{\lambda},\frac{1}{2\varepsilon}[u](x_0)\otimes e_3+\nabla w_s)\,dx$$

$$\geq l\lim_{\rho\to 0}\limsup_{s\to 0}\frac{1}{\text{meas}(A_s)}\inf\{\int_{A_s} b^{\infty,2}(\tilde{x},[u](x_0)\otimes e_3+\nabla\varphi)\,dx : \varphi\in W_0^{1,2}(A_s,\mathbf{R}^3)\}$$

where $A_s := \frac{1}{\lambda}S_{\delta\rho}(x_0)\times]-\frac{t(\varepsilon)}{\lambda},\frac{t(\varepsilon)}{\lambda}[$ but here $\dfrac{t(\varepsilon)}{\lambda}$ tends to zero. Let us set for any Borel bounded subset $\tilde{A}$ of $\mathbf{R}^2$ and any bounded interval I of $\mathbf{R}$:

$$\mathcal{S}_{\tilde{A}\times I} := \inf\{\int_{\overset{0}{\tilde{A}}\times\overset{0}{I}} b^{\infty,2}(\tilde{x},[u](x_0)\otimes e_3+\nabla\varphi)\,dx : \varphi\in W^{1,2}_{\text{per},0}(\overset{0}{\tilde{A}}\times\overset{0}{I},\mathbf{R}^3)\}.$$

By subadditivity and $\mathbf{Z}^2$-invariance of $\tilde{A}\mapsto\mathcal{S}_{\tilde{A}\times I}$ and by the growth condition, it is easy to obtain from (6.28)

$$\lim_{\rho\to 0}\frac{\nu(C_\rho(x_0))}{H_2(S_\rho(x_0))}\geq\limsup_{s\to 0}\frac{\mathcal{S}_{k(s)\tilde{Y}\times]-\frac{t(\varepsilon)}{\lambda},\frac{t(\varepsilon)}{\lambda}[}}{H_2(k(s)\tilde{Y})\;2\frac{t(\varepsilon)}{\lambda}} \tag{6.29}$$

where $k(s)=[\frac{\rho}{\lambda}]+1$. Let us set for every $\tilde{A}\in\mathcal{P}(\mathbf{Z}^2)$ and every $I\in\mathcal{P}(\mathbf{R})$

$$\mathcal{S}^{\#}_{\tilde{A}\times I} := \inf\{\int_{\overset{0}{\tilde{A}}\times\overset{0}{I}} b^{\infty,2}(\tilde{x},[u](x_0)\otimes e_3+\nabla\varphi)\,dx : \varphi\in W^{1,2}_{\text{per},0}(\overset{0}{\tilde{A}}\times\overset{0}{I},\mathbf{R}^3)\}.$$

By convexity of $b^{\infty,2}(\tilde{x},.)$ and the subdifferential inequality, it is straightforward to show (see S. Müller [16])

$$\frac{\mathcal{S}^{\#}_{k(s)\tilde{Y}\times]-\frac{t(\varepsilon)}{\lambda},\frac{t(\varepsilon)}{\lambda}[}}{H_2(k(s)\tilde{Y})\;2\frac{t(\varepsilon)}{\lambda}}=\frac{\mathcal{S}^{\#}_{\tilde{Y}\times]-\frac{t(\varepsilon)}{\lambda},\frac{t(\varepsilon)}{\lambda}[}}{2\frac{t(\varepsilon)}{\lambda}}.$$

The conclusion follows from (6.29) by applying the local Theorem 2.2 to the subadditive set function $I\mapsto\mathcal{S}^{\#}_{\tilde{A}\times I}$.

Second step. To prove (E_2) we also reproduce the outline of the proof of the previous case $\lambda << \varepsilon$. With the same notations, let $u_{s,i}$ be a minimizer of the problem

$$\inf\{\frac{\lambda}{2\varepsilon}\int_{\tilde{Y}\times]-\frac{\varepsilon}{\lambda},\frac{\varepsilon}{\lambda}[} b^{\infty,2}(\tilde{x},[u](a_i)\otimes e_3+\nabla\varphi)\,dx : \varphi\in W^{1,2}_{\text{per},0}(\tilde{Y}\times]-\frac{\varepsilon}{\lambda},\frac{\varepsilon}{\lambda}[,\mathbf{R}^3)\}$$

and set

$$u_{s,\eta}(x) := R_\varepsilon u(x) + \sum_{i \in I(\eta)} \frac{\lambda}{2\varepsilon} u_{s,i}(\frac{x}{\lambda})$$

which defines an element of $W^{1,2}(\mathcal{O}, \mathbf{R}^3)$ if we extend $u_{s,i}$ by $\tilde{Y}$-periodicity with respect to the variable $\tilde{x}$ and by zero with respect to the variable x_3. Taking into account the periodicity assumption and the inclusion $\frac{1}{\lambda}S_i \subset k(s)\tilde{Y} + z_i$ where $k(s) = [\frac{\eta}{\lambda}] + 1$ and $z_i \in \mathbf{Z}^2$,

$$\begin{aligned} &(b^{\infty,2})^{hom}([u](a_i) \otimes e_3) \\ = \; & \lim_{s \to 0} \frac{\lambda}{2\varepsilon} \int_{\tilde{Y} \times]-\frac{\varepsilon}{\lambda}, \frac{\varepsilon}{\lambda}[} b^{\infty,2}(\tilde{x}, [u](a_i) \otimes e_3 + \nabla u_{s,i}) \, dx \\ \geq \; & \limsup_{s \to 0} \frac{1}{H_2(\frac{1}{\lambda}S_i)} \frac{\lambda}{2\varepsilon} \int_{\frac{1}{\lambda}S_i \times]-\frac{\varepsilon}{\lambda}, \frac{\varepsilon}{\lambda}[} b^{\infty,2}(\tilde{x}, [u](a_i) \otimes e_3 + \nabla u_{s,i}) \, dx \end{aligned}$$

thus

$$\begin{aligned} &l H_2(S_i)(b^{\infty,2})^{hom}([u](a_i) \otimes e_3) \\ \geq \; & \limsup_{s \to 0} \frac{\mu}{2\varepsilon} \frac{H_2(S_i)}{H_2(\frac{1}{\lambda}S_i)} \frac{\lambda}{2\varepsilon} \int_{\frac{1}{\lambda}S_i \times]-\frac{\varepsilon}{\lambda}, \frac{\varepsilon}{\lambda}[} b^{\infty,2}(\tilde{x}, [u](a_i) \otimes e_3 + \nabla u_{s,i}) \, dx. \end{aligned}$$

The end of the proof is then identical to that of the previous case.

Third step. In the case when u is not smooth, we reason by a density and a diagonalization argument.

Last step. It remains to establish

$$\begin{aligned} &\lim_{r \to O} \inf\{\frac{1}{2r} \int_{\tilde{Y} \times]-r,r[} b^{\infty,2}(\tilde{x}, Q + \nabla\varphi) \, dx : \varphi \in W^{1,2}_{\text{per},0}(\tilde{Y} \times]-r, r[, \mathbf{R}^3)\} \\ = \; & \int_{\tilde{Y}} b^{\infty,2}(\tilde{x}, Q) \, d\tilde{x}. \end{aligned}$$

A change of scale gives

$$\begin{aligned} &\inf\{\frac{1}{2r} \int_{\tilde{Y} \times]-r,r[} b^{\infty,2}(\tilde{x}, Q + \nabla\varphi) \, dx : \varphi \in W^{1,2}_{\text{per},0}(\tilde{Y} \times]-r, r[, \mathbf{R}^3)\} \\ = \; & \inf\{\int_{\tilde{Y} \times]0,1[} b^{\infty,2}(\tilde{x}, Q + \tilde{\nabla}\varphi + \frac{1}{r}\partial_3\varphi) \, dx : \varphi \in W^{1,2}_{\text{per},0}(\tilde{Y} \times]0, 1[, \mathbf{R}^3)\} \end{aligned}$$

where $\tilde{\nabla}\varphi$ and $\partial_3\varphi$ denote respectively the two matrix valued functions $(\frac{\partial}{\partial x_1}\varphi, \frac{\partial}{\partial x_2}\varphi, 0)$ and $(0, 0, \frac{\partial}{x_3}\varphi)$. Let us set for every φ in $W^{1,2}_{\text{per},0}(\tilde{Y}\times]0,1[,\mathbf{R}^3)$

$$\Phi_r(\varphi) := \int_{\tilde{Y}\times]0,1[} b^{\infty,2}(\tilde{x}, Q + \tilde{\nabla}\varphi + \frac{1}{r}\partial_3\varphi)\ dx,$$

$$\Phi(\varphi) := \int_{\tilde{Y}} b^{\infty,2}(\tilde{x}, Q)\ d\tilde{x} + I_{[\varphi=0]}(\varphi)$$

where $I_{[\varphi=0]}$ denotes the indicator function of the set $[\varphi = 0]$. Noticing that any set $\{\varphi_r\}$ of minimizers of $(\Phi_r)_r$ is relatively weakly compact in $W^{1,2}_{\text{per},0}(\tilde{Y}\times]0,1[,\mathbf{R}^3)$, according to the properties of epiconvergence (see section 5), it suffices to establish the epiconvergence of Φ_r toward Φ in the space $W^{1,2}_{\text{per},0}(\tilde{Y}\times]0,1[,\mathbf{R}^3)$ equipped with its weak topology, when $r \to 0$. Bound (E_2) is trivial (take the sequence $(\varphi_r)_r$ equal to the sequence of null functions). Let us prove (E_1). Let $(\varphi_r)_r$ be a sequence converging to φ weakly in $W^{1,2}_{\text{per},0}(\tilde{Y}\times]0,1[,\mathbf{R}^3)$ and satisfying $\liminf_{r\to 0}\Phi_r(\varphi_r) < +\infty$. Coercivity of $b^{\infty,2}$ implies that $|\tilde{\nabla}\varphi + \frac{1}{r}\partial_3\varphi)|_{L^2(\tilde{Y}\times]0,1[,M^{3,3})} \leq C$ and that $\partial_3\varphi_r$ strongly tends to 0 in $L^2(\tilde{Y}\times]0,1[,M^{3,3})$. We infer that $\partial_3\varphi = 0$ and consequentely $\varphi = 0$. On the other hand, by subdifferential inequality

$$\Phi_r(\varphi_r) \geq \int_{\tilde{Y}} b^{\infty,2}(\tilde{x}, Q)\ d\tilde{x} + \int_{\tilde{Y}\times]0,1[} < \partial b^{\infty,2}(\tilde{x}, Q), \tilde{\nabla}\varphi_r + \frac{1}{r}\partial_3\varphi_r > dx$$

$$= \int_{\tilde{Y}} b^{\infty,2}(\tilde{x}, Q)\ d\tilde{x} + \int_{\tilde{Y}\times]0,1[} < \partial b^{\infty,2}(\tilde{x}, Q), \tilde{\nabla}\varphi_r > dx$$

$$+\frac{1}{r}\int_{\tilde{Y}}\int_0^1 < \partial b^{\infty,2}(\tilde{x}, Q), \partial_3\varphi_r > d\tilde{x}\ dx_3$$

where the last integral in the right hand side is obviously equal to zero. Letting $r \to 0$, we finally obtain (E_1). □

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CHRISTIAN LICHT
UNIVERSITÉ MONTPELLIER 2
LABORATOIRE DE MÉCANIQUE
ET DE GÉNIE CIVIL,UMR 5508 DU CNRS
CASE COURIER 048, PLACE EUGÈNE BATAILLON
34095 MONTPELLIER CEDEX 5
FRANCE
licht@lmgc.univ-montp2.fr

GÉRARD MICHAILLE
UNIVERSITÉ MONTPELLIER 2
LABORATOIRE ACSIOM
DÉPARTEMENT DE MATHÉMATIQUES
CASE COURIER 051, PLACE EUGÈNE BATAILLON
34095 MONTPELLIER CEDEX 5
FRANCE
micha@math.univ-montp2.fr