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Böcher’s theorem in a space of dimension one

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Abstract:

In this paper we express a harmonic function $h$ defined outside a compact set in a B.H. space $\Omega$ as an integral with respect to a signed measure in $\Omega$ assuming $\Omega$ satisfies the axiom of local proportionality. If in particular $h$ is positive and $\Omega$ has harmonic dimension one then this expression leads to an analogue of Böcher's theorem in a space of dimension one.

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§1. Introduction

We consider a harmonic function $h$ defined outside a compact set in a B.H. space $\Omega$. This can be written as the difference of two superharmonic functions in $\Omega$ where both functions have the same compact support in $\Omega$. If we assume the axiom of local proportionality this leads to an integral representation for $h$ with respect to a signed measure which looks like the Riesz representation. This is of interest because the Riesz representation does not give an integral for a harmonic function as the measure associated with a harmonic function is zero. This theorem gives an analogue of Böcher's theorem in a B.H. space of harmonic dimension one if we assume $h$ is positive.

§2. Preliminaries

Let $\Omega$ be a harmonic space satisfying the axioms 1,2,3 of M.Brelot. We assume that constants are harmonic in $\Omega$ in which case $\Omega$ is referred to as a B.H.space. $\Omega$ is called a B.P. or B.S. space according as there exists a positive potential or not in $\Omega$. For a nonlocally polar outer regular compact set $k \subset \Omega$ and a continuous function $f$ on $\partial k$, as in [1], the notation $B_k f$ stands for the Dirichlet solution in $\Omega - k$ with values $f$ or $\partial k$ and 0 at the point at infinity.
In the case of a B.S. space $\Omega$, we fix an outer regular compact set $K$ and a regular domain $\omega$, $K \subset \omega$ with respect to which flux is defined (for definition see [1]). We also fix a harmonic function $H > 0$ in $\Omega - K$ tending to 0 on $\partial K$ with flux at infinity one.

We recall the definition of a B.H. potential in a B.S. space $\Omega$: Let $\{\Omega_i\}$ be a fixed regular exhaustion of $\Omega$. Fix an ultrafilter $\mathcal{e}$ finer than the filter of sections of $\{\Omega_i\}$. Let $\mathcal{D}(\mathcal{u})$ be the limit of $\bar{\mathcal{H}}_{\mathcal{u},\mathcal{e}}^{\Omega_i}$ according to the ultrafilter $\mathcal{e}$. An admissible superharmonic function $u$ in a B.S. space $\Omega$ with flux at infinity $\alpha$ is said to be a B.H. - potential if $\mathcal{D}(u - \alpha H) = 0$.

It can be easily seen that a superharmonic function $u$ with compact support in a B.P. (respectively B.S.) space can be written uniquely as the sum of a potential (respectively B.H. potential) and a harmonic function.

Let $\Omega$ be a B.H. space satisfying the axiom of local proportionality.

Case (i). Let $\Omega$ be a B.P. space. If $\delta$ is a regular domain and $z$ a fixed point in $\delta$, then for any $y$ there exists a unique potential $q_y(x)$ with support $y$ such that $\int q_y(z)d\rho_\delta^y = 1$ where $d\rho_\delta^y$ is the harmonic measure of $\delta$ with respect to $z$.

If $u$ is a potential with compact support $A$ then there exists a unique Radon measure $\mu \geq 0$ supported by $A$ such that $u(z) = \int q_y(z)d\mu(y)$; and conversely if $\mu \geq 0$ is a Radon measure with compact support then $\int q_y(z)d\mu(y)$ is a potential.

Case (ii): Let $\Omega$ be a B.S. space. In this case, for any $y$, there exists a unique B.H. potential $q_y(z)$ with support $y$ and flux $q_y$ at infinity $-1$. Then if $u(z)$ is a B.H. potential with compact support $A$, there exists a unique Radon measure $\mu \geq 0$ supported by $A$ such that $u(z) = \int q_y(z)d\mu(y)$; and conversely, if $\mu \geq 0$ is a Radon measure with compact support, then $u(z) = \int q_y(z)d\mu(y)$ is a B.H. potential with flux $u$ at infinity $= -\int d\mu$.

§ 3. BÖCHER'S THEOREM IN A SPACE OF DIMENSION ONE

Theorem 1.

Let $h$ be a harmonic function defined outside a compact set $X$ in a B.H. space $\Omega$ and $\omega_0$ be any regular domain such that $X \subset \omega_0$. Assume that $\Omega$ has a countable base and satisfies the axiom of local proportionality. Then
there exists a signed measure \( \mu \) with support contained in \( \partial \omega_0 \) and a uniquely determined harmonic function \( u \) in \( \Omega \) such that \( h(x) = \int q_y(x) d\mu(y) + u(x) \) in \( \Omega \sim \bar{\omega}_0 \).

Here \( q_y(x) \) is the potential (respectively B.H. potential) that we fix in \( \Omega \) as explained in \( \S 2 \), if \( \Omega \) is a B.P. (respectively B.S.) space. Moreover if the harmonic dimension at infinity of \( \Omega \) is 1, \( u \) is a constant if and only if \( h \) is bounded on one side near the point at infinity \( A \).

**Proof.**

Let \( x_0 \in X \) and \( s_{x_0} \) be a superharmonic function in \( \Omega \) with point support \( x_0 \).

Choose an outer regular compact set \( K_1 \) such that \( X \subset K_1^0 \subset K_1 \subset \omega_0 \).

Without loss of generality we can assume that \( h \) is harmonic in \( \omega_0 \sim K_1 \) and continuous in \( \omega_0 \sim K_1 \). For a continuous function \( f \) on \( \partial \omega_0 \) let \( Df = H_{f|\omega_0} \) denote the Dirichlet solution in \( \omega_0 \) with boundary value \( f \).

Since \( Ds_{x_0} < s_{x_0} \) in \( \omega_0 \) we have \( \inf_{\partial K_1} (s_{x_0} - Ds_{x_0}) > 0 \).

Choose \( \alpha > 0 \) such that
\[
\alpha(s_{x_0} - Ds_{x_0}) > Dh - h \text{ on } \partial K_1.
\]

Then \( h + \alpha s_{x_0} > D(h + \alpha s_{x_0}) \) on \( \partial K_1 \).

Since \( h + \alpha s_{x_0} = D(h + \alpha s_{x_0}) \) on \( \partial \omega_0 \), by minimum principle of harmonic functions we get
\[
h + \alpha s_{x_0} > D(h + \alpha s_{x_0}) \text{ in } \omega_0 \sim K_1.
\]

Define
\[
h_1 = \begin{cases} 
  h + \alpha s_{x_0} & \text{in } \Omega \sim \omega_0 \\
  D(h + \alpha s_{x_0}) & \text{in } \omega_0
\end{cases}
\]

and
\[
h_2 = \begin{cases} 
  \alpha s_{x_0} & \text{on } \Omega \sim \omega_0 \\
  D(\alpha s_{x_0}) & \text{on } \omega_0
\end{cases}
\]

Then \( h_1 \) and \( h_2 \) are finite, continuous, superharmonic functions in \( \Omega \) with compact support in \( \partial \omega_0 \) such that
\[
h = h_1 - h_2 \text{ on } \Omega \sim \bar{\omega}_0.
\]

Now, \( h_i = p_i + u_i \) \( i = 1, 2 \) where \( p_i \) is a potential (respectively B.H. potential) with support in \( \partial \omega_0 \) if \( \Omega \) is a B.P. (respectively B.S.) space and \( u_i \) is harmonic in \( \Omega \).
Hence \( h = p_1 - p_2 + u \) where \( u = u_1 - u_2 \) is harmonic in \( \Omega \). But \( p_i(x) = \int q_i(y)\mu_i(y) \), \( i = 1, 2 \) where \( \mu_i, i = 1, 2 \) is a Radon measure with support contained in \( \partial \omega_0 \).

Hence \( h(x) = \int q_i(x)\mu_i(y) + u(x) \) where \( \mu = \mu_1 - \mu_2 \) is a signed measure with support contained in \( \partial \omega_0 \).

We shall complete the proof by considering the two cases of a B.P. space and a B.S. space separately.

Case (i). Let \( \Omega \) be a B.P. space.

Suppose \( h(x) = q_i(x)\mu_i(y) + u'(x) \) where \( \mu' \) is also a signed measure with support contained in \( \partial \omega_0 \) and \( u' \) is harmonic in \( \Omega \).

Then \( h \) can be written as

\[
q_i, i = 1, 2 \text{ are potentials in } \Omega \text{ with compact support.}
\]

Thus \( u \) is uniquely determined in \( \Omega \).

Now \( h = p_1 - p_2 + u \) on \( \Omega \sim \partial \omega_0 \).

Since \( p_1 \) and \( p_2 \) are potentials with compact support, they are bounded outside a compact set in \( \Omega \).

Hence if \( h \) is bounded on one side near \( A \) so is \( u \).

Therefore if \( \Omega \) is of harmonic dimension one, we see that \( u \) reduces to a constant [2].

If \( u \) is a constant then clearly \( h \) is bounded on one side near \( A \).

Case (ii): Let \( \Omega \) be a B.S. space.

Let flux \( p_1 = \alpha_1 \) and flux \( p_2 = \alpha_2 \).

Then \( h - (\alpha_1 - \alpha_2)H = (p_1 - \alpha_1 H) - (p_2 - \alpha_2 H) + u \)

gives \( D(h - (\alpha_1 - \alpha_2)H) = u \) by definition of a B.H. potential.

Since \( \alpha_1 - \alpha_2 = \text{flux } h \), we see that given \( h, u \) is uniquely determined in \( \Omega \).
Now since $\mathcal{D}(p_i - \alpha_i H) = 0$ we get $p_i - \alpha_i H, \ i = 1, 2$ are bounded outside a compact set.

Hence $h = u + (\alpha_1 - \alpha_2)H +$ a bounded harmonic function outside a compact set.

If $h$ is bounded on one side near $A$, then $u + (\alpha_1 - \alpha_2)H$ is bounded on one side near $A$.

If $\Omega$ has harmonic dimension one this implies that $u$ is a constant [2].

If $u$ is a constant, $h$ is obviously bounded on one side near $A$.

This completes the proof of the theorem.

Now if we take the function $h$ in the above theorem to be $\geq 0$ we can deduce the analogue of the inverted version of Böcher's theorem, which may be stated as follows, in a space of harmonic dimension one.

Böcher's theorem: (Inverted version). Let $u$ be positive and harmonic in $\mathbb{R}^n - B, n \geq 2$ where $B$ is the unit ball about the origin. Then

$$u(x) = \begin{cases} \alpha \log |x| + b(x) & \text{if } n = 2 \\ \alpha + b(x) & \text{if } n \geq 3 \end{cases}$$

where $b(x)$ is a bounded harmonic function in $\mathbb{R}^n - B$ and $\alpha \geq 0$ is a constant. If $n \geq 3$, $b(x)$ is actually bounded by a bounded potential.

This can be proved by applying the Kelvin's transform to the standard form of Böcher's theorem [3].

**Theorem 2.**

Let $\Omega$ be a B.H. space of harmonic dimension one and $h$ be a positive harmonic function defined outside a compact set $X$. If $\Omega$ is a B.P. space then $h = \alpha + b$ where $\alpha$ is a constant and $b$ is a harmonic function bounded by a bounded potential outside a compact set.

If $\Omega$ is B.S., then $h = \alpha H + b$ outside a compact set where $\alpha$ is a constant and $b$ is a bounded harmonic function outside a compact set.

**Proof.**

**Case (1).** Let $\Omega$ be a B.P. space.

Take $\omega_0, p_1, p_2$ as in Theorem 1.

Since $h \geq 0$, $u$ is a constant say $\alpha$. 
Let $K'$ be an outer regular compact set such that $(K')^{0} \supset \partial \omega_{0}$.

Then for $i = 1, 2$, $v_{i} = \begin{cases} 0 & \text{on } K' \\ p_{i} - B_{K'}p_{i} & \text{on } \Omega \sim K' \end{cases}$
is a subharmonic function on $\Omega$ such that $0 \leq v_{i} \leq p_{i}$.

Since $p_{i}$ is a potential this implies that $v_{i} \equiv 0$ or $p_{i} = B_{K'}p_{i}$ outside the compact set $K'$.

If $p_{i} \leq \lambda$ on $\partial K'$, then $B_{K'}p_{i} \leq \lambda B_{K'}1$.

Hence $h = p_{1} - p_{2} + \alpha = \alpha + b$ where $b = p_{1} - p_{2}$ is such that $|b| \leq 2\lambda B_{K'}1$, a bounded potential outside a compact set.

Case (ii). Let $\Omega$ be a B.S. space.

Then as in the proof of the above theorem since $u$ is a constant we get

$$h = (\alpha_{1} - \alpha_{2})H + \text{a bounded harmonic function}$$

$$= \alpha H + b \quad \text{outside a compact set.}$$

References


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