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ON A THEOREM OF ROLEWICZ FOR SEMIGROUPS OF OPERATORS IN LOCALLY CONVEX SPACES

by
Mihail Megan and Alin Pogan

Abstact. The purpose of this paper is to extend a stability theorem of Rolewicz to the case of semigroups of operators in locally convex topological vector spaces. The results obtained generalize the similar theorems proved by Datko,Pazy,Zabczyk, Rolewicz and Littman for the case of C_0-semigroups of operators in Banach spaces.

1.Introduction

Let X be a locally convex space whose topology is generated by the family of seminorms $\{|\cdot|_\gamma : \gamma \in \Gamma\}$.The space of all continuos linear operators from X into itself will be denoted by $B(X)$. For all $A \in B(X)$ and for all $\beta, \gamma \in \Gamma$ we shall denote

$$\|A\|_{\beta,\gamma} = \sup\{|Ax|_\gamma : |x|_\beta \leq 1\}.$$

It is obviouse that $A \in B(X)$ if and only if for every $\gamma \in \Gamma$ there exists $\beta = \beta(\gamma) \in \Gamma$ such that $\|A\|_{\beta,\gamma} < \infty$.

Recall that a family $\mathbf{S} = (S(t))_{t\geq 0}$ of continuous linear operators from X into itself is a C_0-*semigroup* on X, if

$s_1)S(0) = I$ (the identity operator on X);

$s_2)S(t+s) = S(t)S(s)$, for all $t, s \geq 0$;

$s_3) \lim\limits_{t\to 0} |S(t)x - x|_\gamma = 0$, for all $x \in X$ and all $\gamma \in \Gamma$.

For details about C_0-semigroups in locally convex spaces see for instance [2] and [5].

In what follows we denote by Φ the set of all functions $\varphi : IR_+ \times \Gamma \to \Gamma$ with the properties

$\varphi_1)$ $\varphi(0,\gamma) = \gamma$, for all $\gamma \in \Gamma$;

$\varphi_2)$ $\varphi(t+s,\gamma) = \varphi(t, \varphi(s,\gamma))$, for all $t, s \geq 0$ and all $\gamma \in \Gamma$.

In this paper we consider a particular class of C_0-semigroups defined by

Definition 1.1. A C_0-semigroup $\mathbf{S} = (S(t))_{t\geq 0}$ is called *Φ-semigroup*, if there exists $\varphi \in \Phi$ such that

$$\|S(t)\|_{\varphi(t,\gamma),\gamma} < \infty \quad \text{for all} \quad (t,\gamma) \in I\!R \times \Gamma.$$

Hence if $\mathbf{S}$ is a Φ-semigroup, then there exists $\varphi \in \Phi$ with

$$|S(t)x|_\gamma \leq \|S(t)\|_{\varphi(t,\gamma),\gamma} |x|_{\varphi(t,\gamma)} \quad \text{for all} \quad (t,\gamma,x) \in I\!R \times \Gamma \times X.$$

Definition 1.2. A Φ-semigroup $\mathbf{S} = (S(t))_{t\geq 0}$ is said to be

(i)*exponentially bounded* (and denote *e.b.*) if there exists $\varphi \in \Phi$ and $M, \omega : \Gamma \to I\!R_+^*$ such that

$$\|S(t)\|_{\varphi(t,\gamma),\gamma} \leq M(\gamma) e^{t\omega(\gamma)} \quad \text{for all} \quad (t,\gamma) \in I\!R \times \Gamma;$$

(ii)*uniformly exponentially bounded* (and denote *u.e.b.*) if there exists the functions M and ω from (i) satisfy the conditions:

$$M_0(\gamma) := \sup_{t\geq 0} M(\varphi(t,\gamma)) < \infty \quad \text{and} \quad \omega_0(\gamma) := \sup_{t\geq 0} \omega(\varphi(t,\gamma)) < \infty \quad \text{for all} \quad \gamma \in \Gamma.$$

It is obvious that if $\mathbf{S}$ is u.e.b. then it is u.b.

Remark 1.1. If X is a Banach space then every C_0-semigroup $\mathbf{S}$ is a Φ-semigroup with u.e.b.(see [7]).

Definition 1.3. A Φ-semigroup $\mathbf{S} = (S(t))_{t\geq 0}$ is said to be

(i)*stable* (and denote *s.*) if there are $\varphi \in \Phi$ and $M : \Gamma \to I\!R_+^*$ such that

$$\|S(t)\|_{\varphi(t,\gamma),\gamma} \leq M(\gamma) \quad \text{for all} \quad (t,\gamma) \in I\!R_+ \times \Gamma;$$

(ii)*uniformly stable* (and denote *u.s.*) if it is stable and the function M from (i) satisfies the condition

$$M_0(\gamma) := \sup_{t\geq 0} M(\varphi(t,\gamma)) < \infty \quad \text{for all} \quad \gamma \in \Gamma;$$

(iii)*exponentially stable* (and denote *e.s.*) if there are $\varphi \in \Phi$ and $N, \nu : \Gamma \to I\!R_+^*$ such that

$$\|S(t)\|_{\varphi(t,\gamma),\gamma} \leq N(\gamma) e^{-t\nu(\gamma)} \quad \text{for all} \quad (t,\gamma) \in I\!R_+ \times \Gamma;$$

(iv)*uniformly exponentially stable* (and denote *u.e.s.*) if it is e.s. and the functions N and ν from (iii) satisfy

$$N_0(\gamma) := \sup_{t\geq 0} N(\varphi(t,\gamma)) < \infty \quad \text{and} \quad \nu_0(\gamma) := \inf_{t\geq 0} \nu(\varphi(t,\gamma)) > 0 \quad \text{for all} \quad \gamma \in \Gamma.$$

Remark 1.2. It is obvious that

$$\begin{array}{ccccc} \text{u.e.s.} & \Rightarrow & \text{u.s.} & \Rightarrow & \text{u.e.b.} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{e.s.} & \Rightarrow & \text{s.} & \Rightarrow & \text{e.b.} \end{array}$$

In Banach spaces we have that

$$\text{u.e.s.} \quad \Leftrightarrow \quad \text{e.s} \quad \Rightarrow \quad \text{u.s.} \quad \Leftrightarrow \quad \text{s.}$$

In stability theory in Banach spaces a well-known result due to Rolewicz [8] is

Theorem 1.1. *Let* $\mathbf{S} = (S(t))_{t\geq 0}$ *be a* C_0*-semigroup on the Banach space* X *with the norm* $\|\cdot\|$.$\mathbf{S}$ *is u.e.s. if and only if there exists a non-decreasing continuous function* $R : I\!R_+ \to I\!R_+$ *with* $R(0) = 0, R(t) > 0$ *for all* $t > 0$ *and*

$$\int_0^\infty R(\|S(t)x\|)dt < \infty \quad \text{for all} \quad x \in X.$$

In [8] Rolewicz proved a slightly, more general result valid for so-called evolution families.

A shorter proof of Rolewicz's theorem is given by Q.Zheng ([10]) who removed the continuity assumption.Another proof of (the semigroups case) of Rolewicz's result was offered by Littman in [3].The case $R(t) = t^2$ was originally proved by Datko in [1], and $R(t) = t^p$ by Pazy in [6].

In this paper we attempt to generalize Theorem 1.1. for the case of Φ-semigroups in locally convex spaces.

Firstly, we observe that Theorem 1.1. is not valid in locally convex spaces.

Example 1.1. Let X be the space of all complex continuous functions on $I\!R_+$ and $\Gamma = I\!R_+^* = (0,\infty)$.

The family $\{|\cdot|_\gamma : \gamma \in \Gamma\}$ given by

$$|x|_\gamma = |x(\gamma)| \quad \text{for all} \quad \gamma \in \Gamma$$

determines a structure of locally convex space on X. It is easy to see that

$$\varphi : IR_+ \times \Gamma \to \Gamma, \quad \varphi(t,\gamma) = \gamma e^{2t}$$

belongs to Φ and $\mathbf{S} = (S(t))_{t\geq 0}$ defined by

$$S(t)x(s) = e^{-t}x(se^{2t}) \quad \text{for all} \quad (t,s) \in IR_+ \times IR_+^* \quad \text{and all} \quad x \in X,$$

is a Φ-semigroup with

$$\|S(t)\|_{\varphi(t,\gamma),\gamma} = e^{-t} \quad \text{for all} \quad (t,\gamma) \in IR_+ \times \Gamma.$$

and hence $\mathbf{S}$ is u.e.s. We observe that for $x(s) = s$ we have that

$$\int_0^\infty |S(t)x|_\gamma dt = \int_0^\infty \gamma e^t dt = \infty \quad \text{for all} \quad \gamma \in \Gamma.$$

Example 1.2. Let $\Gamma = IR$ and let X be the space of all complex continuous functions x with the property that there is $M_x > 0$ such that

$$|x(t)| \leq M_x|t| \quad \text{for all} \quad t \in IR.$$

The family $\{|\cdot|_\gamma : \gamma \in \Gamma\}$ defined by

$$|x|_\gamma = |x(\gamma)| \quad \text{for all} \quad \gamma \in \Gamma$$

determines a structure of locally convex space on X. The function

$$\varphi : IR \times \Gamma \to \Gamma, \quad \varphi(t,\gamma) = \gamma e^{-2t}$$

belongs to Φ and $\mathbf{S} = (S(t))_{t\geq 0}$ defined by

$$S(t)x(s) = e^{t}x(se^{-2t}) \quad \text{for all} \quad (t,s) \in IR_+ \times IR \quad \text{and all} \quad x \in X,$$

is a Φ-semigroup on X.

Because

$$\|S(t)\| = e^t \quad \text{for all} \quad (t,\gamma) \in IR \times \Gamma,$$

it follows that $\mathbf{S}$ is not u.e.s.,even if

$$\int_0^\infty |S(t)x|_\gamma dt \leq |\gamma| M_x \int_0^\infty e^{-t} dt = |\gamma| M_x < \infty \quad \text{for all} \quad (t,\gamma,x) \in IR_+ \times \Gamma \times X.$$

We observe that from Theorem 1.1.it results

Theorem 1.2. *A C_0-semigroup* $\mathbf{S}$ *on the Banach space X is u.e.s. if and only if there exists a non-decreasing continuous function $R : I\!R_+ \to I\!R_+$ with the properties:*

$(i) R(0) = 0;$

$(ii) R(t) > 0 \quad \textit{for all} \quad t > 0;$

$(iii) R(ts) \leq R(t)R(s) \quad \textit{for all} \quad (t,s) \in I\!R_+^2;$

$(iv) \int\limits_0^\infty R(\|S(t)\|)dt < \infty.$

In this paper we generalize this theorem for the case of u.e.b. Φ-semigroups in locally convex spaces.

2.Preliminares

We start with the following

Lemma 2.1. *Let $f : I\!R_+ \to I\!R_+$ be a non-decreasing function with $f(0) = 0$ and $f(t) > 0$ for all $t > 0$.Then*

$$F : I\!R_+ \to I\!R_+, \quad F(t) = \int\limits_0^t f(s)ds$$

is a continuous bijection.

Proof. We observe that $F(0) = 0$ and F is a non-decreasing function.If there exists $t_1 < t_2$ such that $F(t_1) = F(t_2)$ then

$$\int\limits_{t_1}^{t_2} f(s)ds = F(t_2) - F(t_1) = 0,$$

wich is a contradiction. Hence F is strictly increasing and because $\lim\limits_{t\to\infty} f(t) > 0$ it folllows that

$$\lim_{t\to\infty} F(t) = \int\limits_0^\infty f(s)ds = \infty,$$

which shows that F is also surjective.

Lemma 2.2. *If* $\mathbf{S} = (S(t))_{t\geq 0}$ *is a Φ-semigroup then*

$$(i) \|S(t+s)\|_{\varphi(t+s,\gamma),\gamma} \leq \|S(t)\|_{\varphi(t,\gamma),\gamma} \|S(s)\|_{\varphi(s,\varphi(t,\gamma)),\varphi(t,\gamma)}$$

for all $(t,s,\gamma)\in IR_+^2\times\Gamma$;

$(ii)\|S(nt)\|_{\varphi(nt,\gamma),\gamma}\leq\prod_{k=1}^{n}\|S(t)\|_{\varphi(kt,\gamma),\varphi((k-1)t,\gamma)}$
for all $(t,n,\gamma)\in IR_+\times IN^*\times\Gamma$.

Proof. (i) We observe that

$$|S(t+s)x|_\gamma=|S(t)S(s)x|_\gamma\leq\|S(t)\|_{\varphi(t,\gamma),\gamma}|S(s)x|_{\varphi(t,\gamma)}\leq$$

$$\leq\|S(t)\|_{\varphi(t,\gamma),\gamma}\|S(s)\|_{\varphi(s,\varphi(t,\gamma)),\varphi(t,\gamma)}|x|_{\varphi(s,\varphi(t,\gamma))}=$$
$$=\|S(t)\|_{\varphi(t,\gamma),\gamma}\|S(s)\|_{\varphi(t+s,\gamma),\varphi(t,\gamma)}|x|_{\varphi(t+s,\gamma)},$$

and hence

$$\|S(t+s)\|_{\varphi(t+s,\gamma)}\leq\|S(t)\|_{\varphi(t,\gamma)}\|S(s)\|_{\varphi(t+s,\gamma),\varphi(t,\gamma)}$$

for all $(t,s,\gamma)\in IR_+^2\times\Gamma$.

(ii) It follows from (i) by induction.

Lemma 2.3. *If* **S** *is a* Φ*-semigroup with u.e.b. then*

$$\|S(t+1)\|_{\varphi(t+1,\gamma),\gamma}\leq M_0(\gamma)e^{\omega_0(\gamma)}\|S(s)\|_{\varphi(s,\gamma),\gamma}$$

for all $(t,s,\gamma)\in IR_+^2\times\Gamma$ *with* $s\in[t,t+1]$.

Proof. Indeed,if $t\geq 0,\gamma\in\Gamma$ and $s\in[t,t+1]$ then by Lemma 2.2., we have that

$$\|S(t+1)\|_{\varphi(t+1,\gamma),\gamma}\leq\|S(s)\|_{\varphi(s,\gamma),\gamma}\|S(t+1-s)\|_{\varphi(t+1,\gamma),\varphi(s,\gamma)}\leq$$

$$\leq M(\varphi(s,\gamma))e^{(t+1-s)\omega(\varphi(s,\gamma))}\|S(s)\|_{\varphi(s,\gamma),\gamma}\leq$$
$$\leq M_0(\gamma)e^{\omega_0(\gamma)}\|S(s)\|_{\varphi(s,\gamma),\gamma}.$$

Lemma 2.4. *Let* **S** *be a* Φ*-semigroup with u.e.b. If there exists*
$P:IR_+\times\Gamma\to IR_+$ *such that*
$(i)\|S(t)\|_{\varphi(t,\gamma),\gamma}\leq P(t,\gamma)$ *for all* $(t,\gamma)\in IR_+\times\Gamma$;

$(ii)\lim_{t\to\infty}P(t,\gamma)=0$ *for all* $\gamma\in\Gamma$;

$(iii)P(t,\varphi(s,\gamma))\leq P(t,\gamma)$ *for all* $(t,s,\gamma)\in IR_+^2\times\Gamma$

then **S** *is u.e.s.*

Proof. Let $A : \Gamma \to \mathcal{P}\ (I\!N^*)$ be the function defined by

$$A(\gamma) = \{n \in I\!N^* : P(n,\gamma) < e^{-1}\}.$$

From (ii) it resullts that $A(\gamma)$ is non-empty for all for every $\gamma \in \Gamma$.

If we denote by $n(\gamma) = \min A(\gamma)$ then from $A(\gamma) \subset A(\varphi(t,\gamma))$ and (iii) it results that

$$n(\varphi(t,\gamma)) \leq n(\gamma) \quad \text{for all} \quad (t,\gamma) \in I\!R_+ \times \Gamma.$$

For all $(t,\gamma) \in I\!R_+ \times \Gamma$ there is a natural number p such that $pn(\gamma) \leq t < (p+1)n(\gamma)$.

If $p = 0$ then

$$\|S(t)\|_{\varphi(t,\gamma),\gamma} \leq M(\gamma)e^{t\omega(\gamma)} \leq M_0(\gamma)e^{n(\gamma)\omega_0(\gamma)}.$$

If $p \geq 1$ then

$$\|S(t)\|_{\varphi(t,\gamma),\gamma} \leq \|S(pn(\gamma))\|_{\varphi(pn(\gamma),\gamma),\gamma}\|S(t-pn(\gamma))\|_{\varphi(t,\gamma),\varphi(pn(\gamma),\gamma)} \leq$$

$$\leq M(\varphi(pn(\gamma),\gamma))e^{(t-pn(\gamma))\omega(\varphi(pn(\gamma),\gamma))}\prod_{k=1}^{p}\|S(n(\gamma))\|_{\varphi(kn(\gamma),\gamma),\varphi((k-1)n(\gamma),\gamma)} \leq$$

$$\leq M_0(\gamma)e^{n(\gamma)\omega_0(\gamma)}\prod_{k=1}^{p}P(n(\gamma),\varphi((k-1)n(\gamma),\gamma)) \leq$$

$$\leq M_0(\gamma)e^{n(\gamma)\omega_0(\gamma)}\prod_{k=1}^{p}P(n(\gamma),\gamma) \leq M_0(\gamma)e^{n(\gamma)\omega_0(\gamma)-p} \leq$$

$$\leq M_0(\gamma)e^{n(\gamma)\omega_0(\gamma)+1}e^{-t\nu(\gamma)} = N(\gamma)e^{-t\nu(\gamma)},$$

where

$$N(\gamma) = M_0(\gamma)e^{n(\gamma)\omega_0(\gamma)+1} \quad \text{and} \quad \nu(\gamma) = \frac{1}{n(\gamma)}.$$

Because

$$N_0(\gamma) = \sup_{t\geq 0} N(\varphi(t,\gamma)) \leq M_0(\gamma)e^{n(\gamma)\omega_0(\gamma)+1} < \infty$$

and

$$\nu_0(\gamma) = \inf_{t\geq 0}\nu(\varphi(t,\gamma)) = \inf_{t\geq 0}\frac{1}{n(\varphi(t,\gamma))} \geq \frac{1}{n(\gamma)} > 0,$$

finally we obtain that **S** is u.e.s.

3.The main results

In what follows for every $\varphi : I\!R_+ \times \Gamma \to \Gamma$ we shall denote by $\mathcal{R}_\varphi$ the set of all functions $R : I\!R_+ \times \Gamma \to I\!R_+$ with the following properties

$r_1)R(0,\gamma) = 0$ for all $\gamma \in \Gamma$;

$r_2)R(t,\gamma) > 0$ for all $t > 0$ and all $\gamma \in \Gamma$;

$r_3)\lim\limits_{t\to\infty} R(t,\gamma) = \infty$ for every $\gamma \in \Gamma$;

$r_4)R(t,\gamma) \leq R(t,\varphi(s,\gamma))$ for all $(t,s,\gamma) \in I\!R_+^2 \times \Gamma$;

$r_5)R(s,\gamma) \leq R(t,\gamma)$ for all $(s,t,\gamma) \in I\!R_+^2 \times \Gamma$ with $s \leq t$.

Lemma 3.1. *Let $\varphi \in \Phi$ and $R \in \mathcal{R}_\varphi$.*

Then for every $(r,\gamma) \in I\!R_+ \times \Gamma$ the set

$$B_r(\gamma) = \{t \geq 0 : R(t,\gamma) \leq r\}$$

is bounded and the function

$$\delta : I\!R_+ \times \Gamma \to I\!R_+^*, \quad \delta(r,\gamma) = 1 + \sup B_r(\gamma)$$

satisfies the inequality

$$\delta(r,\varphi(t,\gamma)) \leq \delta(r,\gamma) \quad \textit{for all} \quad (r,t,\gamma) \in I\!R_+^2 \times \Gamma.$$

Proof. From $\lim\limits_{t\to\infty} R(t,\gamma) = \infty$ it follows that $B_r(\gamma)$ is a bounded set for all $(r,\gamma) \in I\!R_+ \times \Gamma$.

On the other hand,by

$$R(t,\gamma) \leq R(t,\varphi(s,\gamma)) \quad \text{for all} \quad (t,s,\gamma) \in I\!R_+^2 \times \Gamma$$

it results that

$$B_r(\varphi(t,\gamma)) \subset B_r(\gamma) \quad \text{for all} \quad (r,t,\gamma) \in I\!R_+^2 \times \Gamma$$

which proves the lemma.

Theorem 3.1. *Let* **S** *be a Φ-semigroup and let $\varphi \in \Phi$ which satisfies Definition 1.1. If there are $R \in \mathcal{R}_\varphi$ and $K : \Gamma \to I\!R_+^*$ such that for all $\gamma \in \Gamma$*

$$(i) K_0(\gamma) = \sup_{t \geq 0} K(\varphi(t,\gamma)) < \infty$$

and

$$(ii) \sup_{t \geq 0} \int_t^{t+1} R(\|S(s)\|_{\varphi(s,\gamma),\gamma}) ds \leq K(\gamma),$$

then **S** *is u.s.*

Proof. If $R \in \mathcal{R}_\varphi$ then by Lemma 2.3. it follows that

$$R\Big(\frac{\|S(t+1)\|_{\varphi(t+1,\gamma),\gamma}}{M_0(\gamma)e^{\omega_0(\gamma)}}, \gamma\Big) \leq \int_t^{t+1} R(\|S(s)\|_{\varphi(s,\gamma),\gamma}, \gamma) ds \leq K_0(\gamma)$$

for all $(t,\gamma) \in I\!R_+ \times \Gamma$.

This inequality shows that

$$\|S(t+1)\|_{\varphi(t+1,\gamma),\gamma} \leq M_0(\gamma)e^{\omega_0(\gamma)}\delta(K_0(\gamma),\gamma) = M_1(\gamma)$$

and hence

$$\|S(t)\|_{\varphi(t,\gamma),\gamma} \leq M_1(\gamma) \quad \text{for all} \quad (t,\gamma) \in [1,\infty) \times \Gamma.$$

If $t \in [0,1]$ then

$$\|S(t)\|_{\varphi(t,\gamma),\gamma} \leq M_0(\gamma)e^{t\omega_0(\gamma)} \leq M_0(\gamma)e^{\omega_0(\gamma)} < M_1(\gamma),$$

which shows that

$$\|S(t)\|_{\varphi(t,\gamma),\gamma} \leq M_1(\gamma) \quad \text{for all} \quad (t,\gamma) \in I\!R_+ \times \Gamma.$$

Moreover we observe that

$$M_0(\varphi(t,\gamma)) = \sup_{s \geq 0} M(\varphi(s,\varphi(t,\gamma))) = \sup_{s \geq 0} M(\varphi(s+t,\gamma)) =$$

$$= \sup_{s \geq t} M(\varphi(s,\gamma)) \leq M_0(\gamma)$$

and similarly

$$\omega_0(\varphi(t,\gamma)) \leq \omega_0(\gamma) \quad \text{and} \quad K_0(\varphi(t,\gamma)) \leq K_0(\gamma)$$

for all $(t,\gamma) \in IR_+ \times \Gamma$.

These inequalities together with Lemma 3.1. imply that

$$M_1(\varphi(t,\gamma)) \leq M_1(\gamma) \geq \|S(t)\|_{\varphi(t,\gamma),\gamma}$$

for all $(t,\gamma) \in IR_+ \times \Gamma$ and hence **S** is u.s.

The main result of this paper is

Theorem 3.2. *A Φ-semigroup* **S** *is u.e.s. if and only if there exists $\varphi \in \Phi$, $R \in \mathcal{R}_\varphi$ and $K : \Gamma \to IR_+^*$ such that*

$$(i) K_0(\gamma) = \sup_{t\geq 0} K(\varphi(t,\gamma)) < \infty$$

and

$$(ii) \int_0^\infty R(\|S(t)\|_{\varphi(t,\gamma),\gamma}, \gamma)dt \leq K(\gamma)$$

for all $\gamma \in \Gamma$.

Proof.Necessity. It results from Definition 1.3. for

$$R(t,\gamma) = t \quad \text{and} \quad K(\gamma) = \frac{N(\gamma)}{\nu(\gamma)},$$

where N and ν are given by Definition 1.3.

Sufficiency. Because

$$\sup_{t\geq 0} \int_t^{t+1} R(\|S(s)\|_{\varphi(s,\gamma),\gamma}, \gamma)dt \leq \int_0^\infty R(\|S(s)\|_{\varphi(s,\gamma),\gamma}, \gamma)ds \leq K(\gamma),$$

for all $\gamma \in \Gamma$, by Theorem 3.1. it results that there exists $M_1 : \Gamma \to IR_+^*$ such that

$$\|S(t)\|_{\varphi(t,\gamma),\gamma} \leq M_1(\gamma) \quad \text{and} \quad M_1(\varphi(t,\gamma)) \leq M_1(t,\gamma),$$

for all $(t,\gamma) \in IR_+ \times \Gamma$ (i.e. **S** is u.s.).

Let $F : IR_+ \times \Gamma \to IR_+$ be the function defined by

$$F(t,\gamma) = \int_0^t R(s,\gamma)ds.$$

By Lemma 2.1. the function $t \to F(t,\gamma)$ is an increasing continuous bijection for every $\gamma \in \Gamma$. If we denote by $f_\gamma = F(\cdot,\gamma)^{-1}$ then from $R \in \mathcal{R}_\varphi$ it follows that $F \in \mathcal{R}_\varphi$,

$$f_{\varphi(t,\gamma)}(s) \leq f_\gamma(s) \quad \text{for all} \quad (t,s,\gamma) \in I\!R_+^2 \times \Gamma.$$

and

$$\int_0^\infty F(\|S(t)\|_{\varphi(t,\gamma),\gamma},\gamma)dt \leq \int_0^\infty \|S(t)\|_{\varphi(t,\gamma),\gamma} R(\|S(t)\|_{\varphi(t,\gamma),\gamma},\gamma)dt \leq$$

$$\leq M_1(\gamma)\int_0^\infty R(\|S(t)\|_{\varphi(t,\gamma),\gamma},\gamma)dt \leq M_1(\gamma)K(\gamma) \leq M_1(\gamma)K_0(\gamma) = M_2(\gamma)$$

for all $\gamma \in \Gamma$.

If we denote by

$$g(t,\gamma) = \frac{\|S(t)\|_{\varphi(t,\gamma),\gamma}}{M_1(\gamma)}$$

then for $t > 0$ and $\gamma \in \Gamma$ we have

$$tF(g(t,\gamma),\gamma) = \int_0^t F(g(t,\gamma)ds \leq \int_0^t F(g(s,\gamma)\|S(t-s)\|_{\varphi(t,\gamma),\varphi(s,\gamma)},\gamma)ds \leq$$

$$\leq \int_0^t F(g(s,\gamma)M_1(\varphi(s,\gamma),\gamma),\gamma)ds \leq \int_0^t F(\|S(s)\|_{\varphi(s,\gamma),\gamma},\gamma)ds \leq M_2(\gamma)$$

and hence

$$\|S(t)\|_{\varphi(t,\gamma),\gamma} \leq P(t,\gamma)$$

for all $t \geq 0$ and $\gamma \in \Gamma$, where

$$P(t,\gamma) = \begin{cases} M_1(\gamma)f_\gamma\left(\frac{M_2(\gamma)}{t}\right), & \text{if} \quad t > 0 \\ 1, & \text{if} \quad t = 0 \end{cases}$$

It is easy to see that

$$\lim_{t\to\infty} P(t,\gamma) = 0 \quad \text{for every} \quad \gamma \in \Gamma$$

and

$$P(t,\varphi(s,\gamma)) \leq P(t,\gamma), \quad \text{for all} \quad (t,s,\gamma) \in IR_+^2 \times \Gamma.$$

An aplication of Lemma 2.4. proves that **S** is u.e.s.

The discrete variant of Theorem 3.2. is given by

Corollary 3.1. *A Φ-semigroup* **S** *is u.e.s. if and only if there exists $\varphi \in \Phi$, $R \in \mathcal{R}_\varphi$ and $K : \Gamma \to IR_+^*$ such that*

$$(i) K_0(\gamma) = \sup_{t \geq 0} K(\varphi(t,\gamma)) < \infty$$

and

$$(ii) \sum_{n=0}^{\infty} R(\|S(n)\|_{\varphi(n,\gamma),\gamma}, \gamma) \leq K(\gamma)$$

for all $\gamma \in \Gamma$.

Proof.Necessity. It is a simple verification for

$$R(t,\gamma) = t \quad \text{and} \quad K(\gamma) = \frac{N(\gamma)e^{\nu(\gamma)}}{e^{\nu(\gamma)} - 1},$$

where N and ν are given by Definition 1.3.

Sufficiency. Let $M_3 : \Gamma \to IR_+^*$ be the function defined by

$$M_3(\gamma) = \sup_{t \geq 0} M(\varphi(t,\gamma))e^{\omega(\varphi(t,\gamma))},$$

where M and ω are given by Definition 1.2.

We observe that

$$\|S(s)\|_{\varphi(s,\gamma),\gamma} \leq \|S(t)\|_{\varphi(t,\gamma),\gamma}\|S(s-t)\|_{\varphi(s,\gamma),\varphi(t,\gamma)} \leq$$

$$\leq M(\varphi(t,\gamma))e^{(s-t)\omega(\varphi(t,\gamma))}\|S(t)\|_{\varphi(t,\gamma),\gamma} \leq M_3(\gamma)\|S(t)\|_{\varphi(t,\gamma),\gamma}$$

for all $(s,t,\gamma) \in IR_+^2 \times \Gamma$ with $s \in [t, t+1]$.

If we denote by

$$R_1 : IR_+ \times \Gamma \to IR_+, \quad R_1(t,\gamma) = R(\frac{t}{M_3(\gamma)}, \gamma)$$

then $R_1 \in \mathcal{R}_\varphi$ and

$$\int_0^\infty R_1(\|S(t)\|_{\varphi(t,\gamma),\gamma}, \gamma)dt = \sum_{n=0}^{\infty} \int_n^{n+1} R_1(\|S(t)\|_{\varphi(t,\gamma),\gamma}, \gamma)dt =$$

$$= \sum_{n=0}^{\infty} R(\frac{\|S(t)\|_{\varphi(t,\gamma),\gamma}}{M_3(\gamma)}, \gamma)dt \leq \sum_{n=0}^{\infty} R(\|S(n)\|_{\varphi(n,\gamma),\gamma}, \gamma) \leq K(\gamma)$$

for all $\gamma \in \Gamma$.

From Theorem 3.2. it results that **S** is u.e.s.

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