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Abstract

This paper aims to construct adaptedness and stochastic integration on Poisson space in the abstract setting of Hilbert spaces with minimal hypothesis, in particular without use of any notion of time or ordering on index sets. In this framework, several types of stochastic integrals are considered on simple processes and extended to larger domains. The results obtained generalize the existing constructions in the Wiener case, unify them, and apply to multi-parameter time.

1 Introduction

The theory of stochastic integration has been developed in an abstract setting, cf. [9], and [19], [20], [21], for the case of abstract Wiener spaces. This construction is closely related to quantum stochastic calculus, cf. [16]. In this paper we present a construction of Poisson stochastic integration in an Hilbert space setting. We consider both the point of views of quantum and anticipating stochastic calculus and since it is based on the Fock space, our approach can be adapted to deal with the Wiener case. Let $\Gamma(H)$ denote the symmetric Fock space over a given Hilbert space H, and let $\otimes$ and $\circ$ respectively denote the ordinary and symmetric tensor product of Hilbert spaces. A notion of adaptedness is introduced on Fock space, relatively to an abstract measure space $(X, \mathcal{F}_X, \mu)$, and allows to extend the Skorohod integral operator via an Itô type isometry to square-integrable adapted vectors (or "processes") in $\Gamma(H) \otimes H$. Then we define the Poisson integral of simple vectors in $\Gamma(H) \otimes H$, and study its extensions and relationship to the Skorohod integral. No ordering is required on the index set X, moreover we do not make use of a mapping from X to $\mathbb{R}$ or of the real spectrum of a self-adjoint operator to induce

an order on X as in [5], resp. [19]. Our integral operator is originally defined for simple process, and if one removes adaptedness assumptions it still admits several extensions via limiting procedures, although no one is canonical in the sense that they all depend on the particular type of approximation chosen for the integrand. In this sense this type of integral is close to the pathwise integrals of forward, backward and Stratonovich types. Our construction of stochastic integration is also valid in the Wiener interpretation of $\Gamma(H)$, but in the Poisson case it presents several important differences with its Brownian counterpart, due to the particular properties of Poisson measures.

In classical probability, a σ-algebra can be identified to a space of bounded measurable functions in $L^\infty(\Omega)$. In quantum probability, $L^\infty(\Omega)$ is replaced with an operator algebra $\mathcal{A}$, σ-algebras are then identified to a sub-algebras of $\mathcal{A}$, and quantum stochastic integrals act on operator processes. The point of view developed in this paper is intermediate between classical and quantum probability. Namely, $L^2(\Omega)$ is replaced with the Hilbert space $\Gamma(H)$, stochastic integrals act on vectors in $\Gamma(H) \otimes H$, and σ-algebras are identified to vector subspaces of $\Gamma(H)$. Our integral has some similarities with quantum spectral stochastic integrals, but its purpose is different since we map vectors in $\Gamma(H) \otimes H$ to vectors in $\Gamma(H)$.

The first two sections of this paper are only relative to the general symmetric Fock space, not to a particular type of random measure (Gaussian or Poisson). The remaining sections deal with the Poisson case, but each construction is related to its Gaussian counterpart. We proceed as follows. In Sect. 2 we give a definition of measurability of vectors in the Fock space $\Gamma(H)$, using a projection system indexed by a σ-algebra on a measurable space X. In Sect. 3 we introduce a notion of strong adaptedness under which the Skorohod integral operator ∇^+ satisfies the Itô isometry

$$\|\nabla^+(u)\|^2_{\Gamma(H)} = \|u\|^2_{\Gamma(H)\otimes H},$$

which allows to extend ∇^+ to a larger class of square-integrable processes without hypothesis on the Fock gradient $\nabla^- u$. The Poisson "pathwise" integral is defined for simple processes in Sect. 4 and its relation to the Skorohod integral is studied in Sect. 5. Under a weak adaptedness condition, the Poisson stochastic integral operator S coincides with the Skorohod integral ∇^+ on simple weakly adapted processes, hence as the Skorohod integral, it can be extended by density to as space of strongly adapted processes. As an application we construct a Fock space-valued multipa-

rameter Poisson process. Other extensions of S without adaptedness conditions are discussed in Sect. 6, in relation with the quantum spectral stochastic integrals of [2]. Sect. 7 recalls the Poisson probabilistic interpretation of the Fock space $\Gamma(H)$, to be used in Sect. 8, and contains an independent remark on the relation between Guichardet Fock space and Poisson space. In Sect. 8 we work in the Poisson interpretation of $\Gamma(H)$, using a Poisson random measure on an abstract measure space, and show that our integral coincides with the classical Itô-Poisson stochastic integral under adaptedness conditions. Without adaptedness conditions it becomes a Stratonovich type anticipating integral.

2   Measurability in Hilbert space with respect to an abstract index set

In this section we give a definition of measurability for vectors belonging to the symmetric Fock space $\Gamma(H) = \bigoplus_{n=0}^{\infty} H^{\circ n}$ (with $H^{\circ 0} = \mathbb{R}$) over H, by using the notion of projection system. If $h_1, \ldots, h_n \in H$ we denote by $h_1 \circ \cdots \circ h_n \in H^{\circ n}$ the symmetrization of $h_1 \otimes \cdots \otimes h_n \in H^{\otimes n}$, $n \geq 1$, and by S the vector space generated by

$$\{h_1 \circ \cdots \circ h_n \ : \ h_1, \ldots, h_n \in H, \ n \in \mathbb{N}\}.$$

Let $\nabla^- : \Gamma(H) \to \Gamma(H) \otimes H$ and $\nabla^+ : \Gamma(H) \otimes H \to \Gamma(H)$ denote the gradient and Skorohod integral operators, densely defined on S as

$$\nabla^- (h_1 \circ \cdots \circ h_n) = \sum_{i=1}^{i=n} \left(h_1 \circ \cdots \circ \hat{h}_i \circ \cdots \circ h_n \right) \otimes h_i,$$

where $\hat{h}_i$ means that h_i is omitted in the product, and

$$\nabla^+ (h_1 \circ \cdots \circ h_n \otimes h) = h_1 \circ \cdots \circ h_n \circ h,$$

extended by linearity and density as closed operators with domains $\mathrm{Dom}(\nabla^-)$ and $\mathrm{Dom}(\nabla^+)$. Let $\psi(f)$, $f \in H$, denote the exponential vector defined as

$$\psi(f) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{\circ n}.$$

Definition 1 *Let $(X, \mathcal{F}_X)$ be a measurable space, and let $(p_A)_{A \in \mathcal{F}_X}$ be a projection system indexed by $(X, \mathcal{F}_X)$ on an abstract Hilbert space H, i.e. a family of self-adjoint operators $p_A : H \to H$, $A \in \mathcal{F}_X$, that satisfies*

$$p_A p_B = p_B p_A = p_{A \cap B}, \quad \forall A, B \in \mathcal{F}_X, \tag{2.1}$$

and

$$\sum_{k=0}^{\infty} p_{A_k} = p_A,$$

for any $A \in \mathcal{F}_X$ and any countable partition $(A_k)_{k \in \mathbb{N}} \subset \mathcal{F}_X$ of A.

The above assumptions imply that $p_\emptyset = 0$. We may further assume as in Sect. 8 that p_X is the identity of H, but this is not necessary in general. The following definition is apparent in [9], [16], [19].

Definition 2 *A vector $F \in \Gamma(H)$ is said to be $\mathcal{F}_A^p$-measurable, $A \in \mathcal{F}_X$, if $F \in \text{Dom}(\nabla^-)$ and*

$$(\mathrm{I_d} \otimes p_{A^c})\nabla^- F = 0,$$

where A^c stands for the complement of A and $\mathrm{I_d}$ denotes the identity on $\Gamma(H)$.

Each projection system $(p_A)_{A \in \mathcal{F}_X}$ determines a family of vector subspaces of $\Gamma(H)$ which play the role of σ-algebras $\mathcal{F}^p = (\mathcal{F}_A^p)_{A \in \mathcal{F}_X}$. The conditional expectation of a vector

$$F = \sum_{n=0}^{\infty} f_n \in \Gamma(H)$$

with respect to $\mathcal{F}_A^p$ is naturally defined as

$$E[F|\mathcal{F}_A^p] = \sum_{n=0}^{\infty} p_A^{\otimes n} f_n.$$

Condition (2.1) ensures the relation

$$E[E[F \mid \mathcal{F}_A^p] \mid \mathcal{F}_B^p] = E[E[F \mid \mathcal{F}_B^p] \mid \mathcal{F}_A^p] = E[F \mid \mathcal{F}_{A \cap B}^p], \quad F \in \Gamma(H), \ A, B \in \mathcal{F}_X.$$

After fixing $\varphi \in H$, the projection system $(p_A)_{A \in \mathcal{F}_X}$ defines a finite measure μ on $(X, \mathcal{F}_X)$ as

$$\mu(A) = (\varphi, p_A \varphi)_H, \quad A \in \mathcal{F}_X.$$

If H is unitarily equivalent to $L^2(X, \mu)$ via the Hahn-Hellinger theorem (cf. [8], [16]), i.e. there exists a unitary operator $U : H \longrightarrow L^2(X, \mu)$, then p_A can be identified to 1_A with $p_A = U^{-1} 1_A U$, however such an identification may not be always possible. In particular, when working with the probabilistic interpretation of $\Gamma(H)$ under a Poisson random measure on a set M, i.e. $H = L^2(M, \sigma)$, we allow the index set X to be different from M.

3   Adaptedness and Itô isometry in Hilbert space

The definitions introduced in this section are not specific to the Poisson case.

Proposition 1 *The Skorohod integral operator ∇^+ satisfies the isometry*

$$\|\nabla^+(u)\|^2_{\Gamma(H)} = \|u\|^2_{\Gamma(H)\otimes H} + \sum_{i,j=1}^{i=n} ((u_i, \nabla^- F_j)_H, (u_j, \nabla^- F_i)_H)_{\Gamma(H)}, \qquad (3.1)$$

for a simple process of the form $u = \sum_{i=1}^{i=n} F_i \otimes u_i$, where $u_i \in H$ and $F_i \in \mathrm{Dom}(\nabla^-)$, $i = 1, \ldots, n$.

Proof. This result is well-known, however it is generally stated in the case where H is a L^2 space. By bilinearity, orthogonality and density it suffices to choose $u = f^{\circ n} \otimes g \in \Gamma(H) \otimes H$, $f, g \in H$, and to note that

$$\|\nabla^+(u)\|^2_{\Gamma(H)} = \|f^{\circ n} \circ g\|^2_{\Gamma(H)} = \frac{1}{(n+1)^2} \left\| \sum_{i=0}^{i=n} f^{\otimes i} \otimes g \otimes f^{\otimes(n-i)} \right\|^2_{\Gamma(H)}$$

$$= \frac{1}{(n+1)^2}((n+1)!(n+1)\|f\|^{2n}_H\|g\|^2_H + (n+1)!n(n+1)\|f\|^{2n-2}_H(f,g)^2_H)$$

$$= n!\|f\|^{2n}_H\|g\|^2_H + (n-1)!n^2\|f\|^{2n-2}_H(f,g)^2_H$$

$$= \|u\|^2_{\Gamma(H)\otimes H} + ((g, \nabla^- f^{\circ n})_H, (g, \nabla^- f^{\circ n})_H)_{\Gamma(H)}. \quad \square$$

This isometry has been related to the quantum Itô formula in [13]. From this formula, a bound on the norm of $\nabla^+(u)$ can be deduced as

$$\|\nabla^+(u)\|^2_{\Gamma(H)} \le \|u\|^2_{\Gamma(H)\otimes H} + \|\nabla^- u\|^2_{\Gamma(H)\otimes H\otimes H}.$$

This bound allows to extend ∇^+ to a domain in $\Gamma(H) \otimes H$, closed for the $\|\cdot\|_{1,2}$ norm defined as

$$\|u\|^2_{1,2} = \|u\|^2_{\Gamma(H)\otimes H} + \|\nabla^- u\|^2_{\Gamma(H)\otimes H\otimes H},$$

however it imposes some regularity on the process u via its gradient $\nabla^- u$.

Definition 3 *Let $\mathcal{U} \subset \Gamma(H) \otimes H$ denote the space of simple processes of the form*

$$u = \sum_{i=1}^{i=n} F_i \otimes (p_{A_i}\varphi_i), \quad F_1, \ldots, F_n \in \mathrm{Dom}(\nabla^-), \quad A_1, \ldots, A_n \in \mathcal{F}_X, \qquad (3.2)$$

$\varphi_1, \ldots, \varphi_n \in H$, $n \in \mathbf{N}$.

If $H = L^2(X, \mu)$, $X = \mathbb{R}$ and $p_A h = 1_A h$, $A \in \mathcal{F}_X$, a simple process $u = \sum_{i=1}^{i=n} F_i 1_{]t_i, t_{i+1}]}$ is usually said to be adapted if F_i is $\mathcal{F}_{[0,t_i]}^p$-measurable, $i = 1, \ldots, n$. It is well-known that in this case, the term $\sum_{i,j=1}^{i=n} ((u_i, \nabla^- F_j)_H, (u_j, \nabla^- F_i)_H)_{\Gamma(H)}$ in (3.1) vanishes and that the Itô isometry

$$\|\nabla^+(u)\|_{\Gamma(H)}^2 = \|u\|_{\Gamma(H) \otimes H}^2$$

holds in place of the Skorohod isometry, thus allowing to extend the Skorohod integral to square-integrable processes indexed by $\mathbb{R}$. In the general case, due to the lack of ordering on the index set X, there is no analog for the "past" of a set A_i.

We introduce in our abstract setting a definition of adaptedness under which the Itô isometry will hold.

Definition 4 *A simple process $u \in \mathcal{U} \subset \Gamma(H) \otimes H$ is said to be strongly adapted if it has a representation of the form*

$$u = \sum_{i=1}^{i=n} F_i \otimes (p_{A_i} \varphi_i),$$

$A_1, \ldots, A_n \in \mathcal{F}_X$, $F_1, \ldots, F_n \in \mathcal{S}$, $\varphi_1, \ldots, \varphi_n \in H$, *such that*

$$(\mathrm{I_d} \otimes p_{A_i}) \nabla^- F_j = 0, \quad or \quad (\mathrm{I_d} \otimes p_{A_j}) \nabla^- F_i = 0, \quad i, j = 1, \ldots, n,$$

i.e. F_j is $\mathcal{F}_{A_i^c}^p$-measurable or F_i is $\mathcal{F}_{A_j^c}^p$-measurable, $i, j = 1, \ldots, n$. We denote by $\mathcal{U}_{Ad}$ the set of strongly adapted simple processes.

The set $\mathcal{U}_{Ad}$ of simple strongly adapted processes is not a vector space: $F \otimes (p_A \varphi) + G \otimes (p_B \varphi)$ may not be strongly adapted even if $F \otimes (p_A \varphi)$ and $G \otimes (p_B \varphi)$ are. Moreover the strong adaptedness property is not independent of the representation chosen for $u \in \mathcal{U}_{Ad}$. If $X = \mathbb{R}_+$, the usual adapted processes $u = \sum_{i=1}^{i=n} F_i \otimes (p_{[0,t_i]} \varphi)$, $0 \le t_1 \le \cdots \le t_n$, $F_1, \ldots, F_n \in \mathcal{S}$ form a vector space which is contained in the set of strongly adapted processes.

Definition 5 *The set of strongly adapted square-integrable processes is defined to be the completion in $\Gamma(H) \otimes H$ of the set of simple strongly adapted processes.*

The Itô isometry holds on strongly adapted processes because the second term of the Skorohod isometry vanishes for such processes.

Proposition 2 *The Itô isometry*

$$\|\nabla^+(u)\|_{\Gamma(H)} = \|u\|_{\Gamma(H)\otimes H} \tag{3.3}$$

holds for all square-integrable strongly adapted process u.

Proof. If $u \in \mathcal{U}_{Ad}$ is of the form

$$u = \sum_{i=1}^{i=n} F_i \otimes (p_{A_i}\varphi_i),$$

$A_1, \ldots, A_n \in \mathcal{F}_X$, $F_1, \ldots, F_n \in \mathcal{S}$, then the condition

$$(\mathrm{I_d} \otimes p_{A_i})\nabla^- F_j = 0, \quad \text{or} \quad (\mathrm{I_d} \otimes p_{A_j})\nabla^- F_i = 0, \quad i, j = 1, \ldots, n,$$

shows that

$$((p_{A_i}\varphi_i, \nabla^- F_j)_H, (p_{A_j}\varphi_j, \nabla^- F_i)_H)_{\Gamma(H)} = 0, \quad i, j = 1, \ldots, n.$$

Hence (3.3) holds from the Skorohod isometry (3.1), and a density argument concludes the proof. $\quad\square$

Although the sum $u + v$ of two square-integrable strongly adapted processes may not be strongly adapted itself, we have $u + v \in \mathrm{Dom}(\nabla^+) \subset \Gamma(H) \otimes H$, and $\nabla^+(u + v)$ satisfies $\nabla^+(u + v) = \nabla^+(u) + \nabla^+(v)$, however the Itô isometry (3.3) may not hold for $u + v$.

4 Poisson integral

We introduce an integral that will be the analog of a pathwise Poisson integral, and will coincide with the Skorohod integral on strongly adapted processes.

Definition 6 *Let $\varphi \in H$ be such that*

$$p_A\varphi = 0 \implies p_A = 0, \quad \forall A \in \mathcal{F}_X. \tag{4.1}$$

A vector $u \in \Gamma(H) \otimes H$ is said to be a simple φ-process if it can be written as

$$u = \sum_{i=1}^{i=n} F_i \otimes (p_{A_i}\varphi), \quad A_1, \ldots, A_n \in \mathcal{F}_X, \; F_1, \ldots, F_n \in \mathcal{S}, \; n \in \mathbb{N}.$$

We denote by $\mathcal{U}^\varphi$ the vector space of simple φ-processes.

The φ-simple processes are relative to a single $\varphi \in H$. In the Gaussian case, we could work with $\mathcal{U}$ itself, i.e. a simple process could depend on n vectors $\varphi_1, \ldots, \varphi_n$ instead of a single $\varphi \in H$, cf. [19] for the case $X = \mathbb{R}$.

Definition 7 *Let $u \in \mathcal{U}^\varphi$ be a simple φ-process written as*

$$u = \sum_{i=1}^{i=n} F_i \otimes (p_{A_i}\varphi), \quad F_1, \ldots, F_n \in \mathcal{S}, \ A_1, \ldots, A_n \in \mathcal{F}_X, \ n \geq 1. \qquad (4.2)$$

We define the Poisson stochastic integral of u as

$$S(u) = \sum_{i=1}^{i=n} \nabla^+(F_i \otimes (p_{A_i}\varphi)) + \nabla^+((\mathrm{Id} \otimes p_{A_i})\nabla^- F_i) + (p_{A_i}\varphi, \nabla^- F_i)_H. \qquad (4.3)$$

This definition is formulated on the Fock space $\Gamma(H)$ and it will have a concrete meaning under the Poisson probabilistic interpretation of $\Gamma(H)$ in Sect. 8. The operator S is in fact analog to a pathwise Stratonovich type integral.

Proposition 3 *The definition (4.3) of $S(u)$ is independent of the particular representation (4.2) chosen for u. Moreover, S is linear on the vector space $\mathcal{U}^\varphi$ of simple φ-processes:*

$$S(u + v) = S(u) + S(v), \quad u, v \in \mathcal{U}^\varphi.$$

Proof. This proof uses the assumptions of Def. 1 and Def. 6. Let $n \geq 1$ and $F_1, \ldots, F_n, G_1, \ldots, G_n \in \mathcal{S}, \ A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathcal{F}_X$, with

$$A_i \cap A_j = \emptyset, \ B_i \cap B_j = \emptyset, \quad , \ i \neq j, \ i, j = 1, \ldots, n. \qquad (4.4)$$

Assume that u has two representations

$$u = \sum_{i=1}^{i=n} F_i \otimes (p_{A_i}\varphi) = \sum_{i=1}^{i=n} G_i \otimes (p_{B_i}\varphi).$$

Applying successively $\mathrm{Id} \otimes (p_{B_j})$ and $\mathrm{Id} \otimes (p_{A_i})$ on both sides we obtain

$$\sum_{i=1}^{i=n} F_i \otimes \left(p_{B_j \cap A_i}\varphi\right) = G_j \otimes (p_{B_j}\varphi), \quad \text{and} \quad F_i \otimes (p_{A_i}\varphi) = \sum_{j=1}^{j=n} G_j \otimes \left(p_{A_i \cap B_j}\varphi\right).$$

By scalar product with $p_{A_i \cap B_j}\varphi$, this implies from (4.4):

$$F_i = G_j \quad \text{if } p_{A_i \cap B_j}\varphi \neq 0, \ i, j = 1, \ldots, n,$$

and

$$p_{A_i}\varphi = \sum_{j=1}^{j=n} p_{A_i \cap B_j}\varphi, \quad p_{B_j}\varphi = \sum_{i=1}^{i=n} p_{B_j \cap A_i}\varphi,$$

i.e. from the assumption (4.1) on φ:

$$p_{A_i} = \sum_{j=1}^{j=n} p_{A_i \cap B_j}, \quad \text{and} \quad p_{B_j} = \sum_{i=1}^{i=n} p_{B_j \cap A_i}.$$

Consequently,

$$\sum_{i=1}^{i=n} \nabla^+(F_i \otimes (p_{A_i}\varphi)) + \nabla^+((I_d \otimes p_{A_i})\nabla^- F_i) + (p_{A_i}\varphi, \nabla^- F_i)_H$$

$$= \sum_{i,j=1}^{i=n} \nabla^+(F_i \otimes p_{A_i \cap B_j}\varphi) + \nabla^+((I_d \otimes p_{A_i \cap B_j})\nabla^- F_i) + (p_{A_i \cap B_j}\varphi, \nabla^- F_i)_H$$

$$= \sum_{i,j=1}^{i=n} \nabla^+(G_j \otimes p_{A_i \cap B_j}\varphi) + \nabla^+((I_d \otimes p_{A_i \cap B_j})\nabla^- G_j) + (p_{A_i \cap B_j}\varphi, \nabla^- G_j)_H$$

$$= \sum_{j=1}^{i=n} \nabla^+(G_j \otimes (p_{B_j}\varphi)) + \nabla^+((I_d \otimes p_{B_j})\nabla^- G_j) + (p_{B_j}\varphi, \nabla^- G_j)_H.$$

If $A_1, \ldots, A_n$ are not disjoint sets we choose a partition $(C_{i,j})_{1 \le i,j \le n}$ of $A_1 \cup \cdots \cup A_n$ such that for all $i = 1, \ldots, n$, the set A_i is written as $A_i = \bigcup_{j=1}^{j=n} C_{i,j}$. We have

$$u = \sum_{i=1}^{i=n} F_i \otimes (p_{A_i}\varphi) = \sum_{i,j=1}^{i=n} F_i \otimes \left(p_{C_{i,j}}\varphi\right), \quad p_{A_i} = \sum_{j=1}^{j=n} C_{i,j}, \quad (4.5)$$

and

$$\sum_{i=1}^{i=n} \nabla^+(F_i \otimes (p_{A_i}\varphi)) + \nabla^+((I_d \otimes p_{A_i})\nabla^- F_i) + (p_{A_i}\varphi, \nabla^- F_i)_H$$

$$= \sum_{i,j=1}^{i=n} \nabla^+(F_i \otimes (p_{C_{i,j}}\varphi)) + \nabla^+((I_d \otimes p_{C_{i,j}})\nabla^- F_i) + (p_{C_{i,j}}\varphi, \nabla^- F_i)_H$$

$$= \sum_{j=1}^{i=n} \nabla^+(G_j \otimes (p_{B_j}\varphi)) + \nabla^+((I_d \otimes p_{B_j})\nabla^- G_j) + (p_{B_j}\varphi, \nabla^- G_j)_H.$$

In order to show that S is linear on the simple φ-processes $\mathcal{U}^\varphi$ we choose two simple φ-processes

$$u = \sum_{i=1}^{i=n} F_i \otimes (p_{A_i}\varphi), \quad F_1, \ldots, F_n \in \mathcal{S}, \ A_1, \ldots, A_n \in \mathcal{F}_X, \ n \ge 1,$$

and

$$v = \sum_{i=1}^{i=n} G_i \otimes (p_{B_i}\varphi), \quad G_1, \ldots, G_n \in \mathcal{S}, \ B_1, \ldots, B_n \in \mathcal{F}_X, \ n \ge 1,$$

with

$$A_i \cap A_j = \emptyset, \ B_i \cap B_j = \emptyset, \ , \ i \ne j, \ i,j = 1, \ldots, n.$$

We have

$$u + v = \sum_{i=1}^{i=n} F_i \otimes (p_{A_i}\varphi) + \sum_{i=1}^{i=n} G_i \otimes (p_{B_i}\varphi) = \sum_{i=1}^{i=m} H_i \otimes (p_{D_i}\varphi), \qquad (4.6)$$

where $D_1, \ldots, D_m \in \mathcal{F}_X$ are disjoint sets with

$$H_j = \begin{cases} F_i & \text{if } D_j \subset A_i \setminus B_i, \\ G_i & \text{if } D_j \subset B_i \setminus A_i, \\ F_i + G_i & \text{if } D_j \subset A_i \cap B_i, \end{cases}$$

$i = 1, \ldots, n$, $j = 1, \ldots, m$. Moreover,

$$\sum_{j=1}^{j=m} (\mathrm{I_d} \otimes p_{D_j})\nabla^- H_j = \sum_{i=1}^{j=n} (\mathrm{I_d} \otimes p_{A_i})\nabla^- F_i + \sum_{i=1}^{j=n} (\mathrm{I_d} \otimes p_{B_i})\nabla^- G_i,$$

hence $S(u + v) = S(u) + S(v)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

In the Gaussian case, the above proposition is obvious since the analog of S is defined as

$$T(u) = \sum_{i=1}^{i=n} \nabla^+(F_i \otimes (p_{A_i}\varphi_i)) + (p_{A_i}\varphi_i, \nabla^- F_i)_H = \nabla^+(u) + \text{trace } (\nabla^- u), \quad u \in \mathcal{U}. \ (4.7)$$

Remark 1 *Given $\varphi, \psi \in H$ satisfying (4.1), the operator S is linear on $\mathcal{U}^\varphi$ and on $\mathcal{U}^\psi$, but not on the linear space spanned by $\mathcal{U}^\varphi$ and $\mathcal{U}^\psi$.*

More precisely, let $u \in \mathcal{U}^\varphi$, $v \in \mathcal{U}^\psi$, be simple φ-process, resp. ψ-process, with

$$u = \sum_{i=1}^{i=n} F_i \otimes (p_{A_i}\varphi), \qquad v = \sum_{i=1}^{i=n} F_i \otimes (p_{B_i}\psi),$$

then $u + v$ may be a simple $\varphi + \psi$-process (if $A_i = B_i$, $i = 1, \ldots, n$) but even in this case we may not have $S(u + v) = S(u) + S(v)$:

$$\begin{aligned} S(u + v) &= \sum_{i=1}^{i=n} \nabla^+(F_i \otimes p_{A_i}(\varphi + \psi) + (\mathrm{I_d} \otimes p_{A_i})\nabla^- F_i) + (p_{A_i}(\varphi + \psi), \nabla^- F_i)_H \\ &= S(u) + S(v) - \sum_{i=1}^{i=n} \nabla^+((\mathrm{I_d} \otimes p_{A_i})\nabla^- F_i). \end{aligned}$$

The linearity property of S holds on a given class of processes which is determined by φ. This corresponds to the fact that Poisson random measures with same jump sizes are stable by addition but not by linearity. The Gaussian setting, however, is fully linear because in this case the term $\nabla^+((\mathrm{I_d} \otimes p_{A_i})\nabla^- F_i)$ in (4.3) is absent of the definition (4.7) of $T(u)$.

5   Skorohod and Poisson integrals

In this section we deal with the relationship between S and the Skorohod integral operator ∇^+. First we note that S coincides with ∇^+ on a vector space of weakly adapted simple φ-processes which is larger than the set of strongly adapted simple φ-processes. Under the strong adaptedness condition, S can (as ∇^+) be extended to a class of square-integrable processes via the Itô isometry (3.3).

Definition 8 *A simple process $u \in \mathcal{U} \subset \Gamma(H) \otimes H$ is said to be weakly adapted if it has a representation*

$$u = \sum_{i=1}^{i=n} F_i \otimes (p_{A_i}\varphi_i), \quad F_1, \ldots, F_n \in \mathcal{S}, \quad \varphi_1, \ldots, \varphi_n \in H, \ n \in \mathbb{N}, \qquad (5.1)$$

such that

$$(\mathrm{Id} \otimes p_{A_i})\nabla^- F_i = 0, \qquad i = 1, \ldots, n. \qquad (5.2)$$

We denote by $\mathcal{U}_{ad}^{\varphi}$ the set of φ-processes in $\mathcal{U}^{\varphi}$ that are weakly adapted.

This definition states that a simple process $u = \sum_{i=1}^{i=n} F_i \otimes (p_{A_i}\varphi_i)$ is weakly adapted if F_i is $\mathcal{F}_{A_i^c}^p$-measurable, $i = 1, \ldots, n$. The weak adaptedness condition is in general dependent on a particular representation (5.1) of u. In the case of weakly adapted simple φ-processes the situation is different.

Proposition 4 *All representations (5.1) of a given simple weakly adapted φ-process $u \in \mathcal{U}_{ad}^{\varphi}$ satisfy condition (5.2). Moreover, the set $\mathcal{U}_{ad}^{\varphi}$ of weakly adapted simple φ-processes forms a vector space.*

Proof. Let $n \geq 1$ and $F_1, \ldots, F_n, G_1, \ldots, G_n \in \mathcal{S}$, $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathcal{F}_X$ with $A_i \cap A_j = \emptyset$ and $B_i \cap B_j = \emptyset$, $i \neq j$, $i, j = 1, \ldots, n$. Assume that

$$u = \sum_{i=1}^{i=n} F_i \otimes (p_{A_i}\varphi) = \sum_{j=1}^{j=n} G_j \otimes \left(p_{B_j}\varphi\right),$$

and that the representation $u = \sum_{j=1}^{j=n} G_j \otimes \left(p_{B_j}\varphi\right)$ satisfies (5.2). We have as in Prop. 3

$$p_{A_i} = \sum_{j=1}^{j=n} p_{A_i \cap B_j}, \quad p_{B_j} = \sum_{i=1}^{i=n} p_{B_j \cap A_i}, \quad \text{and } F_i = G_j \text{ if } p_{A_i \cap B_j} \neq 0, \ i, j = 1, \ldots, n,$$

hence

$$
\begin{aligned}
(\mathrm{I_d} \otimes p_{A_i})\nabla^- F_i &= \sum_{j=1}^{j=n} (\mathrm{I_d} \otimes p_{A_i \cap B_j})\nabla^- F_i = \sum_{j=1}^{j=n} (\mathrm{I_d} \otimes p_{A_i \cap B_j})\nabla^- G_j \\
&= \sum_{j=1}^{j=n} (\mathrm{I_d} \otimes p_{A_i})(\mathrm{I_d} \otimes p_{B_j})\nabla^- G_j = 0,
\end{aligned}
$$

i.e. (5.2) is also valid for all representations $u = \sum_{i=1}^{i=n} F_i \otimes (p_{A_i}\varphi)$ of u, with $A_i \cap A_j = \emptyset$, $i \neq j$, $i,j = 1,\ldots,n$. If $A_1,\ldots,A_n$ are not disjoint sets we have the expression (4.5):

$$
u = \sum_{i=1}^{i=n} F_i \otimes (p_{A_i}\varphi) = \sum_{j=1}^{j=n} G_j \otimes \left(p_{B_j}\varphi\right) = \sum_{j=1}^{j=m}\sum_{i=1}^{i=n} F_i \otimes \left(p_{C_{i,j}}\varphi\right).
$$

From the above discussion we have $(\mathrm{I_d} \otimes p_{C_{i,j}})\nabla^- F_i = 0$, $i = 1,\ldots,n$, $j = 1,\ldots,m$, hence

$$
(\mathrm{I_d} \otimes p_{A_i})\nabla^- F_i = \sum_{j=1}^{j=m} (\mathrm{I_d} \otimes p_{C_{i,j}})\nabla^- F_i = 0, \quad i = 1,\ldots,n.
$$

In order to show that the weakly adapted processes form a vector space we proceed as in the proof of the linearity of S on the simple φ-processes in $\mathcal{U}^\varphi$ (Prop. 3): we choose two φ-processes

$$
u = \sum_{i=1}^{i=n} F_i \otimes (p_{A_i}\varphi), \quad F_1,\ldots,F_n \in \mathcal{S}, \ A_1,\ldots,A_n \in \mathcal{F}_X, \ n \geq 1,
$$

and

$$
v = \sum_{i=1}^{i=n} G_i \otimes (p_{B_i}\varphi), \quad G_1,\ldots,G_n \in \mathcal{S}, \ B_1,\ldots,B_n \in \mathcal{F}_X, \ n \geq 1,
$$

with

$$
A_i \cap A_j = \emptyset, \quad B_i \cap B_j = \emptyset, \quad i \neq j, \ i,j = 1,\ldots,n.
$$

Writing as in (4.6) of Prop. 3

$$
u + v = \sum_{i=1}^{i=m} H_i \otimes (p_{D_i}\varphi)
$$

we check that if u and v are weakly adapted, then $(\mathrm{I_d} \otimes p_{D_i})\nabla^- H_i = 0$, i.e. $u + v$ is weakly adapted. $\qquad\square$

If $X = \mathbb{R}_+$ and $(E(t))_{t \in \mathbb{R}_+}$ is an adapted operator process in the sense of quantum stochastic calculus, cf. [10], then $(E(t)\mathbf{1})_{t \in \mathbb{R}_+}$ is weakly adapted in our sense, where $\mathbf{1}$ denotes the vacuum vector in $\Gamma(H)$. The operators S and ∇^+ coincide on simple weakly adapted φ-processes but the Itô isometry (and hence the strong adaptedness

condition) is needed to extend this identity to a closed space of processes. The space of square-integrable strongly adapted φ-processes is the completion in $\Gamma(H) \otimes H$ of the space of strongly adapted simple φ-processes.

Proposition 5 *Let $u \in \mathcal{U}_{ad}^{\varphi}$ be a simple weakly adapted φ-process. We have*

$$S(u) = \nabla^+(u),$$

i.e. the Poisson and Skorohod integrals of u coincide. This relation extends to the square-integrable strongly adapted φ-processes via the Itô isometry

$$\|\nabla^+(u)\|_{\Gamma(H)} = \|S(u)\|_{\Gamma(H)} = \|u\|_{\Gamma(H) \otimes H}.$$

Proof. We apply the definition (4.3) of S, the fact that $(\mathrm{I}_d \otimes p_{A_i})\nabla^- F_i = 0$, since F_i is $\mathcal{F}_{A_i^c}^p$-measurable, $i = 1, \ldots, n$, and use the self-adjointness of p_{A_i} which implies $(p_{A_i}\varphi, \nabla^- F_i)_H = (\varphi, (\mathrm{I}_d \otimes p_{A_i})\nabla^- F_i)_H = 0$ in $\Gamma(H)$. Since strongly adapted processes are also weakly adapted and satisfy the Itô isometry for ∇^+, the operators S and ∇^+ coincide on the set of square-integrable strongly adapted φ-processes. □

In the Gaussian case we also have $T(u) = \nabla^+(u)$ if $u \in \mathcal{U}$ is a simple weakly adapted process, and T can be extended as in Prop. 5 to the completion of the set $\mathcal{U}_{Ad}$ of simple strongly adapted processes.

A first application of the operator S is the construction of a multiparameter Poisson process in Fock space. Taking $X = \mathbb{R}^d$ with its canonical (partial) ordering "$\prec$" we let $\bar{t} = (t_1, \ldots, t_d)$,

$$[\bar{s}, \bar{t}] = [s_1, t_1] \times \cdots \times [s_d, t_d], \quad \bar{s}, \bar{t} \in \mathbb{R}^d, \quad \bar{s} \prec \bar{t},$$

and we construct the *d*-parameter compensated Poisson φ-process in $\Gamma(H)$ as

$$\tilde{n}_{\bar{t}} = p_{[0,\bar{t}]}\varphi, \quad \bar{t} \in \mathbb{R}_+^d.$$

This process has intensity $\mu([0, \bar{t}]) = \|p_{[0,\bar{t}]}\varphi\|_H^2$, $\bar{t} \in \mathbb{R}_+^d$, and φ controls the "size" of jumps as noted in Remark 1. The Hilbert space H needs not be equal to $L^2(\mathbb{R}_+^d)$. In this case we can adopt the notation

$$\int_{\mathbb{R}_+^d} u \, d\tilde{n}_{\bar{t}} := S(u)$$

for the compensated Poisson stochastic integral of a simple φ-process $u \in \mathcal{U}^\varphi$ with respect to $(\tilde{n}_{\bar{t}})_{\bar{t} \in \mathbb{R}^d_+}$. Let u be a simple φ-process of the form

$$u = \sum_{i=1}^{i=n} F_i \otimes (p_{[\bar{s}_i, \bar{s}_{i+1}]} \varphi).$$

If $\bar{s}_i \prec \bar{s}_{i+1}$, $i = 1, \ldots, n - 1$, and u is adapted in the usual sense, i.e. F_j is $\mathcal{F}^p_{[0,\bar{s}_i]}$-measurable, $1 \leq j \leq i \leq n$, then F_j is $\mathcal{F}^p_{[\bar{s}_i, \bar{s}_{i+1}]^c}$-measurable, $1 \leq j \leq i \leq n$, and u is strongly adapted (in particular it is also weakly adapted in our sense). In this case,

$$\int_{\mathbb{R}^d_+} u d\tilde{n} := S(u) = \nabla^+(u)$$

and this relation extends to square-integrable strongly adapted processes by density from Prop. 5.

6   Other extensions of S as a linear operator on $\mathcal{U}_\varphi$

In this section we deal with the possibility to extend the operator S without the strong adaptedness condition. Clearly, the operator S is not closable because it involves traces of the gradient ∇^- and there is no canonical way to extend it to a larger domain. We mention two possibilities of extension, one of them being based on the relation of S with quantum spectral stochastic integrals.

a) Let $u \in \text{Dom}(\nabla^-) \subset \Gamma(H) \otimes H$, and let Π be a set of partitions $\pi = \{A_1, \ldots, A_n\} \in \mathcal{F}_X$ of X with $\mu(A_i) > 0$, $i = 1, \ldots, n$, and $|\pi| = \sup_{0 \leq i \leq n-1} \mu(A_i)$, $\pi \in \Pi$, such that

(i) we have

$$\limsup_{\substack{|\pi| \to 0 \\ \pi \in \Pi}} \frac{1}{\mu(A)} \left\| \frac{(u, p_A\varphi)_H}{\mu(A)} \otimes (p_A\varphi) - (\mathrm{I_d} \otimes p_A)u \right\|^2_{1,2} = 0,$$

(ii) $(u, p_A\varphi)_H \in \text{Dom}(\nabla^-)$, $\forall A \in \pi$, and there exists an element of $\Gamma(H) \otimes H$ denoted trace $\nabla^- u$, such that

$$\limsup_{\substack{|\pi| \to 0 \\ \pi \in \Pi}} \frac{1}{\mu(A)} \left\| \frac{1}{\mu(A)} (\mathrm{I_d} \otimes p_A)\nabla^-(u, p_A\varphi)_H - (\mathrm{I_d} \otimes p_A)\text{trace } \nabla^- u \right\|^2_{1,2} = 0.$$

(This requires the existence of at least one sequence $(\pi_n)_{n \in \mathbb{N}} \subset \Pi$ such that $\lim_{n \to \infty} |\pi_n| = 0$). We let

$$u_\pi = \sum_{A \in \pi} \frac{(u, p_A\varphi)_H}{\mu(A)} \otimes (p_A\varphi) \in \mathcal{U}^\varphi, \tag{6.1}$$

and

$$\text{trace } \nabla^- u_\pi = \sum_{A \in \pi} \frac{1}{\mu(A)} (\text{Id} \otimes p_A) \nabla^- (u, p_A \varphi)_H.$$

Under the above assumptions, u_π and $(\text{trace } \nabla^- u_\pi, \varphi)_H$ converge respectively to u and $(\text{trace } \nabla^- u_\pi, \varphi)_H$ in $\Gamma(H)$, and trace $\nabla^- u_\pi$ converges to trace $\nabla^- u$ for the $\|\cdot\|_{1,2}$ norm as $|\pi|$ goes to 0, due to the bounds

$$\|u_\pi - u\|_{1,2}^2 \leq \mu(X) \sup_{A \in \pi} \frac{1}{\mu(A)} \left\| \frac{(u, p_A \varphi)_H}{\mu(A)} \otimes (p_A \varphi) - (\text{Id} \otimes p_A) u \right\|_{1,2}^2,$$

$$\|\text{trace } \nabla^- u_\pi - \text{trace } \nabla^- u\|_{1,2}^2$$
$$\leq \mu(X) \sup_{A \in \pi} \frac{1}{\mu(A)} \left\| \frac{1}{\mu(A)} (\text{Id} \otimes p_A) \nabla^- (u, p_A \varphi)_H - (\text{Id} \otimes p_A) \text{trace } \nabla^- u \right\|_{1,2}^2,$$

and

$$\|(\text{trace } \nabla^- u_\pi, \varphi)_H - (\text{trace } \nabla^- u, \varphi)_H\|_{\Gamma(H)}^2 \leq \|\text{trace } \nabla^- u_\pi - \text{trace } \nabla^- u\|_{\Gamma(H) \otimes H}^2 \|\varphi\|_H^2.$$

Under the above assumptions we can define $S(u)$ as the limit in $\Gamma(H)$

$$S(u) = \lim_{\substack{|\pi| \to 0 \\ \pi \in \Pi}} S(u_\pi) = \nabla^+ (u) + \nabla^+ (\text{trace } \nabla^- u) + (\text{trace } \nabla^- u, \varphi)_H. \qquad (6.2)$$

An example of a vector $u \in \Gamma(H) \otimes H$ satisfying the above assumptions is provided by letting $u = \sum_{n=0}^\infty F_i \otimes p_{B_i} \varphi \in \text{Dom}(\nabla^+) \subset \Gamma(H) \otimes H$, where $(B_i)_{i \in \mathbb{N}} \subset \mathcal{F}_X$ is a disjoint family and $(F_i)_{i \in \mathbb{N}} \subset \text{Dom}(\nabla^-)$, with $\sum_{i=0}^\infty \|(\text{Id} \otimes p_{B_i}) \nabla^- F_i\|_{1,2}^2 < \infty$. In this case, trace $\nabla^- u$ is defined to be $\sum_{i=0}^\infty (\text{Id} \otimes p_{B_i}) \nabla^- F_i$, and (6.2) defines $S(u)$. Conditions (i)-(ii) are naturally satisfied since $\frac{(u, p_A \varphi)_H}{\mu(A)} \otimes (p_A \varphi) = (\text{Id} \otimes p_A) u$ and $\frac{1}{\mu(A)} (\text{Id} \otimes p_A) \nabla^- (u, p_A \varphi)_H = (\text{Id} \otimes p_A) \text{trace } \nabla^- u$, $A \in \pi$, $\pi \in \Pi$. Here, Π can be defined as the set of partitions $\pi = \{A_1, \ldots, A_n\} \in \mathcal{F}_X$ of X with $\mu(A_i) > 0$, $i = 1, \ldots, n$, such that all $A \in \Pi$ is contained in a B_j for some $j \in \mathbb{N}$.

b) The integral $S(u)$ can also be linked to the quantum spectral stochastic integrals of [2], [3], which provide a different extension of S. Given a simple operator process $(E(x))_{x \in X}$ defined as $E(x) = \sum_{i=1}^{i=n} E_i 1_{A_i}(x)$, $x \in X$, where $E_1, \ldots, E_n : \Gamma(H) \longrightarrow \Gamma(H)$ are bounded operators, the quantum spectral stochastic integral of $(E(x))_{x \in X}$ is the operator denoted by

$$\int_X E(x)(a_\varphi^+(p(dx)) + a_\varphi^-(p(dx)) + a_\varphi^\circ(p(dx)))$$

and defined on exponential vectors $\psi(f) := \sum_{n=0}^{\infty} \frac{1}{n!} f^{\circ n}$, $f \in H$, as

$$\left(\int_X E(x)(a_\varphi^+(p(dx)) + a_\varphi^-(p(dx)) + a_\varphi^\circ(p(dx))) \right) \psi(f)$$

$$= \sum_{i=1}^{i=n} \nabla^+(E_i\psi(f) \otimes (p_{A_i}\varphi)) + (\nabla^- E_i\psi(f), p_{A_i}\varphi)_H + \nabla^+((\mathrm{Id} \otimes p_{A_i})\nabla^- E_i\psi(f)).$$

Given u a simple process defined as $u = \sum_{i=1}^{i=n}(E_i\psi(f)) \otimes (p_{A_i}\varphi)$, we have the relation

$$S(u) = \left(\int_X E(x)(a_\varphi^+(p(dx)) + a_\varphi^-(p(dx)) + a_\varphi^\circ(p(dx))) \right) \psi(f), \quad f \in H.$$

If $X = [0,1]$ and $(E(t))_{t\in[0,1]}$ is an adapted operator process such that $t \mapsto E(t)\psi(f)$ is continuous from $[0,1]$ into $\Gamma(H)$, $f \in H$, then as a consequence of Lemma 3.2 of [2] the quantum spectral stochastic integral of $(E(x))_{x\in[0,1]}$ is well-defined on the exponential vector $\psi(f)$ in the sense that

$$\lim_{|\pi|\to 0} \sum_{i=1}^{i=n} S(E(t_i)\psi(f) \otimes (p_{[t_i,t_{i+1}]}\varphi))$$

exists in $\Gamma(H)$, with $\pi = \{0 < t_1 < \cdots < t_n < 1\}$. However, X should be an interval of $\mathbb{R}$ and this requires to consider processes such as $(E(t)\psi(f))_{t\in[0,1]}$, which then satisfy a condition similar to (ii) above, due to the particular form of exponential vectors which implies $\nabla^-\psi(f) = \psi(f) \otimes f$.

7   Isomorphism of Guichardet space and Poisson space

This section recalls the Poisson probabilistic interpretation of the Fock space $\Gamma(H)$, which will be used in the next section, and presents a remark on the relations between Poisson space, Fock space and Guichardet Fock space. The Guichardet interpretation of $\Gamma(H)$, cf. [7], has been used in connection with quantum stochastic integrals, cf. [4], [5], usually in the particular case $M = \mathbb{R}_+$. We assume that $H = L^2(M, \mathcal{G}, \sigma)$, where $(M, \mathcal{G}, \sigma)$ is a measure space with finite diffuse measure σ. In this case, $H^{\circ n}$ is the space $\hat{L}^2(M^n)$ of symmetric square-integrable functions on M^n, $n \geq 1$. The gradient $\nabla^- F \in \Gamma(H) \otimes H$ of $F \in \mathrm{Dom}(\nabla^-) \subset \Gamma(H)$ is denoted as $(\nabla_x F)_{x\in M}$ and $u \in \Gamma(H) \otimes H$ is denoted by $(u(x))_{x\in M}$. Let $\mathcal{P}_0 = \{\emptyset\}$,

$$\mathcal{P}_n = \{(x_1, \ldots, x_n) \in M^n : x_i \neq x_j, \ i \neq j, 1 \leq i, j \leq n\}, \quad n \geq 1$$

and $\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$. Let $\mathcal{A}$ denote the collection of subsets of B of $\mathcal{P}$ such that $B \cap \mathcal{P}_n \in \mathcal{G}^{\otimes n}$, $n \geq 1$. The measure σ defines a measure $\sigma^{\otimes n}$ on $\mathcal{P}_n$, $n \geq 1$, and a measure Q on $(\mathcal{P}, \mathcal{A})$ as

$$Q(B) = \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{\otimes n}(B \cap \mathcal{P}_n),$$

i.e.

$$Q = \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{\otimes n}$$

with the convention $\sigma^{\otimes 0}(\{\emptyset\}) = 1$. In the Guichardet interpretation of $\Gamma(H)$, each element $F = \sum_{n=0}^{\infty} f_n \in \Gamma(H)$ is isometrically identified to a square-integrable function in $L^2(\mathcal{P}, Q)$. The Guichardet interpretation of $\Gamma(H)$ is not a probabilistic interpretation because it does not induce a product on $\Gamma(H)$.

On the other hand, the underlying probability space of a Poisson random measure $(N(A))_{A \in \mathcal{G}}$ on M can be taken equal to $\mathcal{P}$. We identify each element $\omega \in \mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$ to a (finite) sum of Dirac measures

$$\omega \sim \sum_{x \in \omega} \delta_x,$$

and for $A \in \mathcal{G}$ we define $N(A) : \mathcal{P} \longrightarrow \mathbf{R}_+$ as

$$N(A)(\omega) = \omega(A) = \mathrm{card}(\{x \in \omega \ : \ x \in A\}).$$

The compensated random measure $\tilde{N}$ is defined by $\tilde{N}(A) = N(A) - \sigma(A)$. Let P denote the Poisson probability measure with intensity σ on $(\mathcal{P}, \mathcal{A})$, such that for disjoint sets $A_1, \ldots, A_n \in \mathcal{G}$, the applications $N(A_1), \ldots, N(A_n)$ are independent Poisson random variables with respective intensities $\sigma(A_1), \ldots, \sigma(A_n)$, $n \geq 1$. This measure can be written as

$$P(B) = e^{-\sigma(B)} \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{\otimes n}(B \cap \mathcal{P}_n), \quad B \in \mathcal{A}.$$

In this setting, $\omega \in \mathcal{P}_n$ corresponds to a Poisson sample of n points in M, we refer to [14] for this construction of the Poisson measure. The n-th order multiple stochastic integral of a symmetric function $f_n \in \hat{L}^2(M^n)$ is the functional in $L^2(\mathcal{P}, P)$ defined as

$$I_n(f_n) = \int_{\mathcal{P}_n} f_n \, d(\omega - \sigma) \cdots d(\omega - \sigma), \quad \omega \in \mathcal{P},$$

with the isometry

$$\|I_n(f_n)\|_{L^2(\mathcal{P}, P)}^2 = n! \|f_n\|_{L^2(M^n, \sigma^{\otimes n})}^2,$$

cf. e.g. [11], [18]. The isometric isomorphisms between the Fock space $\Gamma(L^2(M, \sigma))$, the Guichardet Fock space $L^2(\mathcal{P}, Q)$ and the Poisson space $L^2(\mathcal{P}, P)$ can be described as follows:

$$L^2(\mathcal{P}, Q) \quad \overset{\mathcal{I}}{\longleftarrow} \quad \Gamma(H) \quad \overset{\mathcal{J}}{\longrightarrow} \quad L^2(\mathcal{P}, P)$$
$$n! f_n \quad \longleftarrow \quad f_n \quad \hookrightarrow \quad I_n(f_n)$$

The isometric isomorphism between $\Gamma(H)$ and the Poisson space $L^2(\mathcal{P}, P)$ will be denoted by $\mathcal{J}$, i.e. $\mathcal{J}$ associates $f_n \in \hat{L}^2(M^n)$ to its multiple stochastic integral $I_n(f_n)$, cf. [12] for a recent account of this isomorphism via the Charlier polynomials. The isometric isomorphism between $\Gamma(H)$ and the Guichardet space $L^2(\mathcal{P}, Q)$ is denoted by $\mathcal{I}$, it maps $f_n \in \hat{L}^2(M^n)$ to the function $n! f_n$ in $L^2(\mathcal{P}, Q)$. The Guichardet interpretation of the operators ∇^- and ∇^+ is

$$\nabla_x^- F(\gamma) = F(\gamma \cup x), \quad \gamma \in \mathcal{P}, \ F \in \mathcal{S}, \tag{7.1}$$

and for $u \in \mathcal{U}$ a simple process,

$$\nabla^+(u)(\gamma) = \sum_{x \in \gamma} u(x, \gamma \setminus x), \quad \gamma \in \mathcal{P}, \tag{7.2}$$

cf. e.g. [5], where we identified respectively $F, \nabla^- F, u, \nabla^+(u)$ to $\mathcal{J} \circ F, \mathcal{J} \circ \nabla^- F,$ $\mathcal{J} \circ u, \mathcal{J} \circ \nabla^+(u)$. On the other hand we know from e.g. [6], [11], [15], [17], that the probabilistic interpretation of ∇^- is given by

$$\nabla_x^- F(\omega) = F(\omega \cup x) - F(\omega), \quad x \in M, \ \omega \in \mathcal{P}, \ F \in \mathcal{S},$$

and

$$\nabla^+(u)(\omega) = \sum_{x \in \omega} u(x, \omega \setminus x) - \int_X u(x, \omega) d\sigma(x), \quad \omega \in \mathcal{P}, \ u \in \mathcal{U},$$

with respectively $F, \nabla^- F, u, \nabla^+(u)$ in place of $\mathcal{I} \circ F, \mathcal{I} \circ \nabla^- F, \mathcal{I} \circ u, \mathcal{I} \circ \nabla^+(u)$.

8 Stochastic integration with respect to a Poisson random measure

In this section the Hilbert space H is $H = L^2(M, \sigma)$, σ is a finite measure, $\Gamma(H)$ is identified via $\mathcal{J}$ to the Poisson space $L^2(\mathcal{P}, P)$, $\Gamma(H) \otimes H$ is identified to $L^2(\mathcal{P} \times M, P \otimes \sigma)$, and $(X, \mathcal{F}_X)$ remains an abstract index set with projection system $(p_A)_{A \in \mathcal{F}_X}$ in the next proposition. We assume that p_X is the identity of H.

Proposition 6 *In the Poisson stochastic interpretation of* $\Gamma(H)$, *the stochastic integral* $S(u)$ *of a simple* 1_M-*process* $u = \sum_{i=1}^{i=n} F_i \otimes (p_{A_i} 1_M) \in \mathcal{U}^{1_M}$ *satisfies*

$$S(u) = \sum_{i=1}^{i=n} F_i \int_M (p_{A_i} 1_M)(x)(N(dx) - \sigma(dx)).$$

Proof. This result follows from the expression on Fock space of the multiplication operator by $\int_M (p_{A_i} 1_M)(x)(N(dx) - \sigma(dx))$, cf. e.g. [16]:

$$F_i \int_M (p_{A_i} 1_M)(x)(N(dx) - \sigma(dx))$$
$$= \nabla^+(F_i \otimes (p_{A_i} 1_M)) + \nabla^+((\mathrm{Id} \otimes p_{A_i})\nabla^- F_i) + (p_{A_i} 1_M, \nabla^- F_i)_H. \qquad \square$$

The Poisson stochastic integral $S(u)$ can be extended to square-integrable strongly adapted processes via the Itô isometry (3.3) with the relation

$$S(u) = \int_M u(x)(N(dx) - \sigma(dx)).$$

The Poisson integral $S(u)$ defined above is for simple processes. It can be extended to a given process u by taking limits from a sequence of simple processes approaching u, however the result depends in general on the choice of an approximating sequence. Considering u_π defined in (6.1), we have

$$
\begin{aligned}
\|u_\pi\|_{L^2(M,\sigma)}^2 &= \left\| \sum_{A \in \pi} \frac{(u, p_A 1_M)_{L^2(M)}}{\mu(A)} \otimes (p_A 1_M) \right\|_{L^2(M,\sigma)}^2 \\
&= \sum_{A \in \pi} \left\| \frac{(u, p_A 1_M)_{L^2(M,\sigma)}}{\mu(A)} \otimes (p_A 1_M) \right\|_{L^2(M,\sigma)}^2 \\
&= \frac{1}{\mu(A)} \sum_{A \in \pi} (u, p_A 1_M)_{L^2(M,\sigma)}^2 = \frac{1}{\mu(A)} \sum_{A \in \pi} (u, p_A p_A 1_M)_{L^2(M,\sigma)}^2 \\
&= \frac{1}{\mu(A)} \sum_{A \in \pi} (p_A u, p_A 1_M)_{L^2(M,\sigma)}^2 \le \sum_{A \in \pi} (p_A u, p_A u)_{L^2(M,\sigma)}^2 \\
&\le \sum_{A \in \pi} (u, p_A u)_{L^2(M,\sigma)}^2 \le (u, u)_{L^2(M,\sigma)}^2,
\end{aligned}
$$

i.e. $\|u_\pi\|_{L^2(M,\sigma)}^2 \le \|u\|_{L^2(M,\sigma)}^2$, P-a.s. On the last line we used the assumption that p_X is the identity of H. Hence in assumption (i) of Sect. 6, L^2 convergence can be replaced by almost sure convergence of the approximation:

$$\limsup_{|\pi| \to 0} \frac{1}{\mu(A)} \sum_{A \in \pi} \left\| \frac{(u, p_A 1_M)_{L^2(M,\sigma)}}{\mu(A)} \otimes (p_A 1_M) - (\mathrm{Id} \otimes p_A)u \right\|_{L^2(M,\sigma)}^2 = 0, \quad a.s.,$$

and this suffices to define

$$S(u) = \lim_{|\pi| \to 0} S(u_\pi),$$

provided assumption (ii) is also satisfied.

As a particular case we can take $X = M$ and the projection p_A, $A \in \mathcal{F}_X$, can be taken equal to the multiplication operator by 1_A. If $X = M = \mathbb{R}^d$ with the canonical partial ordering "$\prec$" and the projection system $\{p_A\}_{A \in \mathcal{B}(\mathbb{R}^d)}$ is defined as $p_A f = 1_A f$, $f \in H = L^2(M, \sigma) = L^2(X, \mu)$, $\mu = \sigma$, then our definition of adaptedness extends the usual definition and for strongly adapted processes the Poisson and Skorohod integrals coincide.

A similar result in the Gaussian case can be obtained by replacing the compensated Poisson random measure $\tilde{N}$ with a Gaussian random measure with variance σ.

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