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P.N. Natarajan

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SOME MORE STEINHAUS TYPE THEOREMS OVER VALUED FIELDS

by P.N. Natarajan

1. Preliminaries :

Throughout this paper, K denotes $\mathbb{R}$ (the field of real numbers) or $\mathbb{C}$ (the field of complex numbers) or a complete, non-trivially valued, non-archimedean field. In the relevant context, we mention explicitly which field is chosen. Entries of infinite matrices and sequences, which occur in the sequel, are in K. If X, Y are sequence spaces over K, by (X, Y) we mean the class of all infinite matrices $A = (a_{nk}),\ n, k = 0, 1, 2, \ldots$ such that $Ax = \{(Ax)_n\} \in Y$ whenever $x = \{x_k\} \in X$, where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k,\ n = 0, 1, 2, \ldots,$$

it being assumed that the series on the right converges. Whenever there is some notion of limit or sum in X, Y, we denote by $(X, Y; P)$ that subclass of (X, Y) consisting of infinite matrices which preserve this limit or sum. Whatever be K, the sequence spaces $\ell, \gamma, c_0, c, \ell_\infty, \gamma_\infty$ are defined as :

$$\begin{aligned}
\ell &= \left\{\{x_k\} \ : \ \sum_{k=0}^{\infty} |x_k| \ \text{converges}\right\};\\
\gamma &= \left\{\{x_k\} \ : \ \sum_{k=0}^{\infty} x_k \ \text{converges}\right\};\\
c_0 &= \left\{\{x_k\} \ : \ \lim_{k\to\infty} x_k = 0\right\};\\
c &= \left\{\{x_k\} \ : \ \lim_{k\to\infty} x_k \ \text{exists}\right\};\\
\ell_\infty &= \left\{\{x_k\} \ : \ \sup_{k\geq 0} |x_k| < \infty\right\};\\
\gamma_\infty &= \left\{\{x_k\} \ : \ \{s_k\} \in \ell_\infty, \ s_k = \sum_{i=0}^{k} x_i, \ k = 0,1,2,\ldots\right\}.
\end{aligned}$$

We note that $\ell \subset \gamma \subset c_0 \subset c \subset \ell_\infty$ and $\gamma_\infty \subset \ell_\infty$.

$(\ell, \gamma; P)$ denotes the class of all infinite matrices $A = (a_{nk})$ in (ℓ, γ) such that $\sum_{n=0}^{\infty}(Ax)_n = \sum_{k=0}^{\infty} x_k, x = \{x_k\} \in \ell$.

2. The case $K = \mathbb{R}$ or $\mathbb{C}$

When $K = \mathbb{R}$ or $\mathbb{C}$, the following result is well-known (see [6], 48, p.7).

Theorem 2.1 *A matrix $A = (a_{nk})$ is in (ℓ, γ) if and only if*

$$\sup_{m,k} \left| \sum_{n=0}^{m} a_{nk} \right| < \infty \quad ; \tag{1}$$

and

$$\sum_{n=0}^{\infty} a_{nk} \ \text{converges}, k = 0, 1, 2, \ldots \ . \tag{2}$$

We now prove the following result when $K = \mathbb{R}$ or $\mathbb{C}$.

Theorem 2.2 *A matrix $A = (a_{nk})$ is in $(\ell, \gamma; P)$ if and only if it satisfies (1) and*

$$\sum_{n=0}^{\infty} a_{nk} = 1, k = 0, 1, 2, \ldots \ . \tag{3}$$

Proof. If A is in $(\ell, \gamma; P)$ then (1) holds. For $k = 0, 1, 2, \dots$, each $e_k = \{0, \dots, 0, 1, 0, \dots\}$, (1 occurring at the kth place), lies in ℓ and so $\sum_{n=0}^{\infty}(Ae_k)_n = 1$, i.e. , $\sum_{n=0}^{\infty} a_{nk} = 1$, $k = 0, 1, 2, \dots$. i.e., (3) holds.

Conversely, let (1) and (3) hold. It follows that A is in (ℓ, γ) in view of Theorem 2.1. Let $B = (b_{mk})$ where

$$b_{mk} = \sum_{n=0}^{m} a_{nk}, \; m, k = 0, 1, 2, \dots .$$

Using (1) and (3), we have

$$\sup_{m,k} |b_{mk}| < \infty \; ; \tag{4}$$

and

$$\lim_{m \to \infty} b_{mk} = 1, \; k = 0, 1, 2, \dots . \tag{5}$$

Thus B is in $(\ell, c\,; P')$ (see [5]). Let, now , $\{x_k\} \in \ell$. So

$$\lim_{m \to \infty} \sum_{k=0}^{\infty} b_{mk} x_k \text{ exists and is equal to } \sum_{k=0}^{\infty} x_k,$$

$$\text{i.e., } \lim_{m \to \infty} \sum_{k=0}^{\infty} \left(\sum_{n=0}^{m} a_{nk} \right) x_k = \sum_{k=0}^{\infty} x_k,$$

$$\text{i.e., } \lim_{m \to \infty} \sum_{n=0}^{m} \left(\sum_{k=0}^{\infty} a_{nk} \; x_k \right) = \sum_{k=0}^{\infty} x_k,$$

$$\text{i.e., } \quad \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} a_{nk} \; x_k \right) = \sum_{k=0}^{\infty} x_k,$$

$$\text{i.e., } \quad \sum_{n=0}^{\infty} (Ax)_n = \sum_{k=0}^{\infty} x_k.$$

In other words, A is in $(\ell, \gamma; P)$, which completes the proof of the theorem.

Maddox [3] proved that $(\gamma, \gamma; P) \cap (\gamma_\infty, \gamma) = \emptyset$. In this context, it is worthwhile to note that the identity matrix (*i.e.*, $I = (i_{nk})$ where $i_{nk} = 1$, if $k = n$ and $i_{nk} = 0$, if $k \neq n$) is in $(\ell, \gamma; P) \cap (\gamma_\infty, \gamma)$ so that $(\ell, \gamma; P) \cap (\gamma_\infty, \gamma) \neq \emptyset$. Since $(\gamma, \gamma) \supset (\gamma_\infty, \gamma)$, it follows that $(\ell, \gamma; P) \cap (\gamma, \gamma) \neq \emptyset$. We note that $(\gamma, \gamma; P) \subset (\ell, \gamma; P)$ and $(c_0, \gamma) \subset (\gamma, \gamma)$. Having "enlarged" the class $(\gamma,\gamma;P)$ to $(\ell,\gamma;P)$, we would like to "contract" the class (γ,γ) to (c_0,γ) and attempt a Steinhaus type theorem involving the classes $(\ell,\gamma;P)$ and (c_0,γ).

Theorem 2.3 $(\ell,\gamma;P)\cap(c_0,\gamma)=\emptyset$.

Proof. Let $A=(a_{nk})$ be in $(\ell,\gamma;P)\cap(c_0,\gamma)$. Since A is in (c_0,γ),

$$\sup_m \sum_{k=0}^{\infty}\left|\sum_{n=0}^{m} a_{nk}\right| \leq M < \infty \tag{6}$$

(see [6], 43, p.6). Now, for $L=0,1,2,\ldots,m=0,1,2\ldots,$

$$\sum_{k=0}^{L}\left|\sum_{n=0}^{m} a_{nk}\right| \leq \sum_{k=0}^{\infty}\left|\sum_{n=0}^{m} a_{nk}\right| \leq M.$$

Taking limit as $m\to\infty$, we have,

$$\sum_{k=0}^{L}\left|\sum_{n=0}^{\infty} a_{nk}\right| \leq M,\ L=0,1,2,\ldots.$$

Taking limit as $L\to\infty$, we get ,

$$\sum_{k=0}^{\infty}\left|\sum_{n=0}^{\infty} a_{nk}\right| \leq M.$$

which is contradiction, since $\sum_{n=0}^{\infty} a_{nk}=1, k=0,1,2\ldots$ in view of (3). This establishes our claim.

Corollary. *Since $c_0\subset c\subset \ell_\infty, (\ell_\infty,\gamma)\subset(c,\gamma)\subset(c_0,\gamma)$ so that $(\ell,\gamma;P)\cap(X,\gamma)=\emptyset$ for $X=c_0,c,\ell_\infty$.*

3. The case when K is a complete, non trivially valued, non-archimedean field.

When K is a complete, non-trivially valued, non-archimedean field, we note that $\gamma=c_0$ and $\gamma_\infty=\ell_\infty$. In this case, it is easy to prove the following results.

Theorem 3.1 $(\ell,\gamma)=(\ell,c_0)=(c_0,c_0)$. *A matrix $A=(a_{nk})$ is in (ℓ,c_0) if and only if it satisfies*

$$\sup_{n,k} |a_{nk}| < \infty \tag{7}$$

and

$$\lim_{n \to \infty} a_{nk} = 0, \; k = 0, 1, 2, \dots \; . \tag{8}$$

Theorem 3.2 $(\ell, \gamma; P) = (\ell, c_0; P) = (c_0, c_0; P) = (\gamma, \gamma; P)$. *A matrix $A = (a_{nk})$ is in $(\ell, c_0; P)$ if and only if it satisfies (3), (7) and (8).*

Theorem 3.3 *A matrix $A = (a_{nk})$ is in (c, c_0) if and only if it satisfies (7), (8) and*

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = 0. \tag{9}$$

Remark 3.4 Theorem 2.3 fails to hold when K is a complete, non-trivially valued, non archimedean field since $(\ell, c_0) = (c_0, c_0)$. We also have

$$(\ell, c_0; P) \cap (c, c_0) \neq \emptyset,$$

as the following example illustrates. Consider the infinite matrix

$$A = (a_{nk}) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & -2 & 0 & 0 & 0 & \dots \\ 0 & 0 & 3 & -3 & 0 & 0 & \dots \\ 0 & 0 & 0 & 4 & -4 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

$$\begin{aligned} \text{i.e.,} \quad a_{nk} &= n+1 \;\text{ if }\; k = n; \\ &= -(n+1), \;\text{ if }\; k = n+1; \\ &= 0, \text{ otherwise }. \end{aligned}$$

Then (3) , (7) , (8) and (9) hold so that A is in $(\ell, c_0; P) \cap (c, c_0)$. These remarks point out significant departure from the case $K = \mathbb{R}$ or $\mathbb{C}$.

The following lemma is needed in the sequel.

Lemma 3.5 *The following statements are equivalent :*

(a) *A matrix $A = (a_{nk})$ is in (ℓ_∞, c_0) ;*

(b) (i) $\lim_{k\to\infty} a_{nk} = 0$; (10)

and

(ii) $\lim_{n\to\infty} \sup_{k\geq 0} |a_{nk}| = 0$; (11)

(c) (i) (8) *holds*

and

(ii) $\lim_{k\to\infty} \sup_{n\geq 0} |a_{nk}| = 0$. (12)

Proof. For the proof of "(a) is equivalent to (b)", see ([4], 422). We now prove that (b) and (c) are equivalent. Let us suppose that (b) holds. For every fixed $k = 0, 1, 2, \ldots$,

$$|a_{nk}| \leq \sup_{k'\geq 0} |a_{nk'}|.$$

Now (8) follows in view of (b) (ii). Again by (b) (ii), given $\varepsilon > 0$, we can choose a positive integer N such that

$$\sup_{k\geq 0} |a_{nk}| < \varepsilon, \ n > N. \tag{13}$$

In view of (b) (i), for $n = 0, 1, 2, \ldots, N$, we can find a positive integer L such that

$$|a_{nk}| < \varepsilon, \ k > L. \tag{14}$$

(13) and (14) imply that

$$|a_{nk}| < \varepsilon, \ n = 0, 1, 2, \ldots, k > L.$$

i.e.,

$$\sup_{n\geq 0} |a_{nk}| < \varepsilon, \ k > L$$

i.e.,

$$\lim_{k\to\infty} \sup_{n\geq 0} |a_{nk}| = 0,$$

so that (c) (ii) holds. Similarly we can prove that (c) implies (b). This establishes the lemma.

We now prove the following Steinhaus type result.

Theorem 3.6 *When K is a complete, non trivially valued, non-archimedean field, then*

$$(\ell, c_0; P) \cap (\ell_\infty, c_0) = \emptyset.$$

Proof. Let $A = (a_{nk})$ be in $(\ell, c_0; P) \cap (\ell_\infty, c_0)$. In view of (3), we have,

$$l = \left|\sum_{n=0}^{\infty} a_{nk}\right| \leq \sup_{n\geq 0} |a_{nk}|.$$

Taking limit as $k \to \infty$ and using (12), we get $1 \leq 0$, which is absurd. This proves the theorem.

In view of Theorem 3.2 and Theorem 3.6 we have the following.

Corollary. $(c_0, c_0; P) \cap (\ell_\infty, c_0) = \emptyset$.

We shall now take up an application of Theorem 3.6 to analytic functions. For the theory of analytic functions in non-archimedean fields, one can refer to [2]. Consider the space of bounded analytic functions inside the disk $d(0,1^-)$ (usually denoted by $\mathcal{A}_b(d(0,1^-))$), provided with the topology of uniform convergence in each disk $d(0,r), r < 1$ and the space of analytic elements in the disk $d(0,1)$ (usually denoted by $H(d(0,1))$), provided with the topology of uniform convergence on $d(0,1)$. Thanks to Lemma 3.5, one can check that (ℓ_∞, c_0) represents the space of continuous linear mappings from $\mathcal{A}_b(d(0,1^-))$ into $H(d(0,1))$. We now have the following result, which follows from Theorem 3.6.

Theorem 3.7 *There exists no continuous linear mapping ϕ from $\mathcal{A}_b(d(0,1^-))$ into $H(d(0,1))$ satisfying $\phi(f)$ $(1) = f(1)$ for all $f \in \mathcal{A}_b(d(0,1^-))$.*

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Department of Mathematics,
Ramakrishna Mission Vivekananda College,
Chennai - 600 004,
INDIA