ANNALES MATHÉMATIQUES BLAISE PASCAL

B. AMIR

L. MANIAR

Composition of pseudo almost periodic functions and Cauchy problems with operator of non dense domain

Annales mathématiques Blaise Pascal, tome 6, nº 1 (1999), p. 1-11

<http://www.numdam.org/item?id=AMBP_1999__6_1_1_0>

Ann. Math. Blaise Pascal, Vol. 6, N° 1, 1999, pp. 1-11

Composition of Pseudo Almost Periodic Functions and Cauchy Problems with Operator of non Dense Domain.

B. Amir[1] and L. Maniar[2]
1-Department of Mathematics,
E.N.S., B.P.S 41, Marrakesh
2 -Department of Mathematics ,
Faculty of Sciences Semlalia,
B.P.S 15, Marrakesh, Morocco.

Abstract

In this work, we give a generalization, to Banach spaces, for Zhang's result concerning the pseudo-almost periodicity of the composition of two pseudo-almost periodic functions. This result is used to investigate the existence of pseudo-almost periodic solutions of semilinear Cauchy problems with operator of non dense domain in original space.

1 Introduction

In this paper, we study the existence, uniqueness and pseudo-almost periodicity of the solution to the following semilinear Cauchy problem

$$x'(t) = Ax(t) + f(t, x(t)), \quad t \in \mathbb{R}, \tag{1}$$

where A is an unbounded linear operator, assumed of Hille-Yosida with negative type and non necessarily dense domain on a Banach space X and $f : \mathbb{R} \times X \longrightarrow X$, is a continuous function.

First, we begin by studying the inhomogeneous Cauchy problem

$$x'(t) = Ax(t) + f(t), \quad t \in \mathbb{R}, \tag{2}$$

which will be used to get our goal.

To study the pseudo-almost periodicity of (1), we need to give a generalization, to Banach spaces, for Zhang's result in which he proved that the composition of two pseudo-almost periodic (p.a.p.) functions in finite dimensional spaces is p.a.p. More precisely, for $f : \mathbb{R} \times Y \longrightarrow X$ and $h : \mathbb{R} \longrightarrow Y$ which are p.a.p. we prove that the function

$$\begin{array}{rccl} g: & \mathbb{R} & \longrightarrow & X \\ & t & \longmapsto & f(t, h(t)) \end{array}$$

is also p.a.p.. One can find this result in Section 3.

The notion of pseudo-almost periodicity has been introduced by Zhang (1992) (see [14]). He has studied in [15] the existence of p.a.p. solutions of (1) in the finite dimensional spaces case. In the case of Banach spaces, in our knowledge, there is only one work [1], concerning the study of the existence of a unique p.a.p. solution of (2), where A is the generator of C_0-semigroup.

2 Preliminaries

One denotes by $AP(\mathbb{R}, X)$ (resp. $AP(\mathbb{R} \times Y, X)$) the set of almost periodic functions from $\mathbb{R}$ into X (resp. from $\mathbb{R} \times Y$ into X), where X and Y are two Banach spaces, and defines the sets $PAP_0(\mathbb{R}, X)$ and $PAP_0(\mathbb{R} \times Y, X)$ by

$$PAP_0(\mathbb{R}, X) := \left\{ \varphi \in C_b(\mathbb{R}, X),\ \lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} \|\varphi(t)\|\, dt = 0 \right\}$$

$$PAP_0(\mathbb{R} \times Y, X) := \left\{ \begin{array}{c} \varphi : \mathbb{R} \times Y \longrightarrow X, \text{ continuous with} \\ \varphi(\cdot, x) \in C_b(\mathbb{R}, X), \text{ for all } x \in Y \text{ and} \\ \lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} \|\varphi(t, x)\|\, dt = 0, \text{ uniformly in } x \in Y. \end{array} \right\}$$

A function $f \in C_b(\mathbb{R}, X)$ (resp. $f \in C(\mathbb{R} \times Y, X)$) is called pseudo-almost periodic if there exist some functions g and φ in $C(\mathbb{R}, X)$ (respectively. in $C(\mathbb{R} \times Y, X)$) such that

(*i*) $g \in AP(\mathbb{R}, X)$ (resp. $g \in AP(\mathbb{R} \times Y, X)$);

(*ii*) $\varphi \in PAP_0(\mathbb{R}, X)$ (resp. $\varphi \in PAP_0(\mathbb{R} \times Y, X)$);

(*iii*) $f = g + \varphi$.

$PAP(\mathbb{R}, X)$ (resp. $PAP(\mathbb{R} \times Y, X)$) denotes the subset of $C_b(\mathbb{R}, X)$ (resp. $C(\mathbb{R} \times Y, X)$) of all pseudo-almost periodic functions from $\mathbb{R}$ into X (resp. from $\mathbb{R} \times Y$ into X).

We have the following result which will be used in the sequel

Proposition 1 *Let* $f \in AP(\mathbb{R} \times Y, X)$ *and* $h \in AP(\mathbb{R}, Y)$, *then the function* $f(\cdot, h(\cdot)) \in AP(\mathbb{R}, X)$.

The proof of this proposition is similar to the one given in ([6], Thm.2.11).

2.1 Extrapolation spaces.

In this subsection, we fix some notations and recall some basic results on extrapolation spaces of Hille-Yosida operators. For more complete account we refer to [10], [11], where the proofs are given.

Let X be a Banach space and A be a linear operator with domain $D(A)$. We say that A is a *Hille-Yosida* operator on X if there exist $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ ($\rho(A)$ is the resolvent set of A) and

$$\sup\{\|(\lambda-\omega)^n R(\lambda, A)^n\| : \lambda > \omega,\, n \geq 0\} < \infty. \tag{3}$$

The infinimum of such ω is called the *type* of A.

It follows from the Hille-Yosida theorem that any Hille-Yosida operator generates a C_0-semigroup on the closure of its domain. More precisely (cf. [7], Thm. 12.2.4), the part $(A_0, D(A_0))$ of A in $X_0 := \overline{D(A)}$ generates a C_0-semigroup $(T_0(t))_{t\geq 0}$.

For the rest of this section we assume without loss of generality that $(A, D(A))$ is a Hille-Yosida operator of negative type on X. This implies that $0 \in \rho(A)$, i.e., $A^{-1} \in \mathcal{L}(X)$.

On the space X_0 we introduce a new norm by

$$\|x\|_{-1} = \left\|A_0^{-1}x\right\|, \quad x \in X_0.$$

The completion of $\left(X_0, \|\cdot\|_{-1}\right)$ will be called the *extrapolation space* of X_0 associated to A_0 and will be denoted by X_{-1}.

One can show easily that, for each $t \geq 0$, the operator $T_0(t)$ can be extended to a unique bounded operator on X_{-1} denoted by $T_{-1}(t)$. The family $(T_{-1}(t))_{t\geq 0}$ is a C_0-semigroup on X_{-1}, which will be called the *extrapolated semigroup* of $(T_0(t))_{t\geq 0}$. The domain of its generator A_{-1} is equal to X_0.

The original space X now fits into this scheme of spaces X_0 and X_{-1} (cf. [11], Thm. 1.7).

Theorem 2 *For the norm*

$$\|x\|_{-1} = \left\|A^{-1}x\right\| \quad x \in X,$$

we have that $X_0 := \overline{D(A)}$ *is dense in* $\left(X, \|\cdot\|_{-1}\right)$. *Hence the extrapolation space is also the completion of* $\left(X, \|\cdot\|_{-1}\right)$ *and* $X \hookrightarrow X_{-1}$. *Moreover, the operator* A_{-1} *is an extension of* A *to* X_{-1}, $(A_{-1})^{-1}X = D(A)$ *and* $(A_{-1})^{-1}X_0 = D(A_0)$.

Abstract extrapolation spaces have been introduced by Da Prato-Grisvard [4] and Nagel [9] and used for various purposes (cf. [2], [3], [8], [11], [12], and [13]).

3 Main results

We state the fundamental lemma, which will be crucial for our aim.

Lemma 3 *Let* A *be a Hille-Yosida operator of negative type,* $\omega \in \rho(A)$, $\omega < 0$ *and* $f \in C_b(\mathbb{R}, X)$. *The following properties hold*

(*i*) $\int_{-\infty}^{t} T_{-1}(t-s)f(s)ds \in X_0,$ *for all* $t \in \mathbb{R}$.

(*ii*) *There exist* C *independent from* f *such that for every* $t \in \mathbb{R}$
$\left\|\int_{-\infty}^{t} T_{-1}(t-s)f(s)ds\right\| \le Ce^{\omega t}\int_{-\infty}^{t} e^{-\omega s}\|f(s)\|\,ds.$

(*iii*) *The operator* $T : C_b(\mathbb{R}, X) \longrightarrow C_b(\mathbb{R}, X_0)$ *defined by*

$$(Tf)(t) := \int_{-\infty}^{t} T_{-1}(t-s)f(s)ds$$

is a linear bounded operator.

Proof. We first prove (*i*) and (*ii*) in the case where f is integrable on $\mathbb{R}^-$ and locally integrable on $\mathbb{R}^+$. In this case, the proof uses the same technics to prove ([11], Prop. 2.1).

For $f \in C_b(\mathbb{R}, X)$, we define the sequence $(f_n)_n$ by $f_n(t) := e^{-\frac{\omega}{n}t}f(t),\ t \in \mathbb{R}$, and $n \in \mathbb{N}^*$. It is clear that f_n is integrable on $\mathbb{R}^-$ and locally integrable on $I\!R^+$. Then (i) is satisfied by $(f_n)_n$. Hence, we have

$$\begin{aligned}
&\left\| \int_{-\infty}^{t} T_{-1}(t-\sigma) f_n(\sigma) d\sigma - \int_{-\infty}^{t} T_{-1}(t-\sigma) f_m(\sigma) d\sigma \right\| \\
\leq\ & M \|f\|_\infty e^{\omega t} \int_{-\infty}^{t} e^{-\omega\sigma} \left| e^{-\frac{\omega}{n}\sigma} - e^{-\frac{\omega}{m}\sigma} \right| d\sigma \underset{n,m\to+\infty}{\longrightarrow} 0.
\end{aligned}$$

Then, by Lebesgue's theorem

$$\lim_{n\to\infty} \int_{-\infty}^{t} T_{-1}(t-\sigma) f_n(\sigma) d\sigma \text{ exists in} X_0.$$

It is easy to see that

$$\int_{-\infty}^{t} T_{-1}(t-\sigma) f_n(\sigma) d\sigma \longrightarrow \int_{-\infty}^{t} T_{-1}(t-\sigma) f(\sigma) d\sigma \text{ in } X_{-1}$$

and consequently, $X_0 \hookrightarrow X_{-1}$ implies

$$\int_{-\infty}^{t} T_{-1}(t-\sigma) f_n(\sigma) d\sigma \underset{n\to\infty}{\longrightarrow} \int_{-\infty}^{t} T_{-1}(t-\sigma) f(\sigma) d\sigma \text{ in } X_0.$$

Then, we obviously have (i). For (ii), it follows immediately from the estimation satisfied by f_n. Finally, (iii) can be obtained easily from (ii). ■

Our main results consists of the study of the existence of a unique bounded and pseudo-almost periodic solution to the inhomogeneous Cauchy problem, the generalization of Zhang's result and to use these results to investigate the semilinear Cauchy problem case.

3.1 Inhomogeneous Cauchy problem

Consider the following Cauchy problem

$$x'(t) = Ax(t) + f(t), \quad t \in I\!R, \tag{4}$$

where A is a Hille-Yosida operator on X of negative type and $f \in C_b(I\!R, X)$. By using the Lemma3, we show easily that the unique bounded mild solution $x(\cdot)$ of this problem is given by

$$\begin{aligned}
x(t) &= (Tf)(t) := \int_{-\infty}^{t} T_{-1}(t-s) f(s) ds, \quad \text{for all } t \in I\!R &(5)\\
&= \int_{-\infty}^{0} T_{-1}(-s) f_t(s) ds. &(6)
\end{aligned}$$

If we assume that $f \in PAP(\mathbb{R}, X)$, then there are $g \in AP(\mathbb{R}, X)$ and $\varphi \in PAP_0(\mathbb{R}, X)$, such that $f = g + \varphi$. It is easy to show that $\varphi \in C_b(\mathbb{R}, X)$, thus $x = Tg + T\varphi$. The operator T is bounded and commutes with translation group, then it's easy to see that $Tg \in AP(\mathbb{R}, X)$. Furthermore, Lemma 3 implies, for $r > 0$, that

$$\begin{aligned} \frac{1}{2r}\int_{-r}^{r} \|T\varphi(t)\|\, dt &\leq \frac{C}{2r}\int_{-r}^{r}\left[e^{\omega t}\int_{-\infty}^{t} e^{-\omega s}\|\varphi(s)\|\, ds\right] dt \\ &\leq \frac{C}{2r}\int_{-r}^{r}\left[\int_{-\infty}^{t} e^{-\omega s}\|\varphi(s+t)\|\, ds\right] dt \\ &\leq C\int_{-\infty}^{0} e^{-\omega s}\left[\frac{1}{2r}\int_{-r}^{r}\|\varphi_s(t)\|\, dt\right] ds, \quad (*) \end{aligned}$$

where $\omega \in \rho(A)$ such that $\omega < 0$.

We show, by simple computation, that the set $PAP_0(\mathbb{R}, X)$ is invariant under the translation group. Hence, using Lebesgue's theorem, $(*)$ goes to zero, as $r \longrightarrow +\infty$. This proves the following theorem.

Theorem 4 *Let A be a Hille-Yosida operator on X of negative type and $f \in C_b(\mathbb{R}, X)$ pseudo almost periodic. Then (4) admits a unique bounded pseudo almost periodic mild solution given by (5).*

3.2 Composition of two pseudo almost periodic functions

Let us consider two Banach spaces X and Y, and a continuous function $f : \mathbb{R} \times Y \longrightarrow X$.
The generalization of Zhang's result is announced in the following theorem.

Theorem 5 *Let $f \in PAP(\mathbb{R} \times Y, X)$ satisfy the Lipschitz condition*

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\|. \quad \text{for all } x,\ y \in Y \text{ and } t \in \mathbb{R}.$$

If $h \in PAP(Y)$, then the function $f(\cdot, h(\cdot)) \in PAP(X)$.

Proof. Since $f \in PAP(\mathbb{R}\times Y, X)$, then $f = g + \varphi$, where $g \in AP(\mathbb{R}\times Y, X)$ and $\varphi \in PAP_0(\mathbb{R} \times Y, X)$. Moreover, $h = h_1 + h_2$, with $h_1 \in AP(\mathbb{R}, Y)$ and $h_2 \in PAP_0(\mathbb{R}, Y)$.

We have

$$\begin{aligned} \|f(t, h(t))\| &\leq L\|h\|_\infty + \|f(t, 0)\| \\ &\leq L\|h\|_\infty + \|g(t, 0)\| + \|\varphi(t, 0)\| \\ &\leq L\|h\|_\infty + \|g(\cdot, 0)\|_\infty + \|\varphi(\cdot, 0)\|_\infty, \end{aligned}$$

i.e., $f(\cdot, h(\cdot)) \in C_b(I\!R, X)$, and

$$\begin{aligned} f(\cdot, h(\cdot)) &= g(\cdot, h_1(\cdot)) + f(\cdot, h(\cdot)) - g(\cdot, h_1(\cdot)) \\ &= g(\cdot, h_1(\cdot)) + f(\cdot, h(\cdot)) - f(\cdot, h_1(\cdot)) + \varphi(\cdot, h_1(\cdot)). \end{aligned}$$

By Proposition 1, the function $g(\cdot, h_1(\cdot)) \in AP(I\!R, X)$. Using the fact that f is lipschitzian and $h_2 \in PAP_0(I\!R, Y)$, it is clear that the function

$$F(\cdot) := f(\cdot, h(\cdot)) - f(\cdot, h_1(\cdot)) \in PAP_0(I\!R, X).$$

To show that $f(\cdot, h(\cdot)) \in PAP(I\!R, X)$, we need to prove

$$\lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} \|\varphi(t, h_1(t))\| \, dt = 0.$$

Since $h_1(I\!R)$ is relatively compact in Y, for $\varepsilon > 0$, one can find finite number of open balls O_k with center $x_k \in h_1(I\!R)$ and radius less than $\frac{\varepsilon}{3L}$, such that $h_1(I\!R) \subset \bigcup_{k=1}^{m} O_k$.

For k $(1 \le k \le m)$, the set

$$B_k = \{t \in I\!R : h_1(t) \in O_k\}$$

is open and $I\!R = \bigcup_{k=1}^{m} B_k$. Let $E_k = B_k \setminus \bigcup_{i=1}^{k-1} B_i$ and $E_1 = B_1$. Then $E_i \cap E_j = \emptyset$, for $i \neq j$. Using the fact that $\varphi \in PAP_0(I\!R \times Y, X)$, there is a number $r_0 > 0$ such that

$$\frac{1}{2r} \int_{-r}^{r} \|\varphi(t, x_k)\| \, dt < \frac{\varepsilon}{3m}, \text{ for all } r \ge r_0 \text{ and } k \in \{1, ..., m\}. \tag{7}$$

Furthermore, since $g \in AP(I\!R \times Y, X)$ is uniformly continuous in $I\!R \times \overline{h_1(I\!R)}$, one can obtain

$$\|g(t, x_k) - g(t, x)\| < \frac{\varepsilon}{3}, \text{ for } x \in O_k \text{ and } k = 1, ..., m; \tag{8}$$

and since $\varphi(\cdot,h_1(\cdot)) = f(\cdot,h_1(\cdot))-g(\cdot,h_1(\cdot))$ and $\varphi(t,x_k) = f(t,x_k)-g(t,x_k)$, we have

$$\begin{aligned}
\frac{1}{2r}\int_{-r}^{r}\|\varphi(t,h_1(t))\|\,dt &= \frac{1}{2r}\sum_{k=1}^{m}\int_{E_k\cap[-r,r]}\|\varphi(t,h_1(t))\|\,dt\\
&\le \frac{1}{2r}\sum_{k=1}^{m}\int_{E_k\cap[-r,r]}(\|\varphi(t,h_1(t))-\varphi(t,x_k)\|+\|\varphi(t,x_k)\|)\,dt\\
&\le \frac{1}{2r}\sum_{k=1}^{m}\int_{E_k\cap[-r,r]}(\|f(t,h_1(t))-f(t,x_k)\|+\|\varphi(t,x_k)\|)\,dt\\
&+\frac{1}{2r}\sum_{k=1}^{m}\int_{E_k\cap[-r,r]}\|g(t,h_1(t))-g(t,x_k)\|\,dt\\
&\le \frac{1}{2r}\sum_{k=1}^{m}\int_{E_k\cap[-r,r]}(L\,\|h_1(t)-x_k\|_Y\,dt+\|g(t,h_1(t))-g(t,x_k)\|)\,dt\\
&+\sum_{k=1}^{m}\frac{1}{2r}\int_{-r}^{r}\|\varphi(t,x_k)\|\,dt.
\end{aligned}$$

For any $t\in E_k\cap[-r,r]$, $h_1(t)\in O_k$ (i.e., $\|h_1(t)-x_k\|_Y < \frac{\varepsilon}{3L}$ $(1\le k\le m)$). It follows from (7) and (8) that

$$\frac{1}{2r}\int_{-r}^{r}\|\varphi(t,h_1(t))\|\,dt \le \varepsilon, \quad \text{for all } r\ge r_0.$$

Hence,

$$\lim_{r\to+\infty}\frac{1}{2r}\int_{-r}^{r}\|\varphi(t,h_1(t))\|\,dt = 0, \tag{9}$$

and the theorem is proved. ■

3.3 Semilinear Cauchy problem

Let A be a Hille-Yosida operator of negative type ω on a Banach space X. Consider the semilinear Cauchy problem

$$x'(t) = Ax(t) + f(t,x(t)), \quad t\in \mathbb{R}, \tag{10}$$

where $f:\mathbb{R}\times X_0 \longrightarrow X$ satisfies

$$\|f(t,x)-f(t,y)\| \le k\,\|x-y\|, \text{ for all } t\in\mathbb{R} \text{ and } x,y\in X_0,$$

with

$$-\frac{kC}{\omega} < 1.$$

We can now state the following main result.

Theorem 6 *Under the above assumptions, if $f \in PAP(\mathbb{R} \times X_0, X)$ then Equation (10) admits one and only one bounded mild solution on $\mathbb{R}$, which is pseudo-almost periodic.*

Proof. Let $f \in PAP(\mathbb{R} \times X_0, X)$ and y be a function in $PAP(\mathbb{R}, X_0)$. Then, using Theorem 5, the function $g(\cdot) := f(\cdot, y(\cdot))$ is in $PAP(\mathbb{R}, X)$. From Theorem 4, the Cauchy problem

$$x'(t) = Ax(t) + g(t), \quad t \in \mathbb{R},$$

has a unique bounded mild solution x in $PAP(\mathbb{R}, X_0)$, which is given by

$$x(t) = \ (Fy)(t) \ := \int_{-\infty}^{t} T_{-1}(t-s) f(s, y(s)) ds, \quad t \in \mathbb{R}.$$

It suffices now to show that this operator F has a unique fixed point in the Banach space $PAP(\mathbb{R}, X_0)$.

For this, let x and y be in $PAP(\mathbb{R}, X_0)$. By using Lemma 3, we have

$$\begin{aligned} \|(Fx)(t) - (Fy)(t)\| &\leq Ce^{\omega t} \int_{-\infty}^{t} e^{-\omega s} \|f(s, x(s)) - f(s, y(s))\| \, ds \\ &\leq Cke^{\omega t} \int_{-\infty}^{t} e^{-\omega s} \|x(s) - y(s)\| \, ds \\ &\leq \left(-\frac{Ck}{\omega}\right) \|x - y\|_{\infty}, \quad t \in \mathbb{R}. \end{aligned}$$

Hence, since $\left(-\dfrac{Ck}{\omega}\right) < 1$, there is a unique bounded and pseudo-almost periodic solution of

$$x(t) = \int_{-\infty}^{t} T_{-1}(t-s) f(s, x(s)) ds, \quad t \in \mathbb{R},$$

which is a bounded pseudo-almost periodic mild solution of (10). ■

To finish this work, we give the following example as an application of our previous abstract results.

Example.

Consider the following partial differential equation

$$\frac{\partial}{\partial t} u(t,x) = \frac{\partial}{\partial x} u(t,x) - \mu u(t,x) + f(t, u(t,x)), \qquad t, x \in \mathbb{R}, \tag{11}$$

where μ is a positive constant and $f : I\!R \times I\!R \longrightarrow I\!R$ is continuous and lipschitzian function with respect to x uniformly in t.

Let $X := L^\infty(I\!R)$ with the supremum norm $\|\cdot\|_\infty$, and the operator A defined on X by

$$Af := f' - \mu f, \quad \text{for } f \in D(A) := \left\{ f \in X : f \text{ is absolutely continuous and } f' \in X \right\}.$$

We can easily show that A is a Hille-Yosida operator of type $\omega = -\mu < 0$, with non dense domain (see [5]).

It is easy to see that (11) can be formulated by the following semilinear Cauchy problem

$$u'(t) = Au(t) + f(t, u(t)), \qquad t \in I\!R, \tag{12}$$

where $u(t) := u(t, \cdot)$ and $f(t, \varphi)(x) := f(t, \varphi(x))$, for all $\varphi \in X$ and $x, t \in I\!R$.

From the above abstract results, if $f(\cdot, \cdot) \in PAP(I\!R \times \overline{D(A)}, X)$, then the semilinear Cauchy problem (12) has one and only one bounded p.a.p. mild solution. Consequently the partial differential equation (11) admits a unique bounded p.a.p. solution with respect to the $L^\infty(I\!R)$-norm.

References

[1] Ait Dads, E. ; Ezzinbi, K. ; Arino, O. : Pseudo almost periodic solutions for some differential equations in a Banach space, Nonlinear Analysis, Theory, Methods & Applications, Vol. 28, No 7, (1997), pp. 1141-1155.

[2] Amann, H. : Linear and Quasilinear Parabolic Problems. Birkhäuser, Berlin 1995.

[3] Amir, B. ; Maniar, L.: Application de la théorie d'extrapolation pour la résolution des équations différentielles à retard homogènes. Extracta Mathematicae Vol**13**, Núm.1, 95-105 (1998).

[4] Da Prato, G. ; Grisvard, E.: On extrapolation spaces. Rend. Accad. Naz. Lincei. **72** (1982), pp. 330-332.

[5] Da Prato, G. ; Sinestrari, E.: Differential operators with non dense domain, Annali Scuola Normale Superiore Pisa **14** (1989), pp. 285-344.

[6] Fink, A. M.: Almost Periodic Differential Equations, Lectures Notes in Mathematics **377**, Springer-Verlag, 1974.

[7] Hille, E. ; Phillips, R.S.: Functional Analysis and Semigroups. Amer. Math. Soc. Providence 1975.

[8] Maniar, L. ; Rhandi, A.: Inhomogeneous retarded differential equation in infinite dimensional space via extrapolation spaces. To appear in Rendiconti Del Circolo Mathematico Di Palermo, Vol.**17** (1998).

[9] Nagel, R.: One Parameter Semigroups of Positive Operators. Lecture Notes in Mathematics **1184**, Springer-Verlag 1986.

[10] Nagel, R.: Sobolev spaces and semigroups, Semesterberichte Funktionalanalysis, Band **4** (1983), pp. 1-20.

[11] Nagel, R. ; Sinestrari, E.: Inhomogeneous Volterra integrodifferential equations for Hille-Yosida operators. Marcel Dekker, Lecture Notes Pure Appl. Math. **150** (1994).

[12] van Neerven, J.: The Adjoint of a Semigroup of Linear Operators. Lecture Notes Math. **1529**, Springer-Verlag 1992.

[13] Nickel, G. ; Rhandi, A.: On the essential spectral radius of semigroups generated by perturbations of Hille-Yosida operators. To appear in Diff. Integ. Equat..

[14] Zhang, C.: Pseudo Almost Periodic Functions and Their Applications. Thesis, the university of Ontario, 1992.

[15] Zhang, C.: Pseudo almost periodic solutions of some differential equations. J.M.A.A.181, (1994), pp. 62-76.