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Non-Archimedean Umbral Calculus

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Abstract

Let K be a non-archimedean valued field which contains $\mathbb{Q}_p$, and suppose that K is complete for the valuation $|\cdot|$, which extends the p-adic valuation. We find many orthonormal bases for $C(\mathbb{Z}_p \to K)$, the Banach space of continuous functions from $\mathbb{Z}_p$ to K, equipped with the supremum norm. To find these bases, we use continuous linear operators on $C(\mathbb{Z}_p \to K)$. Some properties of these continuous linear operators are established. In particular we look at operators which commute with the translation operator.

1. Introduction

Let p be a prime number and let $\mathbb{Z}_p$ be the ring of the p-adic integers, $\mathbb{Q}_p$ the field of the p-adic numbers, and K is a non-archimedean valued field that contains $\mathbb{Q}_p$, and we suppose that K is complete for the valuation $|\cdot|$, which extends the p-adic valuation. $\mathbb{N}$ denotes the set of natural numbers, and $K[x]$ is the set of polynomials with coefficients in K. In this paper we find many orthonormal bases for the Banach space $C(\mathbb{Z}_p \to K)$ of continuous functions from $\mathbb{Z}_p$ to K. To find these bases we use continuous linear operators on $C(\mathbb{Z}_p \to K)$. We also establish some properties of these operators. In particular we look at operators which commute with the translation operator. We start by recalling some definitions and some previous results.

Definition 1.1

A sequence of polynomials (p_n) is called a polynomial sequence if the degree of p_n is n for every $n \in \mathbb{N}$.

In the classical umbral calculus ([3] and [4]) one works with linear operators operating on $\mathbf{R}[x]$, the space of polynomials with coefficients in $\mathbf{R}$. We define the shift-operators E^α on $\mathbf{R}[x]$ by $(E^\alpha p)(x) = p(x+\alpha)$, where $\alpha \in \mathbf{R}$. Linear operators Q which commute with E^α are called shift-invariant operators and they have been studied extensively in the classical umbral calculus. Such a linear operator Q is called a delta-operator if Q commutes with E^α and if Qx is a constant different from zero. If Q is a delta-operator, there exists a unique polynomial sequence (p_n) such that $Qp_n = np_{n-1}$, $p_n(0) = 0 \ (n \geq 1)$, $p_0 = 1$. This sequence is called the sequence of basic polynomials for the delta-operator or simply the basic sequence for Q. If R is a shift-invariant operator and Q is a delta-operator with basic sequence (p_n), then $R = \sum_{k \geq 0} \frac{a_k}{k!} Q^k$ with $a_k = (Rp_k)(0)$. An umbral operator U is an operator which maps a basic sequence (p_n) into another basic sequence (q_n), i.e. $Up_n = q_n$ for all $n \in \mathbf{N}$. Remark that an umbral operator is an operator which is in general not shift-invariant.

Now we look at the non-archimedean case. Let $\mathcal{L}$ be a non-archimedean Banach space over a non-archimedean valued field L, $\mathcal{L}$ equipped with the norm $\|\cdot\|$. A family (f_i) of elements of $\mathcal{L}$ forms an orthonormal basis for $\mathcal{L}$ if each element x of $\mathcal{L}$ has a unique representation $x = \sum_{i=0}^{\infty} x_i f_i$ where $x_i \in L$ and $x_i \to 0$ if $i \to \infty$, and if the norm of x is the supremum of the valuations of x_i. If M is a non-empty compact subset of L whithout isolated points, then $C(M \to L)$ is the Banach space of continuous functions from M to L equipped with the supremum norm $\|\cdot\|_\infty$: $\|f\|_\infty = sup\{|f(x)| \mid x \in M\}$.
Let $\mathbf{Z}_p$, K and $C(\mathbf{Z}_p \to K)$ be as above and let I denote the identity operator on $C(\mathbf{Z}_p \to K)$. All the following results in this section can be found in [6], except mentioned otherwise. The translation operator E and its generalisation E^α are defined on $C(\mathbf{Z}_p \to K)$ as follows

$$(Ef)(x) = f(x+1),$$

$$(E^\alpha f)(x) = f(x+\alpha), \quad \alpha \in \mathbf{Z}_p.$$

The difference operator Δ on $C(\mathbf{Z}_p \to K)$ is defined by

$$(\Delta f)(x) = f(x+1) - f(x) = (Ef)(x) - f(x).$$

The operator Δ has the following properties : if $f : \mathbf{Z}_p \to K$ is a continuous function and $\Delta^n f = 0$, then f is a polynomial of degree not greater dan n. If p is a polynomial of degree n in $K[x]$, then Δp is a polynomial of degree $n-1$. If $f : \mathbf{Z}_p \to K$ is a continuous function then

$$(\Delta^n f)(x) \to 0 \ \ uniformly \ \ in \ \ x \qquad (1.1)$$

([5], exercise 52.D p. 156).
We introduce the polynomial sequence (B_n) defined by

$$B_n(x) = \binom{x}{n},$$

where

$$\binom{x}{0} = 1, \quad \binom{x}{n} = \frac{x(x-1)\dots(x-n+1)}{n!} \quad if \quad n \geq 1.$$

The polynomials $\binom{x}{n}$ are called the binomial polynomials. If Q is an operator on $C(\mathbb{Z}_p \to K)$, we put

$$b_n = (QB_n)(0) \quad n = 0, 1, \dots.$$

L. Van Hamme ([6], proposition) proved the following :

Theorem 1.2

If Q is continuous, linear and commutes with E then the sequence (b_n) is bounded and Q is uniquely determined by the sequence (b_n).

Such an operator Q which is linear, continuous and commutes with E admits an expansion of the form

$$Q = \sum_{i=0}^{\infty} b_i \Delta^i.$$

This expansion is called the Δ-expansion of the operator Q, $\Delta^0 = I$. The equality holds for the pointwise convergenge and not for the convergenge in operatornorm. Conversely, every operator of the form $Q = \sum_{i=0}^{\infty} b_i \Delta^i$ with bounded sequence (b_n) in K is linear, continuous and commutes with E. Further,

$$||Q|| = sup_{n \geq 0}\{|b_n|\} \qquad (1.2)$$

where $||Q||$ denotes the norm of the operator Q :
$||Q|| = inf\{J \in [0, \infty) \; : \; ||Qf||_\infty \leq J||f||_\infty \mid f \in C(\mathbb{Z}_p \to K)\}$.
We remark that in the classical umbral calculus one considers linear operators working on the space of polynomials $\mathbb{R}[x]$, and so there are no convergence problems for operators on $\mathbb{R}[x]$ of the type $R = \sum_{k \geq 0} \frac{a_k}{k!} Q^k$. This is different from what we do here, since here we consider linear operators on the Banach space $C(\mathbb{Z}_p \to K)$ into itself.

Remarks

1) Let $Q = \sum_{i=N}^{\infty} b_i \Delta^i$, $(N \geq 0)$, with $b_N \neq 0$. If p is a polynomial, then Qp is a polynomial. If p is a polynomial of degree $n \geq N$, then the degree of the polynomial Qp is $n - N$. If p is a polynomial of degree $n < N$, then Qp is the zero polynomial.
2) The set of all continuous linear operators on $C(\mathbb{Z}_p \to K)$ that commute with E forms a ring under addition and composition. This ring is isomorphic to the ring of formal power series with bounded coefficients in K.
3) Let Q and R be continuous linear operators that commute with E. Then $QR = RQ$. If Q is a continuous linear operator that commutes with E, then Q also commutes with E^α.
4) If Q is a continuous linear operator that commutes with E, then Q has an inverse

which is also linear, continuous and commutes with E if and only if $||Q|| = |b_0| \neq 0$. If in addition $|b_0| = 1$, then $||Q|| = ||Q^{-1}|| = 1 = |(Q^{-1}B_0)(0)|$. This can be found in [1], corollaire p. 16.06.

Definition 1.3

A delta-operator is a continuous linear operator on $C(\mathbb{Z}_p \rightarrow K)$ which commutes with the translation operator E, and such that the polynomial Qx is a constant different from zero.

L. Van Hamme proved (see [6], theorem)

Theorem 1.4

If Q is a continuous linear operator on $C(\mathbb{Z}_p \rightarrow K)$ that commutes with E, such that $b_0 = 0$, $|b_1| = 1$, $|b_n| \leq 1$ for $n \geq 2$, then

1) there exists a unique polynomial sequence (p_n) such that

$$Qp_n = p_{n-1} \ \ if \ \ n \geq 1, \ \ p_n(0) = 0 \ \ if \ n \geq 1 \ \ and \ \ p_0 = 1,$$

2) every continuous function $f : \mathbb{Z}_p \rightarrow K$ has a uniformly convergent expansion of the form

$$f = \sum_{n=0}^{\infty} (Q^n f)(0) p_n$$

where

$$||f||_\infty = max_{n \geq 0}\{|(Q^n f)(0)|\}.$$

It is easy to see that the operator Q of the theorem is a delta-operator. Just as in the classical case, we'll call the sequence (p_n) the basic sequence for the operator Q. Remark that here we have $Qp_n = p_{n-1}$, instead of $Qp_n = np_{n-1}$ which is used in the classical umbral calculus.

Remarks

1) The sequence (p_n) forms an orthonormal basis for $C(\mathbb{Z}_p \rightarrow K)$. In the classical case, the basic sequence for the delta-operator forms a basis for $\mathbb{R}[x]$. So this theorem is an extension of the classical case.
2) The polynomial sequence that corresponds with the operator Δ is the sequence $(\binom{x}{n})$ which is known as Mahler's basis for $C(\mathbb{Z}_p \rightarrow K)$ ([2]). If f is an element of $C(\mathbb{Z}_p \rightarrow K)$, we have $f(x) = \sum_{n=0}^{\infty} (\Delta^n f)(0)\binom{x}{n}$.

An example

Let Q be the operator $Q = \sum_{i=1}^{\infty} \Delta^i$, then we find for the unique polynomial sequence (p_n) : $p_0(x) = 1$ and $p_n(x) = \sum_{i=1}^{n} (-1)^{n-i}\binom{x}{i}\binom{n-1}{i-1}$ if $n \geq 1$. We show this by (double) induction. For n equal to zero or one this is obvious. Suppose the statement is true for n, then we prove it is also true for $n+1$. We have to prove that

$$\sum_{j=1}^{n+1} \Delta^j \sum_{i=1}^{n+1} (-1)^{n+1-i} \binom{x}{i} \binom{n}{i-1} = \sum_{i=1}^{n} (-1)^{n-i} \binom{x}{i} \binom{n-1}{i-1}$$

Now the expression on the left-hand-side equals

$\sum_{j=1}^{n+1} \sum_{i=j}^{n+1} (-1)^{n+1-i} \binom{x}{i-j} \binom{n}{i-1}$

$= \sum_{j=1}^{n+1} \sum_{k=0}^{n+1-j} (-1)^{n+1-j-k} \binom{x}{k} \binom{n}{k+j-1}$ (where $k = i - j$)

$= \sum_{k=0}^{n} \binom{x}{k} \sum_{j=1}^{n+1-k} (-1)^{n+1-j-k} \binom{n}{k+j-1}$.

And so we have to prove that, if $0 \leq k \leq n$,

$$\sum_{j=1}^{n+1-k} (-1)^{1-j} \binom{n}{k+j-1} = \binom{n-1}{k-1} \tag{1.3}$$

where we put $\binom{n}{-1}$ equal to zero. We prove this by induction on k. For k equal to n this is obvious. Now suppose it holds for $k = s+1$ $(0 \leq s \leq n-1)$, then we show that it holds for $k = s$. Expression (1.3) for k equal to $s+1$ gives us $\sum_{j=1}^{n-s} (-1)^{1-j} \binom{n}{s+j} = \binom{n-1}{s}$ and if we put $j+1 = t$ this gives

$$\sum_{t=2}^{n+1-s} (-1)^{t} \binom{n}{s-1+t} = \binom{n-1}{s} \tag{1.4}$$

The left-hand-side of (1.3) for k equal to s is $-\sum_{j=1}^{n+1-s} (-1)^j \binom{n}{s+j-1}$ and with the aid of (1.4) this equals $\binom{n}{s} - \binom{n-1}{s} = \binom{n-1}{s-1}$ which is the right-hand side for (1.3) for k equal to s. This finishes the proof.

2. Orthonormal Bases for $C(\mathbb{Z}_p \to K)$

In this section we are going to construct some orthonormal bases for the Banach space $C(\mathbb{Z}_p \to K)$. To do this we'll need the following theorem :

Theorem 2.1

Let (p_n) be a polynomial sequence in $C(\mathbb{Z}_p \to K)$, which forms an orthonormal basis for $C(\mathbb{Z}_p \to K)$, and let (r_n) be a polynomial sequence in $C(\mathbb{Z}_p \to K)$ such that

$$r_n = \sum_{j=0}^{n} e_{n,j} p_j, \quad e_{n,j} \in K.$$

Then the following are equivalent :

1) (r_n) *forms an orthonormal basis for* $C(\mathbb{Z}_p \to K)$,
2) $\|r_n\|_\infty = 1, \quad |e_{n,n}| = 1, \quad n = 0, 1, \ldots,$
3) $|e_{n,j}| \le 1, \quad |e_{n,n}| = 1, \quad n = 0, 1, \ldots; 0 \le j \le n.$

Proof

This follows from [7], theorem 3, by putting $M = \mathbb{Z}_p$ □

If (α_n) is a sequence in $\mathbb{Z}_p$, then it is easy to see that the polynomial sequence $(\binom{x-\alpha_n}{n})$ forms an orthonormal basis for $C(\mathbb{Z}_p \to K)$. To see this, put $p_j = B_j = \binom{x}{j}$ in theorem 2.1 ($j = 0, 1, \ldots$). Further, if $k \le n$, $\Delta^k\binom{x-\alpha}{n} = \binom{x-\alpha}{n-k}$ since the sequence $(\binom{x}{n})$ is of *binomial type* (see [1], p.16.06, lemme 1 and théorème 5).

We'll need the next two lemma's to prove the main theorem of this section. *deg p* denotes the degree of the polynomial p.

Lemma 2.2

Let N *be a natural number different from zero, let* α *be a fixed element of* $\mathbb{Z}_p$ *and let* p *be a polynomial in* $K[x]$ *such that* $p(\alpha + i) = 0$ *if* $0 \le i < N$. *Then* $(\Delta^k p)(\alpha) = 0$ *if* $0 \le k < N$.

Proof

If *deg* $p < N$, there is nothing to prove. Now suppose *deg* $p = n \ge N$. We can write p in the following way : $p(x) = \sum_{j=N}^{n} b_j \binom{x-\alpha}{j}$ since $p(\alpha + i) = 0$ if $0 \le i < N$. Then, for $0 \le k < N$, $(\Delta^k p)(x) = \sum_{j=N}^{n} b_j \binom{x-\alpha}{j-k}$ (remarks following theorem 2.1) and so $(\Delta^k p)(\alpha) = 0$ if $0 \le k < N$ □

Lemma 2.3

Let $Q = \sum_{i=N}^{\infty} b_i \Delta^i, \quad b_N \ne 0, \quad N \ge 1, \quad (b_n)$ *a bounded sequence in* K, *and let* α *be a fixed element of* $\mathbb{Z}_p$. *Then there exists a unique polynomial sequence* (p_n) *such that* $(Qp_n) = p_{n-N}$ *if* $n \ge N, \quad p_n(\alpha + i) = 0$ *if* $n \ge N, \quad 0 \le i < N$, *and* $p_n(x) = \binom{x-\alpha}{n}$ *if* $n < N$.

Proof

The series (p_n) is constructed by induction. For $n = 0, 1, \ldots, N-1$ there is nothing to prove. Suppose that $p_0, p_1, \ldots, p_{n-1}$ $(n \ge N)$ have already been constructed. Write p_n in the following way :

$$p_n(x) = a_n x^n + \sum_{i=0}^{n-1} a_i p_i(x).$$

Since p_n is a polynomial of degree $n \geq N$, Qp_n is a polynomial of degree $n-N$. Put $Qx^n = P(x)$, a polynomial of degree $n-N$. So $Qp_n = a_nP + \sum_{i=N}^{n-1} a_i p_{i-N}$ and this equals p_{n-N}. This gives us the coefficients $a_n, a_{n-1}, \ldots, a_N$. The fact that $p_n(\alpha + i)$ must equal zero for $0 \leq i < N$ gives us the coefficients $a_0, a_1, \ldots, a_{N-1}$:

$$0 = p_n(\alpha+i) = a_n(\alpha+i)^n + \sum_{j=0}^{N-1} a_j p_j(\alpha+i) = a_n(\alpha+i)^n + \sum_{j=0}^{N-1} a_j \binom{i}{j}.$$

From this it follows that the polynomial sequence (p_n) exists and is unique □

Now we are ready to prove the main theorem of this section.

Theorem 2.4

Let $Q = \sum_{i=N}^{\infty} b_i \Delta^i$, $N \geq 1$ with $|b_N| = 1$, $|b_n| \leq 1$ if $n > N$, and let α be an arbitrary but fixed element of $\mathbb{Z}_p$.

1) There exists a unique polynomial sequence (p_n) such that

$$Qp_n = p_{n-N} \quad if \quad n \geq N,$$

$$p_n(\alpha+i) = 0 \quad if \quad n \geq N, \quad 0 \leq i < N,$$

$$p_n(x) = \binom{x-\alpha}{n} \quad if \quad n < N.$$

This sequence forms an orthonormal basis for $C(\mathbb{Z}_p \to K)$.
2) If f is an element of $C(\mathbb{Z}_p \to K)$, there exists a unique, uniformly convergent expansion of the form

$$f = \sum_{n=0}^{\infty} c_n p_n$$

where

$$c_n = (\Delta^i Q^k f)(\alpha) \quad if \quad n = i + kN \quad 0 \leq i < N,$$

with

$$||f|| = max_{0 \leq k, 0 \leq i < N}\{|(\Delta^i Q^k f)(\alpha)|\}.$$

Proof

1) The existence and the uniqueness of the sequence follows from lemma 2.3. We only have to prove that the sequence forms an orthonormal basis. We give a proof by induction on n, using theorem 2.1. We put

$$p_n = \sum_{j=0}^{n} c_{n,j} C_j, \quad where \quad C_j(x) = \binom{x-\alpha}{j}.$$

The sequence (C_j) forms an orthonormal basis for $C(\mathbb{Z}_p \to K)$, see the remarks following theorem 2.1. If we apply theorem 2.1 on the sequence (C_j) we find the following :

$$(p_n) \ forms \ an \ orthonormal \ basis \ for \ C(\mathbb{Z}_p \to K)$$

$$if \ and \ only \ if \ |c_{n,j}| \leq 1, \ \ |c_{n,n}| = 1 \ \ n = 0,1,\ldots,0 \leq j \leq n.$$

We prove that $|c_{n,j}| \leq 1$, $|c_{n,n}| = 1$ by induction on n. For $n = 0,1,\ldots,N-1$ the assertion clearly holds. Suppose it holds for $i = 0,\ldots,n-1$, $n \geq N$, then $p_n = \sum_{j=0}^{n} c_{n,j}C_j = \sum_{j=N}^{n} c_{n,j}C_j$ since $p_n(\alpha+i) = 0$ for $0 \leq i < N$. So $|c_{n,j}| \leq 1$ for $0 \leq j < N$. $Qp_n = p_{n-N} = \sum_{j=0}^{n-N} c_{n-N,j}C_j$ where $|c_{n-N,n-N}| = 1$, $|c_{n-N,j}| \leq 1$, $0 \leq j \leq n-N$ by the induction hypothesis.
Now $Qp_n = \sum_{k=N}^{n} b_k\Delta^k \sum_{j=N}^{n} c_{n,j}C_j$
$= \sum_{j=N}^{n} c_{n,j} \sum_{k=N}^{j} b_k C_{j-k}$ (since $\Delta^k C_j = C_{j-k}$)
$= \sum_{j=0}^{n-N} c_{n,j+N} \sum_{k=0}^{j} b_{k+N} C_{j-k}$
$= \sum_{j=0}^{n-N} c_{n,j+N} \sum_{k=0}^{j} b_{j-k+N} C_k$
$= \sum_{k=0}^{n-N} C_k \sum_{j=k}^{n-N} b_{j-k+N}c_{n,j+N}$.
If $k = n-N$, then, since $Qp_n = p_{n-N}$,

$$b_N c_{n,n} = c_{n-N,n-N}$$

so $|c_{n,n}| = 1$. If $n = N$ we may stop here. If $n > N$, we proceed by subinduction. Suppose, if $0 \leq k < n-N$, that then $|c_{n,j+N}| \leq 1$ if $k < j \leq n-N$. Since $Qp_n = p_{n-N}$, it follows that $\sum_{j=k}^{n-N} b_{j-k+N}c_{n,j+N} = c_{n-N,k}$, which implies that

$$b_N c_{n,k+N} = c_{n-N,k} - \sum_{j=k+1}^{n-N} b_{j-k+N}c_{n,j+N}.$$

Then $|c_{n,k+N}| \leq |b_N|^{-1} max\{|c_{n-N,k}|, max_{k<j\leq n-N}|b_{j-k+N}c_{n,j+N}|\} \leq 1$, which we wanted to prove. This finishes the proof of 1).
2) Let f be an element of $C(\mathbb{Z}_p \to K)$. Since the sequence (p_n) forms an orthonormal basis for $C(\mathbb{Z}_p \to K)$, there exists coefficients c_n such that $f = \sum_{n=0}^{\infty} c_n p_n$ uniformly. We prove that c_n equals $(\Delta^i Q^k f)(\alpha)$ if n equals $i + kN$, $0 \leq i < N$. Since $f = \sum_{n=0}^{\infty} c_n p_n$, we have

$$(Q^k f) = \sum_{n=kN}^{\infty} c_n p_{n-kN} = \sum_{n=0}^{N-1} c_{n+kN}\binom{x-\alpha}{n} + \sum_{n=N}^{\infty} c_{n+kN}p_n.$$

If we put $\sum_{n=N}^{\infty} c_{n+kN}p_n = \hat{f}$, then $(\Delta^i \hat{f})(\alpha) = 0$ by lemma 2.2. Further, since $\Delta^i\binom{x-\alpha}{n} = 0$ if $i > n$, $\Delta^i\binom{x-\alpha}{n} = \binom{x-\alpha}{n-i}$ if $i \leq n$, and in particular $\Delta^i\binom{x-\alpha}{i} = 1$, we have $(\Delta^i Q^k f)(\alpha) = c_{i+kN}$. This gives us the coefficients c_n. Since (p_n) forms an orthonormal basis for $C(\mathbb{Z}_p \to K)$, it follows that

$$||f|| = max_{0\leq k, 0\leq i<N}\{|(\Delta^i Q^k f)(\alpha)|\}.$$

□

An example

Let Q be the operator $Q = \sum_{i=2}^{\infty} \Delta^i$ and put $\alpha = 0$. Then we find for the unique polynomial sequence (p_n)

$$p_0(x) = 1$$

and

$$p_{2n+1}(x) = \sum_{k=n+1}^{2n+1} (-1)^{k-1} \binom{x}{k} \binom{n}{2n+1-k} \quad if \quad n \geq 0$$

$$p_{2n+2}(x) = \sum_{k=n+2}^{2n+2} (-1)^{k} \binom{x}{k} \binom{n}{2n+2-k} \quad if \quad n \geq 0$$

The proof is more or less analogous to the proof of the example in the introduction.

We want to construct more orthonormal bases for $C(\mathbb{Z}_p \to K)$. To do this we need the following lemma

Lemma 2.5

Let $Q = \sum_{i=N}^{\infty} b_i \Delta^i$ $(N \geq 0)$ with $1 = |b_N| \geq |b_n|$ if $n > N$, and let p be a polynomial in $K[x]$ of degree $n \geq N$, $p(x) = \sum_{j=0}^{n} c_j \binom{x}{j}$ where $|c_j| \leq 1,\ 0 \leq j < n,\ |c_n| = 1$. Then $Qp = r$ where $r(x) = \sum_{j=0}^{n-N} a_j \binom{x}{j}$ with $|a_j| \leq 1,\ 0 \leq j < n-N,\ |a_{n-N}| = 1$.

Proof

It is clear that r is a polynomial of degree $n - N$. Then

$(Qp)(x) = \sum_{i=N}^{n} b_i \Delta^i \sum_{j=0}^{n} c_j \binom{x}{j} = \sum_{i=N}^{n} b_i \sum_{j=i}^{n} c_j \binom{x}{j-i} = r(x)$. Now $||Qp||_\infty = ||r||_\infty$. Since $|c_j| \leq 1$ and $|b_i| \leq 1$ $(i \geq N, 0 \leq j \leq n)$ we have $||Qp||_\infty \leq 1$ and so $||r||_\infty \leq 1$. If $r(x) = \sum_{j=0}^{n-N} a_j \binom{x}{j}$, then we must have $|a_j| \leq 1$ if $0 \leq j \leq n - N$ (otherwise $||r||_\infty > 1$). So it suffices to prove that $|a_{n-N}| = 1$. Since $Qp = r$ and since the coefficients of $\binom{x}{n-N}$ on both sides must be equal we have $c_n b_N = a_{n-N}$ and so $|a_{n-N}| = 1$ since $|b_N| = 1$ and $|c_n| = 1$ □

And now we immediately have

Theorem 2.6

Let (p_n) be a polynomial sequence which forms an orthonormal basis for $C(\mathbb{Z}_p \to K)$, and let $Q = \sum_{i=N}^{\infty} b_i\Delta^i$ $(N \geq 0)$ with $1 = |b_N| \geq |b_n|$ if $n > N$. If $Qp_n = r_{n-N}$ $(n \geq N)$, then the polynomial sequence (r_k) forms an orthonormal basis for $C(\mathbb{Z}_p \to K)$.

Proof

This follows immediately from theorem 2.1 and lemma 2.5 □

3. Continuous Linear Operators on $C(\mathbb{Z}_p \to K)$

In this section we establish some results on continuous linear operators on $C(\mathbb{Z}_p \to K)$. In particular we look at operators which commute with the translation operator E. Our first theorem in this section concerns delta-operators. To prove this theorem we need the folowing lemma's

Lemma 3.1

$||\Delta^n|| = 1$ *for all* $n \in \mathbb{N}$.

Proof

This follows immediately from (1.2). □

Lemma 3.2

Let Q be an operator such that $Q = \sum_{i=N}^{\infty} b_i\Delta^i$, with $|b_i| \leq 1$ if $i \geq N$,

$b_N \neq 0$ $(N \in \mathbb{N})$. Then we have

1) $||Q^n f||_\infty \leq ||\Delta^{nN} f||_\infty$, $n = 0, 1, \ldots$
2) if $N \geq 1$, then $(Q^n f)(x) \to 0$ uniformly if n tends to infinity.

Proof

1) This follows immediately by considering the corresponding power series $\sum_{i=N}^{\infty} b_i t^i$.

2) $|Q^n f(x)| \leq ||Q^n f||_\infty \leq ||\Delta^{nN} f||_\infty$ and so $(Q^n f)(x)$ tends to zero uniformly if n tends to infinity since $(\Delta^{nN} f)(x)$ tends to zero uniformly if n tends to infinity (by 1.1) □

For delta-operators Q with norm equal to one and with $|QB_1(0)| = 1$ we can prove a theorem analogous to theorem 1.2 of the introduction. Let α be an arbitrary but fixed element of $\mathbb{Z}_p$ and let (p_n) be the polynomial sequence as found in theorem 2.4. If (d_n) $(n = 0, 1, \ldots)$ is a bounded sequence in K, then we can associate an operator T with this sequence such that $(Tp_n)(\alpha) = d_n$. In order to see this we define the operator T in the following way

$$(Tf)(x) = \sum_{n=0}^{\infty} d_n(Q^n f)(x) \qquad (3.1)$$

where $Q^0 = I$ and where f denotes an element of $C(\mathbb{Z}_p \to K)$. Then T is clearly linear and commutes with E since Q commutes with E. The operator T is also continuous. To see this, take $f \in C(\mathbb{Z}_p \to K)$. Since $(Q^n f)(x) \to 0$ uniformly if n tends to infinity (lemma 3.2 2)), the series converges uniformly and defines a continuous function Tf. T is continuous since (lemma 3.1 and lemma 3.2 1))

$$||Tf||_\infty \leq sup_{n\geq 0}\{|d_n|\ ||Q^n f||_\infty\} \leq ||f||_\infty sup_{n\geq 0}\{|d_n|\}. \qquad (3.2)$$

Further, $d_n = (Tp_n)(\alpha)$ since $(Tp_n)(\alpha) = \sum_{k=0}^{n} d_k(Q^k p_n)(\alpha) = \sum_{k=0}^{n} d_k p_{n-k}(\alpha) = d_n$. We'll denote the operator T defined by (3.1) as $T = \sum_{n=0}^{\infty} d_n Q^n$. We remark that the equality holds for the pointwise convergence, and not for the convergence in the operatornorm. We'll call $T = \sum_{i=0}^{\infty} d_i Q^i$ the Q-expansion of the operator T.

We can ask ourselves whether every continuous linear operator that commutes with E is of the form $\sum_{i=0}^{\infty} d_i Q^i$ where the sequence (d_n) is bounded. The answer to this question is given by the following theorem. To prove this theorem we need the following lemma, where $Ker\ T$ denotes the kernel of the linear operator T.

Lemma 3.3

Let T be a continuous linear operator on $C(\mathbb{Z}_p \to K)$ which commutes with the translation operator. If $Ker\ T$ contains a polynomial of degree n, then T lowers the degree of every polynomial with at least $n+1$.

Proof

If $T = \sum_{k=0}^{\infty} b_k \Delta^k$, and $Ker\ T$ contains a polynomial of degree n, then $b_0 = \ldots = b_n = 0$. Suppose that this were not true, let then $k_0 \leq n$ be the smallest index such that $b_{k_0} \neq 0$. Since Δ lowers the degree of a polynomial with one, Tp is a polynomial of degree $n - k_0$ and then p is not in the kernel of T. So $b_0 = \ldots = b_n = 0$ and we conclude that T lowers the degree of every polynomial with at least $n+1$ □

For delta-operators Q with norm equal to one and with $|QB_1(0)| = 1$ we can prove the folowing

Theorem 3.4

Let Q be a delta-operator such that $||Q|| = |QB_1(0)| = 1$, let α be an arbitrary but fixed element of $\mathbb{Z}_p$ and let (p_n) be the polynomial sequence as found in theorem 2.4.

1) Let T be an operator on $C(\mathbb{Z}_p \to K)$ and put $d_n = (Tp_n)(\alpha)$. If T is continuous, linear and commutes with E then the sequence (d_n) is bounded and $T = \sum_{n=0}^{\infty} d_n Q^n$.

2) If (d_n) is a bounded sequence, then the operator defined by $T = \sum_{n=0}^{\infty} d_n Q^n$ is linear, continuous and commutes with E. Furthermore, $d_n = (Tp_n)(\alpha)$.

Proof

We only have to prove 1) since 2) is already proved. This proof is similar to the proof of the proposition in [6]. Suppose that T is a continuous linear operator on $C(\mathbb{Z}_p \to K)$ and $TE = ET$. By the remarks following theorem 1.2 it follows that Tp_0 is a constant. Define

$$d_0 = Tp_0.$$

Then $Ker(T - d_0 I)$ contains p_0 since $Tp_0 - d_0 = 0$. By lemma 3.3, $(T - d_0 I)p_1$ is a constant, and so we can define d_1 by

$$(T - d_0 I)p_1 = d_1.$$

$Ker(T - d_0 I - d_1 Q)$ contains p_1 since $(T - d_0 I)p_1 - d_1 Q p_1 = d_1 - d_1 = 0$. So $Ker(T - d_0 I - d_1 Q)$ contains p_1 etc .... If $d_0, d_1, \ldots, d_{n-1}$ are already defined, then we have that $Ker(T - \sum_{i=0}^{n-1} d_i Q^i)$ contains p_{n-1}. So $(T - \sum_{i=0}^{n-1} d_i Q^i)p_n$ is a constant, hence we can put

$$(T - \sum_{i=0}^{n-1} d_i Q^i)p_n = d_n.$$

Then $d_n = (T - \sum_{i=0}^{n-1} d_i Q^i)p_n = Tp_n - \sum_{i=0}^{n-1} d_i p_{n-i}$.
We now prove that the sequence (d_n) is bounded. Now $|d_0| = ||Tp_0||_\infty \leq ||T|| ||p_0||_\infty \leq ||T||$. By induction : suppose $|d_j| \leq ||T||$ for $j = 0, 1, \ldots, n-1$. Then, since $||p_k||_\infty = 1$ for all k,

$$|d_n| \leq max\{||T||, |d_0|, |d_1| \ldots, |d_{n-1}|\} = ||T||. \qquad (3.3)$$

So the sequence (d_n) is bounded.
It follows from the construction that the kernel of the continuous operator $(T - \sum_{i=0}^{\infty} d_i Q^i)$ contains p_n for all $n \in \mathbb{N}$ and so it contains $K[x]$ (lemma 3.3). Since $K[x]$ is dense in $C(\mathbb{Z}_p \to K)$ ([5], theorem 43.3, Kaplansky's theorem) it is the zero-operator and so

$$T = \sum_{i=0}^{\infty} d_i Q^i.$$

If f is an element of $C(\mathbb{Z}_p \to K)$, then $(Tf)(x) = \sum_{i=0}^{\infty} d_i (Q^i f)(x)$ and the series on the right-hand-side is uniformly convergent since $(Q^i f)(x) \to 0$ uniformly if n tends to infinity (lemma 3.2 2)). Clearly we have $d_n = (Tp_n)(\alpha)$, since $(Tp_n)(\alpha) = \sum_{i=0}^{n} d_i (Q^i p_n)(\alpha) = \sum_{i=0}^{n} d_i p_{n-i}(\alpha) = d_n$ □

Remarks

1) If $T = \sum_{i=0}^{\infty} d_i Q^i$ is a continuous operator, then T satisfies

$$||T|| = sup_{n \geq 0}\{|d_n|\}. \qquad (3.4)$$

This follows immediately from (3.2) and (3.3).
2) The coefficients d_i in the Q-expansion are unique.
3) The composition of two such operators corresponds with multiplication of power series. The set of all continuous operators of the type $\sum_{i=0}^{\infty} d_i Q^i$, (d_i) bounded in K, forms a ring under addition and composition which is isomorphic to the ring of formal power series $\sum_{i=0}^{\infty} d_i t^i$ where (d_i) is bounded.

Let T be a continuous linear operator on $C(\mathbb{Z}_p \to K)$ which commutes with E, and suppose that $T = \sum_{n=0}^{\infty} b_n \Delta^n = \sum_{n=0}^{\infty} d_n Q^n$ where Q is a delta-operator such that $||Q|| = 1 = |(QB_1)(0)|$. Then it is easy to see that T has the following properties $(N \in \mathbb{N})$:

$$b_i = 0 \;\; if \;\; 0 \leq i < N, \;\; b_N \neq 0 \;\; if \;\; and \;\; only \;\; if \;\; d_i = 0 \;\; if \;\; 0 \leq i < N, \;\; d_N \neq 0.$$

Further, if J is a positive real number, then for all $n \geq N$: $|b_n| \leq J$ if and only if for all $n \geq N$: $|d_n| \leq J$. In addition we have that

$$|b_N| = |d_N|.$$

It follows that

$$||T|| = |b_N| \;\; if \;\; and \;\; only \;\; if \;\; ||T|| = |d_N|.$$

If we use the same notation of the theorem, then the operator T is a delta-operator if and only if $d_0 = 0$, $d_1 \neq 0$ i.e. $(Tp_0)(\alpha) = 0$, $(Tp_1)(\alpha) \neq 0$. It also follows that the operator T has an inverse which is also linear, continuous and commutes with E if and only if

$$||T|| = |d_0| \neq 0.$$

In addition,

$$||T|| = ||T^{-1}|| \;\; if \;\; and \;\; only \;\; if \;\; ||T|| = |d_0| = 1.$$

This follows immediately from the properties above and remark 4) following theorem 1.2.

Some examples

Let us consider the delta-operator

$$Q = \sum_{i=1}^{\infty} \Delta^i$$

and put α equal to zero. Then the basic sequence (p_n) for the operator Q is

$$p_0(x) = 1, \quad p_n(x) = \sum_{i=1}^{n} (-1)^{n-i} \binom{x}{i} \binom{n-1}{i-1} \quad if \quad n \geq 1$$

(example following theorem 1.4). For the operators E and Δ^k $(k \geq 1)$ we find

1) $d_0 = (Ep_0)(0) = 1$ and for $n \geq 1$ we have $d_n = (Ep_n)(0) = \sum_{i=1}^{n}(-1)^{n-i}\binom{1}{i}\binom{n-1}{i-1} = (-1)^{n-1}$. This gives us the following expansion for the operator E

$$E = I + \sum_{n=1}^{\infty} (-1)^{n-1} Q^n.$$

2) $d_n = (\Delta^k p_n)(0) = 0$ for $n < k$ and for $n \geq k$ we have $d_n = (\Delta^k p_n)(0) = \sum_{i=k}^{n}(-1)^{n-i}\binom{0}{i-k}\binom{n-1}{i-1} = (-1)^{n-k}\binom{n-1}{k-1}$. This gives us the following expansion for the operator Δ^k

$$\Delta^k = \sum_{n=k}^{\infty} (-1)^{n-k} \binom{n-1}{k-1} Q^n.$$

3) For the operator $\sum_{n=2}^{\infty} \Delta^n$ of the example of section 2 we find $\sum_{n=2}^{\infty} \Delta^n = \sum_{n=1}^{\infty} \Delta^n - \Delta = Q - \sum_{n=1}^{\infty}(-1)^{n-1}Q^n = \sum_{n=2}^{\infty}(-1)^n Q^n$

Theorem 3.5

Let (p_n) and (q_n) be polynomial sequences in $K[x]$ which form orthonormal bases for $C(\mathbb{Z}_p \to K)$ and let N be a natural number.

1) For the linear operator T on $C(\mathbb{Z}_p \to K)$ such that $Tp_n = q_{n-N}$ if $n \geq N$, $Tp_n = 0$ if $n < N$, we have that $\|T\| = 1$ and so T is continuous. If in addition T is of the form $T = \sum_{i=N}^{\infty} b_i \Delta^i$ (by lemma 3.3), then $|b_N| = 1$.
2) If (r_n) is a polynomial sequence which forms an orthonormal basis for $C(\mathbb{Z}_p \to K)$, then the sequence $Tr_N, Tr_{N+1}, \ldots$ also forms an orthonormal basis for $C(\mathbb{Z}_p \to K)$.

Proof

1) If f is an element of $C(\mathbb{Z}_p \to K)$, then since (p_n) forms an orthonormal basis for $C(\mathbb{Z}_p \to K)$, there exists a uniformly convergent expansion of the form

$$f(x) = \sum_{n=0}^{\infty} a_n p_n(x)$$

and then we put

$$(Tf)(x) = \sum_{n=N}^{\infty} a_n q_{n-N}(x).$$

It is then obvious that $Tp_n = q_{n-N}$ if $n \geq N$, $Tp_n = 0$ if $n < N$. Since a_n tends to zero if n tends to infinity, the series on the right-hand-side is uniformly convergent and so Tf is a continuous function. T is clearly linear. Further, $||Tf||_\infty = max_{n\geq N}\{|a_n|\} \leq max_{n\geq 0}\{|a_n|\} = ||f||_\infty$ and so $||T|| \leq 1$. Furthermore, $||Tp_N||_\infty = ||q_0||_\infty = 1 = ||p_N||_\infty$ and so $||T|| = 1$. So T is continuous. If in addition T is of the form $T = \sum_{i=0}^\infty b_i\Delta^i$ then since $Tp_n = q_{n-N}$ if $n \geq N$, $Tp_n = 0$ if $n < N$, we have $T = \sum_{i=N}^\infty b_i\Delta^i$ and then from $Tp_N = q_0$ it immediately follows that $|b_N| = 1$.
2) Since (r_n) and (p_n) are polynomial sequences which form orthonormal bases for $C(\mathbb{Z}_p \to K)$, we can write by theorem 2.1 r_n in the following way :

$r_n = \sum_{j=0}^n b_{n,j}p_j$ $(b_{n,j} \in K)$ with $|b_{n,j}| \leq 1$, $|b_{n,n}| = 1$ for $0 \leq j \leq n$, $n \in \mathbb{N}$, and so if $n \geq N$ we have $Tr_n = \sum_{j=N}^n b_{n,j}q_{j-N}$ and so by theorem 2.1 the sequence $Tr_N, Tr_{N+1}, \ldots$ forms an orthonormal basis for $C(\mathbb{Z}_p \to K)$ since (q_n) forms an orthonormal basis for $C(\mathbb{Z}_p \to K)$ □

We can consider two special cases :

1) Take $p_n = q_n$, $n = 0, 1, \ldots$ where $N > 0$. Then we look for an operator T such that $Tp_n = p_{n-N}$, if $n \geq N$, $Tp_n = 0$ if $n < N$.
2) The other special case is where N is equal to zero. Such an operator T is then called an umbral operator. See definition 3.7.

It is interesting to know whether the operator T of theorem 3.5 is of the form $T = \sum_{i=N}^\infty b_i\Delta^i$, i.e. T commutes with E. The case where $p_n = q_n$ for all n and $N = 1$ can be found in [1], théorème 5, p. 16.10. Another special case is the following

Theorem 3.6

Let (p_n) be a polynomial sequence which forms an orthonormal basis for $C(\mathbb{Z}_p \to K)$ and let Q be a delta-operator with $||Q|| = 1 = |(QB_1)(0)|$. Suppose that the formula

$$Qp_n = \sum_{k=0}^n p_k s_{n-k} \quad n = 0, 1, \ldots$$

holds for some sequence of constants (s_n) in K. Then there exists a continuous linear operator R which commutes with E such that $Rp_n = p_{n-N}$ if $n \geq N$ and $Rp_n = 0$ if $n < N$ $(N \geq 1)$.

Proof

If f is an element of $C(\mathbb{Z}_p \to K)$, there exists coefficients $a_n \in K$ such that

$$f(x) = \sum_{n=0}^\infty a_n p_n(x)$$

where the series on the right-hand-side is uniformly convergent, $||f||_\infty = max_{n\geq 0}\{|a_n|\}$ and a_n tends to zero if n tends to infinity. Let R be the operator defined as follows

$$(Rf)(x) = \sum_{n=N}^{\infty} a_n p_{n-N}(x).$$

It is clear that R satisfies $Rp_n = p_{n-N}$ if $n \geq N$ and $Rp_n = 0$ if $n < N$. Since a_n tends to zero if n tends to infinity and since $||p_n||_\infty = 1$ for all n, the series on the right-hand-side is uniformly convergent and thus Rf is a continuous function. R is clearly linear. We now show that R is continuous. We have $||Rf||_\infty = max_{n\geq N}\{|a_n|\} \leq max_{n\geq 0}\{|a_n|\} = ||f||_\infty$. We conclude that $||R|| \leq 1$ and thus R is continuous. We now show that $RQ^k = Q^kR$ $(k \in \mathbb{N})$. If n is at least N then $RQp_n = R\sum_{k=0}^{n} p_k s_{n-k} = \sum_{k=N}^{n} p_{k-N} s_{n-k} = \sum_{k=0}^{n-N} p_k s_{n-N-k} = Qp_{n-N} = QRp_n$ and if n is strictly smaller than N we have $RQp_n = QRp_n = 0$ so by linearity $QR = RQ$ on $C(\mathbb{Z}_p \to K)$ (since Q and R are continuous and since (p_n) forms an orthonormal basis) and continuing this way we have $RQ^k = Q^kR$ on $C(\mathbb{Z}_p \to K)$ $(k \in \mathbb{N})$. By theorem 3.4, there exists a bounded sequence (d_i) such that $E = \sum_{i=0}^{\infty} d_i Q^i$ and thus R commutes with E □

We now consider the case where $N = 0$ in theorem 3.5. This leads us to the following definition, which is more or less analogous to the definition of the classical umbral calculus (see 1. Introduction)

Definition 3.7

Let (p_n) and (q_n) be polynomial sequences which form orthonormal bases for $C(\mathbb{Z}_p \to K)$, and let U be the linear operator which maps p_n on q_n for all n :

$$Up_n = q_n \quad n = 0, 1, \ldots .$$

Then we will call U the umbral operator which maps p_n on q_n for all n.

Theorem 3.8

Let (p_n) and (q_n) be orthonormal bases for $C(\mathbb{Z}_p \to K)$ consisting of polynomial sequences, and let U be the umbral operator which maps p_n on q_n for all n.

1) Then U is an invertible, continuous operator for which $||U|| = ||U^{-1}|| = 1$.
2) If (r_n) is a polynomial sequence which forms an orthonormal basis for $C(\mathbb{Z}_p \to K)$, then (Ur_n) also forms an orthonormal basis for $C(\mathbb{Z}_p \to K)$.

Proof

1) We already know from theorem 3.5, by putting $N = 0$, that U is continuous and that $||U|| = 1$. If $f(x) = \sum_{n=0}^{\infty} a_n q_n(x)$ $(a_n \in K)$ is an element of $C(\mathbb{Z}_p \to K)$, then we define the operator S as follows $(Sf)(x) = \sum_{n=0}^{\infty} a_n p_n(x)$. Then Sf is a

continuous function for which $||Sf||_\infty = max_{n\geq 0}\{|a_n|\} = ||f||_\infty$ so the operator S is continuous and $||S|| = 1$. S is linear and from the definition of S and U it follows that $SU = US = I$ so $S = U^{-1}$.
2) This follows immediately from 2) of theorem 3.5, by putting $N = 0$ □

The umbral operator U does not necessarily commute with E. In the following special case U commutes with the translation operator :

Theorem 3.9

Let Q be a delta-operator such that $||Q|| = 1 = |(QB_1)(0)|$ and let (p_n) and (q_n) be polynomial sequences which form orthonormal bases for $C(\mathbb{Z}_p \rightarrow K)$ such that $Qq_n = q_{n-1}$ and $Qp_n = p_{n-1}$ $(n \geq 1)$. The umbral operator U which maps p_n on q_n for all n commutes with E. The operator U has an inverse which is also linear continuous and commutes with E.

Proof

The operator U is continuous and invertible (theorem 3.8). We prove that U commuts with E. The operator U commutes with Q : $UQp_n = Up_{n-1} = q_{n-1}$ and $QUp_n = Qq_n = q_{n-1}$ if $n \geq 1$ and if n equals zero we have $UQp_0 = QUp_0$ since both are equal to zero. By linearity, continuity and the fact that (p_n) forms an orthonormal basis, U commutes with Q. Continuing this way we find that U commutes with Q^k for all natural numbers k. By theorem 3.4, there exists an expansion of the form $E = \sum_{n=0}^{\infty} d_n Q^n$, (d_i) bounded, and so U commutes with E. Since $Up_0 = q_0$, it follows that $|(UB_0)(0)| = 1$ and by remark 4) following theorem 1.2 it follows that the operator U has an inverse which is also linear, continuous and commutes with E. In addition, $|(U^{-1}B_0)(0)| = 1$ □

Consider the algebra of continuous linear operators on $C(\mathbb{Z}_p \rightarrow K)$ and let U be an invertible element of this algebra. The map $S \rightarrow USU^{-1}$ is an *inner automorphism* of the algebra of continuous linear operators on $C(\mathbb{Z}_p \rightarrow K)$. Now let U be an umbral operator. Then we are able to prove the following theorem which is more or less similar to [4], section 2.7, proposition 1, p. 29.

Theorem 3.10

Let P and Q be delta-operators on $C(\mathbb{Z}_p \rightarrow K)$, $1 = ||Q|| = ||P|| = |(PB_1)(0)| = |(QB_1)(0)|$, and let p_n and q_n be polynomial sequences which form orthonormal bases for $C(\mathbb{Z}_p \rightarrow K)$ such that $Pp_n = p_{n-1}$ and $Qq_n = q_{n-1}$. Let U be the umbral operator which maps p_n on q_n for all n, and let S be a continuous linear operator which commutes with E. Then we have the following properties :

1) The map $S \rightarrow USU^{-1}$ is an automorphism of the ring of all continuous linear operators on $C(\mathbb{Z}_p \rightarrow K)$ which commute with E. Further, $||S|| = ||USU^{-1}||$.
2) If S is of the form $S = \sum_{n=N}^{\infty} b_n\Delta^n$ $(N \in \mathbb{N})$ with $b_N \neq 0$ then USU^{-1} is of

the form $USU^{-1} = \sum_{n=N}^{\infty} \beta_n \Delta^n$ *with* $\beta_N \neq 0$. *If in addition we have* $||S|| = |b_N|$, *then also* $||USU^{-1}|| = |\beta_N|$. *If* (s_n) *is a polynomial sequence such that* $Ss_n = s_{n-N}$ $(n \geq N)$ *and if* r_n *is the polynomial sequence defined by* $Us_n = r_n$ *then* $Rr_n = r_{n-N}$ $(n \geq N)$ *where* $R = USU^{-1}$.
3) If $S = \sum_{n=0}^{\infty} d_n V^n$, *where* V *is a delta-operator such that* $||V|| = 1 = |(VB_1)(0)|$, *then* $USU^{-1} = \sum_{n=0}^{\infty} d_n W^n$, *where* $W = UVU^{-1}$ *and* W *is a delta-operator such that* $||W|| = 1 = |(WB_1)(0)|$.

Proof

The inverse U^{-1} of U exists and is linear and continuous by theorem 3.8.
1) The map $S \to USU^{-1}$ is an inner automorphism of the algebra of continuous linear operators on $C(\mathbb{Z}_p \to K)$. We have to show that the subalgebra of operators which commute with E is invariant. We have $(n \geq 1)$ $UPp_n = Up_{n-1} = q_{n-1} = Qq_n = QUp_n$ and $UPp_0 = QUp_0 = 0$. So by linearity, continuity and since (p_n) forms an orhonormal basis we have $UP = QU$ on $C(\mathbb{Z}_p \to K)$ thus $UPU^{-1} = Q$. So we also have $UP^kU^{-1} = Q^k$ for all natural numbers k. There exists an expansion of the form $S = \sum_{i=0}^{\infty} d_i P^i$ with $||S|| = sup_{n \geq 0}\{|d_n|\}$ ((3.4) and theorem 3.4) and so $USU^{-1} = \sum_{i=0}^{\infty} d_i Q^i$ and we have $||USU^{-1}|| = sup_{n \geq 0}\{|d_n|\} = ||S||$ (by (3.4)). From the calculations it also follows that the map is onto (again theorem 3.4). So the map is an automorphism from the ring of continuous linear operators which commute with E onto itself.
2) If S is of the form $S = \sum_{n=N}^{\infty} b_n \Delta^n$ with $b_N \neq 0$ then $S = \sum_{n=N}^{\infty} \gamma_n P^n$ with $\gamma_N \neq 0$ (properties following theorem 3.4) and from the calculations in 1) it follows that $USU^{-1} = \sum_{n=N}^{\infty} \gamma_n Q^n$ with $\gamma_N \neq 0$ and so USU^{-1} is of the form $USU^{-1} = \sum_{n=N}^{\infty} \beta_n \Delta^n$, $\beta_N \neq 0$ (properties following theorem 3.4). If in addition $||S|| = |b_N|$, then $|\gamma_N| = ||S|| = ||USU^{-1}||$ ((3.4), 1) and properties folowing theorem 3.4) and so $|\beta_N| = ||USU^{-1}||$ (properties following theorem 3.4). Further we have $Rr_n = USU^{-1}r_n = USs_n = Us_{n-N} = r_{n-N}$.
3) Since $W = UVU^{-1}$, we have $W^k = UV^kU^{-1}$ $(k \in \mathbb{N})$. Thus if $S = \sum_{n=0}^{\infty} d_n V^n$, then $USU^{-1} = \sum_{n=0}^{\infty} d_n UV^nU^{-1} = \sum_{n=0}^{\infty} d_n W^n$. From 1) and 2) it follows that W is a delta-operator and $||W|| = 1 = |(WB_1)(0)|$ □

Finally let us consider the following : let V_q be the subset of $\mathbb{Z}_p$ defined as follows : V_q is the closure of the set $\{aq^n \mid n = 0, 1, \ldots\}$, where a and q are two units of $\mathbb{Z}_p$, q not a root of unity. $C(V_q \to K)$ denotes the Banach space of continuous functions from V_q to K. The operator D_q on $C(V_q \to K)$ is defined by

$$(D_q f)(x) = (f(qx) - f(x))/(x(q-1))$$

We remark that results for the operator D_q on $C(V_q \to K)$ analogous to the results in this paper can be found in [8] and [9], chapter 5.

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