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LOCAL INVERTIBILITY OF NON-ARCHIMEDEAN VECTOR-VALUED FUNCTIONS

Stany De Smedt

Abstract We are well acquainted with the local invertibility of C^n -functions of one variable (see [4]). Surprisingly this is not the case for vector-valued functions. In this paper we will fill this gap in non-archimedean functional analysis.

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1 Introduction

Let K be a complete non-archimedean non-trivially valued field, and $C(X \rightarrow K)$ the space of all continuous functions from X to K , where X is a nonempty subset of K without isolated points.

Let $f : X \rightarrow K$ and $\nabla^2 X = X \times X \setminus \{(x, x) \mid x \in X\}$. The (first) difference quotient $\phi_1 f : \nabla^2 X \rightarrow K$ is defined by

$$\phi_1 f(x, y) = \frac{f(y) - f(x)}{y - x}$$

f is called continuously differentiable (or strictly differentiable, or uniformly differentiable) at $a \in X$ if $\lim_{(x,y) \rightarrow (a,a)} \phi_1 f(x, y)$ exists. We will also say that f is C^1 at a .

In a similar way, we may define C^n -functions as follows :

For $n \in \mathbb{N}$, we define $\nabla^{n+1} X = \{(x_1, \dots, x_{n+1}) \in X^{n+1} \mid x_i \neq x_j \text{ if } i \neq j\}$ and the n -th difference quotient $\phi_n f : \nabla^{n+1} X \rightarrow K$ by $\phi_0 f = f$ and

$$\phi_n f(x_1, x_2, \dots, x_{n+1}) = \frac{\phi_{n-1} f(x_2, x_3, \dots, x_{n+1}) - \phi_{n-1} f(x_1, x_3, \dots, x_{n+1})}{x_2 - x_1}$$

A function f is called a C^n -function if $\phi_n f$ can be extended to a continuous function $\overline{\phi_n f}$ on X^{n+1} .

The set of all C^n -functions from X to K will be denoted by $C^n(X \rightarrow K)$.

For $f : X \times X \rightarrow K$, the first difference quotients $\phi_1^{(1)} f$ and $\phi_1^{(2)} f$ are defined as

$$\phi_1^{(1)} f(x, x', y) = \frac{f(x, y) - f(x', y)}{x - x'}$$

and

$$\phi_1^{(2)} f(x, y, y') = \frac{f(x, y) - f(x, y')}{y - y'}$$

(for $x \neq x'$ and $y \neq y'$).

If $\phi_1^{(1)} f$ and $\phi_1^{(2)} f$ can be extended to continuous functions $\overline{\phi_1^{(1)} f}$, $\overline{\phi_1^{(2)} f}$ respectively, defined on X^3 then f is called a C^1 -function. The space of all these C^1 -functions, will be denoted $C^1(X \times X \rightarrow K)$.

We have obviously

$$\frac{\partial f}{\partial x}(x, y) = \overline{\phi_1^{(1)} f}(x, x, y)$$

$$\frac{\partial f}{\partial y}(x, y) = \overline{\phi_1^{(2)} f}(x, y, y)$$

for all $x, y \in X$.

For the difference quotients of second order, we get

$$\phi_2^{(11)} f(x, x', x'', y) = \frac{\phi_1^{(1)} f(x, x', y) - \phi_1^{(1)} f(x, x'', y)}{x' - x''}$$

$$\phi_2^{(21)} f(x, x', y, y') = \frac{\phi_1^{(1)} f(x, x', y) - \phi_1^{(1)} f(x, x', y')}{y - y'}$$

$$\phi_2^{(12)} f(x, x', y, y') = \frac{\phi_1^{(2)} f(x, y, y') - \phi_1^{(2)} f(x', y, y')}{x - x'}$$

$$\phi_2^{(22)} f(x, y, y', y'') = \frac{\phi_1^{(2)} f(x, y, y') - \phi_1^{(2)} f(x, y, y'')}{y' - y''}$$

and f is a C^2 -function if those four functions can be extended to continuous functions on X^4 . Remark that $\phi_2^{(21)} f(x, x', y, y') = \phi_2^{(12)} f(x, x', y, y')$. Following the notations above, we denote $C^2(X \times X \rightarrow K)$ for the space of all C^2 -functions. Continuing in the same way, we define the difference quotients of n -th order and the C^n -functions.

More information about C^n -functions can be found in [1], [2], [3] and [4].

2 Vector-valued C^1 -functions

Now let us take a look to vector-valued functions.

Let E be an n -dimensional non-archimedean vectorspace, F an m -dimensional non-archimedean vectorspace, and let $f : E \rightarrow F$. Let x be a point of E . For all vectors h such that $|h|$ is small (and $h \neq 0$), the point $x + h$ also lies in E .

However we cannot form a quotient $\frac{f(x+h) - f(x)}{h}$ because it is meaningless to divide by a vector. In order to define what we mean for a function f to be continuously differentiable, we must find a way which does not involve dividing by h . We reconsider the case of functions f of one variable :

f is continuously differentiable at a point a if $\lim_{(x,y) \rightarrow (a,a)} \frac{f(y) - f(x)}{y - x}$ exists

In other words f is C^1 at a if f is differentiable at a and if for each $\epsilon > 0$, there exists $\delta > 0$ such that if $|x-a| < \delta, |y-a| < \delta, x \neq y$ then $\left| \frac{f(y) - f(x)}{y - x} - f'(a) \right| < \epsilon$

This can also be written as :

f is C^1 at a if there exists a function g such that for all $x \neq y$:

$$f(y) - f(x) = f'(a).(y - x) + g(y - x) \text{ whereby } \lim_{(x,y) \rightarrow (a,a)} \frac{g(y - x)}{y - x} = 0$$

We now consider a function of n variables.

Definition

Let X be a nonempty subset of E without isolated points. The function $f : X \rightarrow F$ is *continuously differentiable at a* if there is a linear function L (depending on a) such that for all $x \neq y \in X$:

$$f(x) - f(y) = L(x - y) + g(x - y) \text{ with } \lim_{(x,y) \rightarrow (a,a)} \frac{|g(y - x)|}{|y - x|} = 0$$

The linear function L is then denoted by df_a and is called the *differential of f at a* . Its matrix $f'(a)$ is called *the derivative of f at a* . Thus $f'(a)$ is the unique $m \times n$ -matrix such that $df_a(x) = f'(a).x$ for all $x \in X$. The set of all C^1 -functions $f : X \rightarrow F$ is denoted $C^1(X \rightarrow F)$.

Lemma

The function $f : X \rightarrow F$ is continuously differentiable at a if and only if each of its component functions f_1, \dots, f_m is.

This lemma follows immediately from a componentwise reading of the vector equation in the definition above.

Proposition

The composition of two C^1 -functions is a C^1 -function.

Proof

We prove the proposition for $n = m = 2$. The proof is completely similar for the general case.

$$\begin{aligned}
& \phi_1^{(1)}(g \circ f)_1(x_1, x'_1, x_2) \\
&= \frac{(g \circ f)_1(x_1, x_2) - (g \circ f)_1(x'_1, x_2)}{x_1 - x'_1} \\
&= \frac{g_1(f_1(x_1, x_2), f_2(x_1, x_2)) - g_1(f_1(x'_1, x_2), f_2(x'_1, x_2))}{x_1 - x'_1} \\
&= \frac{g_1(f_1(x_1, x_2), f_2(x_1, x_2)) - g_1(f_1(x'_1, x_2), f_2(x_1, x_2))}{x_1 - x'_1} \\
&\quad + \frac{g_1(f_1(x'_1, x_2), f_2(x_1, x_2)) - g_1(f_1(x'_1, x_2), f_2(x'_1, x_2))}{x_1 - x'_1} \\
&= \frac{g_1(f_1(x_1, x_2), f_2(x_1, x_2)) - g_1(f_1(x'_1, x_2), f_2(x_1, x_2))}{f_1(x_1, x_2) - f_1(x'_1, x_2)} \cdot \frac{f_1(x_1, x_2) - f_1(x'_1, x_2)}{x_1 - x'_1} \\
&\quad + \frac{g_1(f_1(x'_1, x_2), f_2(x_1, x_2)) - g_1(f_1(x'_1, x_2), f_2(x'_1, x_2))}{f_2(x_1, x_2) - f_2(x'_1, x_2)} \cdot \frac{f_2(x_1, x_2) - f_2(x'_1, x_2)}{x_1 - x'_1} \\
&= \phi_1^{(1)} g_1(f_1(x_1, x_2), f_1(x'_1, x_2), f_2(x_1, x_2)) \cdot \phi_1^{(1)} f_1(x_1, x'_1, x_2) \\
&\quad + \phi_1^{(2)} g_1(f_1(x'_1, x_2), f_2(x_1, x_2), f_2(x'_1, x_2)) \cdot \phi_1^{(1)} f_2(x_1, x'_1, x_2)
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \phi_1^{(1)}(g \circ f)_2(x_1, x'_1, x_2) \\
&= \phi_1^{(1)} g_2(f_1(x_1, x_2), f_1(x'_1, x_2), f_2(x_1, x_2)) \cdot \phi_1^{(1)} f_1(x_1, x'_1, x_2) \\
&\quad + \phi_1^{(2)} g_2(f_1(x'_1, x_2), f_2(x_1, x_2), f_2(x'_1, x_2)) \cdot \phi_1^{(1)} f_2(x_1, x'_1, x_2)
\end{aligned}$$

$$\begin{aligned}
& \phi_1^{(2)}(g \circ f)_1(x_1, x_2, x'_2) \\
&= \phi_1^{(1)} g_1(f_1(x_1, x_2), f_1(x_1, x'_2), f_2(x_1, x_2)) \cdot \phi_1^{(2)} f_1(x_1, x_2, x'_2) \\
&\quad + \phi_1^{(2)} g_1(f_1(x_1, x'_2), f_2(x_1, x_2), f_2(x_1, x'_2)) \cdot \phi_1^{(2)} f_2(x_1, x_2, x'_2)
\end{aligned}$$

$$\begin{aligned}
& \phi_1^{(2)}(g \circ f)_2(x_1, x_2, x'_2) \\
&= \phi_1^{(1)} g_2(f_1(x_1, x_2), f_1(x_1, x'_2), f_2(x_1, x_2)) \cdot \phi_1^{(2)} f_1(x_1, x_2, x'_2) \\
&\quad + \phi_1^{(2)} g_2(f_1(x_1, x'_2), f_2(x_1, x_2), f_2(x_1, x'_2)) \cdot \phi_1^{(2)} f_2(x_1, x_2, x'_2)
\end{aligned}$$

And these are continuous extendable since f and g are continuously differentiable functions.

Theorem

Let $f : E \rightarrow E$ be C^1 at a and suppose $\det f'(a) \neq 0$, then f is locally invertible. The local inverse g of f is C^1 at $f(a)$ and $g'(f(a)) = (f'(a))^{-1}$.

Proof

We may assume that $a = 0 = f(a)$. If this is not the case, we first make translations, replacing $f(x)$ by $F(x) = f(x + a) - f(a)$.

Let $L = df_0$ and $\Phi = L^{-1} \circ f$. Note that the inverse L^{-1} exists since $\det f'(0) \neq 0$. Moreover $\Phi(0) = 0$ and $d\Phi_0 = L^{-1} \circ df_0 = L^{-1} \circ L = I$.

L^{-1} is a linear function, so L^{-1} is C^1 and thus also Φ is C^1 . From the definition of C^1 -functions we then get that given ϵ there exists a neighbourhood U of 0 of radius δ such that $|\Phi(s) - \Phi(t) - (s - t)| \leq \epsilon|s - t|$ for every $s, t \in U$.

Choose $\epsilon < 1$, then f is injective on U .

Indeed, suppose $f(s) = f(t)$ with $s, t \in U$, then $\Phi(s) = L^{-1}(f(s)) = L^{-1}(f(t)) = \Phi(t)$. And thus $|s - t| = |\Phi(s) - \Phi(t) - (s - t)| \leq \epsilon|s - t|$, so $s = t$.

Given $y \in U$, define $\varphi_y : E \rightarrow E$ by $\varphi_y(x) = x - \Phi(x) + y$.

We want to show that φ_y is a contraction on U , its unique fixed point will then be the desired point $x \in U$, such that $\Phi(x) = y$.

For $x \in U$:

$$|\varphi_y(x)| = |x - \Phi(x) + y| \leq \max(|x - \Phi(x)|, |y|) < \max(|x|, |y|) < \delta$$

So φ_y maps U into itself.

$$|\varphi_y(s) - \varphi_y(t)| = |s - \Phi(s) - t + \Phi(t)| \leq \epsilon|s - t|$$

Thus $\varphi_y : U \rightarrow U$ is indeed a contraction, and therefore has a unique fixed point $x \in U$ such that $\Phi(x) = y$.

The fixed point $x = \Psi(y)$ is the limit of the sequence (x_m) defined by

$$x_0 = 0, x_{m+1} = x_m - \Phi(x_m) + y$$

We now apply that $|\Phi(s) - \Phi(t) - (s - t)| \leq \epsilon|s - t|$ with $s = \Psi(x), t = \Psi(y)$ for any $\epsilon < 1$, then

$$|\Phi(\Psi(x)) - \Phi(\Psi(y)) - (\Psi(x) - \Psi(y))| \leq \epsilon|\Psi(x) - \Psi(y)| < |\Psi(x) - \Psi(y)|$$

or equivalently

$$|x - y - (\Psi(x) - \Psi(y))| < |\Psi(x) - \Psi(y)|$$

Because of the strict inequality, this means that $|x - y| = |\Psi(x) - \Psi(y)|$, so

$$|\Psi(x) - \Psi(y) - (x - y)| \leq \epsilon|x - y|$$

This proves that Ψ is continuously differentiable with $d\Psi_0 = I$.

Let $\eta = \frac{\delta}{\|L^{-1}\|}$ and let V be the η -neighbourhood of 0.

Define g by $g(x) = \Psi(L^{-1}(x))$ for all $x \in V$. Thus g is C^1 as composition of two C^1 -functions and $g(V) \subset U$.

Moreover,

$$f(g(x)) = f(\Psi(L^{-1}(x))) = L \circ \Phi \circ \Psi \circ L^{-1}(x) = x \text{ for all } x \in V$$

And

$$g(f(t)) = \Psi \circ L^{-1} \circ f(t) = \Psi \circ \Phi(t) = t \text{ for all } t \in U$$

The chain rule gives $g'(f(a)) = (f'(a))^{-1}$, which ends our proof.

3 Local invertibility of C^k -functions

As in classical analysis, C^2 -functions can be defined as follows.

Definition

The function $f : E \rightarrow F$ is C^2 at a if there is a linear function L_1 and a bilinear function L_2 (both depending on a) such that for all $x \neq y \in E$:

$$f(x) - f(y) = L_1(x-y) + L_2(x-y)(x-y) + g(x-y) \text{ with } \lim_{(x,y) \rightarrow (a,a)} \frac{|g(y-x)|}{|y-x|^2} = 0$$

In order to give a similar definition for C^k -functions, we have to formulate an extra assumption on the domain of our function (see [4], section 83). Whereas the proof of the local invertibility theorem for C^1 -functions is based on such a definition, we will prefer the following definition to prove the local invertibility theorem for C^k -functions.

Definiton

The function $f : E \rightarrow F$ is C^k at a if and only if each of its components f_1, \dots, f_m is.

We then get

Theorem

Let $f : E \rightarrow E$ be C^k at a and suppose $\det f'(a) \neq 0$, then f is locally invertible. The local inverse g of f is C^k at $f(a)$.

Proof

We will restrict our proof to the case $k = 2, n = 2$, since otherwise the notations become too complicated.

Since f is a C^2 -function by assumption, f is also a C^1 -function. So the previous theorem tells us that f is locally invertible and the local inverse g of f is C^1 at $f(a)$. To prove that g is also C^2 at $f(a)$ we will give an expression for the second order difference quotients of the components g_1 and g_2 of g .

We start from the formulas in the proposition of section 2, for example

$$\begin{aligned} & \phi_1^{(1)}(f \circ g)_1(x_1, x'_1, x_2) \\ &= \phi_1^{(1)} f_1(g_1(x_1, x_2), g_1(x'_1, x_2), g_2(x_1, x_2)) \cdot \phi_1^{(1)} g_1(x_1, x'_1, x_2) \\ & \quad + \phi_1^{(2)} f_1(g_1(x'_1, x_2), g_2(x_1, x_2), g_2(x'_1, x_2)) \cdot \phi_1^{(1)} g_2(x_1, x'_1, x_2) \end{aligned}$$

Since f and g are inverse functions $f \circ g = I$, the identity function. So all second order difference quotients of $f \circ g$ are identically 0. We thus get

$$\begin{aligned} 0 &= \phi_2^{(11)}(f \circ g)_1(x_1, x'_1, x''_1, x_2) \\ &= \frac{\phi_1^{(1)}(f \circ g)_1(x_1, x'_1, x_2) - \phi_1^{(1)}(f \circ g)_1(x_1, x''_1, x_2)}{x'_1 - x''_1} \\ &= \left(\phi_1^{(1)} f_1(g_1(x_1, x_2), g_1(x'_1, x_2), g_2(x_1, x_2)) \cdot \phi_1^{(1)} g_1(x_1, x'_1, x_2) \right. \\ & \quad + \phi_1^{(2)} f_1(g_1(x'_1, x_2), g_2(x_1, x_2), g_2(x'_1, x_2)) \cdot \phi_1^{(1)} g_2(x_1, x'_1, x_2) \\ & \quad - \phi_1^{(1)} f_1(g_1(x_1, x_2), g_1(x''_1, x_2), g_2(x_1, x_2)) \cdot \phi_1^{(1)} g_1(x_1, x''_1, x_2) \\ & \quad \left. - \phi_1^{(2)} f_1(g_1(x''_1, x_2), g_2(x_1, x_2), g_2(x''_1, x_2)) \cdot \phi_1^{(1)} g_2(x_1, x''_1, x_2) \right) \cdot \frac{1}{x'_1 - x''_1} \\ &= \left(\phi_1^{(1)} f_1(g_1(x_1, x_2), g_1(x'_1, x_2), g_2(x_1, x_2)) \cdot \phi_1^{(1)} g_1(x_1, x'_1, x_2) \right. \\ & \quad - \phi_1^{(1)} f_1(g_1(x_1, x_2), g_1(x'_1, x_2), g_2(x_1, x_2)) \cdot \phi_1^{(1)} g_1(x_1, x''_1, x_2) \\ & \quad + \phi_1^{(1)} f_1(g_1(x_1, x_2), g_1(x'_1, x_2), g_2(x_1, x_2)) \cdot \phi_1^{(1)} g_1(x_1, x''_1, x_2) \\ & \quad + \phi_1^{(2)} f_1(g_1(x'_1, x_2), g_2(x_1, x_2), g_2(x'_1, x_2)) \cdot \phi_1^{(1)} g_2(x_1, x'_1, x_2) \\ & \quad - \phi_1^{(2)} f_1(g_1(x'_1, x_2), g_2(x_1, x_2), g_2(x'_1, x_2)) \cdot \phi_1^{(1)} g_2(x_1, x''_1, x_2) \\ & \quad + \phi_1^{(2)} f_1(g_1(x'_1, x_2), g_2(x_1, x_2), g_2(x'_1, x_2)) \cdot \phi_1^{(1)} g_2(x_1, x''_1, x_2) \\ & \quad - \phi_1^{(1)} f_1(g_1(x_1, x_2), g_1(x''_1, x_2), g_2(x_1, x_2)) \cdot \phi_1^{(1)} g_1(x_1, x''_1, x_2) \\ & \quad - \phi_1^{(2)} f_1(g_1(x''_1, x_2), g_2(x_1, x_2), g_2(x'_1, x_2)) \cdot \phi_1^{(1)} g_2(x_1, x''_1, x_2) \\ & \quad \left. + \phi_1^{(2)} f_1(g_1(x''_1, x_2), g_2(x_1, x_2), g_2(x'_1, x_2)) \cdot \phi_1^{(1)} g_2(x_1, x''_1, x_2) \right) \cdot \frac{1}{x'_1 - x''_1} \\ &= \phi_1^{(1)} f_1(g_1(x_1, x_2), g_1(x'_1, x_2), g_2(x_1, x_2)) \cdot \phi_2^{(11)} g_1(x_1, x'_1, x''_1, x_2) \end{aligned}$$

$$\begin{aligned}
& + \phi_1^{(2)} f_1(g_1(x'_1, x_2), g_2(x_1, x_2), g_2(x'_1, x_2)) \cdot \phi_2^{(11)} g_2(x_1, x'_1, x''_1, x_2) \\
& + \phi_2^{(11)} f_1(g_1(x_1, x_2), g_1(x'_1, x_2), g_1(x''_1, x_2), g_2(x_1, x_2)) \cdot \phi_1^{(1)} g_1(x_1, x'_1, x''_1, x_2) \cdot \phi_1^{(1)} g_1(x'_1, x''_1, x_2) \\
& + \phi_2^{(12)} f_1(g_1(x'_1, x_2), g_1(x''_1, x_2), g_2(x_1, x_2), g_2(x'_1, x_2)) \cdot \phi_1^{(1)} g_2(x_1, x''_1, x_2) \cdot \phi_1^{(1)} g_1(x'_1, x''_1, x_2) \\
& + \phi_2^{(22)} f_1(g_1(x''_1, x_2), g_2(x_1, x_2), g_2(x'_1, x_2), g_2(x''_1, x_2)) \cdot \phi_1^{(1)} g_2(x_1, x'_1, x_2) \cdot \phi_1^{(1)} g_2(x'_1, x''_1, x_2)
\end{aligned}$$

In the same way we can prove that

$$\begin{aligned}
0 & = \phi_2^{(11)} (f \circ g)_2(x_1, x'_1, x''_1, x_2) \\
& = \phi_1^{(1)} f_2(g_1(x_1, x_2), g_1(x'_1, x_2), g_2(x_1, x_2)) \cdot \phi_2^{(11)} g_1(x_1, x'_1, x''_1, x_2) \\
& \quad + \phi_1^{(2)} f_2(g_1(x'_1, x_2), g_2(x_1, x_2), g_2(x'_1, x_2)) \cdot \phi_2^{(11)} g_2(x_1, x'_1, x''_1, x_2) \\
& \quad + \phi_2^{(11)} f_2(g_1(x_1, x_2), g_1(x'_1, x_2), g_1(x''_1, x_2), g_2(x_1, x_2)) \cdot \phi_1^{(1)} g_1(x_1, x'_1, x''_1, x_2) \cdot \phi_1^{(1)} g_1(x'_1, x''_1, x_2) \\
& \quad + \phi_2^{(12)} f_2(g_1(x'_1, x_2), g_1(x''_1, x_2), g_2(x_1, x_2), g_2(x'_1, x_2)) \cdot \phi_1^{(1)} g_2(x_1, x''_1, x_2) \cdot \phi_1^{(1)} g_1(x'_1, x''_1, x_2) \\
& \quad + \phi_2^{(22)} f_2(g_1(x''_1, x_2), g_2(x_1, x_2), g_2(x'_1, x_2), g_2(x''_1, x_2)) \cdot \phi_1^{(1)} g_2(x_1, x'_1, x_2) \cdot \phi_1^{(1)} g_2(x'_1, x''_1, x_2)
\end{aligned}$$

$$\begin{aligned}
0 & = \phi_2^{(21)} (f \circ g)_1(x_1, x'_1, x_2, x'_2) \\
& = \phi_1^{(1)} f_1(g_1(x_1, x_2), g_1(x'_1, x_2), g_2(x_1, x_2)) \cdot \phi_2^{(21)} g_1(x_1, x'_1, x_2, x'_2) \\
& \quad + \phi_1^{(2)} f_1(g_1(x'_1, x_2), g_2(x_1, x_2), g_2(x'_1, x_2)) \cdot \phi_2^{(21)} g_2(x_1, x'_1, x_2, x'_2) \\
& \quad + \phi_2^{(11)} f_1(g_1(x_1, x_2), g_1(x_1, x'_2), g_1(x'_1, x_2), g_2(x_1, x_2)) \cdot \phi_1^{(1)} g_1(x_1, x'_1, x'_2) \cdot \phi_1^{(2)} g_1(x_1, x_2, x'_2) \\
& \quad + \phi_2^{(11)} f_1(g_1(x_1, x'_2), g_1(x'_1, x_2), g_1(x'_1, x'_2), g_2(x_1, x_2)) \cdot \phi_1^{(1)} g_1(x_1, x'_1, x'_2) \cdot \phi_1^{(2)} g_1(x'_1, x_2, x'_2) \\
& \quad + \phi_2^{(21)} f_1(g_1(x_1, x'_2), g_1(x'_1, x'_2), g_2(x_1, x_2), g_2(x_1, x'_2)) \cdot \phi_1^{(1)} g_1(x_1, x'_1, x'_2) \cdot \phi_1^{(2)} g_2(x_1, x_2, x'_2) \\
& \quad + \phi_2^{(12)} f_1(g_1(x'_1, x_2), g_1(x'_1, x'_2), g_2(x_1, x_2), g_2(x'_1, x_2)) \cdot \phi_1^{(1)} g_2(x_1, x'_1, x'_2) \cdot \phi_1^{(2)} g_1(x'_1, x_2, x'_2) \\
& \quad + \phi_2^{(22)} f_1(g_1(x'_1, x'_2), g_2(x_1, x_2), g_2(x_1, x'_2), g_2(x'_1, x_2)) \cdot \phi_1^{(1)} g_2(x_1, x'_1, x'_2) \cdot \phi_1^{(2)} g_2(x_1, x_2, x'_2) \\
& \quad + \phi_2^{(22)} f_1(g_1(x'_1, x'_2), g_2(x_1, x'_2), g_2(x'_1, x_2), g_2(x'_1, x'_2)) \cdot \phi_1^{(1)} g_2(x_1, x'_1, x'_2) \cdot \phi_1^{(2)} g_2(x'_1, x_2, x'_2)
\end{aligned}$$

$$\begin{aligned}
0 & = \phi_2^{(21)} (f \circ g)_2(x_1, x'_1, x_2, x'_2) \\
& = \phi_1^{(1)} f_2(g_1(x_1, x_2), g_1(x'_1, x_2), g_2(x_1, x_2)) \cdot \phi_2^{(21)} g_1(x_1, x'_1, x_2, x'_2) \\
& \quad + \phi_1^{(2)} f_2(g_1(x'_1, x_2), g_2(x_1, x_2), g_2(x'_1, x_2)) \cdot \phi_2^{(21)} g_2(x_1, x'_1, x_2, x'_2) \\
& \quad + \phi_2^{(11)} f_2(g_1(x_1, x_2), g_1(x_1, x'_2), g_1(x'_1, x_2), g_2(x_1, x_2)) \cdot \phi_1^{(1)} g_1(x_1, x'_1, x'_2) \cdot \phi_1^{(2)} g_1(x_1, x_2, x'_2) \\
& \quad + \phi_2^{(11)} f_2(g_1(x_1, x'_2), g_1(x'_1, x_2), g_1(x'_1, x'_2), g_2(x_1, x_2)) \cdot \phi_1^{(1)} g_1(x_1, x'_1, x'_2) \cdot \phi_1^{(2)} g_1(x'_1, x_2, x'_2) \\
& \quad + \phi_2^{(21)} f_2(g_1(x_1, x'_2), g_1(x'_1, x'_2), g_2(x_1, x_2), g_2(x_1, x'_2)) \cdot \phi_1^{(1)} g_1(x_1, x'_1, x'_2) \cdot \phi_1^{(2)} g_2(x_1, x_2, x'_2) \\
& \quad + \phi_2^{(12)} f_2(g_1(x'_1, x_2), g_1(x'_1, x'_2), g_2(x_1, x_2), g_2(x'_1, x_2)) \cdot \phi_1^{(1)} g_2(x_1, x'_1, x'_2) \cdot \phi_1^{(2)} g_1(x'_1, x_2, x'_2) \\
& \quad + \phi_2^{(22)} f_2(g_1(x'_1, x'_2), g_2(x_1, x_2), g_2(x_1, x'_2), g_2(x'_1, x_2)) \cdot \phi_1^{(1)} g_2(x_1, x'_1, x'_2) \cdot \phi_1^{(2)} g_2(x_1, x_2, x'_2) \\
& \quad + \phi_2^{(22)} f_2(g_1(x'_1, x'_2), g_2(x_1, x'_2), g_2(x'_1, x_2), g_2(x'_1, x'_2)) \cdot \phi_1^{(1)} g_2(x_1, x'_1, x'_2) \cdot \phi_1^{(2)} g_2(x'_1, x_2, x'_2)
\end{aligned}$$

$$\begin{aligned}
0 &= \phi_2^{(12)}(f \circ g)_1(x_1, x'_1, x_2, x'_2) \\
&= \phi_1^{(1)} f_1(g_1(x_1, x_2), g_1(x_1, x'_2), g_2(x_1, x_2)) \cdot \phi_2^{(12)} g_1(x_1, x'_1, x_2, x'_2) \\
&\quad + \phi_1^{(2)} f_1(g_1(x_1, x'_2), g_2(x_1, x_2), g_2(x_1, x'_2)) \cdot \phi_2^{(12)} g_2(x_1, x'_1, x_2, x'_2) \\
&\quad + \phi_2^{(11)} f_1(g_1(x_1, x_2), g_1(x'_1, x_2), g_1(x_1, x'_2), g_2(x_1, x_2)) \cdot \phi_1^{(1)} g_1(x_1, x'_1, x_2) \cdot \phi_1^{(2)} g_1(x'_1, x_2, x'_2) \\
&\quad + \phi_2^{(11)} f_1(g_1(x'_1, x_2), g_1(x_1, x'_2), g_1(x'_1, x'_2), g_2(x_1, x_2)) \cdot \phi_1^{(1)} g_1(x_1, x'_1, x'_2) \cdot \phi_1^{(2)} g_1(x'_1, x_2, x'_2) \\
&\quad + \phi_2^{(21)} f_1(g_1(x'_1, x_2), g_1(x'_1, x'_2), g_2(x_1, x_2), g_2(x'_1, x_2)) \cdot \phi_1^{(1)} g_2(x_1, x'_1, x_2) \cdot \phi_1^{(2)} g_1(x'_1, x_2, x'_2) \\
&\quad + \phi_2^{(12)} f_1(g_1(x_1, x'_2), g_1(x'_1, x'_2), g_2(x_1, x_2), g_2(x_1, x'_2)) \cdot \phi_1^{(1)} g_1(x_1, x'_1, x'_2) \cdot \phi_1^{(2)} g_2(x'_1, x_2, x'_2) \\
&\quad + \phi_2^{(22)} f_1(g_1(x'_1, x'_2), g_2(x_1, x_2), g_2(x'_1, x_2), g_2(x_1, x'_2)) \cdot \phi_1^{(1)} g_2(x_1, x'_1, x_2) \cdot \phi_1^{(2)} g_2(x'_1, x_2, x'_2) \\
&\quad + \phi_2^{(22)} f_1(g_1(x'_1, x'_2), g_2(x'_1, x_2), g_2(x_1, x'_2), g_2(x'_1, x'_2)) \cdot \phi_1^{(1)} g_2(x_1, x'_1, x'_2) \cdot \phi_1^{(2)} g_2(x'_1, x_2, x'_2)
\end{aligned}$$

$$\begin{aligned}
0 &= \phi_2^{(12)}(f \circ g)_2(x_1, x'_1, x_2, x'_2) \\
&= \phi_1^{(1)} f_2(g_1(x_1, x_2), g_1(x_1, x'_2), g_2(x_1, x_2)) \cdot \phi_2^{(12)} g_1(x_1, x'_1, x_2, x'_2) \\
&\quad + \phi_1^{(2)} f_2(g_1(x_1, x'_2), g_2(x_1, x_2), g_2(x_1, x'_2)) \cdot \phi_2^{(12)} g_2(x_1, x'_1, x_2, x'_2) \\
&\quad + \phi_2^{(11)} f_2(g_1(x_1, x_2), g_1(x'_1, x_2), g_1(x_1, x'_2), g_2(x_1, x_2)) \cdot \phi_1^{(1)} g_1(x_1, x'_1, x_2) \cdot \phi_1^{(2)} g_1(x'_1, x_2, x'_2) \\
&\quad + \phi_2^{(11)} f_2(g_1(x'_1, x_2), g_1(x_1, x'_2), g_1(x'_1, x'_2), g_2(x_1, x_2)) \cdot \phi_1^{(1)} g_1(x_1, x'_1, x'_2) \cdot \phi_1^{(2)} g_1(x'_1, x_2, x'_2) \\
&\quad + \phi_2^{(21)} f_2(g_1(x'_1, x_2), g_1(x'_1, x'_2), g_2(x_1, x_2), g_2(x'_1, x_2)) \cdot \phi_1^{(1)} g_2(x_1, x'_1, x_2) \cdot \phi_1^{(2)} g_1(x'_1, x_2, x'_2) \\
&\quad + \phi_2^{(12)} f_2(g_1(x_1, x'_2), g_1(x'_1, x'_2), g_2(x_1, x_2), g_2(x_1, x'_2)) \cdot \phi_1^{(1)} g_1(x_1, x'_1, x'_2) \cdot \phi_1^{(2)} g_2(x'_1, x_2, x'_2) \\
&\quad + \phi_2^{(22)} f_2(g_1(x'_1, x'_2), g_2(x_1, x_2), g_2(x'_1, x_2), g_2(x_1, x'_2)) \cdot \phi_1^{(1)} g_2(x_1, x'_1, x_2) \cdot \phi_1^{(2)} g_2(x'_1, x_2, x'_2) \\
&\quad + \phi_2^{(22)} f_2(g_1(x'_1, x'_2), g_2(x'_1, x_2), g_2(x_1, x'_2), g_2(x'_1, x'_2)) \cdot \phi_1^{(1)} g_2(x_1, x'_1, x'_2) \cdot \phi_1^{(2)} g_2(x'_1, x_2, x'_2)
\end{aligned}$$

$$\begin{aligned}
0 &= \phi_2^{(22)}(f \circ g)_1(x_1, x_2, x'_2, x''_2) \\
&= \phi_1^{(1)} f_1(g_1(x_1, x_2), g_1(x_1, x'_2), g_2(x_1, x_2)) \cdot \phi_2^{(22)} g_1(x_1, x_2, x'_2, x''_2) \\
&\quad + \phi_1^{(2)} f_1(g_1(x_1, x'_2), g_2(x_1, x_2), g_2(x_1, x'_2)) \cdot \phi_2^{(22)} g_2(x_1, x_2, x'_2, x''_2) \\
&\quad + \phi_2^{(11)} f_1(g_1(x_1, x_2), g_1(x_1, x'_2), g_1(x_1, x''_2), g_2(x_1, x_2)) \cdot \phi_1^{(2)} g_1(x_1, x_2, x''_2) \cdot \phi_1^{(2)} g_1(x_1, x'_2, x''_2) \\
&\quad + \phi_2^{(12)} f_1(g_1(x_1, x'_2), g_1(x_1, x''_2), g_2(x_1, x_2), g_2(x_1, x'_2)) \cdot \phi_1^{(2)} g_2(x_1, x_2, x''_2) \cdot \phi_1^{(2)} g_1(x_1, x'_2, x''_2) \\
&\quad + \phi_2^{(22)} f_1(g_1(x_1, x''_2), g_2(x_1, x_2), g_2(x_1, x'_2), g_2(x_1, x''_2)) \cdot \phi_1^{(2)} g_2(x_1, x_2, x''_2) \cdot \phi_1^{(2)} g_2(x_1, x'_2, x''_2)
\end{aligned}$$

$$\begin{aligned}
0 &= \phi_2^{(22)}(f \circ g)_2(x_1, x_2, x'_2, x''_2) \\
&= \phi_1^{(1)} f_2(g_1(x_1, x_2), g_1(x_1, x'_2), g_2(x_1, x_2)) \cdot \phi_2^{(22)} g_1(x_1, x_2, x'_2, x''_2) \\
&\quad + \phi_1^{(2)} f_2(g_1(x_1, x'_2), g_2(x_1, x_2), g_2(x_1, x'_2)) \cdot \phi_2^{(22)} g_2(x_1, x_2, x'_2, x''_2) \\
&\quad + \phi_2^{(11)} f_2(g_1(x_1, x_2), g_1(x_1, x'_2), g_1(x_1, x''_2), g_2(x_1, x_2)) \cdot \phi_1^{(2)} g_1(x_1, x_2, x''_2) \cdot \phi_1^{(2)} g_1(x_1, x'_2, x''_2)
\end{aligned}$$

$$\begin{aligned}
& + \phi_2^{(12)} f_2(g_1(x_1, x'_2), g_1(x_1, x''_2), g_2(x_1, x_2), g_2(x_1, x'_2)) \cdot \phi_1^{(2)} g_2(x_1, x_2, x''_2) \cdot \phi_1^{(2)} g_1(x_1, x'_2, x''_2) \\
& + \phi_2^{(22)} f_2(g_1(x_1, x'_2), g_2(x_1, x_2), g_2(x_1, x'_2), g_2(x_1, x''_2)) \cdot \phi_1^{(2)} g_2(x_1, x_2, x''_2) \cdot \phi_1^{(2)} g_2(x_1, x'_2, x''_2)
\end{aligned}$$

We now get a system of 8 ($= n^{k+1}$) equations in the 8 unknowns

$$\begin{aligned}
& \phi_2^{(11)} g_1(x_1, x'_1, x''_1, x_2), \phi_2^{(11)} g_2(x_1, x'_1, x''_1, x_2), \phi_2^{(21)} g_1(x_1, x'_1, x_2, x'_2), \phi_2^{(21)} g_2(x_1, x'_1, x_2, x'_2), \\
& \phi_2^{(12)} g_1(x_1, x'_1, x_2, x'_2), \phi_2^{(12)} g_2(x_1, x'_1, x_2, x'_2), \phi_2^{(22)} g_1(x_1, x_2, x'_2, x''_2), \phi_2^{(22)} g_2(x_1, x_2, x'_2, x''_2)
\end{aligned}$$

with determinant

$$D = \begin{vmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & B \end{vmatrix}$$

where

$$A = \begin{pmatrix} \phi_1^{(1)} f_1(g_1(x_1, x_2), g_1(x'_1, x_2), g_2(x_1, x_2)) & \phi_1^{(2)} f_1(g_1(x'_1, x_2), g_2(x_1, x_2), g_2(x'_1, x_2)) \\ \phi_1^{(1)} f_2(g_1(x_1, x_2), g_1(x'_1, x_2), g_2(x_1, x_2)) & \phi_1^{(2)} f_2(g_1(x'_1, x_2), g_2(x_1, x_2), g_2(x'_1, x_2)) \end{pmatrix}$$

and

$$B = \begin{pmatrix} \phi_1^{(1)} f_1(g_1(x_1, x_2), g_1(x_1, x'_2), g_2(x_1, x_2)) & \phi_1^{(2)} f_1(g_1(x_1, x'_2), g_2(x_1, x_2), g_2(x_1, x'_2)) \\ \phi_1^{(1)} f_2(g_1(x_1, x_2), g_1(x_1, x'_2), g_2(x_1, x_2)) & \phi_1^{(2)} f_2(g_1(x_1, x'_2), g_2(x_1, x_2), g_2(x_1, x'_2)) \end{pmatrix}$$

In the limit that $(x_1, x_2), (x'_1, x_2), (x_1, x'_2)$ tend to $f(a)$, D becomes equal to $(\det f'(a))^4$ which is different from 0. (Herein $4 = n^k$). So $D \neq 0$ in a neighbourhood of $f(a)$.

Since f is a C^2 -function and g is a C^1 -function, and using Cramers rule to solve a system of linear equations, we see that all second order difference quotients of g are continuously extendable. Thus g is a C^2 -function.

Remark

In the proof above we could reduce our system to a system of 6 equations in 6 unknowns, since $\phi_2^{(21)} g_1(x_1, x'_1, x_2, x'_2) = \phi_2^{(12)} g_1(x_1, x'_1, x_2, x'_2)$ and $\phi_2^{(21)} g_2(x_1, x'_1, x_2, x'_2) = \phi_2^{(12)} g_2(x_1, x'_1, x_2, x'_2)$, but then it is more difficult to see the generalization to arbitrary n and k .

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