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Abstract

In the present paper we obtain a few inclusion theorems for the convolution of Nörlund methods in the form \((N, r_n) \subseteq (N, p_n) \ast (N, q_n)\) in complete, non-trivially valued, non-archimedean fields.

Throughout the present paper \(K\) denotes a complete, non-trivially valued, non-archimedean field. Infinite matrices and sequences, which are considered in the sequel, have entries in \(K\). If \(A = (a_{nk}), a_{nk} \in K, n, k = 0, 1, 2, \ldots\) is an infinite matrix, the \(A\)-transform \(A x = \{A x_n\}\) of the sequence \(x = \{x_k\}, x_k \in K, k = 0, 1, 2, \ldots\) is defined by

\[
(A x)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad n = 0, 1, 2, \ldots,
\]

where it is assumed that the series on the right converge. If \(\lim_{n \to \infty} (A x)_n = s\), we say that \(\{x_k\}\) is \(A\)-summable to \(s\), written as \(x_k \to s(A)\) or \(\text{A-lim } x_k = s\). If \(\lim_{n \to \infty} (A x)_n = s\) whenever \(\lim_{k \to \infty} x_k = s\), we say that \(A\) is regular. The following result is well-known (see [2], [4]).
Theorem 1 $A = (a_{nk})$ is regular if and only if

$$\sup_{n,k} |a_{nk}| < \infty; \quad (1)$$

$$\lim_{n \to \infty} a_{nk} = 0, \quad \text{for every fixed } k; \quad (2)$$

and

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = 1. \quad (3)$$

Any matrix $A$ for which (1) holds is called a $K_r$-matrix. If $A$ and $B$ are two infinite matrices such that $x_k \to s(A)$ implies $x_k \to s(B)$, we say that $A$ is included in $B$, written as $A \subseteq B$. $A$ is said to be row-finite if for $n = 0, 1, 2, \ldots$, there exists a positive integer $k_n$ such that $a_{nk} = 0, k > k_n$.

Given two infinite matrices $A$ and $B$, their convolution is defined as the matrix $C = (c_{nk})$, where

$$c_{nk} = \sum_{i=0}^{k} a_{ni}b_{n,k-i}, \quad n, k = 0, 1, 2, \ldots. \quad (4)$$

In such a case we write $C = A \ast B$.

The following properties of the convolution can be easily proved.

1. If $A$ and $B$ are both row-finite or both $K_r$, then their convolution $C$ is row-finite or $K_r$ respectively and their row sums satisfy

$$\sum_{k=0}^{\infty} c_{nk} = \left(\sum_{k=0}^{\infty} a_{nk}\right) \left(\sum_{k=0}^{\infty} b_{nk}\right), \quad n = 0, 1, 2, \ldots. \quad (5)$$

2. If $A, B$ are both regular, then $C$ is regular too.

The Nörlund method of summability i.e., $(N, p_n)$ method in $K$ is defined as follows (see [5]): $(N, p_n)$ is defined by the infinite matrix $(a_{nk})$ where

$$a_{nk} = \begin{cases} \frac{p_{n-k}}{p_n}, & k \leq n; \\ 0, & k > n, \end{cases}$$

where $p_0 \neq 0, |p_0| > |p_j|, j = 1, 2, \ldots$ and $P_n = \sum_{k=0}^{n} p_k, n = 0, 1, 2, \ldots$. It is to be noted that $|P_n| = |p_0| \neq 0$ so that $P_n \neq 0, n = 0, 1, 2, \ldots$. The following result is very useful in the sequel.

Theorem 2 (See [5], Theorem 1.) $(N, p_n)$ is regular if and only if

$$p_n \to 0, n \to \infty. \quad (6)$$
The purpose of the present paper is to prove a few inclusion theorems for the convolution of Nörlund methods in the form \((N, r_n) \subseteq (N, p_n) \ast (N, q_n)\).

We need to define \(\{p_n\}\) by

\[
p_0 p_0 = 1, \quad p_0 p_n + p_1 p_{n-1} + \cdots + p_n p_0 = 0, \quad n \geq 1
\]

i.e., \(p(x) = \sum_{n=0}^{\infty} p_n x^n = \frac{1}{\sum_{n=0}^{\infty} p_n x^n} = \frac{1}{p(x)}\), assuming that these series converge.

The following result is an easy consequence of Kojima-Schur theorem (see [2], [4]).

**Lemma 1** Let \(A = (a_{nk})\) be a row-finite matrix and \((N, p_n)\) be a regular Nörlund method. Then \(A\)-\(\lim x_k\) exists whenever \((N, p_n)\)-\(\lim x_k\) exists if and only if

\[
sup_{0 \leq \gamma \leq k_n} |P_\gamma \sum_{k=\gamma}^{k_n} a_{nk} p_{k-\gamma}| = O(1), \quad n \to \infty; \tag{8}
\]

\[
\lim_{n \to \infty} \sum_{k=\gamma}^{k_n} a_{nk} p_{k-\gamma} = \delta_\gamma, \quad \text{for every fixed } \gamma; \tag{9}
\]

and

\[
\lim_{n \to \infty} \sum_{k=0}^{k_n} a_{nk} = \delta. \tag{10}
\]

**Corollary 1** \((N, p_n) \subseteq A \) if and only if (8), (9) and (10) hold with \(\delta_\gamma = 0, \quad \gamma = 0, 1, 2, \ldots\) and \(\delta = 1\).

**Corollary 2** If \((N, p_n)\) and \((N, q_n)\) are regular Nörlund methods, then \((N, p_n) \subseteq (N, q_n)\) if and only if \(h_n \to 0, \quad n \to \infty\) where

\[
h(x) = \sum_{n=0}^{\infty} h_n x^n = \frac{\sum_{n=0}^{\infty} q_n x^n}{\sum_{n=0}^{\infty} p_n x^n} = \frac{q(x)}{p(x)}
\]

(see [5]).

Let \((N, p_n), (N, q_n), (N, r_n)\) be regular Nörlund methods. Let \(p_n(x) = \sum_{i=n}^{\infty} p_i x^i, \quad p_0(x) = p(x)\) with similar expressions for \(q_n(x), r_n(x)\). Let

\[
\frac{p(x) q(x)}{r(x)} = \sum_{\gamma=0}^{\infty} \theta_\gamma x^\gamma; \tag{11}
\]

\[
\frac{p_{n+1}(x) q(x)}{r(x)} = \sum_{\gamma=0}^{\infty} \alpha_{n\gamma} x^\gamma;
\]

\[
\frac{p(x) q_{n+1}(x)}{r(x)} = \sum_{\gamma=0}^{\infty} \beta_{n\gamma} x^\gamma;
\]
and
\[
\frac{1}{r(x)} \{ p(x)q(x) - p_{n+1}(x)q(x) - p(x)q_{n+1}(x) \} = \sum_{\gamma=0}^{\infty} \varphi_{n\gamma} x^\gamma. \tag{12}
\]
It now follows that
\[
\varphi_{n\gamma} = \theta_\gamma - \alpha_{n\gamma} - \beta_{n\gamma} \tag{13}
\]
and
\[
\alpha_{n\gamma} = \beta_{n\gamma} = 0, \quad 0 \leq \gamma \leq n. \tag{14}
\]
Taking \( C = (N, p_n) * (N, q_n) \), \( C \) is a row-finite matrix with
\[
c_{nk} = \frac{1}{P_n Q_n} \sum_{i=\max(0,k-n)}^{\min(k,n)} p_{n-i} q_{n-k+i} \quad \text{with} \quad k_n = 2n. \tag{15}
\]
We write
\[
f_{n\gamma} = \sum_{k=\max(0,2n-\gamma)}^{2n} c_{nk} \bar{F}_{k+\gamma-2n}, \quad n, \gamma \geq 0. \tag{16}
\]
Lemma 2
\[
P_n Q_n f_{n\gamma} = \varphi_{n\gamma}, \quad 0 \leq \gamma \leq 2n + 1. \tag{17}
\]
Proof. The result follows as in [6].

Theorem 3 \((N, p_n) * (N, q_n)\)-lim \( x_k \) exists whenever \((N, r_n)\)-lim \( x_k \) exists if and only if
\[
\sup_{0 \leq \gamma \leq 2n} |R_{2n-\gamma} \varphi_{n\gamma}| = O(1), \quad n \to \infty; \tag{18}
\]
and
\[
\lim_{n \to \infty} \varphi_{n,2n-\gamma} = \delta_\gamma, \quad \text{for every fixed} \quad \gamma. \tag{19}
\]
Proof. Let \((N, p_n) * (N, q_n)\)-lim \( x_k \) exist whenever \((N, r_n)\)-lim \( x_k \) exists. Applying Lemma 1 with \((N, p_n) = (N, r_n)\) and \( A = (N, p_n) * (N, q_n) = (c_{nk}) \), we have,
\[
\sup_{0 \leq \gamma \leq 2n} |R_\gamma \sum_{k=\gamma}^{2n} c_{nk} \bar{F}_{k-\gamma}| = O(1), \quad n \to \infty \tag{20}
\]
and
\[
\lim_{n \to \infty} \sum_{k=\gamma}^{2n} c_{nk} \bar{F}_{k-\gamma} = \delta_\gamma, \quad \text{for every fixed} \quad \gamma. \tag{21}
\]
Using (16) and (20), we get
\[
\sup_{0 \leq \gamma \leq 2n} |R_{2n-\gamma} f_{n\gamma}| = O(1), \quad n \to \infty. \tag{22}
\]
Using (17), we note that $|f_{\gamma}| = \frac{|\varphi_{n\gamma}|}{|p_0||q_0|}$ since $|P_n| = |p_0|$ and $|Q_n| = |q_0|$. Consequently, in view of (22), we get

$$ \sup_{0 \leq \gamma \leq 2n} |R_{2n-\gamma} \varphi_{n\gamma}| = O(1), \quad n \to \infty. $$

In view of (16) and (21), we have

$$ \lim_{n \to \infty} f_{n,2n-\gamma} = \delta_{\gamma}, \quad \text{for every fixed} \quad \gamma. $$

Now, using (17), we get

$$ \lim_{n \to \infty} \frac{\varphi_{n,2n-\gamma}}{P_n Q_n} = \delta_{\gamma}, \quad \text{for every fixed} \quad \gamma. $$

Thus (18) and (19) hold. Conversely (18) and (19) imply (20) and (21) respectively.

Using (5), we have, $\lim_{n \to \infty} \sum_{k=0}^{2n} c_{nk} = 1$. Using Lemma 1, the result follows, completing the proof of the theorem.

**Corollary 3** $(N, r_n) \subseteq (N, p_n) \ast (N, q_n)$ if and only if (18) and (19) hold with $\delta_{\gamma} = 0$.

**Corollary 4** If $\lim_{n \to \infty} r_n = 0$, then $(N, r_n) \subseteq (N, p_n) \ast (N, q_n)$ if and only if (18) holds.

**Proof.** The result follows using (9) and the fact that $(N, p_n) \ast (N, q_n)$ is regular.

**Theorem 4** If

$$ \varphi_{n,2n-\gamma} = o(1), \quad n \to \infty, \quad \text{for every fixed} \quad \gamma, $$

and either

$$ \varphi_{n\gamma} = O(1), \quad n, \gamma \to \infty, $$

or

$$ \theta_{\gamma}, \alpha_{n\gamma}, \beta_{n\gamma} = O(1), \quad n, \gamma \to \infty, $$

then

$$(N, r_n) \subseteq (N, p_n) \ast (N, q_n).$$

**Proof.** Using (23), (19) follows with $\delta_{\gamma} = 0$ since $|P_n| = |p_0|$ and $|Q_n| = |q_0|$. Because of (13) and (25), (24) holds. So if (24) or (25) holds, (18) holds since $R_n = O(1)$, $n \to \infty$, $(N, r_n)$ being a regular method. The result now follows from Corollary 3.

We shall now take up an application of Theorem 4.
Theorem 5 Let $\bar{p}_n, \bar{q}_n \to 0$, $n \to \infty$ and $t_n = p_0q_n + p_1q_{n-1} + \cdots + p_nq_0$, $n = 0, 1, 2, \ldots$. Then

$$(N, t_n) \subseteq (N, p_n) * (N, q_n)$$

and

$$(N, p_n) \subseteq (N, t_n), \ (N, q_n) \subseteq (N, t_n).$$

Proof. We apply Theorem 4 with $r_n = t_n$. With the usual notation we have $t(x) = p(x)q(x)$ and $\tilde{t}(x) = \bar{p}(x)\bar{q}(x)$. Since $\bar{p}_n, \bar{q}_n \to 0$, $n \to \infty$, $\bar{t}_n \to 0$, $n \to \infty$ (see [3], Theorem 1). Consequently (23) follows using (9). In view of (11), we have,

$$\sum_{\gamma=0}^{\infty} \theta_{\gamma}x^\gamma = \frac{p(x)q(x)}{t(x)} = 1,$$

so that

$$\theta_0 = 1 \text{ and } \theta_\gamma = 0, \ \gamma \geq 1;$$

$$\sum_{\gamma=0}^{\infty} \alpha_{n\gamma}x^\gamma = \frac{p_{n+1}(x)q(x)}{t(x)} = p_{n+1}(x)p(x),$$

so that

$$\alpha_{n\gamma} = \frac{\gamma-(n+1)}{\sum_{\lambda=0}^{\gamma-1}} \bar{p}_\lambda p_{\gamma-\lambda}, \ \gamma \geq n + 1;$$

$$= 0, \ 0 \leq \gamma \leq n.$$

Consequently $\alpha_{n\gamma} = O(1), \ n, \gamma \to \infty$. Similarly $\beta_{n\gamma} = O(1), \ n, \gamma \to \infty$. In view of Theorem 4, $(N, t_n) \subseteq (N, p_n) * (N, q_n)$. Now $t(x) = p(x)q(x)$ and $q_n \to 0, n \to \infty$, $(N, q_n)$ being regular, by (6). So by Corollary 2, $(N, p_n) \subseteq (N, t_n)$. Similarly $(N, q_n) \subseteq (N, t_n)$. The proof of the theorem is now complete.

References


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