ANNALES MATHÉMATIQUES BLAISE PASCAL

GILLES CHRISTOL
ZOGHMAN MEBKHOUT

Topological p-adic vector spaces and index theory

Annales mathématiques Blaise Pascal, tome 2, nº 1 (1995), p. 93-98
<http://www.numdam.org/item?id=AMBP_1995__2_1_93_0>

Ann. Math. Blaise Pascal, Vol. 2, N° 1, 1995, pp.93-98

TOPOLOGICAL P-ADIC VECTOR SPACES

AND INDEX THEORY

Gilles Christol and Zoghman Mebkhout

Abstract. This report is part of a work developed from Robba's ideas whose ultimate goal would be to obtain a general finiteness theorem for p-adic cohomology. The basic question is to prove existence of index for ordinary differential operators. Here we expose continuity properties of index. Although it is apparently of an algebraic nature, the difficulties of index theory are mainly analytic. In particular, it involves a great deal of topological vector spaces, far beyond mere Banach spaces theory. The aim of this report is to illustrate this fact.

1991 Mathematics subject classification : 12H25

I Index and duality.

Let k be a complete ultrametric field, for instance $k = \mathbb{C}_p$, let E be a k-vector space and $u : E \to E$ a linear map.

Definition. The map u is said to have an *index* if both $\ker u$ and $\operatorname{coker} u = E/\operatorname{Im} u$ are finite-dimensional. If so, set :

$$\chi(u) = \chi(u, E) = \dim(\ker u) - \dim(\operatorname{coker} u)$$

Let E be a (locally convex) topological k-vector space, let u be continuous and let E' be the (strong) dual of E.

Question. If u has an index does ${}^t u$ has one ? If true, compare $\chi(u)$ and $\chi({}^t u)$.

The following facts are easy to verify :

$\ker({}^t u) = \operatorname{Im}(u)^{\perp}$ is isomorphic to the dual of the space $\operatorname{coker}(u)$ endowed with the quotient topology.

The pairing $(\bar{x}, y) \stackrel{def}{=} (x, y)$ gives a canonical map $\operatorname{coker}({}^t u\ ) \xrightarrow{i} (\ker u)'$.
To go further, a definition is needed :

Definition. The space E is said to have *Banach's property* if both conditions u continuous and $\operatorname{coker} u$ finite-dimensional imply that $\operatorname{Im} u$ is closed.

For instance, as already noticed by L. Schwarz, every Banach space has Banach's property (this is a straightforward consequence of the open map theorem applied to the map $u \oplus \mathrm{Id} : E \times F \to E$ for some algebraïc, but a priori not topological, supplementary F of $\operatorname{Im} u$).

Proposition 1 : *If E has Banach's property and if u has an index then* $\dim(\ker {}^t u\ ) = \dim(\operatorname{coker} u)$ *and* $\dim(\operatorname{coker} {}^t u\ ) \leq \dim(\ker u)$. *In other words, ${}^t u$ has an index and* $\chi(u) + \chi({}^t u\ ) \leq 0$.

Proof : By hypothesis, $\operatorname{Im} u$ is closed then

1) $\operatorname{coker} u$ is Hausdorff hence :

$$\dim(\ker {}^t u\ ) = \dim(\operatorname{coker} u)' = \dim(\operatorname{coker} u)$$

2) The map $\operatorname{coker}({}^t u\ ) \xrightarrow{i} (\ker u)'$ is injective. Actually, let x in E. If its image $\bar{x}$ in $\operatorname{coker} {}^t u$ belongs to $\ker i$ one has $(x, y) = 0$ for y in $\ker u$. Hence one can define z in $(\operatorname{Im} u)'$ by $(z, u(y)) = (x, y)$. As $\operatorname{Im} u$ is closed, z is the restriction of some element of E' also denoted by z. Then one has $x = {}^t u(z)$ and $\bar{x}=0$.

Remarque. If Hahn-Banach theorem were true for k-vector spaces, one could also prove that i is onto. To bypass this difficulty, additional conditions are needed.

Corollary 2 : *If E is reflexive (E and E'' topologically isomorphic) and both E and E' have Banach's property then u has an index if and only if ${}^t u$ has an index. In that case* $\chi(u) + \chi({}^t u\ ) = 0$.

Proof : By proposition 1 applied to u and ${}^t u$:

$$\dim(\ker {}^t u\ ) = \dim(\operatorname{coker} u),$$

$$\dim(\operatorname{coker} {}^t u\ ) = \dim(\ker {}^{tt} u) = \dim(\ker u).$$

II Spaces.

For each $r > 0$, let $H(r)$ be the ring of power series $\sum_{n \geq 0} a_n x^n$ of $k[[x]]$ which converge in the "closed" disk $|x| \leq r$ (of some enough large extension of k). The k-vector space $H(r)$ is a Banach for the usual norm $\|f\|_r = \max_{|x| \leq r} |f(x)|$.

Let $K(r)$ be the ring of Laurent series $\sum_{n<0} a_n x^n$ of $\frac{1}{x} k[[\frac{1}{x}]]$ which converge in the "closed" disk centered at infinity $|x| \geq r$ and which are zero at infinity. The k-vector space $K(r)$ is a Banach for the norm $\|f\| = \max_{|x| \geq r} |f(x)|$.

Let $\mathcal{A}(r)$ be the ring of power series $\sum_{n\geq 0} a_n x^n$ of $k[[x]]$ which converge in the "open" disk $|x| < r$. The relation :

$$\mathcal{A}(r) = \bigcap_{s<r} H(s) = \varprojlim_{n\in\mathbb{N}} H(r - \frac{1}{n})$$

shows that $\mathcal{A}(r)$ is a countable inverse limit of Banach spaces. Then it is a Frechet space when endowed with inverse limit topology. This is the "usual" topology defined by the family of norms $\{\|.\|_s\}_{s<r}$ and we will use it.

Let $\mathcal{H}^\dagger(r)$ be the ring of Laurent series $f = \sum_{n<0} a_n x^n$ of $\frac{1}{x} k[[\frac{1}{x}]]$ which converge in the disk $s \leq |x|$ for some $s < r$ (depending on f). The relation :

$$\mathcal{H}^\dagger(r) = \bigcup_{s<r} K(s) = \varinjlim_{n\in\mathbb{N}} K(r - \frac{1}{n})$$

shows that $\mathcal{H}^\dagger(r)$ is a countable direct limit of Banach spaces. We will endow it with (locally convex) direct limit topology. Now, by definition of the direct limit topology, the canonical imbedement $\mathcal{H}^\dagger(r) \to H(r)$ is continuous. Hence $\mathcal{H}^\dagger(r)$ is an Hausdorff countable direct limit of Banach spaces, namely an $\mathcal{LF}$ space in the Grothendieck terminology [4].

Let $\mathcal{R}(r)$ be the field of Laurent series $f = \sum_{n\in\mathbb{Z}} a_n x^n$ of $k[[x, \frac{1}{x}]]$ which converge in the annulus $s \leq |x| < r$ for some $s < r$ (depending on f). The Mittag-Leffler decomposition

$$\mathcal{R}(r) = \mathcal{H}^\dagger(r) \oplus \mathcal{A}(r)$$

defines the topology on $\mathcal{R}(r)$ which is then an $\mathcal{LF}$ space.

Theorem 3 [5] : *The space $\mathcal{A}(r)$ is reflexive and its dual is $\mathcal{H}^\dagger(r)$ for the pairing* $(\sum_{n\geq 0} a_n x^n, \sum_{n\geq 0} b_n x^{-n-1}) = \sum_{n\geq 0} a_n b_n$.

So $\mathcal{H}^\dagger(r)$ is both an $\mathcal{LF}$ space and a $\mathcal{DF}$ space (dual of Frechet [4]). Moreover, the space $\mathcal{R}(r)$ is its one dual.

To conclude this section we recall two "classical" results :

Theorem 4 : *Every Frechet space has Banach's property.*

Theorem 5 [4, page 200] : *Every $\mathcal{LF}$ space has Banach's property.*

In fact, the second one is written for real or complexe spaces. It is possible but rather tedious to verify that Hahn-Banach is not used in this long proof. Let take the opportunity to express the wish that this basic theorem and related ones take their deserved places in future account of topological spaces over an ultrametric field.

III Operators.

Let $\mathcal{D}$ be the non-commutative ring $K[x, \frac{d}{dx}]$ of (linear) differential operators with polynomial coefficients. The spaces $\mathcal{A}(r)$, $\mathcal{H}^\dagger(r)$ and $\mathcal{R}(r)$ are stable by derivation. As $\mathcal{A}(r)$ and $\mathcal{R}(r)$ contain $K[x]$, they are $\mathcal{D}$-modules for the scalar multiplication $P\,f = P(f)$. To define a $\mathcal{D}$-module structure on $\mathcal{H}^\dagger(r)$ one uses the exact sequence :

$$0 \longrightarrow \mathcal{A}(r) \longrightarrow \mathcal{R}(r) \xrightarrow{\gamma} \mathcal{H}^\dagger(r) \longrightarrow 0 \qquad (**)$$

For f in $\mathcal{H}^\dagger(r)$ and P in $\mathcal{D}$ we define the scalar multiplication by $P\,f = \gamma(P(f))$. Then (**) becomes an exact sequence of $\mathcal{D}$-modules.

Any differential operator P acts continuously on the Banach spaces $H(r)$ and $K(r)$. Hence it acts continuously on $\mathcal{A}(r)$ and $\mathcal{R}(r)$ and then on $\mathcal{H}^\dagger(r)$.

Basic facts : Let $P = \sum_{i=0}^d a_i(\frac{d}{dx})^i$ be a differential operator of $\mathcal{D}$. The following assertion are easy to check :

A) $\ker(P, \mathcal{A}(r))$, $\ker(P, \mathcal{R}(r))$ and $\ker(P, \mathcal{H}^\dagger(r))$ are finite-dimensional and their dimensions are bounded by $c(P) = d + \max_i \deg(a_i)$.

B) One has ${}^tP = \sum_{i=0}^d(-\frac{d}{dx})^i a_i$ for the three dualities we defined (use the Leibnitz rule to rearrange the terms).

Now the following result is a particular case of corollary 2.

Corollary 6 : *A differential operator P has an index in $\mathcal{A}(r)$ if and only if the differential operator ${}^tP$ has an index in $\mathcal{H}^\dagger(r)$. If so $\chi(P, \mathcal{A}(r)) + \chi({}^tP, \mathcal{H}^\dagger(r)) = 0$.*

If one is only interested in the exitence of index, one can work on $\mathcal{R}$ as shown by the following result.

Proposition 7 : *If an differential operator P has an index in $\mathcal{R}(r)$ then it has index both in $\mathcal{A}(r)$ and $\mathcal{H}^\dagger(r)$ and one has :*

$$\chi(P, \mathcal{R}(r)) = \chi(P, \mathcal{A}(r)) + \chi(P, \mathcal{H}^\dagger(r)) = \chi(P, \mathcal{A}(r)) - \chi({}^tP, \mathcal{A}(r))$$

Proof : The short exact sequence (**) gives rise to a long exact sequence :

$$0 \longrightarrow \underline{\ker(P, \mathcal{A}(r))} \longrightarrow \underline{\ker(P, \mathcal{R}(r))} \longrightarrow \underline{\ker(P, \mathcal{H}^\dagger(r))} \longrightarrow$$
$$\longrightarrow \operatorname{coker}(P, \mathcal{A}(r)) \longrightarrow \underline{\operatorname{coker}(P, \mathcal{R}(r))} \longrightarrow \operatorname{coker}(P, \mathcal{H}^\dagger(r)) \longrightarrow 0$$

where underlined spaces are finite-dimensional by hypothesis or by assertion A). Then the two remaining spaces are also finite-dimensional.

IV The Theorems.

We are now interested by the way the index varies with r.

Theorem 8 : *Let r_n be a growing sequence with limit r and let P be a differential operator. If P has an index in $\mathcal{H}^\dagger(r_n)$ for all n then it has an index in $\mathcal{H}^\dagger(r)$ and $\chi(P, \mathcal{H}^\dagger(r)) = \lim_{n\to\infty} \chi(P, \mathcal{H}^\dagger(r_n))$ (index being integers, that means $\chi(P, \mathcal{H}^\dagger(r)) = \chi(P, \mathcal{H}^\dagger(r_n)$ for n large enough).*

Proof : (see [1]). As $r_n \leq r_{n+1}$, the canonical injection $\mathcal{H}^\dagger(r_n) \to \mathcal{H}^\dagger(r_{n+1})$ has dense image. Then the map $\mathrm{coker}(P, \mathcal{H}^\dagger(r_n)) \to \mathrm{coker}(P, \mathcal{H}^\dagger(r_{n+1}))$, between finite-dimensional Hausdorff spaces, has dense image hence is onto. Therefore the sequence $\dim\left(\mathrm{coker}(P, \mathcal{H}^\dagger(r_n)\right)$ is decreasing and the sequence $\dim\left(\mathrm{coker}(P, \mathcal{H}^\dagger(r_n)\right)$ increasing and bounded by assertion A). Hence both are constant for n large enough. To conclude, suffice it to say that $\mathcal{H}^\dagger(r) = \varinjlim \mathcal{H}^\dagger(r_n)$ and that $\varinjlim$ is an exact functor.

Corollary 9 : *Let r_n be a growing sequence with limit r and let P be a differential operator. If P has an index in $\mathcal{A}(r_n)$ for all n then it has an index in $\mathcal{A}(r)$ and $\chi(P, \mathcal{A}(r)) = \lim_{n\to\infty} \chi(P, \mathcal{A}(r_n))$.*

Proof : By duality (see [1]).

Remark : There are two obstructions to obtain a direct proof of the corollary 9. The first one is that $\varprojlim$ is not an exact functor. This could be overpassed by means of a sophisticated version of Mittag-Leffler condition due to Grothendieck ([3], III-0-13.2.4). The second and deeper one, is that the sequence $\dim(\mathrm{coker}(P, \mathcal{A}(r_n))$ has no a priori reason to be bounded.

We'll explain elsewhere [2] how to define, for each real r, "p-adic exponents for the radius r" of the differential operator P. This definition is far too long to be given here. Then it will be possible to prove the folowing very deep result conjectured by Robba [6] :

Theorem 10 [1,2] : *If p-adic exponents for the radius r of P are not Liouville neither have Liouville differences, then*

1) P has an index in $\mathcal{A}(r)$.

2) $\chi(P, \mathcal{A}(r)) = \chi({}^tP, \mathcal{A}(r))$, hence $\chi(P, \mathcal{R}(r)) = 0$.

BIBLIOGRAPHY :

[1] G. CHRISTOL, Z. MEBKHOUT : Sur le théorème de l'indice des équations différentielles p-adiques I, *Ann. Inst. Fourier*, **43** (1993) 1545-1574.

[2] G. CHRISTOL, Z. MEBKHOUT : Sur le théorème de l'indice des équations différentielles p-adiques II, (en préparation).

[3] A. GROTHENDIECK, J. DIEUDONNE : *Eléments de Géométrie Algébrique*, Publ. Math. IHES **11** (1961).

[4] A. GROTHENDIECK : *Espaces vectoriels topologiques*, Sao Paulo (1954).

[5] Y. MORITA, W. SCHIKHOF : Duality of projective limit spaces and inductive limit spaces over non spherical complete field, *Tohoku Math. J.* **38** (1986) 387-397.

[6] P. ROBBA : Conjectures sur les équations différentielles p-adiques linéaires, *G.E.A.U.* 12ème année (1984-85) n°2, 8 pages.

Gilles CHRISTOL
Equipe d'Arithmétique
Université PARIS 6
4 place Jussieu
F-75252Paris Cedex 5
FRANCE

Zoghman MEBKHOUT
Mathématiques
Université PARIS 7
2 place Jussieu
F-75251 Paris Cedex 5
FRANCE