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DECOMPOSITION OF p -ADIC MEROMORPHIC FUNCTIONS

J.P. Bézivin and A. Boutabaa

Abstract. Given a meromorphic function, we are interested in how many ways it can be expressed as a composite of other meromorphic functions. Is this always possible? And, when it is the case, what about unicity of such an expression? Etc...

In the case of polynomials, the discussion is relatively easy and has been studied in details by Ritt [13]. As soon as one passes to rational functions the problem becomes much more difficult.

Various aspects of the problem concerning the complex meromorphic functions have been studied by many authors [3], [7], [8], [9], [10], [11], [12] and [14].

In this work, we deal with p -adic meromorphic functions. We will see that one neither can always use the same methods that those of the complex case nor obtain the same results. This is due to the fact that the distribution of the singularities is not the same in the two cases. For instance, in C one can consider entire functions whose all zeros lie on a single straight line (real numbers for example). Then many results concerning such functions and based upon a theorem of Edrei [6] are obtained [11], [12]. This result of Edrei plays an important role in the complex decomposition theory and there is no possible p -adic version of it.

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I. p -adic meromorphic functions

We note $\mathcal{A}(\mathbb{C}_p)$ the ring of entire functions in \mathbb{C}_p and $\mathcal{M}(\mathbb{C}_p)$ the field of meromorphic functions in all \mathbb{C}_p .

Definition 1.1. Let $f; f_1, \dots, f_n \in \mathcal{M}(\mathbb{C}_p)$ such that :

$$f = f_1 \circ f_2 \circ \dots \circ f_n.$$

We say that this is a decomposition of f and that f_1, \dots, f_n are factors of f .

Given a rational function $R(x) = \frac{P(x)}{Q(x)}$, where $P(x), Q(x)$ are polynomials of $\mathbb{C}_p[x]$ relatively prime, we call degree of $R(x)$ the number

$$\deg R = \max(\deg P; \deg Q).$$

Definition 1.2. A function $f \in \mathcal{M}(\mathbb{C}_p)$ is said to be indecomposable if, in all decomposition of f , all factors, except at most one, are rational functions of degree one.

Definition 1.3. A function $f \in \mathcal{M}(\mathbb{C}_p)$ is said to be pseudo-indecomposable if in all decomposition, all factors, except at most one, are rational functions.

Remark 1.4. Let f be an entire function. We have naturally a notion of indecomposability (resp. pseudo-indecomposability) depending on whether we regard f as an element of $\mathcal{A}(\mathbb{C}_p)$ or of $\mathcal{M}(\mathbb{C}_p)$. We will see however that (Corollary 2.4.) f is indecomposable (resp. pseudo-indecomposable) in $\mathcal{A}(\mathbb{C}_p)$ if and only if it is indecomposable (resp. pseudo-indecomposable) in $\mathcal{M}(\mathbb{C}_p)$. This result is false in the Complex case (Remark 2.5.).

Definition 1.5. Two decompositions of $h \in \mathcal{M}(\mathbb{C}_p)$, $h = f_1 \circ \dots \circ f_n = g_1 \circ \dots \circ g_n$ are said to be equivalent if there exist rational functions of degree one, $L_1; L_2; \dots; L_{n-1}$ such that :

$$f_1 = g_1 \circ L_1; f_2 = L_1^{-1} \circ g_2 \circ L_2; \dots f_{n-1} = L_{n-2}^{-1} \circ g_{n-1} \circ L_{n-1}; f_n = L_{n-1}^{-1} \circ g_n.$$

We will now recall some basic definitions and results on p -adic meromorphic functions.

Let $f(x) = \sum_{n \geq 0} a_n x^n$ be an entire function in \mathbb{C}_p . For all $R > 0$, we note, $|f|(R) = \max_{n \geq 0} |a_n| R^n$, the maximum modulus function of f . This is extended to meromorphic functions $h = \frac{f}{g}$ by $|h|(R) = \frac{|f|(R)}{|g|(R)}$.

Proposition 1.6. *Let f be an entire function in \mathbb{C}_p . Then the function $(R \mapsto \theta_f(\log R) = \log |f|(R))$ is a convex polygonal curve. Moreover The number of zeros of f in the closed (resp. open) disc of radius ρ and center 0 is given by the right-derivative $\theta_f^+(\log \rho)$ (resp left derivative $\theta_f^-(\log \rho)$) of $\theta_f(u)$ at the point $\log \rho$.*

For the proof see [1] for example :

Proposition 1.7. *Let F, G, H be three meromorphic functions . Suppose that H is not a rational function and that $F = H \circ G$. Then G is entire.*

Proof. We have $H(x) = \frac{A(x)}{B(x)}$ where $A(x)$ and $B(x)$ are entire functions without common zeros. The fact that $H(x)$ is not rational implies that there exist at most one value $\omega_0 \in \mathbb{C}_p$ such that $A(x) - \omega_0 B(x)$ is a polynomial. Let ω be a value different of this eventual one. Then the entire function $A(x) - \omega B(x)$ is transcendental and hence has infinitely many zeros, which we note $c_k, k \in \mathbb{N}$. Suppose that $G(x)$ has a pole x_0 . We set $G(x) = \frac{W(x)}{(x - x_0)^\ell}$ where $W(x)$ is an analytic function that has no zeros in an open disc $|x - x_0| < R$ and ℓ an entire number ≥ 1 . For any k set $T_k(x) = W(x) - c_k(x - x_0)^\ell$. For one of these k let us choice $\rho \in]0, R[$ such that $|c_k| \rho^\ell = |W(x_0)|$. Hence we have :

$$\begin{cases} |c_k| r^\ell \leq |W(x_0)| & \text{if } 0 < r < \rho \\ |c_k| r^\ell \geq |W(x_0)| & \text{if } \rho < r < R. \end{cases}$$

On the other hand, the fact that W has no zero in $|x - x_0| < R$ implies that :

$$|W|(r) = |W(x_0)| \quad \forall r \quad 0 < r < R.$$

Hence we have :

$$|T_k|(r) = \begin{cases} |W(x_0)| & \text{si } 0 < r < \rho \\ |c_k| r^\ell & \text{si } \rho < r < R \end{cases}$$

Therefore the function $T_k(x)$ has at least one zero x_k in the circle $|x - x_0| = \rho$. Thus we have $G(x_k) = c_k$; and so $F(x_k) = \omega$. consequently, since the x_k are infinitely many, the function $F(x) - \omega$ has infinitely many zeros in the disc $|x - x_0| < R$, which is a contradiction. Hence $G(x)$ has no poles and so is entire.

Remark 1.8. The above proposition justifies the fact that, subsequently, in any decomposition of a meromorphic function f in the forme $f = h \circ g$, we suppose h meromorphic and g entire or h rational and g meromorphic.

The following result will enable us to show the existence of indecomposable or pseudo-indecomposable transcendent meromorphic functions .

For an entire fonction f and a positive number ρ , we note $m(f, \rho)$ the number of zeros of f in the open disc $|x| < \rho$, $M(f, \rho)$ the number of zeros of f in the closed disc $|x| \leq \rho$ and $A(f, \rho)$ the number of zeros of f in the circle $|x| = \rho$; each one of these zeros being computed with its multiplicity.

Proposition 1.9. *Let H and G be two entire and non constant functions. Let ρ_0 be a positive real number such that the function $|G|(r)$ is strictly increasing for $r > \rho_0$. Let us put $F = H \circ G$. Then we have for all $\rho > \rho_0$:*

- i) $m(F, \rho) = m(H, |G|(\rho))m(G, \rho)$
- ii) $M(F, \rho) = M(H, |G|(\rho))M(G, \rho)$
- iii) $A(F, \rho) = A(H, |G|(\rho))M(G, \rho) + m(H, |G|(\rho))A(G, \rho)$

Proof. From $F = H \circ G$; we deduce that, for all r , we have : $|F|(r) = |H|(|G|(r))$. Since the function $|G|(r)$ is strictly increasing for $r > \rho_0$, we deduce that, for $u > \log \rho_0$, we have :

$$(1) \theta_F^-(u) = \theta_H^-(\theta_G(u)).\theta_G^-(u), \quad \text{and} \quad (2) \theta_F^+(u) = \theta_H^+(\theta_G(u)).\theta_G^+(u).$$

By proposition 1.6, we have relations i) and ii). Subtracting i) from ii), we obtain relation iii).

Theorem 1.10. *Let $F \in \mathcal{A}(\mathbb{C}_p) \setminus \mathbb{C}_p[x]$ and λ a positive entire number. We suppose that on infinitely many circles of center 0 of \mathbb{C}_p , F has a number of zeros included between 1 and λ . Then all decomposition of F of the form $F = H \circ G$ with $H, G \in \mathcal{A}(\mathbb{C}_p)$ implies that H or G is a polynomial of degree between 1 and λ .*

Proof. If any of the functions H and G is a polynomial of degree between 1 and λ , we have for ρ enough large : $M(G, \rho) \geq \lambda + 1$ and $m(H, |G|(\rho)) \geq \lambda + 1$.

On the other hand, the hypothesis of the Theorem and the formula iii) of the proposition 1.9 show that for an infinity of ρ 's arbitrarily large, we have $A(H, |G|(\rho)) \neq 0$ or $A(G, \rho) \neq 0$. Then for such a ρ we have $A(F, \rho) \geq \lambda + 1$. This is a contradiction with the hypothesis. Hence H or G is necessarily a polynomial of degree between 1 and λ .

Corollary 1.11. *Let $F \in \mathcal{M}(\mathbb{C}_p) \setminus \mathbb{C}_p(x)$ and λ an entire ≥ 1 . We suppose that on an infinity of circles of center 0, F has a number of zeros included between 1 and λ and that on an infinity of circles of center 0, F has a number of poles included between 1 and λ . Then any decomposition of F in the form $F = H \circ G$ with $H \in \mathcal{M}(\mathbb{C}_p)$ and $G \in \mathcal{A}(\mathbb{C}_p)$ implies that :*

- G is a polynomial of degree between 1 and λ ,*
- or H is a rational function of degree between 1 and λ .*

Corollary 1.12. *A function F satisfying the assumptions of theorem 1.10 (resp. the ones of corollary 1.11) is :*

- 1. indecomposable in $\mathcal{A}(\mathbb{C}_p)$ (resp. in $\mathcal{M}(\mathbb{C}_p)$) if $\lambda = 1$.*
- 2. Pseudo-indecomposable in $\mathcal{A}(\mathbb{C}_p)$ (resp. in $\mathcal{M}(\mathbb{C}_p)$) if $\lambda \geq 1$.*

Proof. Let $H(x) = \frac{H_1(x)}{H_2(x)}$ where $H_1, H_2 \in \mathcal{A}(\mathbb{C}_p)$ have no common zeros. Hence : $F = \frac{H_1 \circ G}{H_2 \circ G}$. Then we apply the theorem 1.10 to each of the functions $H_1 \circ G$ and $H_2 \circ G$.

Proposition 1.13. *Let $F \in \mathcal{M}(\mathbb{C}_p)$ possessing the property to have in an infinity of discs of center 0 and radius arbitrarily large a prime number of zeros and in an infinity of discs of center 0 a prime number of poles. Then F is indecomposable in $\mathcal{M}(\mathbb{C}_p)$. In particular, if $P(x) \in \mathbb{C}_p[x]$ and $Q(x) \in \mathbb{C}_p[x]$ have degrees which are prime numbers, the rational*

function $R(x) = \frac{P(x)}{Q(x)}$ is indecomposable.

Proof. We apply the relation ii) of the proposition 1.9.

Corollary 1.14. *Let $F \in \mathcal{A}(\mathbb{C}_p) \setminus \mathbb{C}_p[x]$. We suppose that in an infinity of discs of center 0 and radius arbitrarily large, F has a number of zeros equal to the product of two prime numbers (not necessarily distinct). Then, either F is indecomposable, or it is a composite of two indecomposable factors.*

Proof. We use the previous proposition.

Corollary 1.15. *Let $F \in \mathcal{A}(\mathbb{C}_p) \setminus \mathbb{C}_p[x]$ and $\lambda \in \mathbb{N}^*$. We suppose that in an infinity of discs of center 0 and radius arbitrarily large, F has a number of zeros equal to the product of λ prime numbers (not necessarily distinct). Then F is a composite of at most λ indecomposable factors.*

Proof. We proceed gradually using the corollary 1.14.

Given a meromorphic function, it is not easy to know if it is decomposable or not. The first of the two next examples shows that a decomposable function may have non equivalent decompositions. The second example shows that a meromorphic function can have an infinity of prime factors.

Example 1.16. Let q be a prime number and ℓ a positive entire relatively prime to q . Let $(a_n)_{n \geq 0}$ be a sequence of elements of \mathbb{C}_p^* such that :

$$|a_n| < |a_{n+1}| \quad \text{and} \quad \lim_{n \rightarrow +\infty} |a_n| = +\infty.$$

We consider the following entire functions :

$$F(x) = x^\ell \prod_{n \geq 0} \left(1 - \frac{x^q}{a_n}\right); \quad G(x) = x^\ell \prod_{n \geq 0} \left(1 - \frac{x}{a_n}\right)^\ell;$$

$$H(x) = x^{q\ell} \prod_{n \geq 0} \left(1 - \frac{x^q}{a_n}\right)^q \quad \text{and} \quad \varphi(x) = x^q.$$

Let $r_n = |a_n| \quad \forall n \geq 0$. In the disc of center 0 and radius r_n , each of the functions F and G has $nq + \ell$ zeros. On the other hand the theorem of Dirichlet guarantees that among the terms of the arithmetic progression $(nq + \ell)_{n \geq 0}$ there is an infinity of prime numbers. Hence the functions F and G verify the conditions of the proposition 1.13, and so are indecomposable. The function φ is evidently indecomposable and we have $H = \varphi \circ F = G \circ \varphi$. These two decompositions are clearly not equivalent.

Example 1.17. Let $(a_n)_{n \geq 0}$ be a sequence of elements of \mathbb{C}_p^* such that :

$$|a_n| < |a_{n+1}| \quad \text{and} \quad \lim_{n \rightarrow +\infty} |a_n| = +\infty.$$

Let $f_n(x) = x(1 - \frac{x}{a_n})$. We define the sequence $(\varphi_n(x))$ by :

$$\varphi_0(x) = x \text{ and } \varphi_{n+1}(x) = \varphi_n \circ f_n(x).$$

Let $R > 0$. We will show that the sequence of $\varphi_n(x)$ is of Cauchy in the space of functions analytic in the closed disc $|x| \leq R$. We have :

$$\varphi_{n+1}(x) = \varphi_n(x - \frac{x^2}{a_n}) = \sum_{k \geq 0} (\frac{-x^2}{a_n})^k \frac{\varphi_n^{(k)}(x)}{k!}.$$

On the other hand we have : $|\varphi_n^{(k)}|(R) \leq k! |R^{-k}| \varphi_n|(R)$. Let N be the first index such that $n > N$ implies that $|a_n| > R$. Hence we have : $|\varphi_{n+1} - \varphi_n|(R) \leq |\varphi_n|(R) \frac{R}{|a_n|}$.

Consequently we have, $|\varphi_{n+1}|(R) = |\varphi_n|(R)$ for all $n > N$. It follows that the sequence $(\varphi_n(x))$ is of Cauchy in the space of functions analytic in the closed disc $|x| \leq R$. Hence the sequence $(\varphi_n(x))$ converges to an entire function $\varphi(x)$.

We can write for all k and $n > k$:

$$\varphi_n(x) = f_0 \circ f_1 \circ \dots \circ f_n \circ \theta_n(x),$$

where $\theta_n(x)$ is a function of the same type as $\varphi_n(x)$ and hence will converge to an entire function $\theta(x)$. Hence we will have :

$$\varphi(x) = f_0 \circ f_1 \circ \dots \circ f_n \circ \theta(x);$$

and this relation shows that the $f_k(x)$, which are indecomposable are factors of $\varphi(x)$. Hence $\varphi(x)$ has infinitely many indecomposable factors.

II. Comparison with polynomials and complex case.

We know that $P(x)$ being a given polynomial, we have :

- (1) $P(x)$ has at least one decomposition in indecomposable factors.
- (2) The number of indecomposable factors is finite and, is the same in any decomposition of $P(x)$.
- (3) $P(x)$ has only a finite number of non equivalent decompositions.

Do these properties remain true for entire or meromorphic functions? Example 1.16 shows that (2) is no longer so. In fact, even if we assume that all decomposition of f has a finite number of factors, we can not show if this number is bounded and if it is the same in any decomposition. This is not obvious at all because only for rational functions the following result stated by Ritt [13] in 1922 has only been proved recently [2].

Proposition 2.1. (Ritt). *There exists a rational function which has two decompositions into indecomposable functions each having a different number of factors.*

On the other hand Ritt [13] had shown that for any decomposition of a polynomial into rational functions, there exists an equivalent decomposition into polynomials. The following result shows that this is extended to p -adic entire functions.

Theorem 2.2. *Let $F \in \mathcal{A}(\mathbb{C}_p)$. Assume that $F = G \circ H$ with $G, H \in \mathcal{M}(\mathbb{C}_p)$. Then there exist $g, h \in \mathcal{A}(\mathbb{C}_p)$ such that $F = g \circ h$. Moreover, these two decompositions are equivalent.*

Proof.

1) If $G \notin \mathbb{C}_p(x)$, we have seen (Remark 1.8) that then H must be entire. On the other hand, we have $G(x) = \frac{G_1(x)}{G_2(x)}$, where $G_1, G_2 \in \mathcal{A}(\mathbb{C}_p)$ are without common zeros.

So $F = \frac{G_1(H)}{G_2(H)}$. We can assume that H is not constant. Then $G_2(x)$ has no zeros; because if this is not the case and if x_0 is a zero of $G_2(x)$, there exists $x_1 \in \mathbb{C}_p$ such that $H(x_1) = x_0$. From which $G_1(H(x_1)) = G_1(x_0) \neq 0$ and $G_2(H(x_1)) = G_2(x_0) = 0$ and F would have a pole. Contradiction. Hence G_2 has no zeros and is so equal to a constant α different from zero. Hence $G = \frac{G_1}{\alpha} \in \mathcal{A}(\mathbb{C}_p)$.

2) If $G(x) \in \mathbb{C}_p(x)$, then there exist $P(x), Q(x) \in \mathbb{C}_p[x]$ relatively prime such that $G(x) = \frac{P(x)}{Q(x)}$. So $F = \frac{P(H)}{Q(H)}$. H being non constant and meromorphic, it reaches all values of \mathbb{C}_p except at most one. This implies that $Q(x)$ has at most one zero. If $Q(x)$ has no zero, we have finished. Suppose that $Q(x)$ has a zero α . Set $H(x) = \frac{H_1(x)}{H_2(x)}$, where $H_1, H_2 \in \mathcal{A}(\mathbb{C}_p)$ are without common zeros. Then the function $H(x) - \alpha = \frac{H_1(x)}{H_2(x)} - \alpha$ has no zero. Hence $H_1(x) - \alpha H_2(x) = \beta \neq 0$. So we have $H_1 = \alpha H_2 + \beta$ and $H = L \circ H_2$, where $L = \frac{\alpha x + \beta}{x}$. Hence $F = T \circ H_2$, with $T = \frac{P}{Q} \circ L$. We see that T must be a polynomial; because if not, F should have a pole. We see also that $H_2 = L^{-1} \circ H$, where $L^{-1} = \frac{\beta}{x - \alpha}$.

Corollary 2.3. *If an entire function F is indecomposable (resp pseudo-indecomposable) in $\mathcal{A}(\mathbb{C}_p)$, then F is indecomposable (resp pseudo-indecomposable) in $\mathcal{M}(\mathbb{C}_p)$.*

This result is false in \mathbb{C} . Indeed, Ozawa [11] has shown that the function $F(z) = (e^z - 1)e^{(e^z - 2z)}$ is indecomposable in $\mathcal{A}(\mathbb{C})$, but that $F = f \circ g$ where $f = \frac{x - 1}{x^2} e^x$ and $g = e^x$. This means that in $\mathcal{M}(\mathbb{C})$, the function F is not even pseudo-decomposable.

III. Common right factors of F and $F^{(n)}$.

Suppose that a p -adic meromorphic function $F(z)$ and its derivative $F'(z)$ have an entire function g as their common right factor, it is easily shown from $F = f \circ g$ and $F' = h \circ g$ that g must be a polynomial of degree one. It is no longer a simple problem of searching the possible forms of any common right factor of F and $F^{(n)}$. However, we have the following general result :

Theorem 3.1. *Let $F \in \mathcal{M}(\mathbb{C}_p) \setminus \mathbb{C}_p(x)$ and $g \in \mathcal{A}(\mathbb{C}_p) \setminus \mathbb{C}_p[x]$. Suppose that there exist a positive entire n and two meromorphic functions f and h (which are not rational functions of degree 1) such that :*

$F = f \circ g$ and $F^{(n)} = h \circ g$. Then either f satisfies the following equation :

$$A_n(x)f^{(n)}(x) + \dots + A_{k+1}(x)f^{(k+1)}(x) + A_k(x)f^{(k)}(x) = 0, \quad \text{where } A_i(x) \in \mathbb{C}_p[x];$$

or

$$g'^n = B_0 + B_1g(x) + \dots + B_mg^m(x), \quad \text{where } B_j \in \mathbb{C}_p \text{ and } m \leq n.$$

We need the following results whose proofs are in [4] and [5] :

Lemma 3.2. *Let $h_0, \dots, h_m, F_0, \dots, F_m$ be elements of $\mathcal{M}(\mathbb{C}_p)$ such that $F_i \not\equiv 0$. Let $g \in \mathcal{A}(\mathbb{C}_p)$ such that*

$$T(r, h_0) + T(r, h_1) + \dots + T(r, h_m) = O(\text{Log} |g|(r)).$$

We suppose that :

$$F_0(g)h_0(x) + \dots + F_m(g)h_m(x) = 0.$$

Then there exist polynomials $P_0(x), \dots, P_m(x)$ with $P_i(x)$ not all equal to 0 such that :

$$P_0(g)h_0(x) + \dots + P_m(g)h_m(x) = 0.$$

The function $T(r, \cdot)$ above is the *Nevanlinna characteristic function*. See [4] and [5].

Lemma 3.3. *Let $P(x, y, y', \dots, y^{(n)})$ be a differential polynomial in $y, y', \dots, y^{(n)}$ with coefficients $a_{i_0 i_1 \dots i_n}(x) \in \mathbb{C}_p[x]$. Let $d = \max[i_0 + 2i_1 + \dots + (n+1)i_n]$ for $a_{i_0 i_1 \dots i_n}(x) \not\equiv 0$. Suppose that there exists a transcendental meromorphic solution $f(x)$ of the differential equation :*

$$P(x, y, y', \dots, y^{(n)}) = R(x, y); \quad \text{where } R(x, y) \in \mathbb{C}_p(x, y).$$

Then $R(x, y)$ is a polynomial in y of degree $\leq d$.

For the proof see [5].

Proof of theorem 3.1. From $F^{(n)} = h \circ g$, we have :

$$(1) \quad f^{(n)}(g)D_n(g) + f^{(n-1)}(g)D_{n-1}(g) + \dots + f'(g)D_1(g) - h \circ g = 0;$$

where $D_n(g) = g'^n$; $D_{n-1}(g) = \frac{n(n-1)}{2}g'^{n-2}g''$; ...; $D_1(g) = g^{(n)}$. In general, $D_i(g)$ is a homogeneous differential polynomial in g of degree i . Hence we apply lemma 3.2. with $F_0 = -h, F_1 = f', \dots, F_n = f^{(n)}$ and $h_0 = 1, h_1 = D_i(g), h_n = D_n(g)$. So there exist polynomials $P_{1,1}(x), \dots, P_{n,1}(x)$ which are not all equal to 0 such that :

$$P_{n,1}(g)D_n(g) + \dots + P_{1,1}(g)D_1(g) + P_{0,1}(g) = 0$$

Then if $P_{n,1}(x) \neq 0$, we have :

$$(2) \quad D_n(g) = R_{n-1,1}(g)D_{n-1}(g) + \dots + R_{1,1}(g)D_1(g) + R_{0,1}(g)$$

where the $R_{i,1}(x)$ are rational functions. Multiplying (2) by $f^{(n)}(g)$ and subtracting this from (1) gives us :

$$(3) \quad [f^{(n-1)}(g) + R_{n-1,1}(g)f^{(n)}(g)]D_{n-1}(g) + \dots + [f'(g) + R_{1,1}(g)f^{(n)}(g)]D_1(g) + [R_{0,1}(g)f^{(n)}(g) - h(g)] = 0$$

Applying lemma 3.2. once again we get :

$$(4) \quad D_{n-1}(g) = R_{n-2,2}(g)D_{n-2}(g) + \dots + R_{1,2}(g)D_1(g) + R_{0,2}(g)$$

If we suppose that lemma 3.2. does not break down, this finally gives :

$$(5) \quad D_n(g) = (g')^n = R(g); \text{ where } R(x) \text{ is a rational function. In this case, we apply lemma 3.3 and get : } (g')^n = a_0 + a_1g + \dots + a_mg^m \text{ with } m \leq d = 2n;$$

But $|(g')^n|(r) = |a_mg^m|(r) \leq \frac{|g^n|(r)}{r^n}$; which implies that $m \leq n$. If at some stage lemma 3.2 is not applicable, the procedure breaks down. But this means that one of the functions occurring in equations similar to (3) vanishes identically. Hence we have :

$$A_n(x)f^{(n)}(x) + \dots + A_{k+1}(x)f^{(k+1)}(x) + A_k(x)f^{(k)}(x) = 0 \text{ where } A_i(x) \in \mathbb{C}_p[x].$$

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