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The first conference on p-adic Functional Analysis, held in 1990 at Laredo, Spain, was Javier Martínez Maurica's brainchild, therefore we feel that inaugurating the proceedings of the "grandson" Conference with a warm remembrance of him and his work is most appropriate.

Javier was born in 1952 in Munguía, a small rural and industrial town in the Basque Country, where his father worked as a practitioner. When he was born, the town was predominantly rural, but as he was growing up the town also was developing and changing rapidly to become the home of medium-size industries, thus having a mixed rural/industrial character. Also, its population was mixed, with a lot of "traditional Basques" but also with an important proportion of immigrants from the rest of Spain. On the other hand, in some ways, living in Munguía meant living in a very small town, living in the countryside, so to speak; but Munguía was close enough to the large city of Bilbao to be regarded as having the mixed character of being both large and small.

We wanted to say all this on his native village to draw a parallelism with Javier's academic personality : during his professional life he touched upon the different possible activities that a university professor is usually involved in and he managed to combine them all in a carefully balanced way. Within the international mathematical community he is of course well-known for his research, to which we shall come back later on, but if we had to rate his activities in research, in teaching or in academic service, then the three of them would get equally high marks. We believe that this was an important trait of his personality : his ability to be engaged in multiple things at the same time and doing all of them well.

As a child he spent several years at a boarding school in a town larger than Munguía, for the simple reason that that was about the only opportunity to attend high school for him. He never talked much about those years; it was obvious that there were not many good stories to tell.

His entrance in the University coincided with the beginning of a big expansion of the university system in Spain, during the late sixties. And this was particularly fortunate for him, since a state university was founded in Bilbao for the first time ever, and the opportunity to pursue a degree in Mathematics offered to him. So he could attend the lectures while living at his parents' home; were it not been for that possibility, his family would probably have been unable to support his studies at a different town. He belonged to the second year that graduated in Mathematics in Bilbao, where most of the instructors were young graduates from other (older) Spanish universities, very few of them holding Ph.D. degrees. The atmosphere in the Mathematics Department at that time was not in favor of research, which was done by a very small minority—in practice there was only one professor capable of supervising graduate work, and he had already some students. But when Javier was to start the last year of his *Licenciatura*, Dr. Víctor Onieva became professor at that Department, and a new possibility opened to him : he started his graduate work under the direction of Dr. Onieva, which he finished three years later. When Dr. Onieva decided to go to the very new University of Santander, Javier choose to come along and keep the research group together. In Santander he met his wife, also a mathematician, they had two children, two boys, and he stayed in Santander ever since.

It was during the sixteen years that he spent in Santander when Javier developed his personality as a mathematician; a quick look at the list of his publications clearly shows his belonging to a transition generation within Spanish mathematics, from a time when publishing in the international journals or even publishing at all was not generally regarded as one of the criteria for professional recognition within the university, to the present international standards of "publish in English or perish". Javier contributed his share to this trend with his mathematical publications and his ability to build and maintain professional contacts with mathematicians with similar research interests around the world.

Javier managed to arrive at a difficult balance between his research and teaching dedications. During his academic life, he taught all kinds and levels of courses in Analysis and would teach Algebra or Geometry if necessary, always with the same respect on the part of the students. He was a rigorous and fair grader, according to his own students; after his death, groups of students wrote warm letters about him in the local papers.

During the last years he became involved in academic management, much to our surprise, since he had always kept a low profile in that respect. The real surprise came when he showed his abilities in that field too : he was appointed Director of the Summer Courses of the University, an office of a completely different nature to everything he had been doing before : at our University the Summer Courses cover extramural studies, with no regular catalog, so they have to be planned completely anew every year, and that means dealing and bargaining with all kinds of personnel within the University (from professors to janitors) and outside the University : journalists, politicians, local and national leaders, owners of hotels,... you name it. When he asked for advice about accepting or rejecting that offer, one of us strongly insisted him to reject it, on the grounds that it would be a waist if one of the best researchers of our University devoted too much time to administrative duties, and also, we argued with him, Javier was not fitted for that kind of position, having devoted himself mainly to teaching and research until then. Well, he proved us very wrong. He held that office for two years, he dedicated an enormous amount of energy to it, and when his term finished, everybody—including ourselves—agreed that the Summer Courses under his guidance had attained the highest success by any measure : number of students, appearance in the papers, external financing, etc.

Another characteristic trait of his personality was his total caring for the situations in which he was involved. This was particularly apparent when there was a guest at the Department for whom he considered himself as the responsible person; in those situations he would take care in a particularly dedicated way of everything related to the welfare of the visitor, from meals to socializing or the trips themselves. By the way, Javier's vocation outside mathematics, so to speak, was traveling. Not so much traveling himself, but rather acting as a sort of travel agent for friends and colleagues. He seemed to know by heart full schedules of major airlines, train companies of different countries and all kinds of intercity buses. If asked about anything of a trip that you were preparing, he would tell even minor details, without having ever made that trip himself before.

He did not want to bother his colleagues with information on his illness, which was not openly acknowledged by him till the very end of his life, but our impression is that caring about everything was too heavy a burden for him when he became responsible, not just for organizing the short visit of a friend and colleague, but for the stay of more than one hundred professors and several thousand students that were involved in the Summer Courses.

As for his research activity, it is no easy work to give a picture of his results, since he touched upon several topics within p-adic Mathematics, in cooperation with different people, in what constitutes one of the most characteristic features of Javier's research record : 90 per cent of his work was the joint work of himself with someone else, including his thesis adviser (only at the very beginning), his Ph.D. students (three students did all their graduate work under his guidance), his wife, and several colleagues, both at home and abroad. This also highlights, once more, his ability to cooperate with others, he was a team player.

At the end of this paper we have included a list of the papers he published, and next we point out a part of his results, according to selection criteria that are due to the particular tastes of the authors of this review.

One of the subjects to which J. Martínez-Maurica payed more attention in the course of his research work was the p-adic theory of continuous linear operators. In fact, his Doctoral Thesis was devoted to the study of the "states" of operators between non-archimedean normed spaces.

For non-archimedean normed spaces E, F over a non-archimedean valued field K which is spherically complete (=maximally complete) and whose valuation $| \, . \, |$ is non-trivial, R. Ellis studied in 1968 the relationships concerning the density of the range and the existence and continuity of the inverse, joining a continuous linear operator $T : E \longrightarrow F$ ($T \in L(E,F)$) and its adjoint $T' : F' \longrightarrow E'$ defined between the corresponding duals in the usual way.

The properties of an operator T in relation with its range, $R(T)$, and the continuity of its inverse $T^{-1} : R(T) \longrightarrow E$ are classified in the following way :

I : $R(T) = F$; II : $R(T) \neq F$ and $R(T)$ is dense in F; III : $R(T)$ is not dense in F.

1 : T^{-1} exists and is continuous; 2 : T^{-1} exists and is not continuous; 3 : T^{-1} does not exist.

For instance if T satisfies conditions II and 3 we say that $T \in II_3$ and similarly for T'. If we now consider the ordered pair of operators (T, T'), the conditions of T and T' determine an ordered pair of conditions, which is called the "state" of (T, T') (e.g. if $T \in II_2$ and $T' \in III_3$, we say that the state of (T, T') is (II_2, III_3)).

A priori there are 81 possibilities for the pair (T, T'). However, R. Ellis proved in his work that 65 of them cannot occur and he suggested that, as in the case of normed spaces over the real or complex field (studied by A.E. Taylor and C.J.A. Halberg in 1957) the 16 remaining states would be possible. In 1968 V. Onieva partially completed the diagram of R. Ellis by proving that 13 of the 16 states are indeed possible, leaving as an open question the existence of the other three states : (I_2, II_2), (II_2, II_2) and (III_2, II_3). This question was solved by J. Martínez-Maurica and T. Pellón in 1986, who proved that the three states mentioned above are not possible in the non-archimedean case when K is spherically complete and non-trivially valued. This fact is a direct consequence of the following result :

Theorem 1 ([17], Theorem 1) : *Let E,F be non-archimedean normed spaces over a non-archimedean and non-trivially valued field K, which is spherically complete. If $T \in L(E, F)$ verifies that $R(T')$ is dense in E', then T is injective and its inverse is continuous.*

After the work of R. Ellis and V. Onieva the following tasks arise in a natural way. Study the states of operators between non-archimedean normed spaces E, F over a complete non-archimedean valued field K when :

a) K is not spherically complete.

b) The valuation on K is trivial.

These questions were extensively discussed by J. Martínez-Maurica in his Doctoral Thesis, whose advisor was V. Onieva.

• The case a) was studied in [1] under the additional hypothesis that E and F were pseudoreflexive spaces (i.e., the canonical maps $J_E : E \longrightarrow E''$ and $J_F : F \longrightarrow F''$ are linear isometries from E into E'' and from F into F''). For this kind of spaces J. Martínez-Maurica showed that the situation is very different from the one previously studied for spherically complete fields : There are 54 states which never occur and of the 27 remaining states he gave examples of all of them, except of the following : (III_1, II_2), (III_1, II_3) and (III_1, III_2), leaving as an open problem the existence of such states. He showed that the existence of these three states is equivalent to the existence of (III_1, II_2). On the other hand, the existence of this last state is equivalent to a problem of extension of continuous linear functionals stated in terms of the restriction map. More specifically, he proved the following :

Proposition 2 ([1], Chapter 1, Proposition 11) : *The existence of an operator $T \in (III_1, II_2)$ is equivalent to the existence of a Banach space F (which had to be pseudoreflexive in the context of the Thesis) and a closed subspace M of F such that for the restriction map $\phi_M : f \in F' \longrightarrow f \mid M \in M'$, $\phi_M \in II_2$* holds.

Observe that when K is spherically complete, thanks to Ingleton's Theorem, the map ϕ_M is always surjective. However, for non-spherically complete fields there are several possibilities for the restriction map, which were studied in [4]. In that paper the author also presented, for non-spherically complete fields, the study of the states for the pair (T, T'), but with two novelties with respect to the corresponding one developed in the Thesis. Firstly, the pseudoreflexivity of the spaces was substituted by the weaker condition of "dual separating". Secondly, it was proved in this case that certain states cannot occur, giving examples of all the remaining ones, even (III_1, II_2), (III_1, II_3) and (III_1, III_2). For that they constructed (applying the results proved in [4] about the restriction map) a (non-pseudoreflexive) Banach space F whose dual separates points and a closed subspace M for which $\phi_M \in II_2$ (see Proposition 2).

• The case b) was studied in [1] when E and F are V-spaces in the sense of P. Robert (1968) : A *V-space* is a non-archimedean Banach space E over a trivially valued field K (with characteristic zero), for which there exists a set of integers $W(E)$ and a real number $\rho > 1$ such that

$$\{\|x\| : x \in E - \{0\}\} = \{\rho^{-n} : n \in W(E)\}.$$

The most interesting examples of non-archimedean normed spaces over trivially valued fields (e.g., spaces of asymptotic approximation, spaces of moments,...) are V-spaces. Also it was proved by P. Robert that every V-space E has an orthogonal basis (i.e., there exists a subset A of E whose closed linear hull coincides with E and such that $\|\sum_{i=1}^{n} \alpha_i x_i\| = \max_{i=1}^{n} \|x_i\|$, $\alpha_i \in K$, $\{x_1, \ldots, x_n\}$ a finite subset of A), which provides a useful tool for the study of this kind of spaces.

P. Robert raised in 1968 the question of whether every non-archimedean Banach space over a trivially valued field has an orthogonal basis in the sense exposed above. In 1983 José M. Bayod and J. Martínez-Maurica [7] gave a negative answer to this question.

For V-spaces E, F (over the same K and with the same ρ), the states for the pair (T, T') were studied in [1] when T is norm-bounded, i.e., $\sup\{\|Tx\|/\|x\| : x \in E - \{0\}\} < \infty$ (this implies continuity but not conversely; also one verifies that T norm-bounded $\Rightarrow T'$ norm-bounded, which is not true in general if we substitute "norm-bounded" by "continuous", see [1], p. 120). The results obtained in this case are again very different from the ones previously discussed. For instance, the states (II_2, II_2) and (III_2, II_3) are possible for V-spaces (compare with the situation when K is spherically complete and non-trivially valued). Also, the states (III_1, II_2), (III_1, II_3) and (III_1, III_2) cannot occur in this case (compare with the situation when K is not spherically complete). It is left as an open problem the existence of norm-bounded operators between V-spaces in the following states : (I_2, II_2), (I_2, III_1), (II_2, I_2), (II_2, III_1) and (II_3, III_1), although it is enough to give examples of three of these states to assure the existence of the five ones (see [1], p. 133)

More properties about non-archimedean normed spaces over trivially valued fields were studied by J. Martínez-Maurica (jointly with José M. Bayod) in [2], [3], [5], [6] and

[7]. However, aside from these papers, the rest of his published work on p-adic functional analysis always deals with spaces over a non-archimedean non-trivially complete valued field. So,

FRON NOW ON WE ASSUME THAT K IS A NON-ARCHIMEDEAN COMMUTATIVE AND COMPLETE VALUED FIELD ENDOWED WITH A NON-TRIVIAL VALUATION.

Apart from the usefulness of Theorem 1 for the study of the states of operators between non-archimedean normed spaces, this result was also applied in [23] to prove that Tauberian operators coincide with semi-Fredholm operators when K is spherically complete (which is in sharp contrast with the situation for Banach spaces over the real or complex field, where every semi-Fredholm operator is always Tauberian, but the converse is only true under certain additional hypotheses).

Recall that if E, F are non-archimedean Banach spaces over K, $T \in L(E,F)$ is said to be *Tauberian* if $(T'')^{-1}(F) \subset E$. Also, T is said to be *semi-Fredholm* if its kernel, $N(T)$, is finite-dimensional and its range, $R(T)$, is closed. Taking into account of Theorem 1 and the non-archimedean counterpart of the Closed Range Theorem (*$R(T)$ is closed $\Leftrightarrow$ $R(T')$ is closed*) proved by T. Kiyosawa in 1984 for spherically complete fields, it was proved in [23] that :

Theorem 3 ([23], Theorem 6) : *Let E, F be non-archimedean Banach spaces over a spherically complete field K and let $T \in L(E,F)$. Then,*

$$T \text{ is semi-Fredholm} \quad \Leftrightarrow \quad T \text{ is Tauberian.}$$

(However, if K is not spherically complete the properties "T is semi-Fredholm" and "T is Tauberian" are independent, see [23]).

Besides the study of semi-Fredholm operators in relation with Tauberian operators given in [23], J. Martínez-Maurica contributed (jointly with N. De Grande-De Kimpe) to the development of a non-archimedean Fredholm theory for compact operators. In the sixties, such a theory for compact operators on non-archimedean Banach spaces was developed by R. Ellis, L. Gruson, J.P. Serre and J. van Tiel, under the condition that K is locally compact. One of the important results proved in this Fredholm theory states : *Let E be a non-archimedean Banach space over a locally compact field K;*

if $T : E \longrightarrow E$ is a compact operator, then $I + T$ is a semi-Fredholm operator (*)

where I denotes the identity map on E. (Recall that T is called *compact* if the image of the closed unit ball of E, $T(B_E)$, is precompact in E).

But in p-adic analysis, if the ground field K is not locally compact, convex precompact sets are reduced to be a single point, and so the result mentioned above does not make sense in this case. To overcome this difficulty some variants of the concept of precompactness have been introduced in the non-archimedean literature. For several reasons it seems that the most succesful one is *compactoidity*, defined as follows : If E is a locally convex space over K (i.e., its topology is defined by a family of non-archimedean seminorms), a subset

A of E is called *compactoid* if for every zero-neighbourhood U in E there exists a finite set H in E such that $A \subset U + co(H)$, where $co(H)$ denotes the absolutely convex hull of H (recall that a nonempty subset B of E is called *absolutely convex* if $\lambda x + \mu y \in B$ for all $x, y \in B$, $\lambda, \mu \in K$, $|\lambda|, |\mu| \leq 1$).

Compactoidity was used in [36] to give some interesting descriptions of semi-Fredhom operators. For instance ([36], Corollary 2.6) : *If E, F are infinite dimensional non-archimeedan Banach spaces over K and $T \in L(E,F)$ then, T is semi-Fredholm $\Leftrightarrow$ for every subset D of E, D is compactoid if and only if $T(D)$ is compactoid.*

Also, by using compactoid sets to define compact operators (i.e., a continuous linear operator is called compact when its image of the closed unit ball is compactoid), W.H. Schikhof proved in 1989 that property (*) is also true for Banach spaces over non-locally compact fields.

However, an example was given by N. De Grande-De Kimpe and J. Martínez-Maurica in [34] showing that property (*) is no longer true if E is not complete. This observation led the authors to introduce in [34] a new ideal of operators between non-archimedean normed non-complete spaces over K (and more generally Hausdorff locally convex spaces in [43]) for which property (*) is valid, and such that in the case of Banach spaces these operators coincide with the compact ones. The new ideal is the ideal of *semicompact operators* : If E, F are Hausdorff locally convex spaces over K, a linear map $T : E \longrightarrow F$ is called *semicompact* if there exists a compactoid and completing subset D of F such that $T^{-1}(D)$ is a zero-neighbourhood in E (a bounded set D is said to be *completing* if the linear hull of D, normed with the corresponding Minkowski functional, p_D, is complete). The Fredholm theory for semi-compact operators was developed in [34] (for normed spaces) and in [43] (for general Hausdorff locally convex spaces).

Other properties of compact and semicompact operators T on non-archimedean normed spaces E, F over K were studied in [47]. The aim of this paper is to examine the connection between this kind of operators, their approximation numbers $\alpha_n(T) = \inf\{\|T - A\| : A \in L(E,F), \dim R(A) \leq n\}$ $(n \in N)$ introduced by A.K. Katsaras in 1988, and their Kolmogorov diameters $\delta_n^*(T(B_E))$ introduced by A.K. Katsaras and J. Martínez-Maurica in [41] (if $B \subset F$ is bounded, $\delta_n^*(B) = \inf\{r > 0 : B \subset G + \{x \in F : \|x\| \leq r\}$, G linear subspace of F, $\dim G \leq n\}$; for several descriptions of these diameters see [41]).

As we have seen, starting with his Doctoral Thesis, J. Martínez-Maurica was always interested in questions dealing with the theory of operators between non-archimedean (and locally convex) spaces. Even during the very last months of his life, he employed part of the energies he could have in that moment to study, jointly with W.H. Schikhof, the non-archimedean counterpart of the classical integral operators. The results obtained on this subject were included in [49], whose final version was written after he died. One the main results of this paper states that the p-adic integral operator is compact :

Theorem 4 ([49], Theorem 5.3) : *Let X, Y be compact subsets of K without isolated points and let $C^n(X)$ and $C^m(Y)$ $(n, m \in N)$ be the corresponding Banach spaces of K-*

valued C^n (resp. C^m)-functions on X (resp. Y), as defined by W.H. Schikhof (1984). Then, for every $\mu \in C^m(Y)'$ and for every $G \in C^{n,m}(X \times Y)$, the formula

$$(Tf)(x) = \int G(x,y)f(y)d\mu(y) = \mu(y \longrightarrow G(x,y)f(y))$$

defines a compact operator $T : C^m(Y) \longrightarrow C^n(X)$ (when $m, n = 0$, $C^n(X)$ and $C^m(Y)$ are the corresponding Banach spaces, $C(X)$ (resp. $C(Y)$) of continuous K-valued functions endowed with the supremum norms and in this case the result is true for arbitrary separated zero-dimensional compact spaces X, Y [49], Theorem 1.1).

The key to the proof of Theorem 4 is the p-adic Ascoli Theorem previously proved by J. Martínez-Maurica and S. Navarro, assuring that : *A subset of $C(X)$ is compactoid if and only if it is equicontinuous and pointwise bounded* (see [38], where the relationships between compactoids and equicontinuous sets in some different spaces of contiunous functions are discussed).

Within the p-adic theory of spaces of continuous functions we also point out the contribution of J. Martínez-Maurica (jointly with J. Araujo) to the study (initiated by E. Beckenstein and L. Narici in 1987) of the non-archimedean Banach-Stone Theorem. Among others (see [39] and [46]) they proved the following result wchich improves the ones given earlier by E. Beckenstein and L. Narici.

Theorem 5 ([39], Theorems 2 and 7) : *Let X be a separated zero-dimensional compact space. If X has more than one point, then there are linear surjective isometries $T : C(X) \longrightarrow C(X)$ which are not Banach-Stone maps* (a linear isometry $T : C(X) \longrightarrow C(X)$ is called a Banach-Stone map if there exists a homeomorphism h from X onto X and an $a \in C(X)$, $|a(x)| \equiv 1$, such that $(Tf)(x) = a(x)f(h(x))$, for every $x \in X$, $f \in C(X)$).

Moreover, the set of all surjective isometries $T : C(X) \longrightarrow C(X)$ which are not Banach-Stone maps, is dense in the Banach space of all surjective isometries from $C(X)$ onto $C(X)$ (this last space normed with the norm induced by the usual one in the space $L(C(X), C(X))$).

Theorem 5 is in sharp contrast with the situation when the ground field is the real or complex one. In this case, the Banach-Stone Theorem says that for every separated compact space X, every linear surjective isometry $T : C(X) \longrightarrow C(X)$ is a Banach-Stone map in the sense of Theorem 5. The concept of extreme point and the Krein-Milman Theorem play a very important role in the proof of the Banach-Stone Theorem in the real or complex case. However, for vector spaces over a non-archimedean valued field K several difficulties appear in all the problems in which the boundary of a set seems essential (for instance, it is well known that in a non-archimedean normed space over K every ball has empty boundary). In fact the development of a non-archimedean Krein-Milman Theorem was for a long time an open problem. It was raised by A.F. Monna in 1974 and unsolved until 1985-86, when J. Martínez-Maurica and C. Pérez-García gave in [14] a non-archimedean Krein-Milman Theorem for convex compact sets and locally compact fields which was later extended in [16] for the more general class of c-compact convex sets (introduced by T.A. Springer in 1965) and spherically complete fields (recall that a subset D of a vector space E over K is called *convex* if $D = \emptyset$ or D is the coset of an absolutely convex subset of E). For locally compact fields K, the definition of extreme points was the following :

Definition 6 ([14], Definition 2) : Let E be a vector space over K and let A be a subset of E. A non-empty part S of A is said to be an *extreme set* of A if : i) S is semiconvex (i.e., $\lambda S + (1-\lambda)S \subset S$) for all $\lambda \in K$ with $|\lambda| < 1$), and ii) If the convex hull of a finite set $\{x_1, \ldots, x_n\} \subset A$ has non-empty intersection with S, then there is $i \in \{1, \ldots, n\}$ such that $x_i \in S$. A point $x \in A$ is called an *extreme point* of A if it belongs to some minimal extreme set of A.

This defintion allowed the authors to obtain the following non-archimedean version of the Krein-Milman Theorem.

Theorem 7 ([14], Theorem 3) : *Every non-empty compact convex subset of a separated locally convex space over K is the closed convex hull of its extreme points.*

If K is substituted by the real or complex field and the concept of convex set by the corresponding one in this case, Definition 6 gives the same extreme points as the usual ones, according to the definition of N.J. Kalton (1977).

Besides the development of a non-archimedean Krein-Milman Theorem, A.F. Monna listed in 1974 several open problems in p-adic analysis. One of them was about normed spaces whose norm does not verify the strong triangle inequality (also called A-normed spaces in the literature). Although A-normed spaces appear in a natural way (e.g. ℓ^p-spaces are A-normed spaces for $p \geq 1$) very little was known about their properties. J. Martínez-Maurica (jointly with C. Pérez-García) studied various topics within the context of A-normed spaces. The most interesting results obtained on this subject are contained in [18], [19], [24] and [32]. One of topics discussed in these papers concerns the following question :

PROBLEM. Let E be an A-normed space over K satisfying the Hahn-Banach extension property (i.e., for every closed subspace M of E, each continuous linear functional on M admits a continuous linear extension to the whole space). Does this imply that E is locally convex?

They proved, by using certain techniques on basic sequences, that the above problem has an affirmative answer when :

i) K is locally compact ([18], Teorema 4),

or

ii) E has a denumerable basis, i.e., there exists a sequence (x_n) in E which is a basis (in the usual meaning) for the completion of E ([19], Theorem 5).

However, the answer to this problem in the general case is unknown. In fact, in the Proceedings of the First International Conference on p-adic analysis (p-adic Functional Analysis, Marcel-Dekker, vol. 137, 1992), in the paper writen by A.C.M. van Rooij and W.H. Schikhof entitled "Open problems" and consisting of a selection of some interesting open questions on p-adic analysis, the above problem appears among them. As far as we know, this problem has not been solved yet!

A-normed spaces constitute a particular case of a more general structure : topological vector spaces over a non-archimedean valued field K. J. Martínez-Maurica made some incursions on the study of this kind of spaces (see [12], [48], the latter published after his death). In these papers his attention was centered on the class of topological vector spaces E whose topology is defined by an *ultrametric* d on E, i.e, a metric satisfying the strong triangle inequality $d(x,y) \leq \max(d(x,z), d(z,y))$ for all $x, y, z \in E$ (E is called *ultrametrizable*). Among other things, in contrast with the situation for metrizable spaces, the existence was shown of a topological vector space (more concretely an A-normed space) over K that is ultrametrizable but such that its topology is not induced by an invariant ultrametric ([48], Example 1.2).

Moreover, the study of *ultrametric spaces without any underlying vectorial structure* was another of the subjects in which J. Martínez-Maurica had interesting contributions.

In [25] and [27] he presented (jointly with José M. Bayod) certain relationships among several topics in ultrametric spaces : the problem of extension of Lipschitz (or contractive, or isometric) maps, spherical completness and the analog to tight extensions in arbitrary metric spaces. As a corollary, they obtained many of the properties about ultrametric spaces previously proved by R. Bhaskaran in 1983. Putting together some of the more relevant results given in these two papers we derive the following descriptions of the ultrametric spaces that are spherically complete :

Theorem 8 (see [25] and [27]) : *For an ultrametric space X, the following properties are equivalent :*

i) X is spherically complete (i.e., X satisfies the Nachbin's binary intersection property for balls).

ii) Given any ultrametric space Z, and a subspace Y of Z, any Lipschitz map $Y \longrightarrow X$ can be extended to a Lipschitz map $Z \longrightarrow X$ with the same Lipschitz constant.

iii) Given any ultrametric space Z, and a subspace Y of Z, any contractive map $Y \longrightarrow X$ can be extended to a contractive map $Z \longrightarrow X$.

iv) If Y is an ultrametric space and $Z \supset Y$ is an ultrametrically tight extension of Y, then any isometric embedding $Y \longrightarrow X$ can be extended to an isometric embedding $Z \longrightarrow X$ (an ultrametric space (Z,d) is called an ultrametrically tight extension of Y if $d(z_1, z_2) > \inf\{d(z_1, y) : y \in Y\}$ for all $z_1, z_2 \in Z$).

v) X has no proper ultrametrically tight extensions.

vi) X has no proper immediate extensions (if Y is a subspace of an ultrametric space (Z,d), Z is called an immediate extension of Y if for every $z \in Z - Y$ the distance $d(z,Y)$

is not attained).

Also, an *ultrametric tight span* was constructed for every ultrametric space.

The concept of immediate extension appearing in Theorem 8 is closely related to the concepts of Chebyshev radii and centers in ultrametric spaces (studied by J. Martínez-Maurica, jointly with T. Pellón, in [30]).

Indeed, if (X, d) is an ultrametric space and $W \subset X$, we clearly have that the property

(1) *W has no proper immediate extensions within X.*

is equivalent to

(2) $cent_W(B) \neq \emptyset$ *for all* $B = \{b\}$ *with* $b \in X$,

where, following the standard notation, for every non-empty bounded subset B of X, $cent_W(B)$ denotes the set of all the *relative Chebyshev centers* of B with respect to W, i.e., the set of all $w \in W$ for which the infimum, $\inf_{w \in W}\{\sup d(b, w) : b \in B\}$, is attained (this infimum is called the *relative Chebyshev radius* of B with respect to W). A set W satisfying (2) is called *proximinal.*

In 1985 M.Z.M.C. Soares posed the main problems in the non-archimedean theory of the best approximation, which are :

PROBLEM I : Let $W \subset X$ be given. Determine if W admits Chebyshev centers in X (i.e., $cent_W(B) \neq \emptyset$ for all non-empty bounded set $B \subset X$). In particular, when $W = X$.

PROBLEM II. Let $W \subset X$ be given. Determine if W is proximinal.

In 1987, J. Martínez-Maurica and T. Pellón [30] were able to give a partial answer to Problem I and to establish the equivalence of problems I and II.

Theorem 9 ([30], Theorems 3 and 4) : *Let X be an utrametric space and $W \subset X$. Then,*

i) X admits Chebyshev centers in X.

ii) W is proximinal if and only if W admits Chebyshev centers in X.

(the crucial fact to the proof of this Theorem is the expression for the Chebyshev radius obtained in Theorem 1 of [30] which says that, for every non-empty bounded subset B of X, $rad_W(B) = \max\{\delta(B), d(B, W)\}$, where $\delta(B)$ is the diameter of B and $d(B, W)$ is the distance between B and W).

Within the context of ultrametric spaces without any underlying vectorial structure we also point out the contribution of J. Martínez-Maurica (jointly with José M. Bayod) in [40] concerning the existence of subdominant ultrametrics. In some applications of ultrametric theory to real-world problems in Physics and other sciences, it is of interest to know the precise relation between a given metric d on a set X and the so called *subdominant ultrametric* δ, defined as the largest ultrametric on X among those that are less or equal to d for every pair of points. The existence of such an ultrametric is trivial for any finite metric space X. In [40] they studied the existence problem of subdominant ultrametrics when the finiteness condition is dropped. They proved, among other results that

Theorem 10 ([40], Theorem 2) : *If (X,d) is a locally compact and totally disconnected metric space, then there exists the corresponding subdominant ultrametric and it is topologically equivalent to d.* (Some related examples showing that this statement would not be true under more general conditions, are also included in [40]).

Ultrametric spaces are also becoming important during the last years to the development of a p-adic Quantum Mechanics, which constitutes the research activity of several Russian scientists : A.Yu. Khrenikov, V.S. Vladimirov, I.V. Volovich, E.I. Zelenov,... As in the classical case, one crucial tool to the development of this p-adic Quantum Mechanics is the concept of a Hilbert space over a non-archimedean field K.

A *Springer space* is a linear space over a non-archimedean valued field K, endowed with an anisotropic symetric bilinear form $<,>$ and complete for the associated non-archimedean norm $\|x\| = |<x,x>|^{1/2}$. In 1955 T.A. Springer proved that there are no Springer spaces of dimension biger than 4 over the p-adic fields Q_p. In 1979 H. Gross raised the following problem : *Is there any field over which there is no infinite dimensional Springer space, but admits Springer spaces over it of any finite dimension?.*

In 1989, J. Martínez-Maurica in cooperation with his wife, P. Fernández-Ferreirós, and with José M. Bayod ([35]) studied some questions about Springer spaces over discretely valued fields K. Among other things they proved that : *If K is discretely valued, the existence of some infinite dimensional Springer space over K is equivalent to the existence of a sequence (b_n) in $\{\alpha \in K : |\alpha| = 1\}$ such that $|\sum_n \lambda_n^2 b_n| = \max_n |\lambda_n|^2$ for every sequence (λ_n) in K converging to zero.* As an application of this result they proved that the question of H. Gross mentioned above can be simplified by considering only fields K of power series (see [13], [20] and [35]).

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