ANNALES MATHÉMATIQUES BLAISE PASCAL

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Annales mathématiques Blaise Pascal, tome 2, nº 1 (1995), p. 299-305
<http://www.numdam.org/item?id=AMBP_1995__2_1_299_0>

Ann. Math. Blaise Pascal, Vol. 2, N° 1, 1995, pp.299-305

THE CONSTRUCTION OF NORMAL BASES FOR THE SPACE OF CONTINUOUS FUNCTIONS ON V_q, WITH THE AID OF OPERATORS

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Abstract. Let a and q be two units of $\mathbb{Z}_p$, q not a root of unity, and let V_q be the closure of the set $\{aq^n \mid n = 0, 1, 2, \ldots\}$. K is a non-archimedean valued field, K contains $\mathbb{Q}_p$, and K is complete for the valuation $|\cdot|$, which extends the p-adic valuation. $C(V_q \to K)$ is the Banach space of continuous functions from V_q to K , equipped with the supremum norm. Let $\mathcal{E}$ and D_q be the operators on $C(V_q \to K)$ defined by $(\mathcal{E}f)(x) = f(qx)$ and $(D_q f)(x) = (f(qx) - f(x))/(x(q-1))$. We will find all linear and continuous operators that commute with $\mathcal{E}$ (resp. with D_q), and we use these operators to find normal bases $(r_n(x))$ for $C(V_q \to K)$. If f is an element of $C(V_q \to K)$, then there exist elements α_n of K such that $f(x) = \sum_{n=0}^{\infty} \alpha_n r_n(x)$ where the series on the right-hand-side is uniformly convergent. In some cases it is possible to give an expression for the coefficients α_n.

1991 Mathematics subject classification : 46S10

1. Introduction

Let p be a prime, $\mathbb{Z}_p$ the ring of the p-adic integers, $\mathbb{Q}_p$ the field of the p-adic numbers. K is a non-archimedean valued field, $K \supset \mathbb{Q}_p$, and we suppose that K is complete for the valuation $|\cdot|$, which extends the p-adic valuation. Let a and q be two units of $\mathbb{Z}_p$ (i.e. $|a| = |q| = 1$), q not a root of unity. Let V_q be the closure of the set $\{aq^n \mid n = 0, 1, 2, \ldots\}$. We denote by $C(V_q \to K)$ (resp. $C(\mathbb{Z}_p \to K)$) the set of all continuous functions $f : V_q \to K$ (resp. $f : \mathbb{Z}_p \to K$) equipped with the supremum norm. If f is an element of $C(V_q \to K)$ then we define the operators $\mathcal{E}$ and D_q as follows :

$(\mathcal{E}f)(x) = f(qx)$

$$(D_q f)(x) = \frac{f(qx) - f(x)}{x(q-1)}$$

We remark that the operator $\mathcal{E}$ does not commute with D_q. Furthermore, the operator D_q lowers the degree of a polynomial with one, whereas the operator $\mathcal{E}$ does not.

If $\mathcal{L}$ is a non-archimedean Banach space over a non-archimedean valued field L, and $e_1, e_2, \ldots$ is a finite or infinite sequence of elements of $\mathcal{L}$, then we say that this sequence is orthogonal if $||\epsilon_1 e_1 + \cdots + \epsilon_k e_k|| = \max\{||\epsilon_i e_i|| : i = 1, \ldots, k\}$ for all k in $\mathbb{N}$ (or for all k that do not exceed the length of the sequence) and for all $\epsilon_1, \ldots, \epsilon_k$ in L. An orthogonal sequence $e_1, e_2, \cdots$ is called orthonormal if $||e_i|| = 1$ for all i . A family (e_i) of elements of $\mathcal{L}$ forms a(n) (ortho)normal basis of $\mathcal{L}$ if the family (e_i) is orthonormal and also a basis . We will call a sequence of polynomials $(p_n(x))$ a polynomial sequence if p_n is exactly of degree n for all natural numbers n .

The aim here is to find normal bases for $C(V_q \to K)$, which consist of polynomial sequences. Therefore we will use linear , continuous operators which commute with D_q or with $\mathcal{E}$. If $(r_n(x))$ is such a polynomial sequence , and if f is an element of $C(V_q \to K)$, there exist coefficients α_n in K such that $f(x) = \sum_{n=0}^{\infty} \alpha_n r_n(x)$ where the series on the right-hand-side is uniformly convergent. In some cases it is possible to give an expression for the coefficients α_n.

We remark that all the results (with proofs) in this paper can be found in [5] , except for theorem 5 .

2. Notations.

Let V_q, K and $C(V_q \to K)$ be as in the introduction . The supremum norm on $C(V_q \to K)$ will be denoted by $|| \cdot ||$. We introduce the following :

$A_0(x) = 1, A_n(x) = (x - aq^{n-1})A_{n-1}(x)$ $(n \geq 1)$,

$B_n(x) = A_n(x)/A_n(aq^n)$, $C_n(x) = a^n q^{n(n-1)/2}(q-1)^n B_n(x)$

It is clear that $(A_n(x))$, $(B_n(x))$ and $(C_n(x))$ are polynomial sequences . The sequence $(C_n(x))$ forms a basis for $C(V_q \to K)$ and the sequence $(B_n(x))$ forms a normal basis for $C(V_q \to K)$. From this it follows that $||B_n|| = 1$ and $||C_n|| = |(q-1)^n|$. Let $\mathcal{E}$ and D_q be as in the introduction . Then we introduce the following :

Definition. Let f be a function from V_q to K. We define the following operators :

$(D_q^n f)(x) = (D_q(D_q^{n-1} f))(x)$

$(\mathcal{E}^n f)(x) = f(q^n x)$

$\mathcal{D}f(x) = \mathcal{D}^{(1)} f(x) = f(qx) - f(x) = ((\mathcal{E} - 1)f)(x)$

$\mathcal{D}^{(n)} f(x) = ((\mathcal{E} - 1)..(\mathcal{E} - q^{n-1})f)(x)$, $\mathcal{D}^{(0)} f(x) = f(x)$

The operator D_q does not commute with $\mathcal{D}$. The following properties are easily verified :

$D_q^j C_k(x) = C_{k-j}(x)$ $if k \geq j$, $D_q^j C_k(x) = 0$ if $j > k$. So D_q^j lowers the degree of a polynomial with j

$\mathcal{D}^{(j)}B_k(x) = (x/a)^j q^{j(j-k)} B_{k-j}(x)$ if $j \leq k$, $\mathcal{D}^{(j)}B_k(x) = 0$ if $j > k$
If $p(x)$ is a polynomial of degree n , then $(\mathcal{D}^{(j)}p)(x)$ is a polynomial of degree n if n is at least j , and $(\mathcal{D}^{(j)}p)(x)$ is the zero-polynomial if n is strictly smaller than j .
If f is an element of $C(V_q \to K)$, then we also have

i) $(\mathcal{D}^{(n)}f)(x) = x^n q^{n(n-1)/2}(q-1)^n(D_q^n f)(x)$

ii) $(q-1)^n D_q^n f(x) \to 0$ uniformly

iii) $\mathcal{D}^{(n)}f(x) \to 0$ uniformly

(i) can be found in [1] , p. 60 , ii) can be found in [3] , p. 124-125 , iii) follows from i) and ii)).

3. Linear Continuous Operators which Commute with $\mathcal{E}$ or with D_q

Let us start this section with the following known result :
If f is an element of $C(\mathbb{Z}_p \to K)$, then the translation operator E on $C(\mathbb{Z}_p \to K)$ is the operator defined by $Ef(x) = f(x+1)$.
If we put $G_n(x) = \binom{x}{n}$ (the binomial polynomials), then L. Van Hamme ([4]) proved the following theorem :
A linear , continuous operator Q on $C(\mathbb{Z}_p \to K)$ commutes with the translation operator E if and only if the sequence (g_n) is bounded , where $g_n = QG_n(0)$.
Such an operator Q can be written in the following way : $Q = \sum_{i=0}^{\infty} g_i \Delta^i$, where Δ is the operator defined as follows : $(\Delta f)(x) = f(x+1) - f(x)$

We can prove analogous theorems for the operators $\mathcal{E}$ and D_q on $C(V_q \to K)$:

Theorem 1 *An operator Q on $C(V_q \to K)$ is continuous , linear and commutes with $\mathcal{E}$ if and only if the sequence (b_n) is bounded, where $b_n = (QB_n)(a)$.*

From the proof of the theorem it follows that Q can be written in the form $Q = \sum_{i=0}^{\infty} b_i \mathcal{D}^{(i)}$.

If f is an element of $C(V_q \to K)$, then $(Qf)(x) = \sum_{i=0}^{\infty} b_i(\mathcal{D}^{(i)}f)(x)$ and the series on the right-hand-side is uniformly convergent (since $\mathcal{D}^{(n)}f(x) \to 0$ uniformly) . Clearly we have $b_n = (QB_n)(a)$, since $(QB_n)(a) = (\sum_{i=0}^{\infty} b_i D^{(i)} B_n)(a) = (\sum_{i=0}^{n} b_i (x/a)^i q^{i(i-n)} B_{n-i})(a) = b_n$.
Furthermore , Qx^n is a K-multiple of x^n .
If $b_0 = \ldots = b_{N-1} = 0, b_N \neq 0$, and if p(x) is a polynomial , then x^N divides $(Qp)(x)$.

Some examples

1) For the operator $\mathcal{E}$ we have : $(\mathcal{E}B_n)(x) = B_n(qx)$, so $(\mathcal{E}B_0)(a) = 1$, $(\mathcal{E}B_1)(a) = 1$, and $(\mathcal{E}B_n)(a) = 0$ if $n \geq 2$. This gives us $\mathcal{E} = \mathcal{D}^{(0)} + \mathcal{D}^{(1)}$.

2) The operator $\mathcal{E} \circ \mathcal{D} = \mathcal{E}\mathcal{D}$ clearly commutes with $\mathcal{E}$. We have $((\mathcal{E}\mathcal{D})B_0)(a) = 0$, and since $(n \geq 1)$ $((\mathcal{E}\mathcal{D})B_n)(x) = (\mathcal{E}\ (\frac{x}{a}q^{1-n}B_{n-1}))(x) = \frac{qx}{a}q^{1-n}B_{n-1}(qx)$, we find $((\mathcal{E}\mathcal{D})B_1)(a) = q$, $((\mathcal{E}\mathcal{D})B_2)(a) = 1$ and $((\mathcal{E}\mathcal{D})B_n)(a) = 0$ if $n \geq 3$. We conclude that $\mathcal{E}\mathcal{D} = q\mathcal{D}^{(1)} + D^{(2)}$.

Analogous to theorem 1 we have :

Theorem 2 *An operator Q on $C(V_q \to K)$ is continuous , linear and commutes with D_q if and only if the sequence $(c_n/(q-1)^n)$ is bounded, where $c_n = (QC_n)(a)$.*

Such an operator Q can be written in the form $Q = \sum_{i=0}^{\infty} c_i D_q^i$, and if f is an element of $C(V_q \to K)$ it follows that $(Qf)(x) = \sum_{i=0}^{\infty} c_i(D_q^i f)(x)$, where the series on the right-hand-side converges uniformly (since $(q-1)^n D_q^n f(x) \to 0$ uniformly) . Furthermore , we have $c_n = (QC_n)(a)$ since

$$(QC_n)(a) = (\sum_{i=0}^{\infty} c_i D_q^i C_n)(a) = \sum_{i=0}^{n} c_i C_{n-i}(a) = c_n.$$

Remarks

1) Let R and Q be linear , continuous operators on $C(V_q \to K)$, with R of the form $R = \sum_{i=1}^{\infty} b_i D^{(i)}$ (i.e. R commutes with $\mathcal{E}$, $b_0 = 0$), and Q of the form $Q = \sum_{i=1}^{\infty} c_i D_q^i$ (i.e.Q commutes with D_q, $c_0 = 0$) . The main difference between the operators Q and R is that Q lowers the degree of each polynomial with at least one , where R does not necessarily lowers the degree of a polynomial .

2) If Q_1 and Q_2 both commute with D_q and if $Q_1 = \sum_{i=0}^{\infty} c_{1;i} D_q^i$,

$$Q_2 = \sum_{i=0}^{\infty} c_{2;i} D_q^i \text{ , then } (Q_1 o Q_2)(f) = (Q_2 o Q_1)(f) = \sum_{k=0}^{\infty} D_q^k f \left(\sum_{j=0}^{k} c_{1;j} c_{2;k-j} \right).$$

If we take two formal power series $q_1(t) = \sum_{i=0}^{\infty} c_{1;i} t^i$, $q_2(t) = \sum_{i=0}^{\infty} c_{2;i} t^i$, then

$q_1(t) \cdot q_2(t) = \sum_{k=0}^{\infty} t^k \left(\sum_{j=0}^{k} c_{1;j} c_{2;k-j} \right)$, so the composition of two operators which commute with D_q, corresponds with multiplication of power series .

This is not the case if we take two operators which commute with $\mathcal{E}$: Take e.g. $\mathcal{E} = \mathcal{D}^{(0)} + \mathcal{D}^{(1)}$ and $\mathcal{D}^{(1)}$, then $\mathcal{E}\, o\mathcal{D}^{(1)} = \mathcal{E}\mathcal{D}^{(1)} = q\mathcal{D}^{(1)} + \mathcal{D}^{(2)}$, whereas for power series this gives $q_1(t) = 1 + t$, $q_2(t) = t$ and $q_1(t) \cdot q_2(t) = t + t^2$.

4. Normal bases for $C(V_q \to K)$

We use the operators of theorems 1 and 2 to make polynomials sequences $(p_n(x))$ which form normal bases for $C(V_q \to K)$. If Q is an operator as found in theorem 1 , with b_0 equal to zero , we associate a (unique) polynomial sequence $(p_n(x))$ with Q. We remark that the operator $R = \sum_{i=0}^{\infty} b_i \mathcal{D}^{(i)}$ does not necessarily lowers the degree of a polynomial.

Proposition 1 *Let* $Q = \sum_{i=N}^{\infty} b_i \mathcal{D}^{(i)}$ $(N \geq 1)$ *with* $|b_N| > |b_n|$ *if* $n > N$. *There exists a unique polynomial sequence* $(p_n(x))$ *such that* $(Qp_n)(x) = x^N p_{n-N}(x)$ *if* $n \geq N$, $p_n(aq^i) = 0$ *if* $n \geq N$, $0 \leq i < N$ *and* $p_n(x) = B_n(x)$ *if* $n < N$.

In the same way as in proposition 1 we have.

Proposition 2 *Let* $Q = \sum_{i=N}^{\infty} c_i D_q^i$ $(N \geq 1)$, $c_N \neq 0$, $(c_n/(q-1)^n)$ *bounded.* *Then there exists a unique polynomial sequence* $(p_n(x))$ *such that* $(Qp_n)(x) = p_{n-N}(x)$ *if* $n \geq N$, $p_n(aq^i) = 0$ *if* $n \geq N$, $0 \leq i < N$ *and* $p_n(x) = B_n(x)$ *if* $n < N$.

We use the operators of theorems 1 and 2 to make polynomials sequences $(p_n(x))$ which form normal bases for $C(V_q \to K)$. If f is an element of $C(V_q \to K)$, there exist coefficients α_n such that $f(x) = \sum_{n=0}^{\infty} \alpha_n p_n(x)$ where the series on the right-hand-side is uniformly convergent . In some cases, it is also possible to give an expression for the coefficients α_n.

Theorem 3 *Let* $Q = \sum_{i=N}^{\infty} b_i D^{(i)}$ $(N \geq 1)$ *with* $|b_n| < |b_N| = 1$ *if* $n > N$

1) There exists a unique polynomial sequence $(p_n(x))$ *such that* $(Qp_n)(x) = x^N p_{n-N}(x)$ *if* $n \geq N$, $p_n(aq^i) = 0$ *if* $n \geq N$, $0 \leq i < N$ *and* $p_n(x) = B_n(x)$ *if* $n < N$. *This sequence forms a normal basis for* $C(V_q \to K)$ *and the norm of* Q *equals one .*

2) If f *is an element of* $C(V_q \to K)$, *then* f *can be written as a uniformly convergent series* $f(x) = \sum_{n=0}^{\infty} \beta_n p_n(x)$, $\beta_n = ((D^{(i)}(x^{-N}Q)^k)f)(a)$ *if* $n = i + kN$ $(0 \leq i < N)$, *with* $||f|| = \max_{0 \leq k; 0 \leq i < N} |((D^{(i)}(x^{-N}Q)^k)f)(a)|$, *where* $x^{-N}Q$ *is a linear continuous operator with norm equal to one .*

And analogous to theorem 3 we have

Theorem 4 *Let* $Q = \sum_{i=N}^{\infty} c_i D_q^i$ $(N \geq 1)$ *with* $|c_N| = |(q-1)^N|$, $|c_n| \leq |(q-1)^n|$ *if* $n > N$.
1) There exists a unique polynomial sequence $(p_n(x))$ *such that* $(Qp_n)(x) = p_{n-N}(x)$ *if* $n \geq N$, $p_n(aq^i) = 0$ *if* $n \geq N$, $0 \leq i < N$ *and* $p_n(x) = B_n(x)$ *if* $n < N$. *This sequence forms a normal basis for* $C(V_q \to K)$ *and the norm of* Q *equals one.*
2) If f *is an element of* $C(V_q \to K)$, *there exists a unique , uniformly convergent expansion of the form* $f(x) = \sum_{n=0}^{\infty} \gamma_n p_n(x)$, *where* $\gamma_n = a^i(q-1)^i q^{i(i-1)/2}(D_q^i Q^k f)(a)$ *if* $n = i + kN$ $(0 \leq i < N)$, *with* $||f|| = \max_{0 \leq k; 0 \leq i < N} \{|(q-1)^i (D_q^i Q^k f)(a)|\}$.

Remark. Here we have $|c_n| \leq |c_N|$, in contrast with theorem 3, where we need $|b_n| < |b_N|$ $(n > N)$.

An example
Let us consider the following operator $Q = (q-1)D_q$. Then $c_1 = (q-1)$ and $c_k = 0$ if $k \neq 1$. The polynomials $p_k(x)$ are given by $p_k(x) = C_k(x)/(q-1)^k$, and they form a normal basis for $C(V_q \to K)$. The expansion $f(x) = \sum_{k=0}^{\infty} ((q-1)^k D_q^k f)(a) p_k(x) = \sum_{k=0}^{\infty} (D_q^k f)(a) C_k(x)$ is known as Jackson's interpolation formula ([2],[3]) .

If Q is an operator as found in theorem 4 , with N equal to one, then we can prove a theorem analogous to theorem 2 :

Theorem 5 *Let* Q *be an operator such that* $Q = \sum_{i=1}^{\infty} c_i D_q^i$, *with* $|c_1| = |(q-1)|$, $|c_n| \leq |(q-1)^n|$ *if* $n > 1$, *and let* $p_n(x)$ *be the polynomial sequence as found in theorem 4.*
An operator T *on* $C(V_q \to K)$ *is continuous , linear and commutes with* D_q *if and only if* T *is of the form* $T = \sum_{i=0}^{\infty} d_i Q^i$, *where the sequence* (d_n) *is bounded, where* $d_n = (Tp_n)(a)$.

Remark. In theorem 2 the sequence $(c_n/(q-1)^n)$ must be bounded , whereas here the sequence (d_n) must be bounded. This follows from the fact that the norm of the operator D_q equals $|q-1|^{-1}$, whereas the norm of the operator Q equals 1 .

5. More Normal Bases

We want to make more normal bases , using the ones we found in theorems 3 and 4 . For operators which commute with $\mathcal{E}$ we can prove the following theorem :

Theorem 6 *Let $(p_n(x))$ be a polynomial sequence which forms a normal basis for $C(V_q \to K)$, and let $Q = \sum_{i=N}^{\infty} b_i \mathcal{D}^{(i)}$ $(N \geq 0)$ with $1 = |b_N| > |b_k|$ if $k > N$. If $Qp_n(x) = x^N r_{n-N}(x)$ $(n \geq N)$, then the polynomial sequence $(r_k(x))$ forms a normal basis for $C(V_q \to K)$.*

And analogous for operators which commute with the operator D_q we have :

Theorem 7 *Let $(p_n(x))$ be a polynomial sequence which forms a normal basis for $C(V_q \to K)$, and let $Q = \sum_{i=N}^{\infty} c_i D_q^i$ $(N \geq 0)$ with $|c_N| = |(q-1)^N|$, $|c_n| \leq |(q-1)^n|$ if $n > N$. If $(Qp_n)(x) = r_{n-N}(x)$ $(n \geq N)$, then the polynomial sequence $(r_k(x))$ forms a normal basis for $C(V_q \to K)$.*

We remark that analogous results can be found on the space $C(\mathbb{Z}_p \to K)$ for linear continuous operators which commute with the translation operator E. The result analogous to theorems 3 and 4 for the case N equal to one, was found by L. Van Hamme (see [4]), and the extensive version of theorems 3 and 4, and the analogons of theorems 5, 6 and 7 can be found with proofs similar to the proofs of the theorems in this paper .

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