ANNALES MATHÉMATIQUES BLAISE PASCAL

HANS A. KELLER
HERMINIA OCHSENIUS A.

A spectral theorem for matrices over fields of power series

Annales mathématiques Blaise Pascal, tome 2, nº 1 (1995), p. 169-179
<http://www.numdam.org/item?id=AMBP_1995__2_1_169_0>

Ann. Math. Blaise Pascal, Vol. 2, N° 1, 1995, pp.169-179

A SPECTRAL THEOREM FOR MATRICES OVER FIELDS OF POWER SERIES

Hans A. Keller and Herminia Ochsenius A.

Abstract. Let $K = K = \mathbf{R}((t_1, \dots, t_m))$ be a field of formal power series in one or several variables with real coefficients. We prove that every symmetric square matrix $\mathcal{A} \in Mat_n(K)$ can be diagonalized by means of an orthogonal matrix $\mathcal{U} \in Mat_n(K)$. Our proof is based on a recursive construction and prepares the way for effectively computing the transition matrix $\mathcal{U}$ (and therefore the eigenvalues of $\mathcal{A}$ and their multiplicities). The result carries over to certain Henselian fields of power series in infinitely many variables.

1991 Mathematics subject classification: 12J25, 15A33

Introduction. The most prominent result in the theory of real or complex matrices is the Spectral Theorem which says that every symmetric [resp. hermitian] square matrix can be put into diagonal form by means of an orthogonal [resp. a unitary] matrix. The far-reaching applications of this result and its generalization to bounded linear operators on infinite-dimensional Hilbert spaces have been intensively studied. However, little is known about analogous decompositions of matrices with entries in more general fields. Diarra [2] showed that symmetric matrices over fields of p-adic numbers cannot be diagonalized in general. In turn, Adkins [1] proved a theorem on diagonalization of matrices with entries in discrete hermitian rings.

In the present paper we consider fields $K = \mathbf{R}((t_1, \dots, t_m))$ of formal power series in one or several variables with real coefficients. Our main result states that over these fields K every symmetric matrix can be orthogonally diagonalized. We shall show that this result even carries over to fields of generalized power series in infinitely many variables.

In the classical case of a real symmetrix matrix A the diagonalization is obtained by computing the characteristic polynomial of A and using the fact that the quadratic extension $\mathbf{C} = \mathbf{R}(\sqrt{-1})$ is algebraically closed. In the present case this way of reasoning fails. For, our fields $K = \mathbf{R}((t_1, \dots, t_m))$ are too far from being algebraically closed; in fact these fields admit finite extensions of any degree. Our method of proof combines two ideas and can be outlined as follows. First, write $K = \mathbf{R}((t_1, \dots, t_m)) = K_0((t))$ where $t = t_m$ and $K_0 = \mathbf{R}((t_1, \dots, t_{m-1}))$. The field $K = K_0((t))$ is complete with respect to a non-archimedian, discrete valuation. This allows us to represent a given symmetric matrix $\mathcal{A}$ with entries in K as a convergent power series $\mathcal{A} = A_0 + A_1 \cdot t + A_2 \cdot t^2 + \cdots$ with coefficients A_k in a smaller matrix ring. Secondly, we shall set up a recursive construction that produces an orthogonal transition matrix $\mathcal{U} = U_0 + U_1 \cdot t + U_2 \cdot t^2 + \cdots$ such that $\mathcal{U}^{tr}\mathcal{A}\mathcal{U}$ is decomposed into two blocks of smaller size. The proof is then finished by an easy induction.

It is a remarkable feature of this proof that it does not involve the spectrum. Indeed, the eigenvalues of $\mathcal{A}$ are obtained at the end as a by-product. Thus the proof is potentially a tool to study arithmetical properties of fields of power series.

We should like to mention that the paper has grown out of studies in the theory of orthomodular spaces. These are, by definition, vector spaces E endowed with a hermitean form Φ such that the Projection Theorem holds for (E, Φ) : every orthogonally closed linear subspace $U \subseteq E$ is a direct summand of the whole space. Classical examples are the Hilbert spaces over $\mathbf{R}$ or $\mathbf{C}$ and for a long time there were no others. Then, in 1980, numerous non-classical, infinite-dimensional orthomodular spaces were discovered. They are constructed over certain non-archimedian, complete fields; the valuations in question are of infinite rank. These new spaces carry a natural non-archimedian norm, so there is a notion of "bounded linear operator". The central question is whether a bounded, selfadjoint linear operator $T : E \to E$ always admits an orthogonal decomposition derived from its spectrum. By using the technique of reduction modulo residual spaces the task of decomposing an infinite-dimensional operator $T : E \to E$ is seen to be closely related to the problem of decomposing finite matrices over fields of power series (or of rational functions). For details we refer to [3] and [4].

1. Fields of power series. Given any field K_0 with $char(K_0) \neq 2$ we let $K = K_0((t))$ be the field of formal power series in the indeterminate t with coefficients in K_0, and we let $\varphi : K \to \mathbf{Z}$ be the usual exponential valuation. Thus for a typical $\alpha = \sum_{i \in \mathbf{Z}} a_i t^i$ in K we have $\varphi(\alpha) = min\{i \in \mathbf{Z} \mid a_i \neq 0\}$ if $\alpha \neq 0$, $\varphi(\alpha) = \infty$ if $\alpha = 0$. The valued field (K, φ) is henselian (cf. [4]). To φ there corresponds a valuation ring $R := \{\alpha \in K \mid \varphi(\alpha) \geq 0\}$ with maximal ideal $J = \{\alpha \in K \mid \varphi(\alpha) > 0\}$. The residue field $\hat{K} := R/J$ is isomorphic to K_0, thus there is a canonical epimorphism π from K onto K_0.

We shall need the following fact.

Lemma 1. *K is a purely transcendental extension of K_0.*

Proof : Suppose that $\vartheta \in K$ is algebraic over K_0. We have to show that $\vartheta \in K_0$. Let $p(X) = \sum_{i=0}^{m} a_i X^i \in K_0[X]$ be the irreducible polynomial of ϑ. Then

$$a_0 + a_1\vartheta + a_2\vartheta^2 + \cdots + a_m\vartheta^m = 0.$$

There are at least two indices $0 \le i < j \le m$ such that $\varphi(a_i\vartheta^i) = \varphi(a_j\vartheta^j)$, for otherwise the terms on the lefthand side couldn't cancel. Since $\varphi(a_i) = \varphi(a_j) = 0$ it follows that $\varphi(\vartheta^i) = \varphi(\vartheta^j)$, hence $\varphi(\vartheta) = 0$. Thus $\vartheta \in R$. Applying now the epimorphism $\pi : R \to K_0$ to the above equality and noticing that $\pi(a_i) = a_i$ for all i we obtain $a_0 + a_1\pi(\vartheta) + a_2\pi(\vartheta)^2 + \cdots + a_m\pi(\vartheta)^m = 0$, i.e. $\pi(\vartheta) \in K_0$ is a root of the polynomial $p(X)$. Since $p(X)$ is irreducible this is possible only when $m = 1$. We conclude that $\vartheta \in K_0$, as claimed.

2. Matrices over fields of power series. We consider the ring $Mat_n(K)$ of all square matrices of size $n \times n$ with entries in K along with the subring $Mat_n(K_0)$ consisting of all matrices with entries in the subfield $K_0 \subset K$. We shall denote the matrices in $Mat_n(K)$ by $\mathcal{A}, \mathcal{B}, \ldots \mathcal{U} \ldots$ and those in $Mat_n(K_0)$ by $A, B, \ldots, U \ldots$. The unit matrix is always denoted by I.

A matrix $\mathcal{A} \in Mat_n(K)$ is called *orthogonal* if its transpose $\mathcal{A}^*$ is equal to the inverse $\mathcal{A}^{-1}$, i.e. if $\mathcal{A}^*\mathcal{A} = \mathcal{A}\mathcal{A}^* = I$, We say that $\mathcal{A}$ is *diagonal* if all entries outside the main diagonal are 0, more generally, we say that $\mathcal{A}$ is $(r, n-r)$-*blockdiagonal* if it has the shape

$$\mathcal{A} = \begin{bmatrix} \mathcal{B} & 0 \\ 0 & \mathcal{C} \end{bmatrix}$$

where $\mathcal{B}$ and $\mathcal{C}$ are square matrices of size $r \times r$ and $(n-r) \times (n-r)$ respectively.

Our computations later on will rely on a representation of the elements $\mathcal{A} \in Mat_n(K)$ as a formal power series with coefficients in the (non-commutative) subring $Mat_n(K_0)$. Put

$$\varphi(\mathcal{A}) := \min\{\varphi(\alpha_{ij}) | 1 \le i, j \le n\}$$

and assume, for sake of simplicity, that $\varphi(\mathcal{A}) = 0$. Then each entry α_{ij} can be expressed as

$$\alpha_{ij} = a_{ij}^{(0)} + a_{ij}^{(1)}t + a_{ij}^{(2)}t^2 + \cdots + a_{ij}^{(m)}t^m + \cdots.$$

For $m = 0, 1, \ldots$ we collect the coefficients $a_{ij}^{(m)}$ in a matrix

$$A_m := \begin{bmatrix} a_{11}^{(m)} & \cdots & a_{1n}^{(m)} \\ \vdots & \ddots & \vdots \\ a_{n1}^{(m)} & \cdots & a_{nn}^{(m)} \end{bmatrix} \in Mat_n(K_0).$$

Then we can express $\mathcal{A}$ in the form

$$\mathcal{A} = A_0 + A_1 \cdot t + A_2 \cdot t^2 + \cdots + A_m \cdot t^m + \cdots.$$

This is the representation mentioned above.

3. The main result. Our purpose is to prove

Theorem 1: *Let $K = K_0((t))$ and $n \geq 1$. The following conditions are equivalent:*

(a) Every symmetric matrix $A \in Mat_n(K_0)$ can be diagonalized by means of an orthogonal matrix $U \in Mat_n(K_0)$.

(b) Every symmetric matrix $\mathcal{A} \in Mat_n(K)$ can be diagonalized by means of an orthogonal matrix $\mathcal{U} \in Mat_n(K)$.

The proof will be divided into several steps. We begin with the easy part.

Proof of the implication (b) $\Rightarrow$ (a): Suppose that A in $Mat_n(K_0)$ is symmetric. By hypothesis (b) there exists an orthogonal matrix $\mathcal{U} \in Mat_n(K)$ such that

$$\mathcal{D} := \mathcal{U}^* A \mathcal{U} = \begin{bmatrix} \lambda_{11} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_{nn} \end{bmatrix}$$

Here the diagonal entries λ_{ii} are the eigenvalues of the matrix A, that is, the roots of the characteristic polynomial $p_A(X) = \det(X \cdot I - A)$. Since the coefficients of $p_A(X)$ belong to K_0, the λ_{ii}'s are algebraic over K_0. By Lemma 1 we conclude that $\lambda_{11}, \cdots, \lambda_{nn} \in K_0$.

Consider an eigenvalue λ_{ii} and let m_i be its algebraic multiplicity. Then λ_{ii} is repeated m_i times in $\mathcal{D}$ and consequently

$$A - \lambda_{ii} I = \mathcal{U}(\mathcal{D} - \lambda_{ii} I)\mathcal{U}^*$$

has rank $n - m_i$ over K. But the matrix $A - \lambda_{ii} \cdot I$ is in $Mat_n(K_0)$, so its rank over K_0 is the same as its rank over K as can be easily seen by applying the Gaussian algorithm. Consequently λ_{ii} has geometric multiplicity m_i. Thus for each eigenvalue of A the algebraic and the geometric multiplicity coincide. This entails that there exists an orthogonal matrix U in $Mat_n(K_0)$ such that $U^* A U$ is diagonal, as asserted.

The substantial part is the converse implication to which we now turn.

4. Proof of the implication (a) $\Rightarrow$ (b). In this section we assume throughout that K_0 is a coefficient field satisfying condition (a). Let there be given a symmetric matrix $\mathcal{A} \neq 0$ in $Mat_n(K)$.

4.1. Multiplying $\mathcal{A}$ by a suitable power of t we may assume that $\varphi(\mathcal{A}) = 0$. Then $\mathcal{A}$ can be expressed as a power series

$$\mathcal{A} = A_0 + A_1 \cdot t + + \cdots + A_m \cdot t^m + \cdots.$$

Notice that all the A_m's are symmetric. We first reduce the general case to the special one in which the initial coefficient matrix A_0 of $\mathcal{A}$ satisfies the condition

(1) A_0 is diagonal but not a multiple of the unit matrix I.

In fact, if all the A_m's ($m = 0, 1, \cdots$) are multiples of I then so is $\mathcal{A}$ and there is nothing to prove. Otherwise let $m := \min\{k \in \mathbb{N}_0 |\ A_k$ is not a multiple of $I\}$. Then $\sum_{k=0}^{m-1} A_k \cdot t^k = \lambda \cdot I$ for some $\lambda \in K$. Put

$$\mathcal{B} := t^{-m} \cdot (\mathcal{A} - \lambda I) = t^{-m} \cdot \left(\mathcal{A} - \sum_{k=0}^{m-1} A_k \cdot t^k \right) = A_m + A_{m+1} \cdot t + \cdots$$

Since A_m is symmetric there exists, by (a), an orthogonal matrix $V \in Mat_n(K_0)$ such that $D := V^* A_m V$ is diagonal. Now look at

$$\mathcal{C} := V^* \mathcal{B} V = (V^* A_m V) + (V^* A_{m+1} V) \cdot t + (V^* A_{m+2} V) \cdot t^2 + \cdots$$

The expansion of $\mathcal{C}$ starts with a coefficient matrix $C_0 = V^* A_m V$ that is diagonal but not a multiple of I. Moreover, if we succeed in finding an orthogonal matrix $\mathcal{U} \in Mat_n(K)$ which diagonalizes $\mathcal{C}$ then $V \cdot \mathcal{U}$ will provide a diagonalization of $\mathcal{B}$ and therefore also of $\mathcal{A} = t^m \cdot \mathcal{B} + \lambda I$. Hence we may assume from the start that the initial coefficient of $\mathcal{A}$ satisfies (1).

We should like to point out that it is only in the above preliminary step where the hypothesis (a) is actually needed. However, the condition (a) can hardly be replaced by an assumption on the initial matrix A_0 because it will be used repeatedly in the inductive argument at the end (see section 4.6).

4.2. The idea is to construct recursively an orthogonal transition matrix

$$\mathcal{U} = U_0 + U_1 \cdot t + U_2 \cdot t^2 + \cdots + U_m \cdot t^m + \cdots$$

that diagonalizes $\mathcal{A}$. When trying to do so it turns out that the recursive computation of $U_0, U_1, U_2, \cdots$ can be carried out provided the diagonal entries of A_0 are pairwise different. However, when some diagonal entries of A_0 are repeated there arise serious troubles. The underlying geometric reason for these obstacles in that in the second case the given matrix $\mathcal{A}$ may have multiple eigenvalues and consequently $\mathcal{U}$ is not uniquely determined by $\mathcal{A}$. The way out of the difficulties is as follows: we shall not attempt to put the given matrix $\mathcal{A}$ into diagonal form at once, but we will first decompose $\mathcal{A}$ into blocks the sizes of which are determined by the multiplicities ocurring in A_0.The clue is given by the following result.

Lemma 2: *Let $\mathcal{A} = A_0 + A_1 \cdot t + \cdots + A_m \cdot t^m + \cdots \in Mat_n(K)$ be symmetric. Assume that A_0 satisfies (1). Then there exist an integer r with $1 \le r \le n-1$ along with an orthogonal matrix $\mathcal{U} \in Mat_n(K)$ such that $\mathcal{U}^* \mathcal{A} \mathcal{U}$ is $(r, n-r)$-blockdiagonal.*

The proof will be divided into several steps and will cover the next three subsections.

4.3. Write

$$A_0 = \begin{bmatrix} a_{11} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_{nn} \end{bmatrix}$$

Since A_0 is not a multiple of I the multiplicity r of a_{11} is strictly less than n. After conjugating by some permutation matrix we may assume that

$$a_{ii} = a_{11} \quad \text{for} \quad 1 \le i \le r, \qquad a_{ii} \ne a_{11} \quad \text{for} \quad r+1 \le i \le n.$$

The multiplicity r is the number r referred to in the statement of Lemma 2.

4.4. We shall construct recursively matrices $U_0, U_1, \ldots$ in $Mat_n(K)$ such that

$$\mathcal{U} = U_0 + U_1 \cdot t + U_2 \cdot t^2 + \cdots + U_m \cdot t^m + \cdots$$

satisfies both

$$\mathcal{U}^* \mathcal{U} = I,$$

and

$$\mathcal{U}^* \mathcal{A} \mathcal{U} \quad \text{is } (r, n-r)\text{-blockdiagonal.}$$

The first task is to express the above two conditions in terms of the U_k's. Multiplying out the series $\mathcal{U} = U_0 + U_1 \cdot t + U_2 \cdot t^2 + \cdots$ and $\mathcal{U}^* = U_0^* + U_1^* \cdot t + U_2^* \cdot t^2 + \cdots$ we find

$$\mathcal{U}^* \mathcal{U} = U_0^* U_0 + (U_1^* U_0 + U_0^* U_1) \cdot t + \cdots + \left(\sum_{i+j=k} U_i^* U_j \right) \cdot t^k + \cdots$$

Hence the condition that $\mathcal{U}^* \mathcal{U} = I$ is satisfied if and only if

$$U_0^* U_0 = I, \tag{2}$$

and for all $k \ge 1$ we have

$$\sum_{i+j=k} U_i^* U_j = 0. \tag{3}$$

Next, multiplication of the series for $\mathcal{U}^*$, $\mathcal{A}$ and $\mathcal{U}$ yields

$$\mathcal{U}^* \mathcal{A} \mathcal{U} = V_0 + V_1 \cdot t + V_2 \cdot t^2 + \cdots + V_k \cdot t^k + \cdots$$

where

$$V_0 := U_0^* A_0 U_0, \qquad V_k = \sum_{i+j+h=k} U_i^* A_j U_h. \tag{4}$$

It follows that $\mathcal{U}^* \mathcal{A} \mathcal{U}$ is $(r, n-r)$ -blockdiagonal if and only all the matrices V_k $(k = 0, 1, \ldots)$ given by (4) are $(r, n-r)$ -blockdiagonal.

4.5. We start the recursive construction by putting

$$U_0 := I.$$

Then (2) is satisfied, and the matrix V_0 given by (4) is trivially blockdiagonal since A_0 is diagonal.

Assume that we have already constructed $U_0, \dots, U_{m-1}$ such that (3) holds for $1 \leq k \leq m-1$ and V_k is block-diagonal for $0 \leq k \leq m-1$. Consider then (3) with $k = m$. Since $U_0 = I$ we can rewrite this condition as

$$U_m^* + U_m + \sum_{\substack{i+j=m \\ i \neq m, j \neq m}} U_i^* U_j = 0.$$

Now $S_m := \sum_{\substack{i+j=m \\ i \neq m, j \neq m}} U_i^* U_j$ is symmetric. Hence (3) holds if and only if U_m has the shape

$$U_m = -\frac{1}{2} S_m + Q_m \tag{5}$$

where Q_m is any antisymmetric matrix. Since S_m is determined by the matrices $U_0, \dots, U_{m-1}$ already constructed, the task is to choose Q_m in such a way that the resulting matrix V_m given by (4) is block-diagonal.

Separating in (4) the two summands corresponding to $(i,j,h) = (m,0,0)$ and $(i,j,h) = (0,0,m)$ we obtain

$$V_m = U_m^* A_0 + A_0 U_m + \sum_{\substack{i+j+h=m \\ i \neq m, h \neq m}} U_i^* A_j U_h.$$

Substituting (5) into the above expression we obtain

$$V_m = -Q_m A_0 + A_0 Q_m + T_m \tag{6}$$

where

$$T_m = -\frac{1}{2}(S_m A_0 + A_0 S_m) + \sum_{\substack{i+j+h=m \\ i \neq m, h \neq m}} U_i^* A_j U_h.$$

Since S_m and all the A_k's are symmetric it follows that T_m is symmetric. Notice that T_m is expressed in terms of matrices already determined.

Write

$$V_m = \begin{bmatrix} v_{11} & \dots & v_{1n} \\ \vdots & \ddots & \vdots \\ v_{n1} & \dots & v_{nn} \end{bmatrix}, \quad Q_m = \begin{bmatrix} q_{11} & \dots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \dots & q_{nn} \end{bmatrix}, \quad T_m = \begin{bmatrix} t_{11} & \dots & t_{1n} \\ \vdots & \ddots & \vdots \\ t_{n1} & \dots & t_{nn} \end{bmatrix}.$$

Computing the matrix products in (6) and taking into account that

$$A_0 = \begin{bmatrix} a_{11} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_{nn} \end{bmatrix}$$

is diagonal we find

$$v_{ij} = -q_{ij}a_{jj} + a_{ii}q_{ij} + t_{ij} = -q_{ij}\left(a_{jj} - a_{ii}\right) + t_{ij}$$

for all $1 \le i, j \le n$. Consider now a pair (i, j) outside the blocks, i.e. either $i \in \{1, \dots, r\}$ and $j \in \{r+1, \dots, n\}$ or $i \in \{r+1, \dots, n\}$ and $j \in \{1, \dots, r\}$. Then $a_{ii} \neq a_{jj}$ by definition of r (cf. section 4.3.) Thus if we put

$$q_{ij} := \frac{t_{ij}}{a_{jj} - a_{ii}}$$

then $v_{ij} = 0$, as required. Notice also that $q_{ij} = -q_{ji}$ for these pairs (i, j). If i, j are both in $\{1, \dots, r\}$ or both in $\{r + 1, \dots, n\}$ then we choose q_{ij} arbitrarily but such that $q_{ij} = -q_{ji}$.

The matrix Q_m thus obtained is antisymmetric. Moreover, if we put $U_m := -\frac{1}{2}\, S_m + Q_m$ then the matrix V_m given by (4) is $(r, n-r)$-blockdiagonal. This completes the recursive construction.

By construction the matrix $\mathcal{U} = U_0 + U_1 \cdot t + U_2 \cdot t^2 + \cdots$ is orthogonal and $\mathcal{U}^*\mathcal{A}\mathcal{U}$ is block-diagonal. The proof of Lemma 2 is complete.

4.6. We can now finish the proof of Theorem 1 by an easy induction on the size n. The case $n = 1$ is trivial, so assume $n > 1$. Let there be given a symmetric matrix $\mathcal{A}$ in $Mat_n(K)$ with initial coefficient A_0 satisfying (1). By Lemma 2 there exists a natural number $r < n$ and an orthogonal matrix $\mathcal{U}$ in $Mat_n(K)$ such that

$$\mathcal{U}^*\mathcal{A}\mathcal{U} = \begin{bmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{bmatrix}$$

where $\mathcal{A}_1 \in Mat_r(K), \mathcal{A}_2 \in Mat_{n-r}(K)$. Clearly $\mathcal{A}_1$ and $\mathcal{A}_2$ are symmetric. By induction, there exist orthogonal matrices $\mathcal{V}_1 \in Mat_r(K)$ and $\mathcal{V}_2 \in Mat_{n-r}(K)$ such that $\mathcal{V}_1^*\,\mathcal{A}_1\,\mathcal{V}_1$ and $\mathcal{V}_2^*\,\mathcal{A}_2\,\mathcal{V}_2$ are diagonal. Put

$$\mathcal{W} := \begin{bmatrix} \mathcal{V}_1 & 0 \\ 0 & \mathcal{V}_2 \end{bmatrix}.$$

Then $\mathcal{U}\,\mathcal{W}$ is orthogonal, and

$$(\mathcal{U}\mathcal{W})^*\,\mathcal{A}\,(\mathcal{U}\mathcal{W}) = \begin{bmatrix} \mathcal{V}_1^*\mathcal{A}_1\mathcal{V}_1 & 0 \\ 0 & \mathcal{V}_2^*\mathcal{A}_2\mathcal{V}_2 \end{bmatrix}$$

is diagonal. The proof is complete.

5. Applications. The classical Spectral Theorem (for finite dimensions) states that every symmetric matrix can be orthogonally diagonalized over the field **R** of reals. Applying Theorem 1 repeatedly we deduce the following result.

Theorem 2: *Let $m \geq 0$ and let*

$$K := \mathbf{R}((t_1, \ldots, t_m)) = \mathbf{R}((t_1))((t_2)) \cdots ((t_m))$$

be the field of formal power series in m indeterminates with real coefficients. Then every symmetric matrix can be orthogonally diagonalized over K .

Proof:. By induction on m. The case $m = 0$ is the classical one, and the induction step is just the assertion "(a) $\Rightarrow$ (b)" of Thm. 1.

Corollary: *Let $K := \mathbf{R}((t_1, \ldots, t_m))$. If the matrix $\mathcal{A} \in Mat_n(K)$ is symmetric then its characteristic polynomial*

$$p_{\mathcal{A}}(X) = det(X \cdot I - \mathcal{A}) \in K[X]$$

decomposes into linear factors over K.

The above Spectral Theorem can even be generalized to fields of power series in infinitely many variables as we shall now show. We start with a direct sum

$$\Gamma := \mathbf{Z} \oplus \mathbf{Z} \oplus \cdots \oplus \mathbf{Z} \oplus \cdots$$

of infinitely many copies of the group of integers. Γ is an abelian, additive group under componentwise operations. We order Γ antilexicographically.

Next we form the field

$$K := \mathbf{R}((\Gamma))$$

of generalized power series with exponents in Γ and real coefficients. K can be described as the field of all functions $\xi : \Gamma \to \mathbf{R}$ for which the support

$$supp(\xi) := \min\{\gamma \in \Gamma | \xi(\gamma) \neq 0\}$$

is well-ordered. The operations in K are the obvious ones: $(\xi + \eta)(\gamma) := \xi(\gamma) + \eta(\gamma)$ and $(\xi \cdot \eta)(\gamma) := \sum_{\delta+\delta'=\gamma} \xi(\delta) \cdot \eta(\delta')$. There is a natural valuation

$$\varphi : K \to \Gamma \cup \{\infty\}, \quad \text{given by} \quad \varphi(\xi) := \min \text{supp}(\xi).$$

The valued field (K, φ) is complete and henselian; for details we refer to [5] or [6].

Now we can state

Theorem 3: *Over the field $K := \mathbf{R}((\Gamma))$ every (finite) symmetric matrix can be orthogonally diagonalized.*

Outline of the proof For $m = 0, 1, 2, \ldots$ we define the subgroup $\Delta_m \subset \Gamma$ by

$$\Delta_m := \underbrace{\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}}_{m \text{ times}} \oplus \{0\} \oplus \{0\} \oplus \cdots.$$

The Δ_m's are isolated (or convex) subgroups of Γ. To each Δ_m there corresponds a valuation ring $R_m := \{\xi \in K | \varphi(\xi) \geq \delta \quad \text{for some} \quad \delta \in \Delta_m\}$ with maximal ideal $J_m := \{\xi \in K | \varphi(\xi) > \delta \quad \text{for all} \quad \delta \in \Delta_m\}$ and residue field $\hat{K}_m := R_m / J_m$. It is readily verified that $\hat{K}_m \cong \mathbf{R}((t_1, \ldots, t_m))$. In particular, each residue field $\hat{K}_m$ can be considered as a subfield of K, moreover there is a canonical epimorphism $\pi_m : R_m \to \mathbf{R}((t_1, \ldots, t_m))$.

Now let there be given a symmetric matrix

$$\mathcal{A} = \begin{bmatrix} \alpha_{11} & \ldots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \ldots & \alpha_{nn} \end{bmatrix} \in Mat_n(K).$$

We may suppose that $\varphi(\alpha_{ij}) \geq 0$ for all i, j, thus $\alpha_{ij} \in R_m$ for $m = 0, 1, 2, \ldots$. For each $m \in \mathbf{N}_0$ we form the reduced matrix

$$\hat{\mathcal{A}}_m := \pi_m(\mathcal{A}) = \begin{bmatrix} \pi_m(\alpha_{11}) & \ldots & \pi_m(\alpha_{1n}) \\ \vdots & \ddots & \vdots \\ \pi_m(\alpha_{n1}) & \ldots & \pi_m(\alpha_{nn}) \end{bmatrix}.$$

Applying Theorem 2 we obtain, for each $m \in \mathbf{N}_0$, an orthogonal matrix $\hat{\mathcal{U}}_m \in Mat_n(\hat{K}_m) \subset Mat_n(K)$ such that $\hat{\mathcal{U}}_m^* \hat{\mathcal{A}}_m \hat{\mathcal{U}}_m$ is diagonal.

The point is to show that the orthogonal matrices $\hat{\mathcal{U}}_m$ can be chosen in such a way that

$$\pi_m\left(\hat{\mathcal{U}}_{m+1}\right) = \hat{\mathcal{U}}_m. \tag{7}$$

If all the eigenvalues of the given matrix $\mathcal{A}$ are simple then (7) is automatically satisfied as is shown by a routine verification. In the case where $\mathcal{A}$ has multiple eigenvalues then the orthogonal matrices $\hat{\mathcal{U}}_m$ are not unique and one has to choose a suitable basis in each eigenspace.

Since K is complete we easily deduce from (7) that the sequence $\left(\hat{\mathcal{U}}_m\right)_{m \in \mathbf{N}_0}$ converges in the valuation topology to some matrix $\mathcal{U} \in Mat_m(K)$; by continuity we conclude that $\mathcal{U}$ is orthogonal and $\mathcal{U}^* \mathcal{A} \mathcal{U}$ is diagonal. This completes the proof.

Acknowledgement. The authors have been partially supported by FONDECYT, Proyecto No. 1930513.

References.

[1] W.A. Adkins, *Normal Matrices over Hermitian Discrete Valuation Rings*, Linear Algebra and its Applications 157 (1991), 165-174.

[2] B. Diarra, *Remarque sur les matrices orthogonales (resp. symétriques) à coefficients p-adiques*, Ann. Sci.Univ. Blaise Pascal, Ser. Math., Fasc. 26 (1990), 31-50.

[3] H. Gross and U.M. Künzi, *On a Class of Orthomodular Quadratic Spaces*, L'Enseignement Mathématique, 31 (1985), 187-212.

[4] H. Keller and H. Ochsenius, *Algebras of Bounded Operators on Non-classical Orthomodular Spaces*, Int. Journal of Theor. Physics, Vol. 33, No. 1 (1994), 1-11.

[5] P. Ribenboim, *Théorie des Valuations*, Les Presses de l'Université de Montreal, 1964.

[6] O.F.G. Schilling, *The Theory of Valuations*, Amer. Math. Soc. Surveys, Providence, 1950.

Hans A. Keller
Technikum Luzern,
CH-6048 Horw,
Switzerland.

Herminia Ochsenius A.
Facultad de Matemáticas,
Universidad Católica de Chile, Casilla 306,
Correo 22,
Santiago de Chile
Chile.