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# SEMI-ORTHOGONALITY OF A CLASS OF THE GAUSS' HYPERGEOMETRIC POLYNOMIALS

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**Abstract :** We establish semi-orthogonality for a class of Gauss' hypergeometric polynomials with an elementary weight function and employ it to generate a theory concerning the finite series expansion involving the Gauss' hypergeometric polynomials.

**Key Words :** Gauss' hypergeometric polynomials, Semi-orthogonality, Saalschutz's theorem, Fox's H-function.

**AMS (MOS) :** Subject classification : 33 C 25, 33 C 40

## 1. INTRODUCTION

Gauss' hypergeometric polynomials $x^n {}_2F_1\left(-n, b\ ;\ c\ ; -\frac{1}{x}\right)$ lead to the generalization of the classical polynomials associated with the names of Jacobi, Gegenbauer, Legendre and Chebyshev. Therefore, they appear to represent a very important class of polynomials. In view of this, it seems worthwhile to investigate the matter of their orthogonality and other important aspects.

In this paper, we evaluate an integral involving the Gauss' hypergeometric polynomials and employ it to establish, what we call, semi-orthogonality of the Gauss' hypergeometric polynomials. We further apply this property to develop a theory regarding the finite series expansion involving the Gauss' hypergeometric polynomials. We also employ the integral to evaluate an integral for Fox's H-function [5]. Finally, we present some particular cases of our integral and semi-orthogonality property, some of which are known and others are believed to be new.

We recall that the Jacobi polynomials constitute an important, and a rather wide class of hypergeometric polynomials, from which Chebyshev, Legendre, and Gegenbauer polynomials follow as special cases. Their orthogonality property, with the non-negative weight function $(1-x)^a\ (1+x)^b$ on the interval [-1,1], for $a > -1, b > -1$, is usually derived by the use of the associated differential equation and Rodrigues' formula. In this paper, we introduce a direct method of proof, which is much simpler and more elegant. Our method could be employed to establish orthogonalities of the Jacobi polynomials and other related hypergeometric polynomials.

The class of Gauss' hypergeometric polynomials is defined and represented as follows :

$$x^n {}_2F_1(-n, b\ ; c\ ;\ \frac{1}{x}) = \sum_{r=0}^{n} (-1)^r \frac{(-n)_r (b)_r}{(c)_r r!} x^{n-r}, n = 0, 1, 2, \ldots \tag{1.1}$$

provided $c$ is not zero or a negative integer.

For sake of brevity, the class defined by (1.1) will be denoted by $A_n^{(b,c)}(x)$.

The following integral is required in the proof :

$$\int_0^\infty x^h (1+x)^{b-c-n} A_n^{(b,c)}(x) dx$$
$$= \frac{\Gamma(1+h)\Gamma(c-b-h-1)(1+b+h)_n}{\Gamma(c-b)(c)_n} \tag{1.2}$$

where $Re(c-b-h) > 1,\ Re(h) > -1,\ Re(h+b) > -1.$

**Proof :** The integral (1.2) is established by expressing the hypergeometric polynomials in the integrand in terms of their series representation (1.1), interchanging the order of integration and summation, evaluating the resulting integral with the help of the integral [4, p. 233,(8) ] :

$$\int_0^\infty x^{v-1}(1+x)^{-w}dx = \frac{\Gamma(v)\Gamma(w-v)}{\Gamma(w)}, Re(w) > Re(v) > 0 \tag{1.3}$$

and simplifying by using the following form of the formula [2, p. 3,(4)] :

$$\Gamma(1+a-n) = \frac{(-1)^n\Gamma(1+a)}{(-a)_n} \tag{1.4}$$

whence we get

$$\frac{\Gamma(h+n+1)\,\Gamma(c-b-h-1)}{\Gamma(c-b+n)}\,{}_3F_2\left[\begin{matrix}-n, b, c-b-h-1; 1\\ c, -h-n\end{matrix}\right]. \tag{1.5}$$

It can easily be verified that the generalized hypergeometric series (1.5) is Saalschutzian. Therefore, applying the Saalschutz's theorem [2, p. 188, (3)] :

$${}_3F_2\left[\begin{matrix}-n, a, b & ; & 1\\ 1-c, -c+a+b-n & & \end{matrix}\right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n} \tag{1.6}$$

and simplifying, the right hand side of (1.2) is obtained.

## 2. SEMI-ORTHOGONALITY

The semi-orthogonality property to be established is

$$\int_0^\infty x^{-1-b-m}(1+x)^{b-c-n}A_m^{(b,c)}(x)\; A_n^{(b,c)}\,(x)dx \tag{2.1}$$

$$= 0, \quad \text{if } m < n \tag{2.1a}$$

$$= \frac{(b)_n n!\;\Gamma(c)\Gamma(-b)\Gamma(1+b)}{(c)_n\Gamma(1+b+n)\Gamma(c-b)}, \;\text{ if } m = n \tag{2.1b}$$

where $Re(c) > 0$, $(Re(b) < -m,\; Re(b) > -n \Rightarrow$ when $m = n,\; b \neq -n)$.

**Proof :** To prove (2.1), we write its left hand side in the form

$$\sum_{r=0}^{m}(-1)^r\frac{(-m)_r(b)_r}{r!\;(c)_r}\int_0^\infty x^{-1-b-r}(1+x)^{b-c-n}\;A_n^{(b,c)}\;(x)dx. \tag{2.2}$$

On evaluating the integral in (2.2) by using (1.2), we have

$$\sum_{r=0}^{m}(-1)^r\frac{(-m)_r(b)_r\,\Gamma(-b-r)\,\Gamma(c+r)(-r)_n}{r!\,(c)_r\,\Gamma(c-b)(c)_n}. \tag{2.3}$$

If $r < n$, the numerator of (2.3) vanishes, and since $r$ runs from 0 to $m$, if follows that (2.3) also vanishes when $m < n$. Therefore, it is clear that for $m < n$, all terms of (2.3) vanish, which proves (2.1a).

When $m = n$, on using the standard result $(-n)_n = (-1)^n\,n!$ and simplifying, (2.1b) follows from (2.3).

**Note :** The integral (2.1) exists for $m = n+1,\ n+2,\ n+3,\ \ldots,$ and yields a series of interesting results

### 3. PARTICULAR CASES OF THE INTEGRAL

(i) In (1.2), substituting $x = \dfrac{1-y}{1+y}$, we have

$$\int_{-1}^{1}(1-y)^{h+n}(1+y)^{c-b-h-2}A_n^{(b,c)}\left(\frac{1-y}{1+y}\right)dy$$
$$= \frac{2^{c-b+n-1}\,\Gamma(1+h)\,\Gamma(c-b-h-1)(1+b+h)_n}{\Gamma(c-b)(c)_n} \tag{3.1}$$

where the restrictions on the constants are the same as in (1.2).

(ii) In (1.2), substituting $x = \dfrac{1+y}{1-y}$, we get

$$\int_{-1}^{1}(1+y)^{h+n}(1-y)^{c-b-h-2}A_n^{(b,c)}\left(\frac{1+y}{1-y}\right)dy$$
$$= \frac{2^{c-b+n-1}\,\Gamma(1+h)\,\Gamma(c-b-h-1)(1+b+h)_n}{\Gamma(c-b)(c)_n} \tag{3.2}$$

where the restrictions on the constants are the same as in (1.2).

(iii) In (1.2), substituting $x = \dfrac{2}{y-1}$, we obtain

$$\int_{1}^{\infty}(y-1)^{c-b-h-2}(y+1)^{b-c-n}A_n^{(b,c)}\left(\frac{2}{1-y}\right)dy$$

$$= \frac{\Gamma(1+h)\,\Gamma(c-b-h-1)(1+b+h)_n}{2^{h+n+1}\,\Gamma(c-b)(c)_n} \tag{3.3}$$

where the restrictions on the constants are the same as in (1.2).

(iv) In (3.1), putting $b = -n-a$ and $c = 1+b$, and using the relation [3, p. 170,(16)] :

$$P_n^{(a,b)}(x) = \frac{(1+b)_n}{n!}\left(\frac{x-1}{2}\right)^n {}_2F_1\left(\begin{matrix}-n, -n-a & ; & \frac{x+1}{x-1}\\ 1+b & & \end{matrix}\right), \tag{3.4}$$

we obtain the following integral involving the Jacobi polynomials :

$$\int_{-1}^{1} (1-y)^h (1+y)^{a+b+n-h-1}\, P_n^{(a,b)}(y)dy$$

$$= \frac{(-1)^n 2^{a+b+n}\Gamma(1+h)\Gamma(a+b+n-h)(1-a+h-n)_n}{n!\,\Gamma(1+a+b+n)} \tag{3.5}$$

where $Re(h) > -1,\ Re(a+b-h) > 0,\ Re(h-a) > -1.$

(v) In (3.2) , setting $-n-b$ for $b$ and $1+a$ for $c$, and using the relation [3, p. 170,(16)] :

$$P_n^{(a,b)}(x) = \frac{(1+a)_n}{n!}\left(\frac{1+x}{2}\right)^n {}_2F_1\left(\begin{matrix}-n, -n-b & ; & \frac{x-1}{x+1}\\ 1+a & & \end{matrix}\right), \tag{3.6}$$

we obtain the following integral involving the Jacobi polynomials :

$$\int_{-1}^{1} (1+y)^h (1-y)^{a+b+n-h-1}\, P_n^{(a,b)}(y)dy$$

$$= \frac{2^{a+b+n}\,\Gamma(1+h)\,\Gamma(a+b+n-h)(1-b+h-n)_n}{n!\,\Gamma(1+a+b+n)} \tag{3.7}$$

where $Re(h) > -1,\ Re(a+b-h) > 0,\ Re(h-b) > -1.$

(vi) In (3.3) , setting $n+a+b+1$ for $b$ and $1+a$ for $c$, and using the relation [3, p. 170,(16)] :

$$P_n^{(a,b)}(x) = \frac{(1+a)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, n+a+b+1 & ; & \frac{1-x}{2}\\ 1+a & & \end{matrix}\right), \tag{3.8}$$

we get the following integral involving the Jacobi polynomials :

$$\int_{1}^{\infty} (y-1)^{-2-b-h-n}(y+1)^b\, P_n^{(a,b)}(y)dy$$

$$= \frac{\Gamma(1+h)\,\Gamma(-1-a-b-n)(2+a+b+n+h)_n}{2^{h+n+1}\,n!\,\Gamma(-b-n)} \tag{3.9}$$

where $Re(h) > -1,\ Re(a+b) < -1,\ Re(a+b+h) > -2.$

**Note :** The integrals (3.5), (3.7) and (3.9) appear to be new and may be employed to establish new orthogonalities for the Jacobi polynomials.

## 4. PARTICULAR CASES OF THE SEMI-ORTHOGONALITY

(i) In (2.1), substituting $x = \frac{1-y}{1+y}$, we have

$$\int_{-1}^{1} (1-y)^{-1-b-n}(1+y)^{c-1}\, A_m^{(b,c)}\left(\frac{1-y}{1+y}\right) A_n^{(b,c)}\left(\frac{1-y}{1+y}\right) dy \tag{4.1}$$

$$= 0\,, \quad \text{if } m < n \tag{4.1a}$$

$$= \frac{2^{c-b+n-1}(b)_n n!\, \Gamma(c)\, \Gamma(-b)\, \Gamma(1+b)}{(c)_n\, \Gamma(1+b+n)\, \Gamma(c-b)}, \text{ if } m = n \tag{4.1b}$$

where the restrictions on the constants are the same as in (2.1).

(ii) In (2.1), substituting $x = \frac{1+y}{1-y}$, we obtain

$$\int_{1}^{\infty} (1+y)^{-1-b-n}(1-y)^{c-1}\, A_m^{(b,c)}\left(\frac{1+y}{1-y}\right) A_n^{(b,c)}\left(\frac{1+y}{1-y}\right) dy \tag{4.2}$$

$$= 0\,, \quad \text{if } m < n \tag{4.2a}$$

$$= \frac{2^{c-b+n-1}(b)_n n!\, \Gamma(c)\, \Gamma(-b)\, \Gamma(1+b)}{(c)_n\, \Gamma(1+b+n)\, \Gamma(c-b)}, \text{ if } m = n \tag{4.2b}$$

where the restrictions on the constants are the same as in (2.1).

(iii) In (2.1), putting $x = \frac{2}{y-1}$, we get

$$\int_{1}^{\infty} (y-1)^{c-1}(y-1)^{b-c-n}\, A_m^{(b,c)}\left(\frac{2}{y-1}\right) A_n^{(b,c)}\left(\frac{2}{y-1}\right) dy \tag{4.3}$$

$$= 0\,, \quad \text{if } m < n \tag{4.3a}$$

$$= \frac{2^{b-n}(b)_n n!\, \Gamma(c)\, \Gamma(-b)\, \Gamma(1+b)}{(c)_n\, \Gamma(1+b+n)\, \Gamma(c-b)}, \text{ if } m = n \tag{4.3b}$$

where the restrictions on the constants are the same as in (2.1).

(iv) In (4.1), putting $m = n, b = -n - a$, and $c = 1 + b$, and using (3.4), we obtain a known result [4, p. 285,(6) ].

(v) In (4.2), putting $m = n$, $b = -n - b$ and $c = 1 + a$ and using (3.6), we have

$$\int_{-1}^{1} (1-y)^{a}(1+y)^{b-1}\left[P_n^{(a,b)}(y)\right]^2 dy$$

$$= \frac{2^{a+b}\, \Gamma(1+a+n)\, \Gamma(1+b+n)}{n!\; b\; \Gamma(1+a+b+n)} \tag{4.4}$$

where $Re(a) > -1$, $Re(b) > 0$.

In (4.4), interchanging $a$ and $b$, substituting $-y$ for $y$ and using the relation $P_n^{(a,b)}(x) = (-1)^n\ P_n^{(a,b)}(-x)$, we again obtain a known result [4, p. 285,(6)].

(vi) In (4.3), putting $m = n$, $b = n + a + b + 1$, and $c = 1 + a$ and using the relation (3.8), we obtain a known result [1, p. 3, (2.1)] :

$$\int_1^{\infty} (y-1)^a (y+1)^b \left[P_n^{(a,b)}(y)\right]^2 dy$$
$$= \frac{2^{a+b+1}\,\Gamma(1+a+n)\,\Gamma(1+b+n)\,\Gamma(-1-a-b)\,\Gamma(2+a+b)}{n!\,(1+a+b+2n)\,\Gamma(1+a+b+n)\,\Gamma(-b)\,\Gamma(1+b)} \tag{4.5}$$

where $Re(a) > -1,\ Re(b) > -1$.

## 5. FINITE SERIES EXPANSION INVOLVING THE GAUSS' HYPERGEOMETRIC POLYNOMIALS

Based on the results (2.1a) and (2.1b), we can generate a theory concerning the expansion of arbitrary polynomials, or functions in general, in a finite series expansion of the Gauss' hypergeometric polynomials. If $F(x)$ is a suitable function defined for all $x$, we consider its expansion of the general form :

$$F(x) = \sum_{m=0}^{n} C_m x^{-m} A_m^{(b,c)}(x),\ 0 < x < \infty,\ m \le n \tag{5.1}$$

where the coefficients $C_m$ are given by

$$C_m = \frac{(c)_m\,\Gamma(1+b+m)\,\Gamma(c-b)}{(b)_m m!\,\Gamma(c)\,\Gamma(-b)\,\Gamma(1+b)} \int_0^{\infty} F(x)\ x^{-1-b}(1+x)^{b-c-m}\ A_m^{(b,c)}(x)dx \tag{5.2}$$

with the restrictions on the constants the same as in (2.1).

## 6. INTEGRAL INVOLVING FOX'S H-FUNCTION AND THE GAUSS' HYPERGEOMETRIC POLYNOMIALS

The integral to be established is

$$\int_0^\infty x^h(1+x)^{b-c-n}\, A_n^{(b,c)}(x)\, H_{p,q}^{u,v}\left[zx^k\left|{}_{(b_p,f_q)}^{(a_p,e_p)}\right.\right]dx$$

$$=\frac{1}{\Gamma(c-b)\,(c)_n}H_{p+2,q+2}^{u+1,v+2}\left[z\left|{}_{(c-b-h-1,k),(b_q,f_q),(-b-h,k)}^{(-h,k),(-b-h-n,k),(a_p,e_p)}\right.\right] \tag{6.1}$$

where $k$ is a positive number, and

$$\sum_{j=1}^{p} e_j-\sum_{j=1}^{q} f_j\geq 0,\ \sum_{j=1}^{v} e_j-\sum_{j=v+1}^{p} e_j+\sum_{j=1}^{u} f_j-\sum_{j=u+1}^{p} f_j\equiv B>0, |argz|<{}^1/{}_2B\pi,$$

$Re(h)+k\min\limits_{1\leq j\leq u}[Re(a_j-1)/e_j]<-1$, $Re(h+b)+k\min\limits_{1\leq j\leq u}[Re(a_j-1)/e_j]>-1$, $Re(b+h-c)+k\max\limits_{1\leq j\leq v}[Re(a_j-1)/e_j]<-1$, and $(a_p,e_p)$ represent the set of parameters $(a_1,e_1),\ldots,(a_p,e_p)$.

**Proof :** The integral (6.1) is obtained by expressing the $H$-function in the integrand as a Mellin-Barnes type integral [6, p. 2(1.1.1)], changing the order of integration, evaluating the inner-integral with the help of (1.2), and using the standard method to evaluate such integrals [6].

**Note :** Since on specializing the parameters, the $H$-function may be reduced to almost all special functions appearing in pure and applied mathematics [6, pp.144-159), the integral (6.1) is of a very general character and hence may encompass several cases of interest. Our result (6.1) is a master or key formula from which a large number of results can be derived for Meijer's G-function, MacRobert's E-function, Hypergeometric functions, Bessel functions, Legendre functions, Whittaker functions, orthogonal polynomials, trigonometric functions and other related functions.

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