ANNALES MATHÉMATIQUES
BLAISE PASCAL

Franck Boyer & Morgan Morancey

Distributed null controllability of some 1D cascade parabolic systems

Volume 32, nº 2 (2025), p. 221-280.
https://doi.org/10.5802/ambp.438

Publication éditée par le laboratoire de mathématiques Blaise Pascal de l'université Clermont Auvergne, UMR 6620 du CNRS Clermont-Ferrand — France

CENTRE MERSENNE

Publication membre du centre Mersenne pour l'édition scientifique ouverte
http://www.centre-mersenne.org/
e-ISSN : 2118-7436

Annales mathématiques Blaise Pascal **32**, 221-280 (2025)

Distributed null controllability of some 1D cascade parabolic systems

Franck Boyer
Morgan Morancey

Abstract

We consider several coupled systems of one-dimensional linear parabolic equations where only one equation is controlled with a distributed control. For these systems we study the minimal null-control time that is the minimal time needed to drive any initial condition to zero.

In a previous work [*Comptes Rendus. Mathématique*, 361:1191–1248, 2023] we extended the block moment method to obtain a complete characterization of the minimal null-control time in an abstract setting encompassing such non-scalar controls. In this paper, we push forward the application of this general approach to some classes of 1D parabolic systems with distributed controls whose analysis is out of reach by the usual approaches in the literature like Carleman-based methods, fictitious control and algebraic resolubility, or standard moment method. To achieve this goal, we need to prove refined spectral estimates for Sturm–Liouville operators that have their own interest.

1. Introduction

1.1. Problems under study

In the last 15 years different works exhibited that for some coupled systems of parabolic partial differential equations (see for instance [3, 5, 16, 22, 23]) or degenerate parabolic equations (see for instance [2, 6, 7, 8, 9, 17]) it might be needed to wait for some positive minimal time for null controllability to hold even if, in a parabolic context, the information propagates at infinite velocity.

This phenomenon, quite surprising at first sight since it is not related to any constraint imposed on the state or on the control, is now better understood. It is more related to the geometry of the high frequency eigenelements of the underlying evolution operator relatively to the observation operator. For instance, it may occur in the following (non exclusive) situations: if there is condensation of eigenvalues, if the observation of eigenvectors is too small with respect to the parabolic dissipation rate, or if the norm of suitably chosen generalized eigenvectors is asymptotically too large.

In the previous works [10, 13], we developed the block moment method which is well adapted to study the minimal null-control time for autonomous coupled linear one-dimensional parabolic partial differential equations. Our goal in this paper is to

Both authors were partially supported by the ANR project TRECOS ANR-20-CE40-0009.
Keywords: Control theory, parabolic partial differential equations, minimal null control time, block moment method.
2020 *Mathematics Subject Classification*: 93B05, 93C20, 35K40.

provide several applications of this approach to some classes of such systems whose analysis is out of reach by using other techniques available in the literature. Based on the general results obtained in [13] we first characterize the minimal null-control time of such systems in terms of the asymptotic behavior of some explicit quantities based on the eigenelements of the evolution operator. In a second step, an extra spectral analysis is developed, extending the one given in [1], in order to obtain a tractable expression of the involved quantities that can be computed for actual examples. With this approach, we manage to compute the minimal null-control time for many systems, extending the results in the literature.

To be more precise, the first class of control problems that will be studied in this paper is the following one

$$\begin{cases} \partial_t y + \begin{pmatrix} A & 0 \\ q(x) & A \end{pmatrix} y = \begin{pmatrix} \mathbf{1}_\omega u(t,x) \\ 0 \end{pmatrix}, & (t,x) \in (0,T) \times (0,1), \\ y(t,0) = y(t,1) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1), \end{cases} \tag{1.1}$$

where

- A is the unbounded Sturm–Liouville operator defined in $L^2(0,1)$ by
$$D(A) = H^2(0,1) \cap H_0^1(0,1), \qquad A = -\partial_x\left(\gamma \partial_x \cdot\right) + c\cdot, \tag{1.2}$$
with $c \in L^\infty(0,1)$, $\gamma \in C^1([0,1])$ satisfying $c \geq 0$ and $\inf_{[0,1]} \gamma > 0$.

- the coupling function q belongs to $L^\infty(0,1)$

- $\omega \subset (0,1)$ is a non empty open set.

This system is well-posed in the sense that for every $y_0 \in X = (L^2(0,1))^2$, for every $u \in L^2((0,T) \times (0,1))$ there exists a unique solution in $C^0([0,T];X)$. The null controllability property we shall study for this system is defined as follows.

Definition 1.1. Let $T > 0$. The system (1.1) is said to be null controllable at time T if for any $y_0 \in X$, there exists a control $u \in L^2((0,T) \times (0,1))$ such that the associated solution of (1.1) satisfies $y(T) = 0$.

This problem is not straightforward since the control u is localized in space and only acts in the first equation of the system; therefore controlling both components is only possible through the action of the coupling term corresponding to the function q in the second equation. It is now well known that such system may not be short-time null-controllable and our goal is to go deeper into the understanding of this phenomenon.

Definition 1.2. The minimal null control time for system (1.1) in X is defined as the unique value $T_0(X) \in [0, +\infty]$ such that

- for any $T > T_0(X)$, system (1.1) is null controllable at time T;
- for any $0 < T < T_0(X)$, system (1.1) is not null controllable at time T.

When no confusion is possible, we shall simply denote this minimal time as T_0.

It will be also useful to introduce, for any $y_0 \in X$, the number $T_0(y_0) \in [0, +\infty]$ which is the minimal time that is necessary to drive the system to 0 starting from the particular initial data y_0. Notice that $T_0(X) = \sup_{y_0 \in X} T_0(y_0)$.

The question we address is thus the computation of the minimal null control time (being possibly 0 or infinity) of system (1.1).

This question has already been answered in some particular geometric configurations: when ω intersects the support of the coupling function $\mathrm{Supp}(q)$, by means of Carleman estimates or in the opposite setting when ω is an interval disjoint from the support of q, by solving the associated moment problem. We will discuss those results more in details in Section 1.2.

Our goal in this article concerning (1.1) is twofold. First we prove that applying directly the abstract results on block moment problems from [13] encompasses all the previously known results for this problem even though they were proved with completely different techniques. Then, improving the strategy developed in [1] to study spectral quantities of interest in this problem, we are able to extend these results to any choice of coupling term q and control domain ω. We will emphasize the role of the geometry (that is of the relative position of the connected components of ω with respect to the support of q) in the determination of the minimal null control time for system (1.1). Moreover, contrary to the related results in the literature, our proof does not rely on the explicit expression of the eigenfunctions of A so that it applies for a general Sturm–Liouville operator (instead of the Dirichlet–Laplace operator that was considered in [4, 5]). All these results are precisely stated in Section 1.3.

To point out even more the ability of our approach to determine the minimal null-control time for such problems we propose the study of some other related systems. To begin with, we obtain new results for a similar cascade problem in which coupling terms in the second equation now contain first-order operators, as studied in [16]. More precisely, we consider the following control problem

$$\begin{cases} \partial_t y + \begin{pmatrix} A & 0 \\ q(x)+p(x)\partial_x & A \end{pmatrix} y = \begin{pmatrix} \mathbf{1}_\omega u(t,x) \\ 0 \end{pmatrix}, & (t,x) \in (0,T) \times (0,1), \\ y(t,0) = y(t,1) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1), \end{cases} \tag{1.3}$$

with $q \in L^\infty(0,1)$, $p \in W^{1,\infty}(0,1)$. As a consequence of our analysis we will give an example for which the system is not approximately controllable, even if the coupling terms are active inside the control domain ω.

Finally, we analyze the null-controllability of the following simultaneous control problem which has not been studied in the literature so far

$$\begin{cases} \partial_t y + \begin{pmatrix} A & 0 & 0 \\ q_2(x) & A & 0 \\ q_3(x) & 0 & A \end{pmatrix} y = \begin{pmatrix} \mathbf{1}_\omega u(t,x) \\ 0 \\ 0 \end{pmatrix}, & (t,x) \in (0,T)\times(0,1), \\ y(t,0) = y(t,1) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1), \end{cases} \tag{1.4}$$

with $q_2, q_3 \in L^\infty(0,1)$. This problem can indeed be seen as a simultaneous controllability problem since we look for a single control u that simultaneously controls two systems of the form (1.1): the one satisfied by (y_1, y_2) and the one satisfied by (y_1, y_3).

The expression we obtain for the minimal simultaneous null control-time for (1.4) shows that this time can be strictly larger than the two minimal null-control times associated to the two subsystems. This kind of phenomenon was already observed, for instance, in [22].

In the sequel of this introduction, we will set some notation and present the results available in the literature concerning the analysis of the control problems we are interested in that is (1.1), (1.3) and (1.4), then we will state precisely our main results.

1.2. State of the art

1.2.1. Notation

- For any $\omega \subset (0,1)$, we set for convenience $\|\cdot\|_\omega = \|\cdot\|_{L^2(\omega)}$ and the associated inner product $\langle\cdot,\cdot\rangle_\omega$

- We denote by $(\nu_k)_{k\geq 1}$ the increasing sequence of eigenvalues of the Sturm–Liouville operator A defined in (1.2). Notice that the sign assumption we make on c ensures that for any $k \geq 1$, we have $\nu_k > 0$. The associated normalized eigenvectors are denoted by $(\varphi_k)_{k\geq 1}$; they form a Hilbert basis of $L^2(0,1)$.

- For any $k \geq 1$, we define $\widetilde{\varphi}_k$ as the unique solution of the Cauchy problem

$$\begin{cases} (A - \nu_k)\widetilde{\varphi}_k = 0, \\ \widetilde{\varphi}_k(0) = 1, \\ \widetilde{\varphi}_k'(0) = 0. \end{cases} \tag{1.5}$$

- We will also need to introduce $\psi_{k,q}$, the unique solution of the boundary value problem

$$\begin{cases}(A-\nu_k)\psi_{k,q}=\big(I_k(q)-q\big)\varphi_k,\\ \psi_{k,q}(0)=\psi_{k,q}(1)=0,\\ \langle\varphi_k,\psi_{k,q}\rangle_\omega=0,\end{cases} \tag{1.6}$$

where $I_k(q)$ is the integral defined by

$$I_k(q)=\int_0^1 q(x)\varphi_k^2(x)\mathrm{d}x. \tag{1.7}$$

Such a solution exists since, precisely by (1.7), the right-hand side of the equation is orthogonal to φ_k and it is unique thanks to the choice of normalization $\langle\varphi_k,\psi_{k,q}\rangle_\omega=0$. This particular choice is possible thanks to the fact that $\|\varphi_k\|_\omega>0$ and it implies that $\psi_{k,q}$ is, among all the solutions of the underdetermined problem

$$(A-\nu_k)\psi=(I_k(q)-q)\varphi_k,\qquad \psi(0)=\psi(1)=0,$$

the one with minimal $L^2(\omega)$ norm. This will simplify some computations in the paper since it ensures orthogonality between observations of (generalized) eigenvectors.

- Following [14], for any $F\in L^2(0,1)$ and any $\mathfrak{C}$ connected component of $\overline{(0,1)\backslash\omega}$ we define an element of $\mathbb{R}^2$ as follows

$$M_k(F,\mathfrak{C})=\begin{cases}\begin{pmatrix}\int_{\mathfrak{C}}F\varphi_k\\ 0\end{pmatrix}, & \text{if } \mathfrak{C} \text{ touches the boundary of } (0,1)\\ \begin{pmatrix}\int_{\mathfrak{C}}F\varphi_k\\ \int_{\mathfrak{C}}F\widetilde{\varphi}_k\end{pmatrix}, & \text{otherwise,}\end{cases}$$

that we gather into a single collection defined by

$$\mathfrak{M}_k(F,\omega)=\big(M_k(F,\mathfrak{C})\big)_{\mathfrak{C}\in C\left(\overline{(0,1)\backslash\omega}\right)}, \tag{1.8}$$

where $C(U)$ stands for the set of all connected components of any $U\subset[0,1]$. We finally set

$$\mathcal{M}_k(F,\omega)=\|\mathfrak{M}_k(F,\omega)\|_\infty=\sup\Big\{\big|M_k(F,\mathfrak{C})\big|_\infty\ ;\ \mathfrak{C}\in C\Big(\overline{(0,1)\backslash\omega}\Big)\Big\}. \tag{1.9}$$

1.2.2. **About the cascade system** (1.1)

Approximate controllability. By using the Fattorini–Hautus test (see [20]), it is proved in [14, Theorem 3.2] that, if $\mathrm{Supp}(q) \cap \omega = \emptyset$, approximate controllability of (1.1) holds if and only if

$$\mathcal{M}_k(q\varphi_k, \omega) \neq 0, \qquad \forall\, k \geq 1. \tag{1.10}$$

Notice also that applying [14, Theorem 2.2] we obtain that

- If $\mathrm{Supp}(q) \cap \omega \neq \emptyset$, approximate controllability of (1.1) holds without any other condition.

- If $\mathrm{Supp}(q) \cap \omega = \emptyset$, the necessary and sufficient condition (1.10) for approximate controllability of (1.1) can be rewritten as

$$\mathcal{M}_k((I_k(q) - q)\varphi_k, \omega) \neq 0, \qquad \forall\, k \geq 1, \tag{1.11}$$

  where $I_k(q)$ is introduced in (1.7). Rewriting the approximate controllability condition as (1.11) is more coherent with the expression of the minimal null control time that we obtain below (see Section 2.2). The equivalence between conditions (1.10) and (1.11) is proved in Lemma 3.3 (choosing there $p = 0$).

Null controllability under a sign assumption. If there exists $\omega_0 \subset \omega$ such that q has a strict sign inside ω_0 then it follows from [21] that null controllability holds in any arbitrary small time. The proof is based on Carleman estimates.

Null controllability with disjoint control and coupling domains. System (1.1) was then studied in the case where $A = -\partial_{xx}$ and $\omega = (a, b)$ is an interval such that $\mathrm{Supp}(q) \cap \omega = \emptyset$.

First, it was proved in [4] that if $\mathrm{Supp}(q) \subset (b, 1)$ then, approximate controllability holds if and only if

$$I_k(q) \neq 0, \qquad \forall\, k \geq 1.$$

This condition is just a rephrasing of (1.10). In this case the authors proved that the minimal null-control time $T_{0,q}$ for this system is given by

$$T_{0,q} = \limsup_{k\to+\infty} \frac{-\ln|I_k(q)|}{\nu_k}.$$

Later on, it was proved in [5] that if $\mathrm{Supp}(q)$ is included in $(0, a) \cup (b, 1)$, then approximate controllability holds if and only if

$$\left|I_{1,k}(q)\right| + \left|I_{2,k}(q)\right| \neq 0, \qquad \forall\, k \geq 1,$$

where

$$I_{1,k}(q) = \int_0^a q(x)\varphi_k^2(x)\mathrm{d}x, \qquad I_{2,k}(q) = \int_b^1 q(x)\varphi_k^2(x)\mathrm{d}x. \tag{1.12}$$

In that situation, this condition is also a rephrasing of (1.10) and it was also proved in [5] that the minimal null-control time is

$$T_{0,q} = \limsup_{k\to+\infty} \frac{-\ln\max\big\{|I_k(q)|,\ |I_{1,k}(q)|,\ |I_{2,k}(q)|\big\}}{\nu_k}. \tag{1.13}$$

Moreover, it is proved that for any $\tau_0 \in [0,+\infty]$ there exists a coupling function $q \in L^\infty(0,1)$ such that the corresponding minimal time is actually $T_{0,q} = \tau_0$. Let us underline that these results are the first results exhibiting a positive minimal null control time for a system of coupled parabolic equations with a distributed control.

The proofs of those results are based on the moment method since, due to the assumption $\mathrm{Supp}(q) \cap \omega = \varnothing$, the strategy based on Carleman estimates is inefficient.

1.2.3. **About the system with a first-order coupling term**

Null controllability of system (1.3) with a coupling term of order one has been studied in [11, 16, 18, 19]. Among other things, the author proves in [16] that, when approximate controllability holds, the minimal null-control time $T_{0,q,p}$ of system (1.3) when $\omega = (a,b)$ is an interval and A is the Dirichlet Laplace operator is given by

$$T_{0,q,p} = \limsup_{k\to+\infty} \frac{-\ln\max\big\{|I_k(r)|,\ \big|I_{1,k}(r)\big|,\ \big|I_{2,k}(r)\big|\big\}}{\nu_k},$$

where $r = q - \frac{1}{2}p'$. Note that the value of $T_{0,q,p}$ only depends on r and is equal to $T_{0,r}$ as defined in (1.13). As proved in Section 3.4.1, this feature is specific to the case where ω is an interval since in general $T_{0,q,p}$ really depend on both q and p, and not only on r.

The proof given in [16] is also based on the moments method and follows that of [5]. More precisely, the analysis in this reference is reduced, thanks to well-suited manipulations, to the one of a scalar moment problem despite the fact that the control space is, by nature, infinite dimensional. Those computations are thus specific to the problem under study and makes use of the explicit formulas for the eigenfunctions of the 1D Laplace operator, which is not the case of our proof.

In [18], the authors give a sufficient condition for null controllability for general parabolic systems in any dimension with first-order coupling terms. They deal with coefficients depending both on space and time but their analysis does not apply when $p = 0$ in ω. In [19], the same authors study the influence of the position of the control domain on controllability for one dimensional parabolic systems with first-order coupling terms. Their result can hardly be compared with our study since the two equations they consider are associated to different evolution operators.

1.2.4. About the simultaneous control problem

To the best of our knowledge, the only available result in the literature concerning the controllability of (1.4) is the necessary and sufficient condition for approximate controllability given in [14, Theorem 3.2] that we recall now: approximate controllability for system (1.4) holds if and only if, for any $k \geq 1$,

$$\mathfrak{M}_k(q_2\varphi_k, \omega) \text{ and } \mathfrak{M}_k(q_3\varphi_k, \omega) \text{ are linearly independent in } (\mathbb{R}^2)^{C\left(\overline{(0,1)\backslash\omega}\right)}, \quad (1.14)$$

where the notation $\mathfrak{M}_k$ is introduced in (1.8).

This gave rise to unexpected geometric control conditions for this problem. For instance, if ω is an interval that does not touch the boundary of $(0, 1)$, approximate controllability of system (1.4) never holds when $\mathrm{Supp}(q_2)$ and $\mathrm{Supp}(q_3)$ are located in the same connected component of $\overline{(0, 1)\backslash\omega}$. However, if there are located in two distinct connected components then approximate controllability holds if and only if the two subsystems are approximately controllable (see [14, Section 3.4]).

1.3. Main results of this paper

First, we obtain the following characterization of the minimal null-control time for system (1.1).

Theorem 1.3. *Let $\omega \subset (0, 1)$ be a non empty open set and let $q \in L^\infty(0, 1)$. Assume that either* $\mathrm{Supp}(q) \cap \omega \neq \varnothing$ *or that* (1.11) *holds. Then, the minimal null-control time for system* (1.1) *is given by*

$$T_{0,q} = \limsup_{k\to+\infty} \frac{-\ln\left(I_k(q)^2\|\varphi_k\|_\omega^2 + \|\psi_{k,q}\|_\omega^2\right)}{2\nu_k}.$$

We recall that $\psi_{k,q}$ is defined in (1.6) and $I_k(q)$ is defined by (1.7).

This theorem is proved in Section 2.1. It is valid without any geometric assumptions on the control domain ω nor on the support of the coupling term q. In this respect, it unifies the different results obtained in the literature for the study of null controllability of system (1.1) recalled in Section 1.2.2. For example, even if it is not clear at first sight, we manage to prove, in Section 2.3, that the formula above reduces to $T_{0,q} = 0$ when q has a strict sign on $\omega_0 \subset \omega$ as proved in [21].

Theorem 1.3 will be obtained as a consequence of [13, Theorems 11, 14 and 18] where the minimal null control time issue is analyzed in an abstract general setting.

It relies on a careful estimate of the cost of resolution for block moment problems associated to a general admissible control operator. With some additional work, based on the method developed in [1] to obtain spectral estimates for the eigenelements of A, we also obtain the following characterization of the minimal null control time in the case where $\mathrm{Supp}(q) \cap \omega = \emptyset$.

Theorem 1.4. *Let $\omega \subset (0,1)$ be a non empty open set with a finite number of connected components. Let $q \in L^\infty(0,1)$ be such that $\mathrm{Supp}(q) \cap \omega = \emptyset$. Assume that (1.11) holds. Then, the minimal null-control time for system (1.1) is given by*

$$T_{0,q} = \limsup_{k \to +\infty} \frac{-\ln \mathcal{M}_k\big((I_k(q) - q)\varphi_k,\ \omega\big)}{\nu_k}$$

where $\mathcal{M}_k$ is defined by (1.9).

The proof is given in Section 2.2. Compared to the one in Theorem 1.3, the expression for $T_{0,q}$ above is more convenient to deal with since it does not involve the function $\psi_{k,q}$. As we shall prove in Section 2.3, this formula is a natural extension of the ones obtained in the literature in some particular cases. However it holds true in more general situations, so that we are able to compute $T_{0,q}$ in cases that were not covered in the literature (see Proposition 2.5 as an example).

This theorem also extends the previous works in the field by considering for A a general Sturm–Liouville operator (and not only the Dirichlet–Laplace operator) since it does not make use of the explicit expressions of its eigenvalues and eigenfunctions.

Remark 1.5. Notice that the assumption $\mathrm{Supp}(q) \cap \omega = \emptyset$ is necessary for this theorem to be true. For instance, if $q = 1$ and ω is an interval then, from [21], null controllability holds in any time $T > 0$ whereas we have $I_k(q) - q = 0$ for any $k \geq 1$.

However, this is not restrictive for our study since it is well-known that when $\mathrm{Supp}(q) \cap \omega \neq \emptyset$, the system is indeed null-controllable at every time $T > 0$ (see for instance [21]).

The tools used to prove Theorems 1.3 and 1.4 allow for a similar analysis for system (1.3). The corresponding results are stated in Section 3.

We now turn to the simultaneous controllability problem (1.4). For $q_2, q_3 \in L^\infty(0,1)$, we set $\boldsymbol{q} = (q_2, q_3)$ and denote by $T_{0,\boldsymbol{q}}$ the minimal null control time for system (1.4) in $(L^2(0,1))^3$. Since simultaneous null controllability at a given time implies null controllability at the same time for both subsystems (1.1) with $q = q_2$ and $q = q_3$ it directly comes that

$$T_{0,\boldsymbol{q}} \geq \max(T_{0,q_2}, T_{0,q_3}). \tag{1.15}$$

Actually, by linearity of the system, simultaneous null controllability at a given time implies null controllability at the same time for system (1.1) with any q in $\mathrm{Span}(q_2, q_3)\backslash\{0\}$ that is

$$T_{0,\boldsymbol{q}} \geq \sup_{\substack{q\in \mathrm{Span}(q_2,q_3)\\ q\neq 0}} T_{0,q}. \tag{1.16}$$

We give below general characterizations of $T_{0,\boldsymbol{q}}$ similar to those obtained in Theorems 1.3 and 1.4 for system (1.1) and give, in Section 4.4, an explicit example of system (1.4) for which the inequality in (1.16) is strict.

In order to state the results, it will be convenient to use some extra notation. For any $q \in L^\infty(0,1)$, we set

$$\zeta_{k,q} = \psi_{k,q} + I_k(q)\varphi_k, \tag{1.17}$$

where $\psi_{k,q}$ was introduced in (1.6), and

$$\vartheta_{k,q} = \big(I_k(q) - q\big)\varphi_k. \tag{1.18}$$

The formulas obtained in Theorems 1.3 and 1.4 for the minimal null-control time of system (1.1) can now be rephrased respectively as follows

$$T_{0,q} = \limsup_{k\to+\infty} \frac{-\ln \|\zeta_{k,q}\|_\omega}{\nu_k}$$

and

$$T_{0,q} = \limsup_{k\to+\infty} \frac{-\ln \mathcal{M}_k\big(\vartheta_{k,q},\ \omega\big)}{\nu_k}.$$

We shall generalize those expressions for system (1.4) as follows.

Theorem 1.6. *Let $\omega \subset (0,1)$ be a non empty open set and let $q_2, q_3 \in L^\infty(0,1)$. Assume that* (1.14) *holds. Then, the minimal null control time $T_{0,\boldsymbol{q}}$ for system* (1.4) *in* $(L^2(0,1))^3$ *is given by*

$$T_{0,\boldsymbol{q}} = \limsup_{k\to+\infty} \frac{1}{2\nu_k} \ln \frac{\max\Big(\big\|\zeta_{k,q_2}\big\|_\omega^2, \big\|\zeta_{k,q_3}\big\|_\omega^2\Big)}{\big\|\zeta_{k,q_2}\big\|_\omega^2\big\|\zeta_{k,q_3}\big\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3}\rangle_\omega^2}.$$

This theorem is proved in Section 4.2. Though it is not obvious at first sight, we will show in the proof of Proposition 4.3 that the approximate controllability assumption (1.14) actually implies that

$$\big\|\zeta_{k,q_2}\big\|_\omega^2\big\|\zeta_{k,q_3}\big\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3}\rangle_\omega^2 > 0, \qquad \forall\ k \geq 1$$

and thus the formula defining $T_{0,\boldsymbol{q}}$ is well-defined. Since for any $k \geq 1$ we clearly have

$$\big\|\zeta_{k,q_2}\big\|_\omega^2\big\|\zeta_{k,q_3}\big\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3}\rangle_\omega^2 \leq \big\|\zeta_{k,q_2}\big\|_\omega^2\big\|\zeta_{k,q_3}\big\|_\omega^2,$$

we immediately see, by Theorem 1.3, that this formula is compatible with the expected property (1.15).

Notice that, when $\big(\mathrm{Supp}(q_2) \cup \mathrm{Supp}(q_3)\big) \cap \omega = \emptyset$, we have

$$I_k(q_j) = \sum_{\mathfrak{C} \in C\left(\overline{(0,1)\setminus\omega}\right)} \int_{\mathfrak{C}} q_j \varphi_k^2, \quad \forall\, j \in \{1,2\}.$$

Hence, it comes that the approximate controllability condition (1.14) is equivalent in that case to the condition

$$\mathfrak{M}_k\big(\vartheta_{k,q_2}, \omega\big) \text{ and } \mathfrak{M}_k\big(\vartheta_{k,q_3}, \omega\big) \text{ are linearly independent} \tag{1.19}$$

where ϑ_{k,q_2} and ϑ_{k,q_3} are defined by (1.18).

Theorem 1.7. *Let $\omega \subset (0,1)$ be a non empty open set with a finite number of connected components. Let $q_2, q_3 \in L^\infty(0,1)$ be such that*

$$\big(\mathrm{Supp}(q_2) \cup \mathrm{Supp}(q_3)\big) \cap \omega = \emptyset.$$

Assume that (1.19) *holds. Then, the minimal null control time for system* (1.4) *in* $(L^2(0,1))^3$ *is given by*

$$T_{0,\boldsymbol{q}} = \limsup_{k\to+\infty} \frac{-\ln \min_{q\in\mathbb{S}[\boldsymbol{q}]} \mathcal{M}_k\big(\vartheta_{k,q},\ \omega\big)}{\nu_k}$$

where $\mathcal{M}_k$ is defined by (1.9)*, $\vartheta_{k,q}$ is defined by* (1.18) *and*

$$\mathbb{S}[\boldsymbol{q}] = \{q \in \mathrm{Span}(q_2, q_3),\, \|q\|_\infty = 1\}.$$

This theorem is proved in Section 4.3. Notice that, by compactness of $\mathbb{S}[\boldsymbol{q}]$, the min appearing in this formula is actually achieved and moreover, since the approximate controllability condition (1.19) implies that

$$\mathcal{M}_k\big(\vartheta_{k,q},\ \omega\big) \neq 0, \quad \forall\, q \in \mathbb{S}[\boldsymbol{q}],$$

we know that this min is positive. Thus, the formula for $T_{0,\boldsymbol{q}}$ in the above theorem is well defined.

This formulation is more convenient to deal with than the one of Theorem 1.6 on actual systems. For instance, with this formulation, we prove that the minimal null control time is not related to the minimal null control times of the subsystems. Indeed, in Section 4.4, for any $\tau_0 \in [0, +\infty]$, we design a couple of functions $\boldsymbol{q} = (q_2, q_3)$ such that $T_{0,\boldsymbol{q}} = \tau_0$ and

$$\sup_{q\in\mathbb{S}[\boldsymbol{q}]} T_{0,q} = \sup_{\substack{q\in\mathrm{Span}(q_2,q_3)\\ q\neq 0}} T_{0,q} = 0$$

which proves that the inequality in (1.16) can be strict.

1.4. **Outline of the article**

Section 2 is dedicated to the proof of the two formulations of the minimal null control time for system (1.1) stated in Theorems 1.3 and 1.4.

In Section 2.3, we give some applications of the obtained formulas: we prove that they encompass previously known results and let us get precise results in more general new configurations.

Section 3 is dedicated to the analysis of system (1.3). We show that taking into account first-order coupling terms in our methodology is relatively straightforward, compared to the original proofs in [16].

In Section 4, we determine the minimal null control time for the simultaneous controllability problem (1.4) as stated in Theorems 1.6 and 1.7.

Finally, we have gathered in Appendix A some spectral properties of Sturm–Liouville operators that are used all along this article.

2. **A system with a space varying zero order coupling term**

In this section we prove the characterizations of the minimal null control time for system (1.1). We prove Theorem 1.3 in Section 2.1 as an application of the results of [13]. Then, analyzing the behaviour of the spectral quantities arising in Theorem 1.3, we prove Theorem 1.4 in Section 2.2.

2.1. **A first formula for the minimal time**

First, let us check that our system (1.1) fits in the formalism of [13]. There, we considered abstract control problems of the form

$$\begin{cases} y'(t) + \mathcal{A}y(t) = \mathcal{B}u(t), \\ y(0) = y_0. \end{cases}$$

Thus, for system (1.1), the evolution operator $\mathcal{A}$ is defined by

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ q & A \end{pmatrix}, \qquad D(\mathcal{A}) = D(A)^2$$

and the control operator $\mathcal{B}$ is defined by

$$\mathcal{B} : u \in U = L^2(0,1) \longmapsto \begin{pmatrix} \mathbf{1}_\omega u \\ 0 \end{pmatrix}.$$

In [13] the results involve a Gelfand triple of Hilbert spaces $X_\diamond^* \subset X \subset X_{-\diamond}$ in order to deal with possibly unbounded control operators. In the present article we only consider distributed control operators which implies that there are no particular subtleties on the functional framework and we shall set here $X_{-\diamond} = X_\diamond^* = X = L^2(0,1;\mathbb{R})^2$ (see [13, Section 2.1.1]). This implies the wellposedness of system (1.1) in the sense of [13, Proposition 2].

Thus, to use the characterizations of the minimal null control time obtained in [13] we shall prove that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumption (H) defined in [13, Section 2.1.2]. Roughly speaking this assumption states that the operator $\mathcal{A}^*$ admits a complete family of generalized eigenvectors which are observable (i.e. not in the kernel of $\mathcal{B}^*$). It also requires that the associated family of eigenvalues, each of them having finite geometric multiplicity and globally bounded algebraic multiplicity, satisfies a weak-gap assumption (i.e. they can be gathered in well separated blocks of bounded diameter and cardinality) and appropriate estimates on its counting function (see (A.4) and (A.5)).

Let us detail the spectral analysis of the operator $\mathcal{A}^*$.

Its spectrum is given by $\Lambda = (\nu_k)_{k\geq 1}$. Recall that $I_k(q)$ is defined by (1.7) and $\psi_{k,q}$ is defined by (1.6). We distinguish the following cases.

- If $I_k(q) \neq 0$ then ν_k is algebraically double and geometrically simple. A Jordan chain is given by

$$\phi_k^0 = \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix}, \qquad \phi_k^1 = \frac{1}{I_k(q)} \begin{pmatrix} \psi_{k,q} \\ \varphi_k \end{pmatrix}. \tag{2.1}$$

- If $I_k(q) = 0$ then ν_k is geometrically double and a basis of eigenvectors is given by

$$\phi_{k,1}^0 = \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix}, \qquad \phi_{k,2}^0 = \begin{pmatrix} \psi_{k,q} \\ \varphi_k \end{pmatrix}. \tag{2.2}$$

Properties of eigenvalues. The eigenvalues of $\mathcal{A}^*$ are real and, due to the assumption $c \geq 0$ they satisfy $\nu_k > 0$.

From (A.1), these eigenvalues satisfy a gap condition with parameter ϱ and thus a grouping in the sense of [13, Proposition 6] is given by $G_k = \{\nu_k\}$.

The associated counting function N satisfies (A.4) and (A.5).

Gathering all these properties, we have that the sequence of eigenvalues of $\mathcal{A}^*$ satisfies

$$\Lambda \in \mathcal{L}_w\left(1, \varrho, 0, \frac{1}{2}, \kappa\right)$$

as defined in [13, Section 2.1.2].

Properties of eigenvectors. The eigenvalue ν_k is either geometrically simple and algebraically double or semi-simple with geometric multiplicity 2. Due to the expressions (2.1) and (2.2) we obtain that the family of (generalized) eigenvectors of $\mathcal{A}^*$ forms a complete family in X.

As stated in Section 1.2, the approximate controllability assumption

$$\operatorname{Ker}(\mathcal{A}^* - \lambda) \cap \operatorname{Ker} \mathcal{B}^* = \{0\}, \qquad \forall\, \lambda \in \Lambda$$

follows from (1.11) and [14, Theorem 2.2].

Thus, the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumption (H) stated in [13, Section 2.1.2].

Proof of Theorem 1.3. From [13, Theorem 11], for any $y_0 \in X$ the minimal null control time from y_0 is given by

$$T_{0,q}(y_0) = \limsup_{k \to +\infty} \frac{\ln^+ C(G_k, y_0)}{2\nu_k}$$

where $\ln^+ s = \max(0, \ln s)$, for any $s \geq 0$ and the cost of the k-th block is given by

$$C(G_k, y_0) = \inf \left\{ \left\| \Omega^0 \right\|_U^2 + \left\| \Omega^1 \right\|_U^2 \; ; \; \begin{array}{c} \Omega^0, \Omega^1 \in U \text{ with } \langle \Omega^0, \mathcal{B}^* \phi_k^0 \rangle_U = \langle y_0, \phi_k^0 \rangle_X \\ \text{and } \langle \Omega^0, \mathcal{B}^* \phi_k^1 \rangle_U + \langle \Omega^1, \mathcal{B}^* \phi_k^0 \rangle_U = \langle y_0, \phi_k^1 \rangle_X \end{array} \right\}$$

if $I_k(q) \neq 0$ and

$$C(G_k, y_0) = \inf \left\{ \|\Omega\|_U^2 \; ; \; \Omega \in U \text{ with } \langle \Omega, \mathcal{B}^* \phi_{k,j}^0 \rangle_U = \langle y_0, \phi_{k,j}^0 \rangle_X \text{ for } j \in \{1,2\} \right\}$$

if $I_k(q) = 0$.

To compute $C(G_k, y_0)$ we distinguish the two cases.

Case 1: Assume that $I_k(q) \neq 0$. Then, from [13, Theorem 14], it comes that

$$C(G_k, y_0) = \langle M^{-1} \xi, \xi \rangle$$

where

$$M = \operatorname{Gram}\big(\mathcal{B}^* \phi_k^0, \mathcal{B}^* \phi_k^1\big) + \operatorname{Gram}\big(0, \mathcal{B}^* \phi_k^0\big) = \begin{pmatrix} \|\varphi_k\|_\omega^2 & 0 \\ 0 & \|\varphi_k\|_\omega^2 + \frac{1}{I_k(q)^2} \|\psi_{k,q}\|_\omega^2 \end{pmatrix}$$

and

$$\xi = \begin{pmatrix} \langle y_0, \phi_k^0 \rangle_X \\ \langle y_0, \phi_k^1 \rangle_X \end{pmatrix}.$$

Thus,

$$C(G_k, y_0) = \frac{1}{\|\varphi_k\|_\omega^2} \left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix} \right\rangle_X^2 + \frac{1}{I_k(q)^2 \|\varphi_k\|_\omega^2 + \|\psi_{k,q}\|_\omega^2} \left\langle y_0, \begin{pmatrix} \psi_{k,q} \\ \varphi_k \end{pmatrix} \right\rangle_X^2.$$

Case 2: Assume that $I_k(q) = 0$. Then, from [13, Theorem 18], it comes that

$$C(G_k, y_0) = \langle M^{-1} \xi, \xi \rangle$$

where

$$M = \operatorname{Gram}\big(\mathcal{B}^* \phi_{k,1}^0, \mathcal{B}^* \phi_{k,2}^0\big) = \begin{pmatrix} \|\varphi_k\|_\omega^2 & 0 \\ 0 & \|\psi_{k,q}\|_\omega^2 \end{pmatrix}$$

and

$$\xi = \begin{pmatrix} \langle y_0, \phi_{k,1}\rangle_X \\ \langle y_0, \phi_{k,2}\rangle_X \end{pmatrix}.$$

Thus,

$$C(G_k, y_0) = \frac{1}{\|\varphi_k\|_\omega^2}\left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix}\right\rangle_X^2 + \frac{1}{\|\psi_{k,q}\|_\omega^2}\left\langle y_0, \begin{pmatrix} \psi_{k,q} \\ \varphi_k \end{pmatrix}\right\rangle_X^2.$$

Finally, in both cases, the cost corresponding to the group G_k is given by

$$C(G_k, y_0) = \frac{1}{\|\varphi_k\|_\omega^2}\left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix}\right\rangle_X^2 + \frac{1}{I_k(q)^2\|\varphi_k\|_\omega^2 + \|\psi_{k,q}\|_\omega^2}\left\langle y_0, \begin{pmatrix} \psi_{k,q} \\ \varphi_k \end{pmatrix}\right\rangle_X^2. \quad (2.3)$$

We now evaluate the different contributions of the terms in the right-hand side of (2.3).

Recall that $\|\varphi_k\|_{(0,1)} = 1$, that $(\psi_{k,q})_k$ is bounded thanks to Lemma A.2, and that, from (A.3), we have

$$\|\varphi_k\|_\omega \geq C > 0, \qquad \forall\, k \geq 1.$$

Thus, getting back to (2.3), we obtain that

$$C(G_k, y_0) \leq C\|y_0\|_X^2\left(1 + \frac{1}{I_k(q)^2\|\varphi_k\|_\omega^2 + \|\psi_{k,q}\|_\omega^2}\right), \quad \forall\, k \geq 1,\ \forall\, y_0 \in X,$$

which proves that

$$T_{0,q}(y_0) \leq \limsup_{k\to+\infty} \frac{-\ln\big(I_k(q)^2\|\varphi_k\|_\omega^2 + \|\psi_{k,q}\|_\omega^2\big)}{2\nu_k}.$$

This estimate holds for every y_0, which gives the upper bound for $T_{0,q}$.

To prove the converse inequality let us choose

$$y_0 = \sum_{k\geq 1} \frac{1}{\nu_k}\begin{pmatrix} 0 \\ \varphi_k \end{pmatrix},$$

which is indeed a converging series in X. From (2.3) we obtain that for this particular choice of y_0,

$$C(G_k, y_0) = \frac{1}{\nu_k^2}\frac{1}{I_k(q)^2\|\varphi_k\|_\omega^2 + \|\psi_{k,q}\|_\omega^2}, \quad \forall\, k \geq 1.$$

Thus,

$$T_{0,q} \geq T_{0,q}(y_0) = \limsup_{k\to+\infty} \frac{-\ln\big(I_k(q)^2\|\varphi_k\|_\omega^2 + \|\psi_{k,q}\|_\omega^2\big)}{2\nu_k}.$$

This ends the proof of Theorem 1.3. □

2.2. A second formula for the minimal time with disjoint control and coupling domains

The minimal null control time has been characterized in Theorem 1.3. Thus, the proof of Theorem 1.4 consists in comparing the asymptotic behaviors of

$$\mathcal{M}_k\big((I_k(q) - q)\varphi_k, \omega\big)$$

and

$$I_k(q)^2\|\varphi_k\|_\omega^2 + \|\psi_{k,q}\|_\omega^2.$$

To do so we will use the following result whose proof is postponed to the end of the section, to improve the readability.

Proposition 2.1. *Let $\omega \subset (0,1)$ be a non empty open set with a finite number of connected components.*

(i) *There exists $K \in \mathbb{N}^*$ and $C > 0$ such that for any $k \geq K$, any $F \in L^2(0,1)$ and any u satisfying the differential equation*

$$(A - \nu_k)u = F,$$

we have

$$\mathcal{M}_k(F, \omega) \leq C\Big(\sqrt{\nu_k}\|u\|_\omega + \sqrt{\nu_k}(|u(0)| + |u(1)|) + \|F\|_\omega\Big).$$

(ii) *There exists $K \in \mathbb{N}^*$ and $C > 0$ such that for any $k \geq K$ and any $F \in L^2(0,1)$ such that $\int_0^1 F(x)\varphi_k(x)\mathrm{d}x = 0$, there exists u satisfying the boundary value problem*

$$\begin{cases} (A - \nu_k)u = F, \\ u(0) = u(1) = 0, \end{cases}$$

as well as the estimate

$$\sqrt{\nu_k}\|u\|_\omega \leq C(\mathcal{M}_k(F, \omega) + \|F\|_\omega).$$

We now turn to the proof of Theorem 1.4.

Proof of Theorem 1.4. Recall that, from Theorem 1.3,

$$T_{0,q} = \limsup_{k\to+\infty} \frac{-\ln\big(I_k(q)^2\|\varphi_k\|_\omega^2 + \|\psi_{k,q}\|_\omega^2\big)}{2\nu_k}.$$

By (1.6), we can apply point (i) of Proposition 2.1 to $u = \psi_{k,q}$ and $F = (I_k(q) - q)\varphi_k$ to get, for $k \geq K$,

$$\begin{aligned}\mathcal{M}_k((I_k(q) - q)\varphi_k, \omega)^2 &\leq C\Big(\nu_k \|\psi_{k,q}\|_\omega^2 + \|(I_k(q) - q)\varphi_k\|_\omega^2\Big)\\ &\leq C\nu_k\Big(\|\psi_{k,q}\|_\omega^2 + I_k(q)^2\|\varphi_k\|_\omega^2\Big),\end{aligned}$$

since $\mathrm{Supp}(q) \cap \omega = \emptyset$. Thus,

$$T_{0,q} \leq \limsup_{k\to+\infty} \frac{-\ln \mathcal{M}_k\big((I_k(q) - q)\varphi_k,\ \omega\big)}{\nu_k}.$$

We now prove the converse inequality. For k large enough, let u be the function given by the point (ii) of Proposition 2.1 with $F = \big(I_k(q) - q\big)\varphi_k$ (which, by definition of $I_k(q)$, satisfies $\int_0^1 F(x)\varphi_k(x)\mathrm{d}x = 0$). We observe that there exists $\alpha \in \mathbb{R}$ such that we can write $u = \psi_{k,q} + \alpha\varphi_k$. Recall that we have imposed in (1.6), that $\langle \varphi_k, \psi_{k,q}\rangle_\omega = 0$, so that we have $\|\psi_{k,q}\|_\omega \leq \|u\|_\omega$. Thus, using the estimate given by point (ii) of Proposition 2.1 and the assumption $\mathrm{Supp}(q) \cap \omega = \emptyset$, we obtain that, for any $k \geq K$,

$$\nu_k\|\psi_{k,q}\|_\omega^2 \leq C\Big(\mathcal{M}_k(F,\omega)^2 + \|F\|_\omega^2\Big) \leq C\Big(\mathcal{M}_k(F,\omega)^2 + I_k(q)^2\|\varphi_k\|_\omega^2\Big). \tag{2.4}$$

We denote by $\mathfrak{C}_1, \dots, \mathfrak{C}_N$ the connected components of $\overline{(0,1)\backslash\omega}$. As $\mathrm{Supp}(q) \cap \omega = \emptyset$, notice that

$$\begin{aligned}\sum_{j=1}^N \int_{\mathfrak{C}_j} F(x)\varphi_k(x)\mathrm{d}x &= I_k(q)\sum_{j=1}^N \int_{\mathfrak{C}_j} \varphi_k^2(x)\mathrm{d}x - \sum_{j=1}^N \int_{\mathfrak{C}_j} q(x)\varphi_k^2(x)\mathrm{d}x\\ &= I_k(q)\Big(1 - \|\varphi_k\|_\omega^2\Big) - I_k(q)\\ &= -I_k(q)\|\varphi_k\|_\omega^2.\end{aligned}$$

Thus, from (A.3) we deduce that

$$|I_k(q)|\|\varphi_k\|_\omega \leq C\mathcal{M}_k(F,\omega).$$

Plugging this inequality into (2.4) we obtain

$$\|\psi_{k,q}\|_\omega^2 + I_k(q)^2\|\varphi_k\|_\omega^2 \leq C\mathcal{M}_k(F,\omega)^2.$$

This implies that

$$T_{0,q} \geq \limsup_{k\to+\infty} \frac{-\ln \mathcal{M}_k\big((I_k(q) - q)\varphi_k,\ \omega\big)}{\nu_k}$$

and ends the proof of Theorem 1.4. □

To conclude this section, it remains to prove Proposition 2.1. To do so, we start with the following result that comes from Lemma A.1.

Lemma 2.2. *Let A be the Sturm–Liouville operator defined by* (1.2) *and let $\lambda_0 > 0$. There exists $C > 0$ depending on γ, c and λ_0 such that, for any $\lambda \geq \lambda_0$, for any $F \in L^2(0,1)$, for any $0 < a < b < 1$, for any u satisfying*

$$(A - \lambda)u = F \quad \text{on } [a,b],$$

and for any $x \in [a,b]$, we have

$$|u(x)|^2 + \frac{\gamma(x)}{\lambda}|u'(x)|^2 \leq \frac{C}{b-a}\left(1 + \frac{1}{\lambda(b-a)^2}\right)\|u\|^2_{(a,b)} + C\frac{(b-a)}{\lambda}\|F\|^2_{(a,b)}.$$

Proof. Let $\chi_0 \in C^\infty(\mathbb{R};\mathbb{R})$ be a cut-off function such that $0 \leq \chi_0 \leq 1$ and

- $\chi_0(x) = 1$ for every $x \in [1/4, 3/4]$,
- $\chi_0(x) = 0$ for every $x \notin (0,1)$.

We then set

$$\chi(x) = \chi_0\Big(\frac{x-a}{b-a}\Big),$$

in such a way that, if we set $\alpha = a + \frac{b-a}{4}$ and $\beta = b - \frac{b-a}{4}$, we have

- $\chi(x) = 1$ for every $x \in [\alpha, \beta]$,
- $\chi(x) = 0$ for every $x \notin (a,b)$.

Let $C_1 > 0$ be the constant given by Lemma A.1 and assume that $\lambda \geq \lambda_0$.
Let $x \in [a,b]$. We apply Lemma A.1 to obtain for any $y \in (a,b)$

$$\left(|u(x)|^2 + \frac{\gamma(x)}{\lambda}|u'(x)|^2\right) \leq C_1\left(|u(y)|^2 + \frac{\gamma(y)}{\lambda}|u'(y)|^2 + \frac{b-a}{\lambda}\|F\|^2_{(a,b)}\right).$$

Integrating in the variable $y \in (\alpha, \beta)$ gives

$$\frac{b-a}{2}\left(|u(x)|^2 + \frac{\gamma(x)}{\lambda}|u'(x)|^2\right) \leq C_1\left(\|u\|^2_{(a,b)} + \frac{(b-a)^2}{2\lambda}\|F\|^2_{(a,b)}\right) + \frac{C_1}{\lambda}\int_\alpha^\beta \gamma(y)|u'(y)|^2 \mathrm{d}y. \quad (2.5)$$

Then integrating by parts, using $(A-\lambda)u = 0$ and Cauchy–Schwarz inequality yield

$$\begin{aligned}
&\frac{1}{\lambda}\int_\alpha^\beta \gamma(y)|u'(y)|^2 \mathrm{d}y \\
&\qquad \le \frac{1}{\lambda}\int_a^b \chi(y)\gamma(y)|u'(y)|^2\mathrm{d}y \\
&\qquad = -\frac{1}{\lambda}\int_a^b \chi'(y)(\gamma u')(y)u(y)\mathrm{d}y + \frac{1}{\lambda}\int_a^b \chi(y)(\lambda - c(y))|u(y)|^2\mathrm{d}y \\
&\qquad \le \frac{\|\chi'\|_{L^\infty}\|\sqrt{\gamma}\|_{L^\infty}}{\lambda}\|\sqrt{\gamma}u'\|_{(a,b)}\|u\|_{(a,b)} + \int_a^b \chi(y)\left|1 - \frac{c(y)}{\lambda}\right||u(y)|^2\mathrm{d}y \\
&\qquad \le \left(\frac{1}{\sqrt{\lambda}}\|\sqrt{\gamma}u'\|_{(a,b)}\right)\left(\frac{\|\chi'\|_{L^\infty}\|\sqrt{\gamma}\|_{L^\infty}}{\sqrt{\lambda}}\|u\|_{(a,b)}\right) + \left(1 + \frac{\|c\|_{L^\infty}}{\lambda_0}\right)\|u\|^2_{(a,b)}.
\end{aligned}$$

Thus, for any $\overline{C} > 0$,

$$\frac{1}{\lambda}\int_\alpha^\beta \gamma(y)|u'(y)|^2\mathrm{d}y \le \left(1 + \frac{\|c\|_{L^\infty}}{\lambda_0} + \frac{\|\sqrt{\gamma}\|^2_{L^\infty}}{4\overline{C}}\frac{\|\chi'\|^2_{L^\infty}}{\lambda}\right)\|u\|^2_{(a,b)} + \frac{\overline{C}}{\lambda}\|\sqrt{\gamma}u'\|^2_{(a,b)}.$$

Plugging it into estimate (2.5) and using that $\|\chi'\|_{L^\infty} = \left\|\chi_0'\right\|_{L^\infty}(b-a)^{-1}$, we obtain

$$\begin{aligned}
\frac{b-a}{2}\left(|u(x)|^2 + \frac{\gamma(x)}{\lambda}|u'(x)|^2\right) &\le C_1\left(2 + \frac{\|c\|_{L^\infty}}{\lambda_0} + \frac{\|\sqrt{\gamma}\|^2_{L^\infty}\left\|\chi_0'\right\|^2_{L^\infty}}{4\overline{C}\lambda(b-a)^2}\right)\|u\|^2_{(a,b)} \\
&\quad + \frac{C_1(b-a)^2}{2\lambda}\|F\|^2_{(a,b)} + \frac{C_1\overline{C}}{\lambda}\|\sqrt{\gamma}u'\|^2_{(a,b)}. \qquad (2.6)
\end{aligned}$$

Applying again Lemma A.1 gives that, for any $y \in (a,b)$,

$$\frac{\gamma(y)}{\lambda}|u'(y)|^2 \le C_1\left(|u(x)|^2 + \frac{\gamma(x)}{\lambda}|u'(x)|^2\right) + C_1\frac{b-a}{\lambda}\|F\|^2_{(a,b)}.$$

Integrating in the variable $y \in (a,b)$ and setting $\overline{C} = \frac{1}{4C_1^2}$, we obtain

$$\begin{aligned}
\frac{C_1\overline{C}}{\lambda}\|\sqrt{\gamma}u'\|^2_{(a,b)} &\le C_1^2\overline{C}(b-a)\left(|u(x)|^2 + \frac{\gamma(x)}{\lambda}|u'(x)|^2\right) + \frac{C_1^2\overline{C}}{\lambda}(b-a)^2\|F\|^2_{(a,b)} \\
&\le \frac{b-a}{4}\left(|u(x)|^2 + \frac{\gamma(x)}{\lambda}|u'(x)|^2\right) + \frac{1}{4}\frac{(b-a)^2}{\lambda}\|F\|^2_{(a,b)}.
\end{aligned}$$

Plugging it into (2.6) ends the proof of Lemma 2.2. □

We now have all the ingredients to prove Proposition 2.1.

Proof of Proposition 2.1. We denote by $\omega_1, \dots, \omega_N$ the connected components of ω labeled such that

$$\sup \omega_j \leq \inf \omega_{j+1}, \qquad \forall\, j \in [\![1, N-1]\!].$$

Let

$$\lambda_0 = \max_{j \in [\![1,N]\!]} \frac{1}{|\omega_j|^2} \tag{2.7}$$

and, let $K > 0$ be such that

$$k \geq K \quad \Longrightarrow \quad \nu_k \geq \lambda_0.$$

We start with the proof of item (i). Let $\mathfrak{C} = [a, b]$ be a connected component of $\overline{(0,1)\backslash\omega}$. Integrating by parts we obtain

$$\int_{\mathfrak{C}} F(x)\varphi_k(x)\mathrm{d}x = -(\gamma u' \varphi_k)(b) + (\gamma u' \varphi_k)(a) + (u\gamma\varphi_k')(b) - (u\gamma\varphi_k')(a).$$

Recall that from (A.6),

$$|\varphi_k(x)| + \frac{1}{\sqrt{\nu_k}}|\varphi_k'(x)| \leq C, \qquad \forall\, x \in (0,1),\ \forall\, k \geq 1.$$

Similarly, applying Lemma A.1 with $y = 0$ we obtain

$$|\widetilde{\varphi}_k(x)| + \frac{1}{\sqrt{\nu_k}}|\widetilde{\varphi}_k'(x)| \leq C, \qquad \forall\, x \in (0,1),\ \forall\, k \geq 1. \tag{2.8}$$

Thus,

$$\frac{1}{\sqrt{\nu_k}}\left|\int_{\mathfrak{C}} F(x)\varphi_k(x)\mathrm{d}x\right| \leq C\left(|u(a)| + \frac{\sqrt{\gamma(a)}}{\sqrt{\nu_k}}|u'(a)|\right) + C\left(|u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}}|u'(b)|\right).$$

If $\mathfrak{C} \cap \{0,1\} = \varnothing$, then there exists $j \in [\![2, N]\!]$ such that $a \in \overline{\omega_{j-1}}$ and $b \in \overline{\omega_j}$. Applying twice Lemma 2.2 (recall that λ_0 is defined by (2.7)) we obtain

$$|u(a)| + \frac{\sqrt{\gamma(a)}}{\sqrt{\nu_k}}|u'(a)| \leq C\left(\|u\|_{\omega_{j-1}} + \frac{1}{\sqrt{\nu_k}}\|F\|_{\omega_{j-1}}\right),$$

and

$$|u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}}|u'(b)| \leq C\left(\|u\|_{\omega_j} + \frac{1}{\sqrt{\nu_k}}\|F\|_{\omega_j}\right)$$

where C now also depends on ω. This implies

$$\left|\int_{\mathfrak{C}} F(x)\varphi_k(x)\mathrm{d}x\right| \leq C\big(\sqrt{\nu_k}\|u\|_\omega + \|F\|_\omega\big).$$

The same computations hold for $\left|\int_{\mathfrak{C}} F(x)\widetilde{\varphi}_k(x)\mathrm{d}x\right|$.

Now, if $a = 0$, taking into account the boundary condition $\varphi_k(a) = 0$, the same computations yields

$$\frac{1}{\sqrt{\nu_k}}\left|\int_{\mathfrak{C}} F(x)\varphi_k(x)\mathrm{d}x\right| \leq C|u(0)| + C\left(|u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}}|u'(b)|\right).$$

As $b \in \overline{\omega}$, applying Lemma 2.2 (recall that λ_0 is defined by (2.7)) we obtain

$$|u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}}|u'(b)| \leq C\left(\|u\|_\omega + \frac{1}{\sqrt{\nu_k}}\|F\|_\omega\right)$$

where C now also depends on ω. This implies

$$\left|\int_{\mathfrak{C}} F(x)\varphi_k(x)\mathrm{d}x\right| \leq C\big(\sqrt{\nu_k}\|u\|_\omega + \sqrt{\nu_k}|u(0)| + \|F\|_\omega\big).$$

Similarly, if $b = 1$, we prove that

$$\left|\int_{\mathfrak{C}} F(x)\varphi_k(x)\mathrm{d}x\right| \leq C\big(\sqrt{\nu_k}\|u\|_\omega + \sqrt{\nu_k}|u(1)| + \|F\|_\omega\big).$$

Gathering these results proves item (i).

We now turn to the proof of item (ii). We start designing u a solution of

$$\begin{cases}(A - \nu_k)u = F,\\ u(0) = u(1) = 0,\end{cases}$$

such that

$$|u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}}|u'(x)| \leq \frac{C}{\sqrt{\nu_k}}\big(\mathcal{M}_k(F,\omega) + \|F\|_{\omega_1}\big), \qquad \forall\, x \in \overline{\omega_1}. \tag{2.9}$$

To this end let us take any solution $\bar{u}$ of

$$\begin{cases}(A - \nu_k)\bar{u} = F,\\ \bar{u}(0) = \bar{u}(1) = 0.\end{cases}$$

Such a solution exists since $\int_0^1 F(x)\varphi_k(x)\mathrm{d}x = 0$.

If $0 \notin \overline{\omega_1}$ we set $b = \inf \omega_1$ whereas if $0 \in \overline{\omega_1}$ we set $b \in \omega_1$. Notice that in both cases

$$\int_0^b F(x)\varphi_k(x)\mathrm{d}x = \bar{u}(b)\gamma(b)\varphi_k'(b) - \gamma(b)\bar{u}'(b)\varphi_k(b).$$

Applying Lemma A.1 with $y = b$, integrating with respect to the variable $x \in (0,1)$ and using $\|\varphi_k\|_{(0,1)} = 1$ we obtain that there exists $C > 0$ such that

$$|\varphi_k(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}}|\varphi_k'(b)| \geq C.$$

- If $|\varphi_k(b)| \geq \frac{C}{2}$, we set $u = \bar{u} - \frac{\bar{u}(b)}{\varphi_k(b)}\varphi_k$.

  Thus, we have $u(b) = 0$ which implies

$$\sqrt{\gamma(b)}u'(b) = \frac{-1}{\sqrt{\gamma(b)}\varphi_k(b)} \int_0^b F(x)\varphi_k(x)\mathrm{d}x.$$

  Thus,

$$|u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}}|u'(b)| \leq \frac{C}{\sqrt{\nu_k}}\left|\int_0^b F(x)\varphi_k(x)\mathrm{d}x\right|. \tag{2.10}$$

- Otherwise, we have $\frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}}|\varphi_k'(b)| \geq \frac{C}{2}$. Setting $u = \bar{u} - \frac{\bar{u}'(b)}{\varphi_k'(b)}\varphi_k$, the same computations also imply (2.10).

We now prove that (2.10) implies (2.9).
As $b \in \overline{\omega_1}$, applying Lemma A.1 and (2.10) we obtain for any $x \in \overline{\omega_1}$,

$$\begin{aligned}
|u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}}|u'(x)| &\leq C\left(|u(b)| + \frac{\sqrt{\gamma(b)}}{\sqrt{\nu_k}}|u'(b)| + \frac{1}{\sqrt{\nu_k}}\|F\|_{\omega_1}\right) \\
&\leq \frac{C}{\sqrt{\nu_k}}\left(\left|\int_0^b F(x)\varphi_k(x)\mathrm{d}x\right| + \|F\|_{\omega_1}\right).
\end{aligned}$$

- Assume first that $0 \notin \overline{\omega_1}$ and recall that $b = \inf \omega_1$. Then, by definition of $\mathcal{M}_k(F,\omega)$ (see (1.9)), we have

$$\left|\int_0^b F(x)\varphi_k(x)\mathrm{d}x\right| \leq \mathcal{M}_k(F,\omega).$$

  Thus, for any $x \in \overline{\omega_1}$,

$$|u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}}|u'(x)| \leq \frac{C}{\sqrt{\nu_k}}\big(\mathcal{M}_k(F,\omega) + \|F\|_{\omega_1}\big).$$

- Otherwise, $0 \in \overline{\omega_1}$ and we have set $b \in \omega_1$. Then, since $(0,b) \subset \omega_1$ and $\|\varphi_k\|_{(0,1)} = 1$, we have

$$\left|\int_0^b F(x)\varphi_k(x)\mathrm{d}x\right| \leq \|F\|_{\omega_1}.$$

  Thus, for any $x \in \overline{\omega_1}$,

$$|u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}}|u'(x)| \leq \frac{C}{\sqrt{\nu_k}}\|F\|_{\omega_1}.$$

Gathering these two cases proves (2.9).

We prove by induction that the function u designed at the previous step satisfies

$$|u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}}|u'(x)| \leq \frac{C}{\sqrt{\nu_k}}\big(\mathcal{M}_k(F,\omega) + \|F\|_\omega\big), \qquad \forall\, x \in \overline{\omega_j}. \tag{2.11}$$

The case $j = 1$ is exactly (2.9) that was proved in the previous step. Let $j \in [\![2, N]\!]$ be such that

$$|u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}}|u'(x)| \leq \frac{C}{\sqrt{\nu_k}}\big(\mathcal{M}_k(F,\omega) + \|F\|_\omega\big), \qquad \forall\, x \in \overline{\omega_{j-1}}.$$

Let $a_j = \sup \omega_{j-1}$ and $b_j = \inf \omega_j$. Integrating by parts we obtain

$$\begin{aligned}\frac{1}{\sqrt{\nu_k}}\int_{a_j}^{b_j} F(x)\varphi_k(x)\mathrm{d}x = u(b_j)\frac{\gamma(b_j)\varphi_k'(b_j)}{\sqrt{\nu_k}} - \frac{\gamma(b_j)u'(b_j)}{\sqrt{\nu_k}}\varphi_k(b_j)\\ - u(a_j)\frac{\gamma(a_j)\varphi_k'(a_j)}{\sqrt{\nu_k}} + \frac{\gamma(a_j)u'(a_j)}{\sqrt{\nu_k}}\varphi_k(a_j).\end{aligned}$$

The same computations hold replacing φ_k by $\widetilde{\varphi}_k$. Using the notation in Appendix A, this can be rewritten in matrix form as

$$W_k(b_j)\begin{pmatrix} u(b_j) \\ \frac{\gamma(b_j)u'(b_j)}{\sqrt{\nu_k}}\end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\nu_k}}\int_{a_j}^{b_j} F(x)\varphi_k(x)\mathrm{d}x \\ \frac{1}{\sqrt{\nu_k}}\int_{a_j}^{b_j} F(x)\widetilde{\varphi}_k(x)\mathrm{d}x\end{pmatrix} + W_k(a_j)\begin{pmatrix} u(a_j) \\ \frac{\gamma(a_j)u'(a_j)}{\sqrt{\nu_k}}\end{pmatrix}. \tag{2.12}$$

Using Lemma A.3 and the definition of $\mathcal{M}_k(F,\omega)$ (see (1.9)), we deduce that

$$|u(b_j)| + \frac{\sqrt{\gamma(b_j)}}{\sqrt{\nu_k}}|u'(b_j)| \leq \frac{C}{\sqrt{\nu_k}}\mathcal{M}_k(F,\omega) + C|u(a_j)| + C\frac{\sqrt{\gamma(a_j)}}{\sqrt{\nu_k}}|u'(a_j)|.$$

As $a_j \in \overline{\omega_{j-1}}$ the induction hypothesis imply

$$|u(a_j)| + \frac{\sqrt{\gamma(a_j)}}{\sqrt{\nu_k}}|u'(a_j)| \leq \frac{C}{\sqrt{\nu_k}}\big(\mathcal{M}_k(F,\omega) + \|F\|_\omega\big)$$

and thus we conclude that

$$|u(b_j)| + \frac{\sqrt{\gamma(b_j)}}{\sqrt{\nu_k}}|u'(b_j)| \leq \frac{C}{\sqrt{\nu_k}}(\mathcal{M}_k(F,\omega) + \|F\|_\omega).$$

As $b_j \in \overline{\omega_j}$, applying Lemma A.1 we obtain for any $x \in \overline{\omega_j}$

$$\begin{aligned}|u(x)| + \frac{\sqrt{\gamma(x)}}{\sqrt{\nu_k}}|u'(x)| &\leq C\left(|u(b_j)| + \frac{\sqrt{\gamma(b_j)}}{\sqrt{\nu_k}}|u'(b_j)| + \frac{1}{\sqrt{\nu_k}}\|F\|_{\omega_j}\right)\\ &\leq \frac{C}{\sqrt{\nu_k}}\big(\mathcal{M}_k(F,\omega) + \|F\|_\omega\big).\end{aligned}$$

This proves (2.11).

Conclusion. From (2.11) we obtain

$$|u(x)| \leq \frac{C}{\sqrt{\nu_k}}\big(\mathcal{M}_k(F,\omega) + \|F\|_\omega\big), \qquad \forall\, x \in \omega_j,\ \forall\, j \in [\![1,N]\!].$$

This leads to

$$\|u\|_{\omega_j} \leq \frac{C}{\sqrt{\nu_k}}\big(\mathcal{M}_k(F,\omega) + \|F\|_\omega\big), \qquad \forall\, j \in [\![1,N]\!]$$

with a new value of C and ends the proof of item (ii). □

2.3. Application of the minimal null control time formulas

In this section we apply the characterizations of the minimal null control time obtained in Theorems 1.3 and 1.4 to different specific configurations.

In Section 2.3.1, we recover previous characterizations of the minimal null control time proved in [4, 5] when ω is an interval. Note however that in the above references, explicit computations of eigenelements when A is the Laplace Dirichlet operator are used. Our analysis does not make use of such computations and thus extend those results to any Sturm–Liouville operator as defined in (1.2).

In Section 2.3.2, we recover null controllability in arbitrary time when q has a strict sign on a part of ω as proved in [21].

Finally, in Section 2.3.3 we prove a new null controllability result for an explicit q when ω is the union of two intervals.

2.3.1. Unification of previous formulas for the minimal null control time

Let us prove that the obtained results unifies previous characterizations given in the literature and stated in Section 1.2.

• Let us consider the setting studied in [4] i.e., $\omega = (a,b)$ and $\mathrm{Supp}(q) \subset (b,1)$.

In this case, $\overline{(0,1)\backslash\omega}$ has at most two connected components both touching the boundary of $(0,1)$. Thus, setting

$$F = (I_k(q) - q)\varphi_k$$

we obtain

$$\mathcal{M}_k\big(F,\omega\big) = \max\left\{\left|\int_0^a F(x)\varphi_k(x)\mathrm{d}x\right|, \left|\int_b^1 F(x)\varphi_k(x)\mathrm{d}x\right|\right\}.$$

Using the assumption $\mathrm{Supp}(q) \subset (b,1)$ we get

$$\left|\int_0^a F(x)\varphi_k(x)\mathrm{d}x\right| = |I_k(q)| \int_0^a \varphi_k^2(x)\mathrm{d}x,$$

and

$$\left|\int_b^1 F(x)\varphi_k(x)\mathrm{d}x\right| = \left|I_k(q)\int_b^1 \varphi_k^2(x)\mathrm{d}x - I_k(q)\right| = |I_k(q)| \int_0^b \varphi_k^2(x)\mathrm{d}x.$$

This gives

$$\mathcal{M}_k\big(F,\omega\big) = |I_k(q)| \int_0^b \varphi_k^2(x)\mathrm{d}x.$$

Recall that from (A.3)

$$\inf_{k\geq 1} \int_0^b \varphi_k^2(x)\mathrm{d}x > 0.$$

This implies that approximate controllability holds if and only if

$$I_k(q) \neq 0, \qquad \forall\ k \geq 1,$$

and in this case that

$$T_{0,q} = \limsup_{k\to+\infty} \frac{-\ln|I_k(q)|}{\nu_k}.$$

Thus we have extended the result proved in [4] for the Dirichlet–Laplace operator to a general Sturm–Liouville operator.

• Let us now consider the setting studied in [5] i.e., $\omega = (a,b)$ and $\mathrm{Supp}(q) \cap \omega = \varnothing$. Again, setting

$$F = (I_k(q) - q)\varphi_k$$

we obtain

$$\mathcal{M}_k\big(F,\omega\big) = \max\left\{\left|\int_0^a F(x)\varphi_k(x)\mathrm{d}x\right|, \left|\int_b^1 F(x)\varphi_k(x)\mathrm{d}x\right|\right\}.$$

Using the notations introduced in (1.12) we have

$$\int_0^a F(x)\varphi_k(x)\mathrm{d}x = I_k(q) \int_0^a \varphi_k^2(x)\mathrm{d}x - I_{1,k}(q) \tag{2.13}$$

and

$$\int_b^1 F(x)\varphi_k(x)\mathrm{d}x = I_k(q) \int_b^1 \varphi_k^2(x)\mathrm{d}x - I_{2,k}(q). \tag{2.14}$$

Thus,

$$\mathcal{M}_k\big(F,\omega\big) \leq 2\max\big\{|I_k(q)|, |I_{1,k}(q)|, |I_{2,k}(q)|\big\}.$$

Conversely, using (2.13) and (2.14) we have

$$\begin{aligned}\int_0^a F(x)\varphi_k(x)\mathrm{d}x + \int_b^1 F(x)\varphi_k(x)\mathrm{d}x \\ &= I_k(q)\left(\int_0^a \varphi_k^2(x)\mathrm{d}x + \int_b^1 \varphi_k^2(x)\mathrm{d}x\right) - \big(I_{1,k}(q) + I_{2,k}(q)\big) \\ &= -I_k(q)\int_a^b \varphi_k(x)^2\mathrm{d}x\end{aligned}$$

where we have used that $I_k(q) = I_{1,k}(q) + I_{2,k}(q)$. Thus, from (A.3) we get

$$|I_k(q)| \leq C\mathcal{M}_k\big(F,\omega\big).$$

Using (2.13) or (2.14) and the previous inequality we obtain

$$|I_{j,k}(q)| \leq C\mathcal{M}_k\big(F,\omega\big), \qquad \forall\, j \in \{1,2\}.$$

Thus,

$$\max\big\{|I_k(q)|, |I_{1,k}(q)|, |I_{2,k}(q)|\big\} \leq C\mathcal{M}_k\big(F,\omega\big).$$

This implies that approximate controllability holds if and only if

$$\max\big\{|I_k(q)|, |I_{1,k}(q)|, |I_{2,k}(q)|\big\} \neq 0, \qquad \forall\, k \geq 1$$

and in this case

$$T_{0,q} = \limsup_{k\to+\infty} \frac{-\ln\max\big\{|I_k(q)|, |I_{1,k}(q)|, |I_{2,k}(q)|\big\}}{\nu_k}.$$

Thus we have extended the result proved in [5] for the Dirichlet–Laplace operator to a general Sturm–Liouville operator.

2.3.2. Null controllability in arbitrary time with intersecting control and coupling regions

Let us here consider the setting studied in [21].

Proposition 2.3. *Assume that there exists an open set $\omega_0 \subset \omega$ and $q_0 > 0$ such that*

$$\inf_{\omega_0} q \geq q_0 \quad \text{or} \quad \sup_{\omega_0} q \leq -q_0,$$

then, system (1.1) *is null controllable in any time $T > 0$.*

Even though this result is already known from [21], we provide here a proof without Carleman estimates.

Proof. We assume that $\inf_{\omega_0} q \geq q_0$ since the other case is similar. Here we consider the minimal time characterization given by Theorem 1.3 and we shall prove that $I_k(q)^2\|\varphi_k\|_\omega^2 + \|\psi_{k,q}\|_\omega^2$ does not tend to zero exponentially fast, with respect to ν_k, as k goes to infinity.

We split $\psi_{k,q}$ into two parts $\psi_{k,q} = \psi_{k,q,1} + \psi_{k,q,2}$ where $\psi_{k,q,1}$ is the unique solution of the Cauchy problem

$$\begin{cases}(A - \nu_k)\psi_{k,q,1} = I_k(q)\varphi_k,\\ \psi_{k,q,1}(0) = 0,\\ \psi'_{k,q,1}(0) = 0.\end{cases} \tag{2.15}$$

From Lemma A.1, there exists $C > 0$ depending only on γ and c such that

$$\|\psi_{k,q,1}\|_{L^\infty(0,1)} \leq \frac{C}{\sqrt{\nu_k}}|I_k(q)|.$$

Then, from (A.3), we deduce that, when $\nu_k \geq 1$,

$$\begin{aligned}\|\psi_{k,q,2}\|_\omega^2 &\leq 2\big(\|\psi_{k,q}\|_\omega^2 + \|\psi_{k,q,1}\|_{L^\infty(0,1)}^2\big)\\ &\leq 2\left(\|\psi_{k,q}\|_\omega^2 + \frac{C}{\nu_k}|I_k(q)|^2\right)\\ &\leq C\Big(\|\psi_{k,q}\|_\omega^2 + |I_k(q)|^2\|\varphi_k\|_\omega^2\Big).\end{aligned}$$

Thus, in order to prove the result, it is enough to find some explicit lower bound $r_k > 0$ such that

$$\|\psi_{k,q,2}\|_\omega \geq r_k \quad \text{with} \quad \limsup_{k\to+\infty} \frac{-\ln r_k}{\nu_k} = 0. \tag{2.16}$$

As we seek for a lower bound for $\psi_{k,q,2}$ on ω, and thanks to our assumption on q, we can restrict ω to an interval (a, b) such that $q(x) \geq q_0 > 0$ for almost every $x \in \omega$. Taking some $\ell > 0$ small enough to be determined later (see (2.19)), we introduce the following subsets of ω:

- $\omega_1 = (a, a + \ell)$;
- $\omega_2 = (b - \ell, b)$;
- $\widetilde{\omega} = \omega_1 \cup \omega_2$;
- $\mathfrak{C}_0 = \left[\frac{a+b}{2} - \frac{b-a}{6}, \frac{a+b}{2} + \frac{b-a}{6}\right]$;
- $\mathfrak{C} = [a + \ell, b - \ell]$.

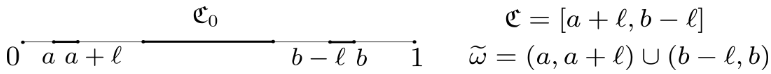

FIGURE 2.1. Splitting of $\omega = (a, b)$

This configuration is pictured in Figure 2.1. Notice that $\widetilde{\omega}$ is a subset of ω and thus for any $k \geq 1$,

$$\|\psi_{k,q,2}\|_{\omega}^2 \geq \|\psi_{k,q,2}\|_{\widetilde{\omega}}^2.$$

From (A.3), there exists $\alpha_1 > 0$ depending on γ, c and $\mathfrak{C}_0$ such that

$$\int_{\mathfrak{C}_0} \varphi_k^2(x)\mathrm{d}x \geq \alpha_1, \qquad \forall\, k \geq 1. \tag{2.17}$$

Following closely the proof of item i of Proposition 2.1 with a careful tracking of the dependency with respect to ℓ we can obtain the following lemma whose proof is postponed at the end of the section.

Lemma 2.4. *There exists $\alpha_2 > 0$ depending on γ and c such that for any $\ell < \frac{b-a}{3}$, any $k \geq 1$ such that $\nu_k \geq 1$, any $F \in L^2(0, 1; \mathbb{R})$ and any u satisfying the differential equation*

$$(A - \nu_k)u = F,$$

we have

$$\sqrt{\ell}\left|\int_{\mathfrak{C}} F(x)\varphi_k(x)\mathrm{d}x\right| \leq \alpha_2\sqrt{\nu_k}\left(1 + \frac{1}{\ell\sqrt{\nu_k}}\right)\|u\|_{\widetilde{\omega}} + \alpha_2\ell\|F\|_{\widetilde{\omega}}. \tag{2.18}$$

It is important to notice that the norms in the right-hand side of (2.18) are taken on the small set $\widetilde{\omega}$ whereas the left-hand side is an integral on the large set $\mathfrak{C} = \omega \setminus \widetilde{\omega}$. Hence, this inequality can be understood as an estimate of cancellations that occur in this integral.

Let $\alpha_2 > 0$ be the constant given in the above lemma and assume in all what follows that $\ell > 0$ is fixed such that

$$\sqrt{\ell} < \min\left\{\sqrt{\frac{b-a}{3}}, \frac{q_0\alpha_1}{2\alpha_2\|q\|_{L^\infty(0,1)}}\right\}. \tag{2.19}$$

There exists $K \in \mathbb{N}^*$ such that

$$\nu_k \geq \frac{1}{\ell^2}, \qquad \forall\, k \geq K.$$

In the rest of the proof, we assume that $k \geq K$.

Thanks to the equations (1.6) and (2.15) satisfied respectively by $\psi_{k,q}$ and $\psi_{k,q,1}$, we see that $\psi_{k,q,2}$ solves

$$(A - \nu_k)\psi_{k,q,2} = -q\varphi_k, \qquad \text{in } (0, 1).$$

Applying Lemma 2.4, with $u = \psi_{k,q,2}$ and $F = -q\varphi_k$ we obtain

$$\sqrt{\ell}\left|\int_{\mathfrak{C}} q(x)\varphi_k^2(x)\mathrm{d}x\right| \leq 2\alpha_2\sqrt{\nu_k}\|\psi_{k,q,2}\|_{\widetilde{\omega}} + \alpha_2\ell\|q\varphi_k\|_{\widetilde{\omega}}.$$

As $\ell < \frac{b-a}{3}$ we have $\mathfrak{C}_0 \subset \mathfrak{C}$ and thus

$$\left|\int_{\mathfrak{C}} q(x)\varphi_k^2(x)\mathrm{d}x\right| \geq q_0\int_{\mathfrak{C}_0}\varphi_k^2(x)\mathrm{d}x \geq q_0\alpha_1.$$

Notice also that, since $\|\varphi_k\|_{(0,1)} = 1$, we have

$$\|q\varphi_k\|_{\widetilde{\omega}} \leq \|q\|_{L^\infty(0,1)}.$$

Gathering these estimates and using (2.19) we obtain the lower bound

$$\begin{aligned}2\alpha_2\sqrt{\nu_k}\|\psi_{k,q,2}\|_{\widetilde{\omega}} &\geq \sqrt{\ell}\left|\int_{\mathfrak{C}} q(x)\varphi_k^2(x)\mathrm{d}x\right| - \alpha_2\ell\|q\varphi_k\|_{\widetilde{\omega}}\\ &\geq \sqrt{\ell}\Big(q_0\alpha_1 - \alpha_2\sqrt{\ell}\|q\|_{L^\infty(0,1)}\Big) \geq \sqrt{\ell}\,\frac{q_0\alpha_1}{2}\end{aligned}$$

which leads to (2.16) and ends the proof of Proposition 2.3. □

It remains to prove the lemma.

Proof of Lemma 2.4. From (A.6), there exists $C > 0$ depending on γ and c such that

$$\|\varphi_k\|_{L^\infty(0,1)} + \frac{1}{\sqrt{\nu_k}}\|\varphi_k'\|_{L^\infty(0,1)} \leq C, \qquad \forall\, k \geq 1. \tag{2.20}$$

Integrating by parts, we obtain

$$\begin{aligned}\int_{\mathfrak{C}} F(x)\varphi_k(x)\mathrm{d}x &= \int_{a+\ell}^{b-\ell}(A-\nu_k)u(x)\varphi_k(x)\mathrm{d}x\\ &= -\big(\gamma u'\varphi_k\big)(b-\ell) + \big(\gamma u'\varphi_k\big)(a+\ell)\\ &\quad + \big(u\gamma\varphi_k'\big)(b-\ell) - \big(u\gamma\varphi_k'\big)(a+\ell).\end{aligned}$$

Using (2.20) we obtain

$$\begin{aligned}\frac{1}{\sqrt{\nu_k}}\left|\int_{\mathfrak{C}} F(x)\varphi_k(x)\mathrm{d}x\right| \leq{}& C\|\sqrt{\gamma}\|_{L^\infty}\left(|u(a+\ell)| + \frac{\sqrt{\gamma(a+\ell)}}{\sqrt{\nu_k}}|u'(a+\ell)|\right)\\ &+ C\|\sqrt{\gamma}\|_{L^\infty}\left(|u(b-\ell)| + \frac{\sqrt{\gamma(b-\ell)}}{\sqrt{\nu_k}}|u'(b-\ell)|\right).\end{aligned}$$

Let $\lambda_0 = 1$ and let $K \in \mathbb{N}^*$ be such that

$$k \geq K \quad \Longrightarrow \quad \nu_k \geq \lambda_0.$$

Assume that $k \geq K$. As $a+\ell \in \overline{\omega_1}$ the application of Lemma 2.2 (recall that $\lambda_0 = 1$) yields

$$|u(a+\ell)| + \frac{\sqrt{\gamma(a+\ell)}}{\sqrt{\nu_k}}|u'(a+\ell)| \leq \frac{C}{\sqrt{\ell}}\left(1+\frac{1}{\ell\sqrt{\nu_k}}\right)\|u\|_{\omega_1} + \frac{C\sqrt{\ell}}{\sqrt{\nu_k}}\|F\|_{\omega_1}.$$

As $b-\ell \in \overline{\omega_2}$ the application of Lemma 2.2 yields

$$|u(b-\ell)| + \frac{\sqrt{\gamma(b-\ell)}}{\sqrt{\nu_k}}|u'(b-\ell)| \leq \frac{C}{\sqrt{\ell}}\left(1+\frac{1}{\ell\sqrt{\nu_k}}\right)\|u\|_{\omega_2} + \frac{C\sqrt{\ell}}{\sqrt{\nu_k}}\|F\|_{\omega_2}$$

which concludes the proof. □

2.3.3. **Dealing with new geometric configurations**

We now illustrate that the minimal time formula obtained in Theorem 1.4 can be successfully exploited to give an explicit value of this minimal time in more general geometric configurations than the one available in the literature, for example when ω is not an interval and $\mathrm{Supp}(q) \cap \omega = \emptyset$. We provide below an example inspired by [14].

Proposition 2.5. *Let A be the Dirichlet Laplace operator (i.e., $\gamma = 1$ and $c = 0$) and let*

$$q : x \in (0,1) \longmapsto \left(x - \frac{1}{2}\right)\mathbf{1}_{\left(\frac{1}{4},\frac{3}{4}\right)}(x).$$

(i) *If $\omega \subset \left(\frac{3}{4}, 1\right)$, then approximate controllability for system* (1.1) *does not hold.*

(ii) *If $\omega = \left(0, \frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right)$, then system* (1.1) *is null controllable from X in any time $T > 0$.*

Proof. In this case, we have for any $k \geq 1$,

$$\nu_k = k^2\pi^2, \qquad \varphi_k = \sqrt{2}\sin(k\pi\cdot), \qquad \widetilde{\varphi}_k = \cos(k\pi\cdot).$$

The proof of item (i) can be found in [14, Section 3.3.1] and relies on explicit computations: due to symmetry it comes that $I_k(q) = 0$ for any $k \geq 1$. This implies that

$$\int_0^{\inf(\omega)} q(x)\varphi_k(x)\varphi_k(x)\mathrm{d}x = I_k(q) = 0.$$

Let $\mathfrak{C}$ be any other connected component of $\overline{(0,1)\backslash\omega}$ than $[0, \inf(\omega)]$. Then $\mathfrak{C} \subset \left(\frac{3}{4}, 1\right)$. This means that $q = 0$ on $\mathfrak{C}$ which gives

$$\int_{\mathfrak{C}} q(x)\varphi_k(x)\varphi_k(x)\mathrm{d}x = \int_{\mathfrak{C}} q(x)\varphi_k(x)\widetilde{\varphi}_k(x)\mathrm{d}x = 0.$$

Thus,

$$\mathcal{M}_k(q\varphi_k, \omega) = 0, \qquad \forall\, k \geq 1.$$

We now turn to item (ii). In this case $\overline{(0,1)\backslash\omega}$ has only one connected component which is $\left[\frac{1}{4}, \frac{3}{4}\right]$ but the key point is that it does not touch the boundary of $(0,1)$. Approximate controllability in this case was also studied in [14, Section 3.3.1]. Again for symmetry reasons we have

$$\int_{\frac{1}{4}}^{\frac{3}{4}} q(x)\varphi_k(x)\varphi_k(x)\mathrm{d}x = 0, \qquad \forall\, k \geq 1,$$

but

$$\int_{\frac{1}{4}}^{\frac{3}{4}} q(x)\varphi_k(x)\widetilde{\varphi}_k(x)\mathrm{d}x = \begin{cases} -\dfrac{(-1)^{\frac{k-1}{2}}}{2\sqrt{2}\pi^2 k^2}, & \text{if } k \text{ is odd,} \\ -\dfrac{(-1)^{\frac{k}{2}}}{4\sqrt{2}\pi k}, & \text{if } k \text{ is even.} \end{cases}$$

This implies that for any $k \geq 1$,

$$\mathcal{M}_k\big((I_k(q) - q)\varphi_k, \omega\big) = \begin{cases} \dfrac{1}{2\sqrt{2}\pi^2 k^2}, & \text{if } k \text{ is odd,} \\ \dfrac{1}{4\sqrt{2}\pi k}, & \text{if } k \text{ is even.} \end{cases}$$

Thus, from Theorem 1.4, we get

$$T_{0,q} = \limsup_{k\to+\infty} \frac{-\ln \mathcal{M}_k\big((I_k(q) - q)\varphi_k,\ \omega\big)}{\nu_k} = 0,$$

which means that null-controllability holds at any time $T > 0$. □

3. Cascade system with a first order coupling term

In this section we describe how the analysis conducted in Section 2 can be directly extended to system (1.3) that is when the coupling between the two equations operates through a zero order term and a first order term. This is for instance the setting studied in [16] and that we complete here.

3.1. Setting and spectral analysis

To fit in the formalism of [13], we define

- the evolution operator $\mathcal{A}$ by

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ q + p\partial_x & A \end{pmatrix}, \qquad D(\mathcal{A}) = D(A)^2,$$

and the control operator $\mathcal{B}$ by

$$\mathcal{B} : u \in U = L^2(0,1) \longmapsto \begin{pmatrix} \mathbf{1}_\omega u \\ 0 \end{pmatrix}.$$

It will be convenient to separate the symmetric and skew-symmetric parts of the coupling terms in $\mathcal{A}$. In order to do so, we define a function r and an operator S_p as follows

$$r = q - \frac{1}{2}p', \quad \text{and} \quad S_p = \frac{1}{2}p' + p\partial_x. \tag{3.1}$$

We observe that S_p is skew-symmetric in $D(A)$ and that we can write

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ r + S_p & A \end{pmatrix}.$$

- The adjoint operator of $\mathcal{A}$ is given by

$$\mathcal{A}^* = \begin{pmatrix} A & q - \partial_x(p\cdot) \\ 0 & A \end{pmatrix} = \begin{pmatrix} A & r - S_p \\ 0 & A \end{pmatrix}, \qquad D(\mathcal{A}^*) = D(\mathcal{A}).$$

Recall that $I_k(r)$ is defined by (1.7). In this section, $\psi_{k,r,p}$ denotes the unique solution of

$$\begin{cases} (A - \nu_k)\psi_{k,r,p} = \big(I_k(r) - r\big)\varphi_k + S_p\varphi_k, \\ \psi_{k,r,p}(0) = \psi_{k,r,p}(1) = 0, \\ \langle \varphi_k, \psi_{k,r,p} \rangle_\omega = 0. \end{cases} \tag{3.2}$$

This system has indeed a unique solution since, due to the definition of $I_k(r)$ and the fact that S_p is skew-symmetric, the right-hand side of this equation is orthogonal to φ_k.

Let us detail the spectral analysis of the operator $\mathcal{A}^*$: its spectrum is given by $\Lambda = (\nu_k)_{k\geq 1}$ and we can distinguish the following cases.

- If $I_k(r) \neq 0$ then ν_k is algebraically double and geometrically simple. An associated Jordan chain is given by

$$\phi_k^0 = \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix}, \qquad \phi_k^1 = \frac{1}{I_k(r)} \begin{pmatrix} \psi_{k,r,p} \\ \varphi_k \end{pmatrix}. \tag{3.3}$$

- If $I_k(r) = 0$ then ν_k is geometrically double and an associated basis of eigenvectors is given by

$$\phi_{k,1}^0 = \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix}, \qquad \phi_{k,2}^0 = \begin{pmatrix} \psi_{k,r,p} \\ \varphi_k \end{pmatrix}. \tag{3.4}$$

Except from the definition of $\psi_{k,r,p}$, the spectral analysis is the same as for system (1.1) (see Section 2.1). Thus, for the operators $\mathcal{A}$ and $\mathcal{B}$ to satisfy the assumption (H) stated in [13, Section 2.1.2] it only remains to study the approximate controllability condition

$$\operatorname{Ker}(\mathcal{A}^* - \lambda) \cap \operatorname{Ker} \mathcal{B}^* = \{0\}, \qquad \forall\, \lambda \in \Lambda.$$

This is the goal of the following section.

3.2. Approximate controllability

From the Fattorini–Hautus test, we obtain the following characterization for approximate controllability of system (1.3).

Proposition 3.1. *Let $\omega \subset (0,1)$ be a non empty set and let $q \in L^\infty(0,1)$ and $p \in W^{1,\infty}(0,1)$. Approximate controllability of system* (1.3) *holds if and only if*

$$\mathcal{M}_k\big(r\varphi_k - S_p\varphi_k, \omega\big) \neq 0, \qquad \forall\, k \geq 1 \text{ such that } r\varphi_k - S_p\varphi_k = 0 \text{ in } \omega. \tag{3.5}$$

The proof follows directly from [14, Theorems 2.1 and 2.2].

Notice that, for approximate controllability to hold, we have two very different situations.

- When $\big(\operatorname{Supp}(q) \cup \operatorname{Supp}(p)\big) \cap \omega = \varnothing$ condition (3.5) has to be checked for any $k \geq 1$.
- Whereas, when $\big(\operatorname{Supp}(q) \cup \operatorname{Supp}(p)\big) \cap \omega \neq \varnothing$ condition (3.5) has to be checked for at a most a single $k \geq 1$.

Remark 3.2. The question of approximate controllability for system (1.3) was already studied in [16, Theorems 1.1 and 1.2]. There it is stated that, if

$$(\operatorname{Supp}(p) \cup \operatorname{Supp}(q)) \neq \varnothing,$$

approximate controllability holds in any time. In fact, this result is not correct since there can exist $k \geq 1$ such that

$$r\varphi_k - S_p\varphi_k = 0 \text{ in } \omega \quad \text{ and } \quad \mathcal{M}_k\big(r\varphi_k - S_p\varphi_k, \omega\big) = 0.$$

Such a counter-example was constructed by A. Dupouy in her Master Thesis [15], under the supervision of the first author.

We set $q = 0$, which implies that $r - S_p = -\partial_x(p\bullet)$. For a given $k \geq 1$, the idea is to select an interval $\omega = (a,b)$ such that $\varphi_k \neq 0$ on $\overline{\omega}$, which is possible since φ_k has only finitely many zeros in $(0,1)$. Then, we choose $p = \frac{1}{\varphi_k}$ in ω so that, by construction

$r\varphi_k - S_p\varphi_k = -\partial_x(p\varphi_k) = 0$ in ω. Finally, it is possible to extend p outside ω with appropriate regularity such that

$$\int_0^a \varphi_k(x)\partial_x(p\varphi_k)(x)\mathrm{d}x = \int_b^1 \varphi_k(x)\partial_x(p\varphi_k)(x)\mathrm{d}x = 0,$$

i.e., $\mathcal{M}_k\big(r\varphi_k - S_p\varphi_k, \omega\big) = 0$.

Due to the analysis conducted in Section 3.1, under the assumption (3.5), the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumption (H) stated in [13, Section 2.1.2].

For more coherence with the expression of the minimal null control time obtained in Theorem 3.6 below, instead of the approximate controllability condition (3.5), we use the following characterization.

Lemma 3.3. *Let $\omega \subset (0,1)$ be a non empty set and let $q \in L^\infty(0,1)$ and $p \in W^{1,\infty}(0,1)$. Assume that $\big(\mathrm{Supp}(q) \cup \mathrm{Supp}(p)\big) \cap \omega = \emptyset$. Then, for any $k \geq 1$,*

$$\mathcal{M}_k\big(r\varphi_k - S_p\varphi_k,\ \omega\big) = 0 \quad \Longleftrightarrow \quad \mathcal{M}_k\big((I_k(r) - r)\varphi_k + S_p\varphi_k,\ \omega\big) = 0.$$

Thus, approximate controllability of system (1.3) *holds if and only if*

$$\mathcal{M}_k\big((I_k(r) - r)\varphi_k + S_p\varphi_k, \omega\big) \neq 0, \qquad \forall\, k \geq 1 \text{ such that } r\varphi_k - S_p\varphi_k = 0 \text{ in } \omega. \tag{3.6}$$

Proof. Let $k \geq 1$. First of all notice that for any connected component $\mathfrak{C}$ of $\overline{(0,1)\backslash\omega}$ we have

$$\int_{\mathfrak{C}} (S_p\varphi_k)(x)\varphi_k(x)\mathrm{d}x = 0. \tag{3.7}$$

Indeed, for any $a, b \in [0,1]$ such that $p(a) = p(b) = 0$, integrating by parts we obtain

$$\begin{aligned}\int_a^b (S_p\varphi_k)(x)\varphi_k(x)\mathrm{d}x &= \int_a^b \left(\frac{1}{2}\partial_x p(x)\varphi_k(x) + p(x)\partial_x\varphi_k(x)\right)\varphi_k(x)\mathrm{d}x \\ &= -\int_a^b p(x)\partial_x\varphi_k(x)\varphi_k(x)\mathrm{d}x + \int_a^b p(x)\partial_x\varphi_k(x)\varphi_k(x)\mathrm{d}x \\ &= 0.\end{aligned}$$

Thus, the assumption $\mathrm{Supp}(p) \cap \omega = \emptyset$ proves (3.7).

Now assume that $\mathcal{M}_k\big(r\varphi_k - S_p\varphi_k,\ \omega\big) = 0$. Then, using (3.7), for any connected component $\mathfrak{C}$ of $\overline{(0,1)\backslash\omega}$ we have

$$\int_{\mathfrak{C}} r(x)\varphi_k^2(x)\mathrm{d}x = \int_{\mathfrak{C}} (S_p\varphi_k)(x)\varphi_k(x)\mathrm{d}x = 0.$$

Since $\mathrm{Supp}(r) \cap \omega = \emptyset$, this gives

$$I_k(r) = \sum_{\mathfrak{C}\in C\left(\overline{(0,1)\backslash\omega}\right)} \int_{\mathfrak{C}} r(x)\varphi_k(x)^2\mathrm{d}x = 0$$

which proves that $\mathcal{M}_k\big((I_k(r) - r)\varphi_k + S_p\varphi_k,\ \omega\big) = 0$.

Finally assume that $\mathcal{M}_k\big((I_k(r) - r)\varphi_k + S_p\varphi_k,\ \omega\big) = 0$. Then, using (3.7), for any connected component $\mathfrak{C}$ of $\overline{(0,1)\backslash\omega}$ we have

$$I_k(r)\int_{\mathfrak{C}} \varphi_k(x)^2 \mathrm{d}x = \int_{\mathfrak{C}} r(x)\varphi_k(x)^2 \mathrm{d}x.$$

Since $\mathrm{Supp}(r) \cap \omega = \varnothing$, this gives

$$\begin{aligned} I_k(r)\Big(1 - \|\varphi_k\|_\omega^2\Big) &= I_k(r) \sum_{\mathfrak{C}\in C\left(\overline{(0,1)\backslash\omega}\right)} \int_{\mathfrak{C}} \varphi_k(x)^2\mathrm{d}x \\ &= \sum_{\mathfrak{C}\in C\left(\overline{(0,1)\backslash\omega}\right)} \int_{\mathfrak{C}} r(x)\varphi_k(x)^2\mathrm{d}x = I_k(r). \end{aligned}$$

Using (A.3) we obtain $I_k(r) = 0$ and thus $\mathcal{M}_k\big(r\varphi_k - S_p\varphi_k,\ \omega\big) = 0$. □

3.3. **Minimal null control time**

We now turn to the determination of the minimal null control time. For this system, we have a result which is similar to Theorem 1.3 and that reads as follows.

Theorem 3.4. *Let $\omega \subset (0,1)$ be a non empty open set and let $q \in L^\infty(0,1)$ and $p \in W^{1,\infty}(0,1)$. Assume that* (3.5) *holds. Then, the minimal null control time $T_{0,q,p}$ for system* (1.3) *is given by*

$$T_{0,q,p} = \limsup_{k\to+\infty} \frac{-\ln\big(I_k(r)^2\|\varphi_k\|_\omega^2 + \|\psi_{k,r,p}\|_\omega^2\big)}{2\nu_k}$$

where $\psi_{k,r,p}$ is given by (3.2).

The proof follows exactly the proof of Theorem 1.3 and is left to the reader. The only difference is that, due to the change of definition of $\psi_{k,r,p}$ one cannot use Lemma A.2 but shall instead use the following lemma.

Lemma 3.5. *There exists $C > 0$ such that*

$$\|\psi_{k,r,p}\|_{(0,1)} \le C, \qquad \forall\ k \ge 1$$

where $\psi_{k,r,p}$ is given by (3.2).

The proof follows the proof of Lemma A.2 with the use of the estimate

$$\int_0^1 \partial_x(p\varphi_k)(x)^2\mathrm{d}x \le 2\|p'\|_\infty^2 + 2\int_0^1 p(x)^2\varphi_k'(x)^2\mathrm{d}x \le C\nu_k$$

due to (A.2).

Then, as in Theorem 1.4, we can simplify the formula in the case where the coupling terms are not active in the control domain.

Theorem 3.6. *Let $\omega \subset (0,1)$ be a non empty open set with a finite number of connected components. Let $q \in L^\infty(0,1)$ and $p \in W^{1,\infty}(0,1)$ be such that*

$$\big(\mathrm{Supp}(q) \cup \mathrm{Supp}(p)\big) \cap \omega = \varnothing.$$

Assume that (3.6) *holds. Then, the minimal null control time for system* (1.1) *is given by*

$$T_{0,q,p} = \limsup_{k\to+\infty} \frac{-\ln \mathcal{M}_k\big((I_k(r)-r)\varphi_k + S_p\varphi_k,\ \omega\big)}{\nu_k}$$

where $\mathcal{M}_k$ is defined by (1.9).

The proof follows exactly the proof of Theorem 1.4 and is left to the reader.

3.4. Applications of the minimal null control time formulas

3.4.1. When the coupling is not active in the control region

In this section, we assume that

$$\big(\mathrm{Supp}(p) \cup \mathrm{Supp}(q)\big) \cap \omega = \varnothing. \tag{3.8}$$

Assume first that $\omega = (a,b)$ is an interval.

In that case, and when A is the Dirichlet–Laplace operator, it is proved in [16, Theorem 1.4] that, under the condition (3.8), when approximate controllability holds, the minimal null control time is given by

$$T_{0,q,p} = \limsup_{k\to+\infty} \frac{-\ln \max\big\{|I_k(r)|,\ \big|I_{1,k}(r)\big|,\ \big|I_{2,k}(r)\big|\big\}}{\nu_k}, \tag{3.9}$$

where $I_{1,k}$ and $I_{2,k}$ are defined in (1.12).

Let us show that the formulation given in Theorem 3.6 allows to recover this result, for a general diffusion operator A.

Since ω is an interval, setting

$$F = (I_k(r)\varphi_k - r\varphi_k) + S_p\varphi_k,$$

we have

$$\mathcal{M}_k\big(F,\omega\big) = \max\left\{\left|\int_0^a F(x)\varphi_k(x)\mathrm{d}x\right|, \left|\int_b^1 F(x)\varphi_k(x)\mathrm{d}x\right|\right\}.$$

Due to the assumption $\mathrm{Supp}(p)\cap\omega = \varnothing$ we can use (3.7) to get

$$\int_0^a (S_p\varphi_k)\varphi_k \mathrm{d}x = \int_b^1 (S_p\varphi_k)\varphi_k \mathrm{d}x = 0.$$

Thus, it follows that

$$\int_0^a F(x)\varphi_k(x)\mathrm{d}x = I_k(r)\int_0^a \varphi_k^2(x)\mathrm{d}x - I_{1,k}(r)$$

and

$$\int_b^1 F(x)\varphi_k(x)\mathrm{d}x = I_k(r)\int_b^1 \varphi_k^2(x)\mathrm{d}x - I_{2,k}(r).$$

The rest of the proof follows that of Section 2.3.1, by using Theorem 3.6.

In the previous setting it appears that the minimal control time given in (3.9) only depends on the quantity r. We will show now that when the control domain ω is not an interval, this may not be true any more. More precisely, we shall design an example such that $r = 0$, but nevertheless null controllability holds for any time $T > 0$.

Assume that $\omega = (0, a) \cup (b, 1)$ with $0 < a < b < 1$. The main difference with the previous situation comes from the fact that $\overline{(0,1)\backslash\omega}$ has a (unique) connected component that does not touch the boundary of the domain, which makes an important difference in the definition of the quantities $\mathfrak{M}_k$, see Section 1.2.1.

We build our example as follows. We first choose a smooth function p supported in (a, b) and such that

$$\int_a^b \frac{p(x)}{\gamma(x)}\,\mathrm{d}x \neq 0. \tag{3.10}$$

We now set $q = \frac{p'}{2}$ in such a way that $r = q - \frac{1}{2}p' = 0$. Moreover, by assumption on p, the condition (3.8) holds.

For any k, since r and p are supported outside ω, we immediately have that

$$r\varphi_k - S_p\varphi_k = 0, \text{ in } \omega,$$

and, by (1.8) and (3.7), we get

$$\mathcal{M}_k\big((I_k(r) - r)\varphi_k + S_p\varphi_k,\ \omega\big) = \left|\int_a^b (S_p\varphi_k)(x)\widetilde{\varphi}_k(x)\mathrm{d}x\right|.$$

By definition of S_p we can integrate by parts, using that $p(a) = p(b) = 0$, to find

$$\int_a^b (S_p\varphi_k)(x)\widetilde{\varphi}_k(x)\mathrm{d}x = \frac{1}{2}\int_a^b \frac{p(x)}{\gamma(x)}W_k(x)\mathrm{d}x,$$

where $W_k = (\gamma\varphi_k')\widetilde{\varphi}_k - \varphi_k(\gamma\widetilde{\varphi}_k')$ is the Wronskian of φ_k and $\widetilde{\varphi}_k$. Since φ_k and $\widetilde{\varphi}_k$ solve the same second order linear ODE, this Wronskian is constant and we get

$$\int_a^b (S_p\varphi_k)(x)\widetilde{\varphi}_k(x)\mathrm{d}x = \frac{\gamma(0)\varphi_k'(0)}{2}\int_a^b \frac{p(x)}{\gamma(x)}\mathrm{d}x.$$

Thanks to the assumption (3.10) we see that this quantity is not zero, which proves the approximate controllability condition (3.6). In addition, by using Theorem 3.6 and the asymptotics (A.2), it follows that the minimal null control time for our system is simply given by

$$T_{0,q,p} = \limsup_{k\to+\infty} \frac{-\ln|\varphi_k'(0)|}{\nu_k} = 0.$$

In this case, despite the fact that $r = 0$, we get that the system is null-controllable at any time $T > 0$.

Observe that if the control domain is restricted to $\omega = (0, a)$ (or $\omega = (b, 1)$) then this particular system is not even approximately controllable.

3.4.2. **When the coupling is active in the control region**

We now use the formulation given in Theorem 3.4 and the computations done in Section 2.3.2 to get the following sufficient condition for null controllability in arbitrary small time.

Proposition 3.7. *Assume that the coefficients defining the Sturm–Liouville operator A in* (1.2) *are sufficiently regular, i.e., $\gamma \in C^2([0,1])$ and $c \in C^0([0,1])$. Assume that there exists an open set $\omega_0 \subset \omega$ and $r_0 > 0$ such that*

$$\inf_{\omega_0} r \geq r_0 \quad \textit{or} \quad \sup_{\omega_0} r \leq -r_0 \tag{3.11}$$

and that the approximate controllability condition (3.5) *holds. Then, system* (1.3) *is null controllable at any time $T > 0$.*

We observe that the approximate controllability condition is crucial in this result. For instance, the example shown in Remark 3.2 is not approximately controllable even if we have $r = -\frac{1}{2}\left(\frac{1}{\phi_k}\right)'$ which clearly satisfies (3.11).

Proof. The proof follows closely the one in Section 2.3.2 but needs to be adapted to handle some boundary terms coming from integration by parts in integrals involving the first order coupling terms. We assume that $\inf_{\omega_0} r \geq r_0$, the other case being similar.

From Theorem 3.4, it is sufficient to prove that the quantity $I_k(r)\|\varphi_k\|_\omega^2 + \left\|\psi_{k,r,p}\right\|_\omega^2$ does not tend exponentially fast to zero with respect to the eigenvalue ν_k.

The contribution of $I_k(r)$ is dealt with as in Section 2.3.2 by writting $\psi_{k,r,p} = \psi_{k,r,p,1} + \psi_{k,r,p,2}$ with $\psi_{k,r,p,1}$ solving the Cauchy problem

$$\begin{cases} (A - \nu_k)\psi_{k,r,p,1} = I_k(r)\varphi_k, \\ \psi_{k,r,p,1}(0) = 0, \\ \psi_{k,r,p,1}'(0) = 0. \end{cases}$$

It is thus sufficient to obtain a lower bound of the following form

$$\left\|\psi_{k,r,p,2}\right\|_\omega^2 \geq R_k \text{ with } \limsup_{k\to+\infty} \frac{-\ln R_k}{\nu_k} = 0, \tag{3.12}$$

where $\psi_{k,r,p,2}$ satisfies the equation

$$(A - \nu_k)\psi_{k,r,p,2} = -r\varphi_k + S_p\varphi_k.$$

As we seek for a lower bound it is sufficient to assume that $\omega = (a, b)$ is an interval and that $r(x) \geq r_0$ for almost every $x \in \omega$.

Due to Sturm oscillation theorem (see for instance [12, Corollary A.4.33]), there exists $\ell_0 \in (0, 1)$ and $K \in \mathbb{N}^*$ depending on γ, c and $b - a$, such that for any $k \geq K$ there exists $c_k, d_k \in (a, b)$ satisfying

$$\begin{cases} \varphi_k(c_k) = \varphi_k(d_k) = 0, \\ |d_k - c_k| \geq \frac{3}{4}|b - a| \quad \text{and} \quad \min(|b - d_k|, |c_k - a|) \geq \ell_0, \\ \nu_k^2 \geq \frac{2}{\ell_0}. \end{cases} \tag{3.13}$$

For every $k \geq K$, we now set

$$\ell_k = \frac{1}{\nu_k^2}, \qquad \forall\, k \geq 1. \tag{3.14}$$

To mimic the proof of Section 2.3.2, we introduce a_k and b_k such that $a_k + \ell_k = c_k$ and $b_k - \ell_k = d_k$. By the last point of (3.13) we see that $\ell_k \leq \frac{1}{2}\ell_0$ so that we have $(a_k, b_k) \subset (a, b)$.

We now operate a splitting of the interval (a_k, b_k) similar to that of Section 2.3.2 that is we set

- $\widetilde{\omega}_k = (a_k, a_k + \ell_k) \cup (b_k - \ell_k, b_k)$,
- $\mathfrak{C}_k = [a_k + \ell_k, b_k - \ell_k]$
- and $\mathfrak{C}_0 = \left[\frac{a+b}{2} - \frac{b-a}{6}, \frac{a+b}{2} + \frac{b-a}{6}\right]$.

Notice that, by construction, we have $\mathfrak{C}_0 \subset \mathfrak{C}_k$ for every $k \geq 1$.

From (A.3), there exists $\alpha_1 > 0$ depending on γ, c and $\mathfrak{C}_0$ such that

$$\int_{\mathfrak{C}_0} \varphi_k^2(x)\mathrm{d}x \geq \alpha_1, \qquad \forall\, k \geq 1. \tag{3.15}$$

Applying Lemma 2.4, with $u = \psi_{k,r,p,2}$ and $F = -r\varphi_k + S_p\varphi_k$ we obtain

$$\sqrt{\ell_k}\left|\int_{\mathfrak{C}_k} F(x)\varphi_k(x)\mathrm{d}x\right| \leq \alpha_2\sqrt{\nu_k}\left(1 + \frac{1}{\ell_k\sqrt{\nu_k}}\right)\|\psi_{k,r,p,2}\|_{\widetilde{\omega}_k} + \alpha_2\ell_k\|F\|_{\widetilde{\omega}_k}.$$

Using (A.6) we obtain the existence of $\overline{C} > 0$ depending on γ, c, q and p such that

$$\|F\|_{\widetilde{\omega}_k} \leq 2\left(\int_0^1 r(x)^2\varphi_k(x)^2 + p'(x)^2\varphi_k(x)^2 + p(x)^2\varphi_k'(x)^2\mathrm{d}x\right)^{\frac{1}{2}} \leq \overline{C}\sqrt{\nu_k}.$$

Thus,

$$\sqrt{\ell_k}\left|\int_{\mathfrak{C}_k} F(x)\varphi_k(x)\mathrm{d}x\right| \leq \alpha_2\sqrt{\nu_k}\left(1 + \frac{1}{\ell_k\sqrt{\nu_k}}\right)\|\psi_{k,r,p,2}\|_{\widetilde{\omega}_k} + \alpha_2\overline{C}\ell_k\sqrt{\nu_k}.$$

Since $\mathfrak{C}_0 \subset \mathfrak{C}_k$ we have

$$\int_{\mathfrak{C}_k} F(x)\varphi_k(x)\mathrm{d}x = -\int_{\mathfrak{C}_k} r(x)\varphi_k(x)^2\mathrm{d}x,$$

because the contribution of $S_p\varphi_k$ in this integral is zero, by integration by parts using the first point in (3.13). This integration by parts is the reason of the adjustments needed compared to Section 2.3.2. Thus,

$$\left|\int_{\mathfrak{C}_k} F(x)\varphi_k(x)\mathrm{d}x\right| \geq r_0\int_{\mathfrak{C}_0} \varphi_k^2(x)\mathrm{d}x \geq r_0\alpha_1.$$

Gathering these estimates we obtain

$$\alpha_2\sqrt{\nu_k}\left(1 + \frac{1}{\ell_k\sqrt{\nu_k}}\right)\|\psi_{k,r,p,2}\|_{\widetilde{\omega}_k} \geq \sqrt{\ell_k}\left(r_0\alpha_1 - \alpha_2\overline{C}\sqrt{\ell_k}\sqrt{\nu_k}\right).$$

Using the definition of ℓ_k in (3.14), it follows

$$\|\psi_{k,r,p,2}\|_{\widetilde{\omega}_k} \geq \frac{1}{\alpha_2\nu_k^{3/2}(1+\nu_k^{3/2})}\left(r_0\alpha_1 - \frac{\alpha_2\overline{C}}{\sqrt{\nu_k}}\right).$$

This proves (3.12) and ends the proof of Proposition 3.7. □

4. Simultaneous controllability of systems with a space varying zero order coupling term

This section is dedicated to the analysis of the minimal null control time for the simultaneous null controllability problem stated in (1.4). In Section 4.1 we detail the spectral analysis of the underlying evolution operator. Section 4.2 is dedicated to the proof of the first formulation for the minimal null control time given in Theorem 1.6. Using the computations done in Section 2.2, we then deduce in Section 4.3 the second formulation given in Theorem 1.7. Finally an example is considered in Section 4.4.

4.1. **Spectral analysis**

To fit again in the formalism of [13], we define the evolution operator $\mathcal{A}$ in the state space $X = (L^2(0,1))^3$ by

$$\mathcal{A} = \begin{pmatrix} A & 0 & 0 \\ q_2 & A & 0 \\ q_3 & 0 & A \end{pmatrix}, \qquad D(\mathcal{A}) = D(A)^3$$

and the control operator $\mathcal{B}$ by

$$\mathcal{B} : u \in U = L^2(0,1) \longmapsto \begin{pmatrix} \mathbf{1}_\omega u \\ 0 \\ 0 \end{pmatrix}.$$

The spectrum of $\mathcal{A}^*$ is given by $\Lambda = \{\nu_k \ ; \ k \geq 1\}$ and thus, as proved in Section 2.1,

$$\Lambda \in \mathcal{L}_w\left(1, \varrho, 0, \frac{1}{2}, \kappa\right)$$

as defined in [13, Section 2.1.2].

In any case,

$$\phi_{k,1}^0 = \begin{pmatrix} \varphi_k \\ 0 \\ 0 \end{pmatrix}$$

is an eigenvector of $\mathcal{A}^*$ associated to the eigenvalue ν_k. Recall that, for any $q \in L^\infty(0,1)$, the function $\psi_{k,q}$ is defined by (1.6).

Case i. If $I_k(q_2) = I_k(q_3) = 0$ then ν_k is geometrically triple. A basis of associated eigenvectors of $\mathcal{A}^*$ is given by

$$\phi_{k,1}^0, \qquad \phi_{k,2}^0 = \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix}, \qquad \phi_{k,3}^0 = \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix}. \tag{4.1}$$

Case ii (a). If $I_k(q_2) = 0$ and $I_k(q_3) \neq 0$ then ν_k is geometrically double and algebraically double. A basis of the generalized eigenspace of $\mathcal{A}^*$ is given by

$$\phi_{k,1}^0, \qquad \phi_{k,2}^0 = \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix}, \qquad \phi_{k,1}^1 = \frac{1}{I_k(q_3)} \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix} \tag{4.2}$$

where $\phi_{k,1}^0$ and $\phi_{k,2}^0$ are eigenvectors and the generalized eigenvector $\phi_{k,1}^1$ satisfies

$$(\mathcal{A}^* - \nu_k)\phi_{k,1}^1 = \phi_{k,1}^0.$$

Case ii (b). If $I_k(q_2) \neq 0$ and $I_k(q_3) = 0$ then ν_k is geometrically double and algebraically double. A basis of the generalized eigenspace of $\mathcal{A}^*$ is given by

$$\phi^0_{k,1}, \qquad \phi^0_{k,2} = \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix}, \qquad \phi^1_{k,1} = \frac{1}{I_k(q_2)} \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix} \tag{4.3}$$

where $\phi^0_{k,1}$ and $\phi^0_{k,2}$ are eigenvectors and the generalized eigenvector $\phi^1_{k,1}$ satisfies

$$(\mathcal{A}^* - \nu_k)\phi^1_{k,1} = \phi^0_{k,1}.$$

Case iii. If $I_k(q_2) \neq 0$ and $I_k(q_3) \neq 0$ then ν_k is geometrically double and algebraically double. A basis of the generalized eigenspace of $\mathcal{A}^*$ is given by

$$\phi^0_{k,1}, \qquad \phi^0_{k,2} = I_k(q_3) \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix} - I_k(q_2) \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix}, \qquad \phi^1_{k,1} = \frac{1}{I_k(q_2)} \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix} \tag{4.4}$$

where $\phi^0_{k,1}$ and $\phi^0_{k,2}$ are eigenvectors and the generalized eigenvector $\phi^1_{k,1}$ satisfies

$$(\mathcal{A}^* - \nu_k)\phi^1_{k,1} = \phi^0_{k,1}.$$

Thus, using (4.1)–(4.4), we obtain that the family of (generalized) eigenvectors forms a complete family in X.

From [14, Theorem 3.2], the approximate controllability assumption

$$\operatorname{Ker}(\mathcal{A}^* - \lambda) \cap \operatorname{Ker} \mathcal{B}^* = \{0\}, \qquad \forall\, \lambda \in \Lambda$$

is equivalent to (1.14).

Thus, the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumption (H) stated in [13, Section 2.1.2].

4.2. Characterization of the minimal null control time

This section is devoted to the proof of Theorem 1.6.

4.2.1. An abstract characterization of the minimal null control time

Since the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumption (H) stated in [13, Section 2.1.2] it comes from [13, Theorem 11] that, for any $y_0 \in X$, the minimal null control time for system (1.4) from y_0 is given by

$$T_{0,\boldsymbol{q}}(y_0) = \limsup_{k \to +\infty} \frac{\ln^+ C(G_k, y_0)}{2\nu_k} \tag{4.5}$$

where $\ln^+ s = \max(0, \ln s)$, for any $s \geq 0$ and the cost of the k-th block is given by

- in Case *i*

$$C(G_k, y_0) = \inf\left\{ \|\Omega\|_U^2 \, ; \, \begin{array}{l} \Omega \in U \\ \text{with } \langle \Omega, \mathcal{B}^* \phi_{k,j}^0 \rangle_U = \langle y_0, \phi_{k,j}^0 \rangle_X \text{ for } j \in \{1,2,3\} \end{array} \right\} \tag{4.6}$$

- and in Cases *ii (a)*, *ii (b)* and *iii*

$$C(G_k, y_0) = \inf\left\{ \left\|\Omega^0\right\|_U^2 + \left\|\Omega^1\right\|_U^2 \, ; \, \begin{array}{c} \Omega^0, \Omega^1 \in U \\ \text{with } \langle \Omega^0, \mathcal{B}^* \phi_{k,j}^0 \rangle_U = \langle y_0, \phi_{k,j}^0 \rangle_X \text{ for } j \in \{1,2\} \\ \text{and } \langle \Omega^0, \mathcal{B}^* \phi_{k,1}^1 \rangle_U + \langle \Omega^1, \mathcal{B}^* \phi_{k,1}^0 \rangle_U = \langle y_0, \phi_{k,1}^1 \rangle_X \end{array} \right\}. \tag{4.7}$$

The proof of Theorem 1.6 consists in computing the quantity $C(G_k, y_0)$ and evaluating its asymptotic behaviour.

From [13, Theorem 18], in Case *i*, an explicit expression of the cost $C(G_k, y_0)$ of the block is given by

$$C(G_k, y_0) = \langle M^{-1}\xi, \xi \rangle \tag{4.8}$$

where

$$\xi = \begin{pmatrix} \langle y_0, \phi_{k,1}^0 \rangle_X \\ \langle y_0, \phi_{k,2}^0 \rangle_X \\ \langle y_0, \phi_{k,3}^0 \rangle_X \end{pmatrix} \quad \text{and} \quad M = \mathrm{Gram}_U\left(\mathcal{B}^* \phi_{k,1}^0, \mathcal{B}^* \phi_{k,2}^0, \mathcal{B}^* \phi_{k,3}^0 \right)$$

$$= \begin{pmatrix} \|\varphi_k\|_\omega^2 & 0 & 0 \\ 0 & \left\|\psi_{k,q_2}\right\|_\omega^2 & \langle \psi_{k,q_2}, \psi_{k,q_3} \rangle_\omega \\ 0 & \langle \psi_{k,q_2}, \psi_{k,q_3} \rangle_\omega & \left\|\psi_{k,q_3}\right\|_\omega^2 \end{pmatrix}.$$

Since Cases *ii (a)*, *ii (b)* and *iii* involve algebraic and geometric multiplicities occuring simultaneously inside the same block, we cannot apply [13, Theorem 14] nor [13, Theorem 18] to get a similar expression. We compute such an explicit expression in the next subsection.

4.2.2. An intermediate optimization argument

As detailed in [13, Section 5.4], when both algebraic and geometric multiplicities appear in the same group, one can repeat the arguments developed there to obtain an explicit expression of the cost of the block. This is what we do in the following proposition.

Proposition 4.1. *Let U be a real Hilbert space. Let $b_1^0, b_2^0, b_1^1 \in U$ be such that b_1^0 and b_2^0 are linearly independent. Then, for any $\omega_1^0, \omega_2^0, \omega_1^1 \in \mathbb{R}$,*

$$\inf\left\{ \left\|\Omega^0\right\|_U^2 + \left\|\Omega^1\right\|_U^2 \, ; \, \begin{array}{l} \Omega^0, \Omega^1 \in U \text{ with } \langle \Omega^0, b_j^0 \rangle_U = \omega_j^0 \text{ for } j \in \{1,2\} \\ \text{and } \langle \Omega^0, b_1^1 \rangle_U + \langle \Omega^1, b_1^0 \rangle_U = \omega_1^1 \end{array} \right\} = \langle M^{-1}\xi, \xi \rangle$$

where

$$M = \mathrm{Gram}_U\left(b_1^0, b_2^0, b_1^1\right) + \mathrm{Gram}_U\left(0, 0, b_1^0\right) \quad \textit{and} \quad \xi = \begin{pmatrix} \omega_1^0 \\ \omega_2^0 \\ \omega_1^1 \end{pmatrix}.$$

Proof. First of all, notice that by projection the infimum can be computed for

$$\Omega^0, \Omega^1 \in \mathrm{Span}\Big(b_1^0, b_2^0, b_1^1\Big).$$

Thus, we are solving a finite dimensional optimization problem with a quadratic coercive functional and linear constraints. It admits a unique solution characterized by the existence of multipliers $m_1^0, m_2^0, m_1^1 \in \mathbb{R}$ such that

$$\begin{aligned} &\langle \Omega^0, H^0\rangle_U + \langle \Omega^1, H^1\rangle_U \\ &\qquad = m_1^0 \langle H^0, b_1^0\rangle_U + m_2^0 \langle H^0, b_2^0\rangle_U + m_1^1 \Big(\langle H^0, b_1^1\rangle_U + \langle H^1, b_1^0\rangle_U\Big) \end{aligned} \tag{4.9}$$

for any $H^0, H^1 \in U$.

Using the constraints $\langle \Omega^0, b_j^0\rangle_U = \omega_j^0$ for $j \in \{1,2\}$ and $\langle \Omega^0, b_1^1\rangle_U + \langle \Omega^1, b_1^0\rangle_U = \omega_1^1$ and choosing successively

- $H^0 = b_1^0$ and $H^1 = 0$,
- $H^0 = b_2^0$ and $H^1 = 0$,
- $H^0 = b_1^1$ and $H^1 = b_1^0$

yields

$$\begin{pmatrix} \omega_1^0 \\ \omega_2^0 \\ \omega_1^1 \end{pmatrix} = M \begin{pmatrix} m_1^0 \\ m_2^0 \\ m_1^1 \end{pmatrix} \tag{4.10}$$

with

$$M = \mathrm{Gram}_U\left(b_1^0, b_2^0, b_1^1\right) + \mathrm{Gram}_U\left(0, 0, b_1^0\right).$$

We now prove that M is invertible. Let $x \in \mathbb{R}^3$ be such that $Mx = 0$. Then,

$$0 = \langle Mx, x\rangle = \left\|x_1 b_1^0 + x_2 b_2^0 + x_3 b_1^1\right\|_U^2 + x_3^2 \left\|b_1^0\right\|_U^2.$$

This implies $x_3 = 0$. Then, since b_1^0 and b_2^0 are assumed to be linearly independent, we obtain $x_1 = x_2 = 0$. Getting back to (4.10), this gives

$$\begin{pmatrix} m_1^0 \\ m_2^0 \\ m_1^1 \end{pmatrix} = M^{-1}\xi.$$

Finally, choosing $H^0 = \Omega^0$ and $H^1 = \Omega^1$ in (4.9) yields that the seeked infimum is

$$\left\|\Omega^0\right\|_U^2 + \left\|\Omega^1\right\|_U^2 = \left\langle \begin{pmatrix} m_1^0 \\ m_2^0 \\ m_1^1 \end{pmatrix}, \xi \right\rangle = \langle M^{-1}\xi, \xi\rangle$$

which ends the proof of Proposition 4.1. □

4.2.3. **Spectral characterization of the minimal null control time**

To prove Theorem 1.6 we now give a more explicit expression for the quantity $C(G_k, y_0)$.

Lemma 4.2. *For any $k \geq 1$, let $C(G_k, y_0)$ be defined by* (4.6)-(4.7). *Then,*

$$C(G_k, y_0) = \frac{\left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \\ 0 \end{pmatrix} \right\rangle_X^2}{\|\varphi_k\|_\omega^2} + \frac{\left\| \left\langle y_0, \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix} \right\rangle_X \zeta_{k,q_3} - \left\langle y_0, \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix} \right\rangle_X \zeta_{k,q_2} \right\|_\omega^2}{\left\|\zeta_{k,q_2}\right\|_\omega^2 \left\|\zeta_{k,q_3}\right\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3}\rangle_\omega^2} \tag{4.11}$$

where $\zeta_{k,\cdot}$ is defined in (1.17).

Proof. The explicit expression of $C(G_k, y_0)$ is given either by (4.8) or by Proposition 4.1.

In all cases, we have

$$C(G_k, y_0) = \langle M^{-1}\xi, \xi\rangle$$

where, due to the choice of normalization $\langle \psi_{k,q_j}, \varphi_k\rangle_\omega = 0$, the matrix M has the form

$$M = \begin{pmatrix} m_{1,1} & 0 & 0 \\ 0 & m_{2,2} & m_{2,3} \\ 0 & m_{2,3} & m_{3,3} \end{pmatrix}.$$

Thus, explicit computations yields

$$\langle M^{-1}\xi, \xi\rangle = \frac{1}{m_{1,1}}\xi_1^2 + \frac{1}{m_{2,2}m_{3,3} - m_{2,3}^2}\left(m_{3,3}\xi_2^2 - 2m_{2,3}\xi_2\xi_3 + m_{2,2}\xi_3^2\right).$$

We now distinguish the different cases.

Case i. We have

$$\xi = \begin{pmatrix} \langle y_0, \phi_{k,1}^0\rangle_X \\ \langle y_0, \phi_{k,2}^0\rangle_X \\ \langle y_0, \phi_{k,3}^0\rangle_X \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} \|\varphi_k\|_\omega^2 & 0 & 0 \\ 0 & \left\|\psi_{k,q_2}\right\|_\omega^2 & \langle \psi_{k,q_2}, \psi_{k,q_3}\rangle_\omega \\ 0 & \langle \psi_{k,q_2}, \psi_{k,q_3}\rangle_\omega & \left\|\psi_{k,q_3}\right\|_\omega^2 \end{pmatrix}$$

where $\phi^0_{k,1}$, $\phi^0_{k,2}$ and $\phi^0_{k,3}$ are defined in (4.1). Thus,

$$\langle M^{-1}\xi,\xi\rangle = \frac{\left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \\ 0 \end{pmatrix}\right\rangle_X^2}{\|\varphi_k\|^2_\omega} + \frac{\left\| \left\langle y_0, \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix}\right\rangle_X \psi_{k,q_3} - \left\langle y_0, \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix}\right\rangle_X \psi_{k,q_2} \right\|^2_\omega}{\left\|\psi_{k,q_2}\right\|^2_\omega \left\|\psi_{k,q_3}\right\|^2_\omega - \langle \psi_{k,q_2}, \psi_{k,q_3}\rangle^2_\omega}$$

Notice that, due to the approximate controllability assumption

$$\operatorname{Ker}(\mathcal{A}^* - \lambda) \cap \operatorname{Ker}\mathcal{B}^* = \{0\}, \qquad \forall\, \lambda \in \mathbb{R},$$

we have $\left\|\psi_{k,q_2}\right\|^2_\omega \left\|\psi_{k,q_3}\right\|^2_\omega - \langle \psi_{k,q_2}, \psi_{k,q_3}\rangle^2_\omega > 0$.

Case ii (a). We have

$$\xi = \begin{pmatrix} \langle y_0, \phi^0_{k,1}\rangle_X \\ \langle y_0, \phi^0_{k,2}\rangle_X \\ \langle y_0, \phi^1_{k,1}\rangle_X \end{pmatrix}$$

and

$$M = \begin{pmatrix} \|\varphi_k\|^2_\omega & 0 & 0 \\ 0 & \left\|\psi_{k,q_2}\right\|^2_\omega & \frac{1}{I_k(q_3)}\langle \psi_{k,q_2}, \psi_{k,q_3}\rangle_\omega \\ 0 & \frac{1}{I_k(q_3)}\langle \psi_{k,q_2}, \psi_{k,q_3}\rangle_\omega & \frac{1}{I_k(q_3)^2}\left\|\psi_{k,q_3}\right\|^2_\omega + \|\varphi_k\|^2_\omega \end{pmatrix}$$

where $\phi^0_{k,1}$, $\phi^0_{k,2}$ and $\phi^1_{k,1}$ are defined in (4.2). Thus,

$$\begin{aligned} \langle M^{-1}\xi,\xi\rangle = {}& \frac{\left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \\ 0 \end{pmatrix}\right\rangle_X^2}{\|\varphi_k\|^2_\omega} + \frac{I_k(q_3)^2 \left\langle y_0, \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix}\right\rangle_X^2 \|\varphi_k\|^2_\omega}{\left\|\psi_{k,q_2}\right\|^2_\omega \left(\left\|\psi_{k,q_3}\right\|^2_\omega + I_k(q_3)\|\varphi_k\|^2_\omega \right) - \langle \psi_{k,q_2}, \psi_{k,q_3}\rangle^2_\omega} \\ & + \frac{\left\| \left\langle y_0, \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix}\right\rangle_X \psi_{k,q_3} - \left\langle y_0, \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix}\right\rangle_X \psi_{k,q_2} \right\|^2_\omega}{\left\|\psi_{k,q_2}\right\|^2_\omega \left(\left\|\psi_{k,q_3}\right\|^2_\omega + I_k(q_3)\|\varphi_k\|^2_\omega \right) - \langle \psi_{k,q_2}, \psi_{k,q_3}\rangle^2_\omega}. \end{aligned}$$

Using the normalization condition $\langle \psi_{k,q_j}, \varphi_k\rangle_\omega = 0$, this can be rewritten as

$$\langle M^{-1}\xi,\xi\rangle = \frac{\left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \\ 0 \end{pmatrix}\right\rangle_X^2}{\|\varphi_k\|^2_\omega} + \frac{\left\| \left\langle y_0, \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix}\right\rangle_X \zeta_{k,q_3} - \left\langle y_0, \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix}\right\rangle_X \psi_{k,q_2} \right\|^2_\omega}{\left\|\psi_{k,q_2}\right\|^2_\omega \left\|\zeta_{k,q_3}\right\|^2_\omega - \langle \psi_{k,q_2}, \zeta_{k,q_3}\rangle^2_\omega}.$$

Notice that, from Cauchy–Schwarz inequality,

$$
\begin{aligned}
\big\|\psi_{k,q_2}\big\|_\omega^2\big\|\zeta_{k,q_3}\big\|_\omega^2 - \langle\psi_{k,q_2},\zeta_{k,q_3}\rangle_\omega^2 \\
&= \big\|\psi_{k,q_2}\big\|_\omega^2\Big(\big\|\psi_{k,q_3}\big\|_\omega^2 + I_k(q_3)^2\|\varphi_k\|_\omega^2\Big) - \langle\psi_{k,q_2},\psi_{k,q_3}\rangle_\omega^2 \\
&\geq I_k(q_3)^2\big\|\psi_{k,q_2}\big\|_\omega^2\|\varphi_k\|_\omega^2.
\end{aligned}
$$

Then, due to the approximate controllability assumption

$$
\operatorname{Ker}(\mathcal{A}^* - \lambda) \cap \operatorname{Ker}\mathcal{B}^* = \{0\}, \qquad \forall\, \lambda \in \mathbb{R},
$$

we have $\big\|\psi_{k,q_2}\big\|_\omega^2\big\|\zeta_{k,q_3}\big\|_\omega^2 - \langle\psi_{k,q_2},\zeta_{k,q_3}\rangle_\omega^2 > 0$.

Case ii (b). This case is exactly Case *ii (a)* when exchanging the roles of q_2 and q_3. Thus,

$$
\langle M^{-1}\xi,\xi\rangle = \frac{\Big\langle y_0, \Big(\begin{smallmatrix}\varphi_k\\0\\0\end{smallmatrix}\Big)\Big\rangle_X^2}{\|\varphi_k\|_\omega^2} + \frac{\Big\|\Big\langle y_0, \Big(\begin{smallmatrix}\psi_{k,q_2}\\\varphi_k\\0\end{smallmatrix}\Big)\Big\rangle_X \psi_{k,q_3} - \Big\langle y_0, \Big(\begin{smallmatrix}\psi_{k,q_3}\\0\\\varphi_k\end{smallmatrix}\Big)\Big\rangle_X \zeta_{k,q_2}\Big\|_\omega^2}{\big\|\zeta_{k,q_2}\big\|_\omega^2\big\|\psi_{k,q_3}\big\|_\omega^2 - \langle\zeta_{k,q_2},\psi_{k,q_3}\rangle_\omega^2}
$$

and $\big\|\zeta_{k,q_2}\big\|_\omega^2\big\|\psi_{k,q_3}\big\|_\omega^2 - \langle\zeta_{k,q_2},\psi_{k,q_3}\rangle_\omega^2 > 0$.

Case iii. Recall that the eigenvectors are defined in (4.4). To preserve symmetry, we consider here the generalized eigenvector given by

$$
\phi_{k,1}^1 = \frac{1}{2I_k(q_2)}\begin{pmatrix}\psi_{k,q_2}\\ \varphi_k\\ 0\end{pmatrix} + \frac{1}{2I_k(q_3)}\begin{pmatrix}\psi_{k,q_3}\\ 0\\ \varphi_k\end{pmatrix}.
$$

We have

$$
\xi = \begin{pmatrix}\langle y_0, \phi_{k,1}^0\rangle_X\\ \langle y_0, \phi_{k,2}^0\rangle_X\\ \langle y_0, \phi_{k,1}^1\rangle_X\end{pmatrix} \quad \text{and} \quad M = M_1 + \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \|\varphi_k\|_\omega^2\end{pmatrix}
$$

with

$$
M_1 = \operatorname{Gram}_U\bigg(\varphi_k\ ,\ I_k(q_3)\psi_{k,q_2} - I_k(q_2)\psi_{k,q_3}\ ,\ \frac{1}{2I_k(q_2)}\psi_{k,q_2} + \frac{1}{2I_k(q_3)}\psi_{k,q_3}\bigg).
$$

As in the previous cases, straightforward computations (which are left to the reader) give

$$
\begin{aligned}
\langle M^{-1}\xi,\xi\rangle = {} & \frac{\left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \\ 0 \end{pmatrix}\right\rangle_X^2}{\|\varphi_k\|_\omega^2} \\
& + \frac{\left\| \left\langle y_0, \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix}\right\rangle_X \psi_{k,q_3} - \left\langle y_0, \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix}\right\rangle_X \psi_{k,q_2} \right\|_\omega^2}{\left\| I_k(q_3)\psi_{k,q_2} - I_k(q_2)\psi_{k,q_3}\right\|_\omega^2 \|\varphi_k\|_\omega^2 + \left\|\psi_{k,q_2}\right\|_\omega^2 \left\|\psi_{k,q_3}\right\|_\omega^2 - \langle \psi_{k,q_2}, \psi_{k,q_3}\rangle_\omega^2} \\
& + \frac{\left(I_k(q_3)^2 \left\langle y_0, \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix}\right\rangle_X^2 + I_k(q_2)^2 \left\langle y_0, \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix}\right\rangle_X^2 \right) \|\varphi_k\|_\omega^2}{\left\| I_k(q_3)\psi_{k,q_2} - I_k(q_2)\psi_{k,q_3}\right\|_\omega^2 \|\varphi_k\|_\omega^2 + \left\|\psi_{k,q_2}\right\|_\omega^2 \left\|\psi_{k,q_3}\right\|_\omega^2 - \langle \psi_{k,q_2}, \psi_{k,q_3}\rangle_\omega^2}.
\end{aligned}
$$

Using the normalization condition $\langle \psi_{k,q_j}, \varphi_k\rangle_\omega = 0$, this can be rewritten as

$$
\begin{aligned}
\langle M^{-1}\xi,\xi\rangle = {} & \frac{\left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \\ 0 \end{pmatrix}\right\rangle_X^2}{\|\varphi_k\|_\omega^2} \\
& + \frac{\left\| \left\langle y_0, \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix}\right\rangle_X \zeta_{k,q_3} - \left\langle y_0, \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix}\right\rangle_X \zeta_{k,q_2} \right\|_\omega^2}{\left\| I_k(q_3)\psi_{k,q_2} - I_k(q_2)\psi_{k,q_3}\right\|_\omega^2 \|\varphi_k\|_\omega^2 + \left\|\psi_{k,q_2}\right\|_\omega^2 \left\|\psi_{k,q_3}\right\|_\omega^2 - \langle \psi_{k,q_2}, \psi_{k,q_3}\rangle_\omega^2}.
\end{aligned}
$$

Using again the normalization condition $\langle \psi_{k,q_j}, \varphi_k\rangle_\omega = 0$, we obtain

$$
\begin{aligned}
\left\|\zeta_{k,q_2}\right\|_\omega^2 \left\|\zeta_{k,q_3}\right\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3}\rangle_\omega^2 = {} & \left\| I_k(q_3)\psi_{k,q_2} - I_k(q_2)\psi_{k,q_3}\right\|_\omega^2 \|\varphi_k\|_\omega^2 \\
& + \left\|\psi_{k,q_2}\right\|_\omega^2 \left\|\psi_{k,q_3}\right\|_\omega^2 - \langle \psi_{k,q_2}, \psi_{k,q_3}\rangle_\omega^2.
\end{aligned}
$$

Thus, from Cauchy–Schwarz inequality and the approximate controllability condition

$$
\operatorname{Ker}(\mathcal{A}^* - \lambda) \cap \operatorname{Ker} \mathcal{B}^* = \{0\}, \qquad \forall\, \lambda \in \mathbb{R},
$$

it comes that $\left\|\zeta_{k,q_2}\right\|_\omega^2 \left\|\zeta_{k,q_3}\right\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3}\rangle_\omega^2 > 0$ and

$$
\langle M^{-1}\xi,\xi\rangle = \frac{\left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \\ 0 \end{pmatrix}\right\rangle_X^2}{\|\varphi_k\|_\omega^2} + \frac{\left\| \left\langle y_0, \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix}\right\rangle_X \zeta_{k,q_3} - \left\langle y_0, \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix}\right\rangle_X \zeta_{k,q_2} \right\|_\omega^2}{\left\|\zeta_{k,q_2}\right\|_\omega^2 \left\|\zeta_{k,q_3}\right\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3}\rangle_\omega^2}.
$$

Notice that the last formula obtained in Case *iii* degenerates as expected when $I_k(q_2) = 0$ and / or $I_k(q_3) = 0$. Thus, gathering all cases proves (4.11) and ends the proof of Lemma 4.2. □

We now have all the ingredients to prove Theorem 1.6.

Proof of Theorem 1.6. Recall that from (4.5) we have

$$T_{0,\boldsymbol{q}}(y_0) = \limsup_{k\to+\infty} \frac{\ln^+ C(G_k, y_0)}{2\nu_k}$$

where, due to Lemma 4.2, we have for any $k \geq 1$,

$$C(G_k, y_0) = \frac{\left\langle y_0, \begin{pmatrix} \varphi_k \\ 0 \\ 0 \end{pmatrix} \right\rangle_X^2}{\|\varphi_k\|_\omega^2} + \frac{\left\| \left\langle y_0, \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix} \right\rangle_X \zeta_{k,q_3} - \left\langle y_0, \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix} \right\rangle_X \zeta_{k,q_2} \right\|_\omega^2}{\left\|\zeta_{k,q_2}\right\|_\omega^2 \left\|\zeta_{k,q_3}\right\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3} \rangle_\omega^2}.$$

We now estimate the previous right-hand side. As proved in Lemma A.2, we have

$$\left\|\psi_{k,q_2}\right\|_{(0,1)} + \left\|\psi_{k,q_3}\right\|_{(0,1)} \leq C, \qquad \forall\, k \geq 1.$$

Thus,

$$\left\| \left\langle y_0, \begin{pmatrix} \psi_{k,q_2} \\ \varphi_k \\ 0 \end{pmatrix} \right\rangle_X \zeta_{k,q_3} - \left\langle y_0, \begin{pmatrix} \psi_{k,q_3} \\ 0 \\ \varphi_k \end{pmatrix} \right\rangle_X \zeta_{k,q_2} \right\|_\omega^2 \leq C\|y_0\|_X^2 \max\left(\left\|\zeta_{k,q_2}\right\|_\omega^2, \left\|\zeta_{k,q_3}\right\|_\omega^2 \right).$$

Recall that φ_k satisfies (A.3). This implies that

$$C(G_k, y_0) \leq C\|y_0\|_X^2 \left(1 + \frac{\max\left(\left\|\zeta_{k,q_2}\right\|_\omega^2, \left\|\zeta_{k,q_3}\right\|_\omega^2 \right)}{\left\|\zeta_{k,q_2}\right\|_\omega^2 \left\|\zeta_{k,q_3}\right\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3} \rangle_\omega^2} \right)$$

for any $k \geq 1$ and any $y_0 \in X$ which gives

$$T_{0,\boldsymbol{q}} \leq \limsup_{k\to+\infty} \frac{1}{2\nu_k} \ln \frac{\max\left(\left\|\zeta_{k,q_2}\right\|_\omega^2, \left\|\zeta_{k,q_3}\right\|_\omega^2 \right)}{\left\|\zeta_{k,q_2}\right\|_\omega^2 \left\|\zeta_{k,q_3}\right\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3} \rangle_\omega^2}.$$

We now prove the converse inequality. We define for all $k \geq 1$

$$\epsilon_k = \begin{cases} 1 & \text{if } \left\|\zeta_{k,q_2}\right\|_\omega > \left\|\zeta_{k,q_3}\right\|_\omega \\ 0 & \text{otherwise} \end{cases}$$

and we choose the particular initial condition

$$y_0 = \sum_{k\geq 1} \frac{1}{\nu_k} \left(\epsilon_k \begin{pmatrix} 0 \\ 0 \\ \varphi_k \end{pmatrix} + (1-\epsilon_k) \begin{pmatrix} 0 \\ \varphi_k \\ 0 \end{pmatrix} \right).$$

From the expression (4.11) we obtain

$$C(G_k, y_0) = \frac{1}{\nu_k^2} \frac{\max\left(\left\|\zeta_{k,q_2}\right\|_\omega^2, \left\|\zeta_{k,q_3}\right\|_\omega^2 \right)}{\left\|\zeta_{k,q_2}\right\|_\omega^2 \left\|\zeta_{k,q_3}\right\|_\omega^2 - \langle \zeta_{k,q_2}, \zeta_{k,q_3} \rangle_\omega^2}.$$

This gives that

$$T_{0,\boldsymbol{q}} \geq T_{0,\boldsymbol{q}}(y_0) = \limsup_{k\to+\infty} \frac{1}{2\nu_k} \ln \frac{\max\left(\left\|\zeta_{k,q_2}\right\|_\omega^2, \left\|\zeta_{k,q_3}\right\|_\omega^2\right)}{\left\|\zeta_{k,q_2}\right\|_\omega^2\left\|\zeta_{k,q_3}\right\|_\omega^2 - \langle\zeta_{k,q_2},\zeta_{k,q_3}\rangle_\omega^2}$$

which ends the proof of Theorem 1.6. □

4.3. A second characterization of the minimal null control time

The goal of this section is to prove Theorem 1.7.

We first notice that, by (1.19), we have that q_2 and q_3 are linearly independent and thus there exists $\underline{C}, \overline{C} > 0$ such that

$$\underline{C}(|\alpha_2| + |\alpha_3|) \leq \|\alpha_2 q_2 + \alpha_3 q_3\|_\infty \leq \overline{C}(|\alpha_2| + |\alpha_3|), \quad \forall\, \alpha_2, \alpha_3 \in \mathbb{R}. \tag{4.12}$$

Proof. From Theorem 1.6 we now estimate, for any $k \geq 1$,

$$\max\left(\frac{\left\|\zeta_{k,q_2}\right\|_\omega^2}{\left\|\zeta_{k,q_2}\right\|_\omega^2\left\|\zeta_{k,q_3}\right\|_\omega^2 - \langle\zeta_{k,q_2},\zeta_{k,q_3}\rangle_\omega^2}, \frac{\left\|\zeta_{k,q_3}\right\|_\omega^2}{\left\|\zeta_{k,q_2}\right\|_\omega^2\left\|\zeta_{k,q_3}\right\|_\omega^2 - \langle\zeta_{k,q_2},\zeta_{k,q_3}\rangle_\omega^2}\right).$$

Let $k \geq 1$ and assume that $\left\|\zeta_{k,q_3}\right\|_\omega > \left\|\zeta_{k,q_2}\right\|_\omega$. Notice that

$$\left\|\zeta_{k,q_3}\right\|_\omega^2 \left\|\zeta_{k,q_2} - \frac{\langle\zeta_{k,q_2},\zeta_{k,q_3}\rangle_\omega}{\left\|\zeta_{k,q_3}\right\|_\omega^2}\zeta_{k,q_3}\right\|_\omega^2 = \left\|\zeta_{k,q_3}\right\|_\omega^2\left\|\zeta_{k,q_2}\right\|_\omega^2 - \langle\zeta_{k,q_2},\zeta_{k,q_3}\rangle_\omega^2.$$

Thus,

$$\begin{aligned}\frac{\max\left(\left\|\zeta_{k,q_2}\right\|_\omega^2, \left\|\zeta_{k,q_3}\right\|_\omega^2\right)}{\left\|\zeta_{k,q_2}\right\|_\omega^2\left\|\zeta_{k,q_3}\right\|_\omega^2 - \langle\zeta_{k,q_2},\zeta_{k,q_3}\rangle_\omega^2} &= \frac{1}{\left\|\zeta_{k,q_2} - \frac{\langle\zeta_{k,q_2},\zeta_{k,q_3}\rangle_\omega}{\left\|\zeta_{k,q_3}\right\|_\omega^2}\zeta_{k,q_3}\right\|_\omega^2} \\ &= \frac{1}{\min_{\tau\in\mathbb{R}}\left\|\zeta_{k,q_2} - \tau\zeta_{k,q_3}\right\|_\omega^2}.\end{aligned} \tag{4.13}$$

By linearity we have, for any $\tau \in \mathbb{R}$,

$$\zeta_{k,q_2} - \tau\zeta_{k,q_3} = \zeta_{k,q_2-\tau q_3}.$$

We proved in Section 2.2 that there exists $K \in \mathbb{N}^*$, $C_1, C_2 > 0$ such that, for any $k \geq K$ and any $q \in L^\infty(0,1)$ such that $\operatorname{Supp}(q) \cap \omega = \varnothing$, we have

$$C_1\left\|\zeta_{k,q}\right\|_\omega^2 \leq \mathcal{M}_k\left(\vartheta_{k,q},\omega\right)^2 \leq C_2\nu_k\left\|\zeta_{k,q}\right\|_\omega^2. \tag{4.14}$$

where $\vartheta_{k,q}$ is defined by (1.18). The analysis is the same in the symmetric case $\left\|\zeta_{k,q_2}\right\|_\omega > \left\|\zeta_{k,q_3}\right\|_\omega$.

Thus, from Theorem 1.6, (4.13) and (4.14), it comes that

$$T_{0,\boldsymbol{q}} = \limsup_{k\to+\infty} \frac{-\ln\min\left\{\min_{\tau\in\mathbb{R}} \mathcal{M}_k\left(\vartheta_{k,q_2-\tau q_3},\ \omega\right), \min_{\tau\in\mathbb{R}} \mathcal{M}_k\left(\vartheta_{k,q_3-\tau q_2},\ \omega\right)\right\}}{\nu_k}. \tag{4.15}$$

To conclude the proof of Theorem 1.7, let us prove that the quantity

$$\min\left\{\min_{\tau\in\mathbb{R}} \mathcal{M}_k\left(\vartheta_{k,q_2-\tau q_3},\ \omega\right), \min_{\tau\in\mathbb{R}} \mathcal{M}_k\left(\vartheta_{k,q_3-\tau q_2},\ \omega\right)\right\}$$

appearing in the formula above has the same asymptotic behaviour as

$$\min_{q\in\mathbb{S}[\boldsymbol{q}]} \mathcal{M}_k\left(\vartheta_{k,q},\ \omega\right).$$

Notice that, for any $\tau\in\mathbb{R}$, the function $q_\tau = \frac{q_2-\tau q_3}{\|q_2-\tau q_3\|_\infty}$ belongs to $\mathbb{S}[\boldsymbol{q}]$ and thus

$$\begin{aligned}\mathcal{M}_k\left(\vartheta_{k,q_2-\tau q_3},\ \omega\right) &= \|q_2-\tau q_3\|_\infty \mathcal{M}_k\left(\vartheta_{k,q_\tau},\ \omega\right)\\ &\geq \underline{C} \min_{q\in\mathbb{S}[\boldsymbol{q}]} \mathcal{M}_k\left(\vartheta_{k,q},\ \omega\right),\end{aligned}$$

where we have used (4.12). It follows that

$$\min_{\tau\in\mathbb{R}} \mathcal{M}_k\left(\vartheta_{k,q_2-\tau q_3},\ \omega\right) \geq \underline{C} \min_{q\in\mathbb{S}[\boldsymbol{q}]} \mathcal{M}_k\left(\vartheta_{k,q},\ \omega\right)$$

and the exact same computation holds for $q_3-\tau q_2$.

Conversely, let $q = \alpha_2 q_2 + \alpha_3 q_3 \in \mathbb{S}[\boldsymbol{q}]$. If $|\alpha_2| \geq |\alpha_3|$, then by (4.12), we have $|\alpha_2| \geq \frac{1}{2\overline{C}}$ and thus

$$\begin{aligned}\mathcal{M}_k\left(\vartheta_{k,q},\ \omega\right) &= |\alpha_2| \mathcal{M}_k\left(\vartheta_{k,q_2+\frac{\alpha_3}{\alpha_2}q_3},\ \omega\right)\\ &\geq \frac{1}{2\overline{C}} \min_{\tau\in\mathbb{R}} \mathcal{M}_k\left(\vartheta_{k,q_2-\tau q_3},\ \omega\right).\end{aligned}$$

Otherwise, we have $|\alpha_3| > |\alpha_2|$ and a symmetric analysis gives

$$\mathcal{M}_k\left(\vartheta_{k,q},\ \omega\right) \geq \frac{1}{2\overline{C}} \min_{\tau\in\mathbb{R}} \mathcal{M}_k\left(\vartheta_{k,q_3-\tau q_2},\ \omega\right).$$

Finally, from the expression of the minimal null control time given in (4.15), the claim of Theorem 1.7 is proved. □

4.4. An explicit example

In this section we consider A to be the Dirichlet–Laplace operator (i.e., $\gamma = 1$ and $c = 0$ in (1.2)) and $\omega = \left(0, \frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right)$.

Proposition 4.3. *Let A and ω be defined as above. Let $\tau_0 \in [0, +\infty]$. There exists $q_2, q_3 \in L^\infty(0,1)$ such that*

(i) *approximate controllability of system* (1.4) *holds,*

(ii) *for any $(\alpha_2, \alpha_3) \in \mathbb{R}^2 \backslash \{0\}$, the minimal null control time for system* (1.1) *with $q = \alpha_2 q_2 + \alpha_3 q_3$ is $T_{0,q} = 0$. In particular $T_{0,q_2} = T_{0,q_3} = 0$.*

(iii) *the minimal null control time for system* (1.4) *is $T_{0,\boldsymbol{q}} = \tau_0$.*

Proof. For $j \in \{2,3\}$, we set $q_j = \mathbf{1}_{O_j}$ with

$$O_2 = \left(\frac{1}{2} - \delta_2, \frac{1}{2} + \delta_2\right) \text{ and } O_3 = (\eta_3 - \delta_3, \eta_3 + \delta_3),$$

where η_3, δ_2 and δ_3 are chosen such that

$$\mathrm{Supp}(q_2) \cap \omega = \varnothing \quad \text{and} \quad \mathrm{Supp}(q_3) \cap \omega = \varnothing. \tag{4.16}$$

The approximate controllability of system (1.4) with these coupling functions has been studied in [14, Section 3.4.2]. It is proved that approximate controllability holds if and only if

$$\eta_3 \notin \mathbb{Q} \quad \text{and} \quad \delta_3 \notin \mathbb{Q}. \tag{4.17}$$

Using for instance [5, Lemma 7.1], we can find $\eta_3 \notin \mathbb{Q}$ and $\delta_2, \delta_3 \notin \mathbb{Q}$ such that $2\eta_3$ and $2\delta_2$ are irrational algebraic numbers of degree 2 and

$$\limsup_{k\to+\infty} \frac{-\ln|\sin(2k\pi\delta_3)|}{k^2\pi^2} = \tau_0. \tag{4.18}$$

These choices prove (i).

Let us now focus on (ii) that is the determination of the minimal null control time for system (1.1). Under the considered assumptions, we have explicit formulas for φ_k and $\widetilde{\varphi}_k$ as follows:

$$\varphi_k = \sqrt{2}\sin(k\pi\cdot) \quad \text{and} \quad \widetilde{\varphi}_k = \cos(k\pi\cdot).$$

From Theorem 1.4, for any $q \in L^\infty(0,1)$, we have

$$T_{0,q} = \limsup_{k\to+\infty} \frac{-\ln \mathcal{M}_k\big((I_k(q) - q)\varphi_k,\ \omega\big)}{k^2\pi^2}.$$

Since $\overline{(0,1) \setminus \omega}$ has only one connected component $\mathfrak{C} = \left[\frac{1}{4}, \frac{3}{4}\right]$ it comes that

$$\mathcal{M}_k\big((I_k(q) - q)\varphi_k,\ \omega\big) = \max\left\{\left|\int_{\mathfrak{C}} (I_k(q) - q(x))\varphi_k(x)^2 \mathrm{d}x\right|, \left|\int_{\mathfrak{C}} (I_k(q) - q(x))\varphi_k(x)\widetilde{\varphi}_k(x)\mathrm{d}x\right|\right\}.$$

Then, for $j \in \{2,3\}$, since $\mathrm{Supp}(q_j) \subset \mathfrak{C}$, we have

$$\int_{\mathfrak{C}} (I_k(q_j) - q_j(x))\varphi_k(x)^2 \mathrm{d}x = I_k(q_j)\Big(1 - \|\varphi_k\|_{\omega}^2\Big) - I_k(q_j) = -\|\varphi_k\|_{O_j}^2 \|\varphi_k\|_{\omega}^2.$$

where we have used

$$I_k(q_j) = \int_{O_j} \varphi_k(x)^2 \mathrm{d}x = \|\varphi_k\|_{O_j}^2.$$

From (A.3) it comes that there exists $C > 0$ such that for any $k \geq 1$ and any $j \in \{2,3\}$,

$$C \leq \left| \int_{\mathfrak{C}} (I_k(q_j) - q_j(x))\varphi_k(x)^2 \mathrm{d}x \right| = \|\varphi_k\|_{O_j}^2 \|\varphi_k\|_{\omega}^2 \leq 1. \tag{4.19}$$

This already implies that $T_{0,q_2} = T_{0,q_3} = 0$. Let $(\alpha_2, \alpha_3) \in \mathbb{R}^2 \backslash \{0\}$ and $q = \alpha_2 q_2 + \alpha_3 q_3$. We prove that

$$\limsup_{k \to +\infty} k^2 |I_k(q)| > 0 \tag{4.20}$$

which implies $T_{0,q} = 0$ since

$$\left| \int_{\mathfrak{C}} (I_k(q) - q(x))\varphi_k(x)^2 \mathrm{d}x \right| = |I_k(q)| \|\varphi_k\|_{\omega}^2.$$

Explicit computations yield

$$\begin{aligned} I_k(q) &= \alpha_2 \int_{O_2} \sin^2(k\pi x)\mathrm{d}x + \alpha_3 \int_{O_3} \sin^2(k\pi x)\mathrm{d}x \\ &= \alpha_2\delta_2 + \alpha_3\delta_3 + \frac{(-1)^{k+1}\alpha_2}{2k\pi}\sin(2k\pi\delta_2) - \frac{\alpha_3}{2k\pi}\cos(2k\pi\eta_3)\sin(2k\pi\delta_3). \end{aligned}$$

If $\alpha_2\delta_2 + \alpha_3\delta_3 \neq 0$, the property (4.20) follows directly. Otherwise, we necessarily have $\alpha_2 \neq 0$ and since $2\delta_2$ is an irrational algebraic number of degree 2 we have (see for instance [5, Lemma 7.1])

$$\inf_{k \geq 1} k|\sin(2k\pi\delta_2)| > 0.$$

Together with the choice of δ_3 in (4.18) this proves (4.20) and thus gives $T_{0,q} = 0$.

We now turn to (iii) that is the determination of the minimal null control time for system (1.4). From Theorem 1.7 we have that the minimal null control time is given by

$$T_{0,\boldsymbol{q}} = \limsup_{k \to +\infty} \frac{-\ln \min_{q \in \mathbb{S}[\boldsymbol{q}]} \mathcal{M}_k\big(\vartheta_{k,q},\ \omega\big)}{\nu_k}. \tag{4.21}$$

Let $k \geq 1$. Since $\mathfrak{C}$ is symmetric with respect to $\frac{1}{2}$, we have

$$\int_{\mathfrak{C}} \varphi_k(x)\widetilde{\varphi}_k(x)\mathrm{d}x = \sqrt{2}\int_{\frac{1}{4}}^{\frac{3}{4}} \sin(k\pi x)\cos(k\pi x)\mathrm{d}x = 0.$$

Thus, for $j \in \{2,3\}$, we have

$$M_k\Big(\vartheta_{k,q_j}, \mathfrak{C}\Big) = \begin{pmatrix} -I_k(q_j)\|\varphi_k\|_\omega^2 \\ -\int_{\mathfrak{C}} q_j(x)\varphi_k(x)\widetilde{\varphi}_k(x)\mathrm{d}x \end{pmatrix}.$$

Again a symmetry argument shows that

$$\int_{\mathfrak{C}} q_2(x)\varphi_k(x)\widetilde{\varphi}_k(x)\mathrm{d}x = \sqrt{2}\int_{O_2} \sin(k\pi x)\cos(k\pi x)\mathrm{d}x = 0.$$

It follows that for any $q = \alpha_2 q_2 + \alpha_3 q_3 \in \mathbb{S}[\boldsymbol{q}]$, we have

$$M_k\Big(\vartheta_{k,q}, \mathfrak{C}\Big) = \begin{pmatrix} -I_k(q)\|\varphi_k\|_\omega^2 \\ -\alpha_3\int_{O_3} \varphi_k(x)\widetilde{\varphi}_k(x)\mathrm{d}x \end{pmatrix}$$

and thus

$$\mathcal{M}_k\Big(\vartheta_{k,q}, \omega\Big) = \max\left\{|I_k(q)|\|\varphi_k\|_\omega^2, |\alpha_3|\left|\int_{O_3} \varphi_k(x)\widetilde{\varphi}_k(x)\mathrm{d}x\right|\right\}. \tag{4.22}$$

Let us now prove that (4.21) reduces to

$$T_{0,\boldsymbol{q}} = \limsup_{k\to+\infty} \frac{-\ln\left|\int_{O_3}\varphi_k(x)\widetilde{\varphi}_k(x)\mathrm{d}x\right|}{\nu_k}. \tag{4.23}$$

We set

$$\widetilde{q}_k = I_k(q_3)q_2 - I_k(q_2)q_3, \quad \bar{q}_k = \frac{\widetilde{q}_k}{\|\widetilde{q}_k\|_\infty}$$

in such a way that $I_k(\bar{q}_k) = 0$ and $\|\bar{q}_k\|_\infty = 1$. By (4.22) and (4.12), we get

$$\mathcal{M}_k\Big(\vartheta_{k,\bar{q}_k}, \omega\Big) \leq \frac{1}{\underline{C}}\left|\int_{O_3} \varphi_k(x)\widetilde{\varphi}_k(x)\mathrm{d}x\right|,$$

so that

$$\min_{q\in\mathbb{S}[\boldsymbol{q}]} \mathcal{M}_k\big(\vartheta_{k,q},\ \omega\big) \leq \frac{1}{\underline{C}}\left|\int_{O_3} \varphi_k(x)\widetilde{\varphi}_k(x)\mathrm{d}x\right|. \tag{4.24}$$

Recall that $\underline{C}$ is the constant appearing in (4.12).

We now prove that, for some $C > 0$ that does not depend on k, we have

$$\min_{q\in\mathbb{S}[\boldsymbol{q}]} \mathcal{M}_k\big(\vartheta_{k,q},\ \omega\big) \geq C\left|\int_{O_3} \varphi_k(x)\widetilde{\varphi}_k(x)\mathrm{d}x\right|. \tag{4.25}$$

If it were not the case, we would have, up to a subsequence, the inequality

$$\min_{q\in\mathbb{S}[\boldsymbol{q}]} \mathcal{M}_k\big(\vartheta_{k,q},\ \omega\big) \leq \varepsilon_k\left|\int_{O_3} \varphi_k(x)\widetilde{\varphi}_k(x)\mathrm{d}x\right|,$$

for some $\varepsilon_k \to 0$.

In particular, from (4.22), it would exist for each k, a function $\widetilde{q}_k = \alpha_{2,k} q_2 + \alpha_{3,k} q_3 \in \mathbb{S}[\boldsymbol{q}]$, such that

$$|I_k(\widetilde{q}_k)| \|\varphi_k\|_\omega^2 \le \varepsilon_k \left| \int_{O_3} \varphi_k(x) \widetilde{\varphi}_k(x) \mathrm{d}x \right|, \tag{4.26}$$

and

$$|\alpha_{3,k}| \left| \int_{O_3} \varphi_k(x) \widetilde{\varphi}_k(x) \mathrm{d}x \right| \le \varepsilon_k \left| \int_{O_3} \varphi_k(x) \widetilde{\varphi}_k(x) \mathrm{d}x \right|. \tag{4.27}$$

From (4.27), we deduce first that $|\alpha_{3,k}| \le \varepsilon_k$, and in particular $\alpha_{3,k} \to 0$. Since $\|\widetilde{q}_k\|_\infty = 1$, it follows that $|\alpha_{2,k}| \to \frac{1}{\|q_2\|_\infty}$, from which we deduce that

$$\lim_{k\to\infty} |I_k(\widetilde{q}_k)| = \frac{1}{\|q_2\|_\infty} \lim_{k\to\infty} |I_k(q_2)| = \frac{|O_2|}{\|q_2\|_\infty} > 0.$$

By using (A.3), we obtain a contradiction with (4.26).

Using (4.24) and (4.25) in (4.21) exactly proves (4.23).

Finally, explicit computations yield

$$\int_{O_3} \varphi_k(x) \widetilde{\varphi}_k(x) \mathrm{d}x = \int_{O_3} \sin(k\pi x) \cos(k\pi x) \mathrm{d}x = \frac{\sin(2k\pi\eta_3)\sin(2k\pi\delta_3)}{2k\pi}.$$

Since $2\eta_3$ is an irrational algebraic number of degree 2 we have (see for instance [5, Lemma 7.1])

$$\inf_{k\ge 1} k |\sin(2k\pi\eta_3)| > 0.$$

Together with the choice of δ_3 in (4.18) this ends the proof of Proposition 4.3. □

Appendix A. Spectral properties of the Sturm–Liouville operator

Let A be the Sturm–Liouville operator defined by (1.2). We recall here some spectral properties that will be used in our study.

From [1, Theorem 1.1 and Remark 2.1], there exist $\varrho > 0$ and $C > 0$ such that

$$\varrho < \nu_{k+1} - \nu_k, \qquad \forall\, k \ge 1, \tag{A.1}$$

$$\frac{1}{C}\sqrt{\nu_k} \le |\varphi_k'(x)| \le C\sqrt{\nu_k}, \quad \forall\, x \in \{0,1\},\ \forall\, k \ge 1, \tag{A.2}$$

and, for any non-empty open set $\omega \subset (0,1)$,

$$\inf_{k\ge 1} \|\varphi_k\|_\omega > 0. \tag{A.3}$$

Let N be the counting function associated with the sequence of eigenvalues $(\nu_k)_{k\ge 1}$ i.e.,

$$N : r \in (0,+\infty) \longmapsto \sharp\{\nu_k \,;\, \nu_k \le r\}.$$

Using [12, Theorem IV.1.3], this counting function satisfies for some $\kappa > 0$,

$$N(r) \leq \kappa\sqrt{r}, \quad \forall\, r > 0, \tag{A.4}$$

and

$$|N(r) - N(s)| \leq \kappa\Big(1 + \sqrt{|r-s|}\Big), \quad \forall\, r, s > 0. \tag{A.5}$$

To estimate various quantities, we will make an intensive use of the following lemma proved in [1, Lemma 2.3].

Lemma A.1. *Let A be the Sturm–Liouville operator defined by* (1.2) *and let $\lambda_0 > 0$. There exists $C > 0$ depending on γ, c and λ_0 such that, for any $\lambda \geq \lambda_0$, for any $F \in L^2(0,1)$, for any $x, y \in [0,1]$, for any u satisfying*

$$(A - \lambda)u = F \quad \textit{in } [0,1],$$

we have

$$|u(x)|^2 + \frac{\gamma(x)}{\lambda}|u'(x)|^2 \leq C\left(|u(y)|^2 + \frac{\gamma(y)}{\lambda}|u'(y)|^2 + \frac{1}{\lambda}\left|\int_x^y |F(s)|\mathrm{d}s\right|^2\right).$$

Applying Lemma A.1 with $u = \varphi_k$, $F = 0$, $\lambda = \nu_k$ and integrating with respect to the variable $y \in (0,1)$ we obtain

$$|\varphi_k(x)|^2 + \frac{1}{\nu_k}|\varphi_k'(x)|^2 \leq C\left(1 + \frac{1}{\nu_k}\int_0^1 \gamma(y)|\varphi_k'(y)|^2\mathrm{d}y\right), \quad \forall\, x \in (0,1),\ \forall\, k \geq 1.$$

Integrating by parts leads to

$$\int_0^1 \gamma(y)|\varphi_k'(y)|^2\mathrm{d}y = \int_0^1 (\nu_k - c(y))\varphi_k(y)^2\mathrm{d}y \leq \nu_k + \|c\|_\infty$$

which yields the existence of $C > 0$ such that

$$|\varphi_k(x)|^2 + \frac{1}{\nu_k}|\varphi_k'(x)|^2 \leq C, \qquad \forall\, x \in (0,1),\ \forall\, k \geq 1. \tag{A.6}$$

We shall also use this lemma to estimate $\psi_{k,q}$ (defined in (1.6)) as follows:

Lemma A.2. *There exists $C > 0$ such that*

$$\left\|\psi_{k,q}\right\|_{(0,1)} \leq C, \quad \forall\, k \geq 1.$$

Proof. The function $\widetilde{\psi}_{k,q}$ defined by

$$\widetilde{\psi}_{k,q} := \psi_{k,q} - \frac{\psi_{k,q}'(0)}{\varphi_k'(0)}\varphi_k,$$

satisfies

$$\begin{cases} (A - \nu_k)\widetilde{\psi}_{k,q} = \big(I_k(q) - q\big)\varphi_k, \\ \widetilde{\psi}_{k,q}(0) = \widetilde{\psi}_{k,q}(1) = 0, \\ \widetilde{\psi}'_{k,q}(0) = 0. \end{cases}$$

From Lemma A.1 with $y = 0$ it comes that

$$\left|\widetilde{\psi}_{k,q}(x)\right|^2 + \frac{\gamma(x)}{\nu_k}\left|\widetilde{\psi}'_{k,q}(x)\right|^2 \leq \frac{C}{\nu_k}, \quad \forall\, x \in (0,1),\ \forall\, k \geq 1.$$

which yields

$$\left\|\widetilde{\psi}_{k,q}\right\|_{(0,1)} \leq C, \qquad \forall\, k \geq 1.$$

Notice that, by definition of $\widetilde{\psi}_{k,q}$, we have $\big(\psi_{k,q} - \widetilde{\psi}_{k,q}\big) \in \mathbb{R}\varphi_k$. Then, multiplying by φ_k, integrating over ω and recalling that $\langle \psi_{k,q}, \varphi_k\rangle_\omega = 0$, we obtain that

$$\psi_{k,q} = \widetilde{\psi}_{k,q} - \frac{\langle \widetilde{\psi}_{k,q}, \varphi_k\rangle_\omega}{\|\varphi_k\|_\omega^2}\varphi_k.$$

This implies that

$$\left\|\psi_{k,q}\right\|_{(0,1)} \leq \left\|\widetilde{\psi}_{k,q}\right\|_{(0,1)}\left(1 + \frac{1}{\|\varphi_k\|_\omega}\right), \qquad \forall\, k \geq 1.$$

Then, estimate (A.3) ends the proof of Lemma A.2. □

By definition, φ_k and $\widetilde{\varphi}_k$ are solutions of the same linear second order ODE $(A-\nu_k)\varphi_k = (A - \nu_k)\widetilde{\varphi}_k = 0$. It is therefore natural to introduce the associated Wronskian matrix

$$W_k(x) = \begin{pmatrix} \frac{\gamma(x)\varphi'_k(x)}{\sqrt{\nu_k}} & -\varphi_k(x) \\ \frac{\gamma(x)\widetilde{\varphi}'_k(x)}{\sqrt{\nu_k}} & -\widetilde{\varphi}_k(x) \end{pmatrix},$$

for which we can prove the following estimate.

Lemma A.3. *There exists $C > 0$ such that*

$$\|W_k(x)\| + \|W_k(x)^{-1}\| \leq C, \quad \forall\, x \in [0,1],\ \forall\, k \geq 1.$$

Proof. Let us fix a $k \geq 1$. Applying Lemma A.1 to $u = \widetilde{\varphi}_k$ and $y = 0$, we obtain

$$|\widetilde{\varphi}_k(x)|^2 + \frac{1}{\nu_k}|\widetilde{\varphi}'_k(x)|^2 \leq C, \quad \forall\, x \in (0,1),\ \forall\, k \geq 1.$$

Together with (A.6), this shows the uniform estimate on $\|W_k(x)\|$.

Moreover, the determinant of $W_k(x)$ does not depend on x and is thus equal to the determinant of $W_k(0)$ that is to $-\gamma(0)\varphi'_k(0)/\sqrt{\nu_k}$. By (A.2), we know that this quantity is uniformly bounded from below. The bound for $W_k(x)^{-1}$ follows. □

References

[1] Damien Allonsius, Franck Boyer, and Morgan Morancey. Spectral analysis of discrete elliptic operators and applications in control theory. *Numer. Math.*, 140(4):857–911, 2018.

[2] Damien Allonsius, Franck Boyer, and Morgan Morancey. Analysis of the null controllability of degenerate parabolic systems of Grushin type via the moments method. *J. Evol. Equ.*, 21(4):4799–4843, 2021.

[3] Farid Ammar Khodja, Assia Benabdallah, Manuel González-Burgos, and Luz de Teresa. Minimal time for the null controllability of parabolic systems: The effect of the condensation index of complex sequences. *J. Funct. Anal.*, 267(7):2077–2151, 2014.

[4] Farid Ammar Khodja, Assia Benabdallah, Manuel González-Burgos, and Luz de Teresa. Minimal time of controllability of two parabolic equations with disjoint control and coupling domains. *Comptes Rendus. Mathématique*, 352(5):391–396, 2014.

[5] Farid Ammar Khodja, Assia Benabdallah, Manuel González-Burgos, and Luz de Teresa. New phenomena for the null controllability of parabolic systems: minimal time and geometrical dependence. *J. Math. Anal. Appl.*, 444(2):1071–1113, 2016.

[6] Karine Beauchard, Piermarco Cannarsa, and Roberto Guglielmi. Null controllability of Grushin-type operators in dimension two. *J. Eur. Math. Soc.*, 16(1):67–101, 2014.

[7] Karine Beauchard, Jérémi Dardé, and Sylvain Ervedoza. Minimal time issues for the observability of Grushin-type equations. *Ann. Inst. Fourier*, 70(1):247–312, 2020.

[8] Karine Beauchard, Bernard Helffer, Raphael Henry, and Luc Robbiano. Degenerate parabolic operators of Kolmogorov type with a geometric control condition. *ESAIM, Control Optim. Calc. Var.*, 21(2):487–512, 2015.

[9] Karine Beauchard, Luc Miller, and Morgan Morancey. 2D Grushin-type equations: minimal time and null controllable data. *J. Differ. Equations*, 259(11):5813–5845, 2015.

[10] Assia Benabdallah, Franck Boyer, and Morgan Morancey. A block moment method to handle spectral condensation phenomenon in parabolic control problems. *Ann. Henri Lebesgue*, 3:717–793, 2020.

[11] Assia Benabdallah, Michel Cristofol, Patricia Gaitan, and Luz De Teresa. Controllability to trajectories for some parabolic systems of three and two equations by one control force. *Math. Control Relat. Fields*, 4(1):17–44, 2014.

[12] Franck Boyer. Controllability of linear parabolic equations and systems, 2023. Lecture Notes, `https://hal.archives-ouvertes.fr/hal-02470625v4`.

[13] Franck Boyer and Morgan Morancey. Analysis of non scalar control problems for parabolic systems by the block moment method. *Comptes Rendus. Mathématique*, 361:1191–1248, 2023.

[14] Franck Boyer and Guillaume Olive. Approximate controllability conditions for some linear 1D parabolic systems with space-dependent coefficients. *Math. Control Relat. Fields*, 4(3):263–287, 2014.

[15] Amélie Dupouy. Approximate and null controllability of a parabolic system with coupling terms of order one, 2024. Master's thesis, University Paul Sabatier - Toulouse 3, `https://arxiv.org/abs/2410.10307`.

[16] Michel Duprez. Controllability of a 2×2 parabolic system by one force with space-dependent coupling term of order one. *ESAIM, Control Optim. Calc. Var.*, 23(4):1473–1498, 2017.

[17] Michel Duprez and Armand Koenig. Control of the Grushin equation: non-rectangular control region and minimal time. *ESAIM, Control Optim. Calc. Var.*, 26: article no. 3 (18 pages), 2020.

[18] Michel Duprez and Pierre Lissy. Indirect controllability of some linear parabolic systems of m equations with $m - 1$ controls involving coupling terms of zero or first order. *J. Math. Pures Appl. (9)*, 106(5):905–934, 2016.

[19] Michel Duprez and Pierre Lissy. Positive and negative results on the internal controllability of parabolic equations coupled by zero- and first-order terms. *J. Evol. Equ.*, 18(2):659–680, 2018.

[20] Hector O. Fattorini. Some remarks on complete controllability. *SIAM J. Control*, 4:686–694, 1966.

[21] Manuel González-Burgos and Luz de Teresa. Controllability results for cascade systems of m coupled parabolic PDEs by one control force. *Port. Math.*, 67(1):91–113, 2010.

[22] Lydia Ouaili. Minimal time of null controllability of two parabolic equations. *Math. Control Relat. Fields*, 10(1):89–112, 2020.

[23] El Hadji Samb. Boundary null-controllability of two coupled parabolic equations: simultaneous condensation of eigenvalues and eigenfunctions. *ESAIM, Control Optim. Calc. Var.*, 27: article no. S29 (43 pages), 2021.

Franck Boyer
Institut de Mathématiques de Toulouse, UMR 5219
Université de Toulouse, CNRS, UPS IMT
31062 Toulouse Cedex 9
FRANCE
franck.boyer@math.univ-toulouse.fr

Morgan Morancey
Aix Marseille Univ, CNRS, I2M, UMR 7373
Marseille
FRANCE
morgan.morancey@univ-amu.fr