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# $Nil$-closed Noetherian sub-algebras of $H^*(W)$ and their centres

Ouriel Blœdé

**Abstract**

For $\mathcal{G}$ some groupoid whose objects are the sub-vector spaces of a $\mathbb{F}_p$-vector space $W$, we define $H^*(W)^{\mathcal{G}}$ a $nil$-closed, noetherian, unstable sub-algebra of $H^*(W)$ over the Steenrod algebra. The application on the appropriate ordered set of groupoids, that maps $\mathcal{G}$ to $H^*(W)^{\mathcal{G}}$ defines an isomorphism of posets to the set of noetherian, $nil$-closed, unstable sub-algebras of $H^*(W)$ of transcendence degree $\dim(W)$, ordered by inclusion.

Since any noetherian and integral unstable algebra of transcendence degree $\dim(W)$ admits an injection into $H^*(W)$, any such $nil$-closed unstable algebra is isomorphic to some $H^*(W)^{\mathcal{G}}$.

We prove that $\mathcal{G}$ encodes the centre, in the sense of Heard, of $H^*(W)^{\mathcal{G}}$. Also, there is a $H^*(C)$-comodule structure on $K$ that is associated with the centre of $K$. For $K = H^*(W)^{\mathcal{G}}$, we explain how the sub-algebra of primitive elements of $H^*(W)^{\mathcal{G}}$ for this comodule structure is also encoded in $\mathcal{G}$. Along the way, we prove that this algebra of primitive elements is also noetherian.

## 1. Introduction

### 1.1. The two theorems of Adams–Wilkerson

We consider $p$ a prime number and $W$ a $\mathbb{F}_p$-vector space. The $\mathbb{F}_p$-algebra $H^*(W) := H^*(BW)$, where $BW$ is the classifying space of $W$ and $H^*$ denote the singular cohomology with $\mathbb{F}_p$-coefficients, is an unstable algebra over $\mathcal{A}$, the Steenrod algebra over $\mathbb{F}_p$. The category $\mathcal{U}$ of unstable modules over $\mathcal{A}$ admits a localizing sub-category $\mathcal{N}$il and an unstable module $M$ is called $nil$-closed if its localization away from $\mathcal{N}$il is an isomorphism.

The first theorem of Adams–Wilkerson [1, Theorem 1.1] states that any integral and noetherian unstable algebra $K$ is isomorphic to some sub-algebra of $H^*(W)$, with $\dim(W)$ equal to the transcendence degree of $K$. The first aim of this article is to describe the poset of unstable sub-algebras of $H^*(W)$ which are noetherian, $nil$-closed and whose transcendence degree is the dimension of $W$. Combined with the theorem of Adams–Wilkerson, this would give us a description of any $nil$-closed, integral and noetherian unstable algebra.

We define $Groupoid(W)$, a poset of groupoids $\mathcal{G}$, whose objects are the sub-vector spaces of $W$ and whose morphisms are isomorphisms, and with $\mathcal{G}$ satisfying a so-called restriction property.

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**Theorem 1.1.** *For all finite dimensional $\mathbb{F}_p$-vector space $W$, there is an isomorphism of posets between the poset of nil-closed and noetherian sub-algebras of $H^*(W)$ whose transcendence degree is* $\dim(W)$ *and Groupoid$(W)$.*

This isomorphism is given by $\mathcal{G} \mapsto H^*(W)^{\mathcal{G}}$, where the sub-algebra $H^*(W)^{\mathcal{G}}$ is a generalisation of the algebra of invariants $H^*(W)^G$ in the case where $G$ is a sub-group of $\mathrm{Gl}(W)$. Theorem 1.1 therefore generalizes the second theorem of Adams–Wilkerson, which states that a noetherian, *nil*-closed sub-algebra of $H^*(W)$ whose transcendance degree is $\dim(W)$ and which is integrally closed in its field of fraction is some $H^*(W)^G$ for some sub-group $G$ of $\mathrm{Gl}(W)$.

## 1.2. The centre of an unstable algebra

In [3], Dwyer and Wilkerson introduced the notion of a central element of an unstable algebra; this notion allowed them to exhibit the only exotic finite loop space at prime 2 in [4]. In the case where $K$ is noetherian and connected, the set of central elements of $K$ coincides with the set of pairs $(V, \phi)$ such that

(1) $\phi \in \mathrm{Hom}_{\mathcal{K}}(K, H^*(V))$,

(2) $K$ admits a structure $\kappa$ of $H^*(V)$-comodule in $\mathcal{K}$, such that the following diagram commutes:

$$\begin{array}{ccc} K & \xrightarrow{\kappa} & K \otimes H^*(V) \\ & \searrow^{\phi} & \downarrow{\scriptstyle \epsilon_K \otimes \mathrm{id}} \\ & & H^*(V), \end{array}$$

where $\epsilon_K$ denotes the augmentation of $K$ (which is uniquely defined because of the connectedness of $K$).

In [7], Heard showed that for $K$ noetherian, $K$ admits a unique (up to isomorphism) central element $(C, \gamma)$ such that $\gamma$ induces a structure of finitely generated $K$-module on $H^*(C)$ and $\dim(C)$ is maximal among such central elements. Heard called this central element the centre of $K$. The centre of an unstable algebra has been shown to be an important invariant. In [11] and [12], Kuhn used it to approximate the depth of $K$ as well as invariants $d_0(K)$ and $d_1(K)$ introduced by Henn, Lannes, and Schwartz in [9], in the case where $K$ is the cohomology of a group. Heard generalised those results for $K$ noetherian in [6] and [7].

For $K$ noetherian, since a central element of $K$ is associated with a $H^*(V)$-comodule structure on $K$, it gives rise to a second invariant: the sub-algebra of primitive elements of $K$ under this $H^*(V)$-comodule structure. The second objective of this article is to explain how central elements of $H^*(W)^{\mathcal{G}}$ and their associated sub-algebra of primitive elements are determined by $\mathcal{G}$. We prove the following theorem, which gives a complete description of central elements of $H^*(W)^{\mathcal{G}}$, since any morphism from $H^*(W)^{\mathcal{G}}$ to $H^*(T)$ in $K$ factors through the inclusion of $H^*(W)^{\mathcal{G}}$ in $H^*(W)$.

**Theorem 1.2.** *For $\mathcal{G} \in Groupoid(W)$ and $\delta$ a morphism from some vector space $T$ to $W$, the induced morphism in $\mathcal{K}$ from $H^*(W)^{\mathcal{G}}$ to $H^*(T)$ is central if and only if $V := \delta(T)$ satisfies the two following conditions:*

(1) *for any $U$ and $U'$ subspaces of $W$, $\alpha \in \mathcal{G}(U, U')$ and $v \in V \cap U$, we have $v \in U'$ and $\alpha(v) = v$,*

(2) *for any $U$ and $U'$ subspaces of $W$ and $\alpha$ an isomorphism from $U$ to $U'$ such that $\alpha(v) = v$ for all $v \in V \cap U$, $\alpha \in \mathcal{G}(U, U')$ if and only if $\bar{\alpha} \in \mathcal{G}(V + U, V + U')$, where $\bar{\alpha}$ is the morphism that maps $v \in V$ to itself and $u \in U$ to $\alpha(u)$.*

## 1.3. The algebra of primitive elements associated with a central element

We prove the following theorem, for $\mathcal{G} \in Groupoid(W)$, $\phi$ being a central element of $H^*(W)^{\mathcal{G}}$, and $P(H^*(W)^{\mathcal{G}}, \phi)$ denoting the sub-algebra of primitive elements of $H^*(W)^{\mathcal{G}}$ with respect to the comodule structure associated to $\phi$.

**Theorem 1.3.** *$P(H^*(W)^{\mathcal{G}}, \phi)$ is nil-closed and noetherian.*

Furthermore, taking $\delta$ a morphism of vector spaces with codomain $W$, such that $\phi$ is the restriction of $\delta^*$ to $H^*(W)^{\mathcal{G}}$, $P(H^*(W)^{\mathcal{G}}, \phi)$ identifies with a sub-algebra of $H^*(W/\mathrm{Im}(\delta))$ with transcendence degree $\dim(W/\mathrm{Im}(\delta))$. Therefore, by Theorem 1.1, there is $\mathcal{G}' \in Groupoid(W/\mathrm{Im}(\delta))$ such that $P(H^*(W)^{\mathcal{G}}, \phi)$ identifies with $H^*(W/\mathrm{Im}(\delta))^{\mathcal{G}'}$. In Theorem 5.29, we will explain how to compute $\mathcal{G}'$ from $\mathcal{G}$.

## 1.4. Organisation of the paper

In Section 2, we recall some known facts about unstable algebras and their centres.

In [8], the authors described an equivalence of categories between $\mathcal{K}/\mathcal{N}\mathrm{il}$, which is the localization of $\mathcal{K}$ in morphisms whose kernels and cokernels are nilpotent, and some category of functors, given by $K \mapsto \mathrm{Hom}_{\mathcal{K}}(K, H^*(\_))$. The main idea of this article is to classify the categories of elements of functors of the form $W \mapsto \mathrm{Hom}_{\mathcal{K}}(K, H^*(W))$. In Section 3, we define the notion of a formal category of elements and characterize those that can be obtained as the category of elements of a functor $\mathrm{Hom}_{\mathcal{K}}(K, H^*(\_))$ with $K$ noetherian. Then, we study the properties of such formal categories of elements.

In Section 4, we define and study the notion of a central element in a formal category of elements. We show that, for $\mathfrak{S}_K$ the category of elements of the functor $\mathrm{Hom}_{\mathcal{K}}(K, H^*(\_))$ with $K$ $nil$-closed and noetherian, the central elements of $\mathfrak{S}_K$ are the central elements of $K$.

Finally, in Section 5, we define the sub-algebras $H^*(W)^{\mathcal{G}}$ and prove our different classification results.

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## 2. Recollections on unstable algebras over the Steenrod algebra

In the following, $\mathcal{A}$ denotes the Steenrod algebra over $\mathbb{F}_p$ with $p$ a prime number, $\mathcal{U}$ and $\mathcal{K}$ denote the categories of unstable modules and unstable algebras over $\mathcal{A}$, and $\mathcal{N}$il denotes the Serre class of nilpotent objects in $\mathcal{U}$. Recollections about unstable algebras, unstable modules, and nilpotent objects can be found in [15].

We start this section, by recalling some known facts about the localization $\mathcal{K}/\mathcal{N}\mathrm{il}$, which is the localization of $\mathcal{K}$ by morphisms whose kernel and cokernel are nilpotent.

Then, we recall a definition of central elements of a noetherian unstable algebra $K$ over $\mathcal{A}$.

### 2.1. $\mathcal{N}$il-localisation of unstable algebras

In [8], Henn, Lannes, and Schwartz constructed a localized category $\mathcal{K}/\mathcal{N}\mathrm{il}$ with respect to the morphisms whose kernels and cokernels are in $\mathcal{N}$il, in the sense of [10]. Then, they described an equivalence of categories between $\mathcal{K}/\mathcal{N}\mathrm{il}$ and some category whose objects are contravariant functors from $\mathcal{V}^f$, the category of finite dimensional vector spaces, to $\mathcal{P}\mathrm{fin}^{(\mathcal{V}^f)^{\mathrm{op}}}$, the category of profinite sets.

*Notation 2.1.* We denote by $r\colon \mathcal{K} \to \mathcal{K}/\mathcal{N}\mathrm{il}$ the localization functor. It admits a right-adjoint (cf [8]) that we denote by $m$. Finally, we denote by $l_1$ the composition $m \circ r$ from $\mathcal{K}$ to itself.

**Definition 2.2.** An unstable algebra $K$ is called *nil*-closed if the unit of the adjunction $K \to l_1(K)$ is an isomorphism.

**Proposition 2.3** ([8, Proposition 4.4]). *An unstable algebra $K$ is nil-closed if and only if* $\mathrm{Ext}^0_{\mathcal{U}}(N,K) \cong \mathrm{Ext}^1_{\mathcal{U}}(N,K) \cong 0$ *for any nilpotent module $N$, where the* Ext-*groups are computed in the abelian category $\mathcal{U}$ of unstable modules over the Steenrod algebra.*

**Theorem 2.4.** *For $V \in \mathcal{V}^f$, $H^*(V)$ is nil-closed.*

*Proof.* It is a direct consequence of Proposition 2.3 and the injectivity of $H^*(V)$ ([13]). □

**Proposition 2.5.** *For $K \in \mathcal{K}$ nil-closed and for $V \in \mathcal{V}^f$, $K \otimes H^*(V)$ is also nil-closed.*

*Proof.* The tensor product of *nil*-closed modules is *nil*-closed (see [5, Proposition 3.4]). Therefore, this result follows from Theorem 2.4. □

For $K \in \mathcal{K}$, $\mathrm{Hom}_{\mathcal{K}}(K, H^*(V))$ has a structure of profinite set which comes from the fact that $K$ is the direct limit of the unstable sub-algebras of $K$ which are finitely generated as $\mathcal{A}$-algebras. In particular, if $K$ is noetherian, the profinite set structure of $\mathrm{Hom}_{\mathcal{K}}(K, H^*(V))$ is that of a finite set.

**Definition 2.6.** Let $\mathcal{P}\mathrm{fin}^{(\mathcal{V}^f)^{\mathrm{op}}}$ be the category of functors from $(\mathcal{V}^f)^{\mathrm{op}}$ to $\mathcal{P}\mathrm{fin}^{(\mathcal{V}^f)^{\mathrm{op}}}$ and $g\colon \mathcal{K} \to \mathcal{P}\mathrm{fin}^{(\mathcal{V}^f)^{\mathrm{op}}}$ be the contravariant functor that maps $K$ to $V \mapsto \mathrm{Hom}_{\mathcal{K}}(K, H^*(V))$.

We denote by $\mathcal{P}\mathrm{fin}_{\omega}^{(\mathcal{V}^f)^{\mathrm{op}}}$ the essential image of $g$ in $\mathcal{P}\mathrm{fin}^{(\mathcal{V}^f)^{\mathrm{op}}}$.

**Theorem 2.7** ([8, Theorem 1.5 of Part II]). *The functor $g$ induces an equivalence of categories between $\mathcal{K}/\mathcal{N}\mathrm{il}$ and $(\mathcal{P}\mathrm{fin}_{\omega}^{(\mathcal{V}^f)^{\mathrm{op}}})^{op}$.*

*Remark 2.8.* The category $\mathcal{P}\mathrm{fin}_{\omega}^{(\mathcal{V}^f)^{\mathrm{op}}}$ is described in more detail in [8], and $\mathcal{F}\mathrm{in}^{(\mathcal{V}^f)^{\mathrm{op}}}$, the full subcategory of $\mathcal{P}\mathrm{fin}^{(\mathcal{V}^f)^{\mathrm{op}}}$ of contravariant functors with values in finite sets, is included in $\mathcal{P}\mathrm{fin}_{\omega}^{(\mathcal{V}^f)^{\mathrm{op}}}$.

*Notation 2.9.* We denote by $m_1$ the composition of $m$ with the equivalence of categories from $\mathcal{P}\mathrm{fin}_{\omega}^{(\mathcal{V}^f)^{\mathrm{op}}}$ to $\mathcal{K}/\mathcal{N}\mathrm{il}$.

The following lemma will be of importance in the following.

**Lemma 2.10.** *The functor $g$ turns injections into surjections and finite inverse limits into direct limits.*

*Proof.* There is an exact functor $f$ from the category $\mathcal{U}$ to the category of functors from $\mathcal{V}^f$ to the category $\mathcal{V}$ of vector spaces of any dimension (cf [8]). For $K \in \mathcal{K}$, it satisfies $f(K) \cong \mathbb{F}_p^{g(K)}$, where $\mathbb{F}_p^{g(K)(V)}$ denote the set of continuous maps from the profinite set $g(K)(V)$ to the discrete topological space $\mathbb{F}_p$. Since $f$ is exact it sends injections to injections and commutes with finite inverse limits, which concludes the proof. □

## 2.2. Central elements of a noetherian unstable algebra

The notion of a central element of an unstable algebra $K$ is defined by Dwyer and Wilkerson in [3] and they used it in [4] to exhibit the only exotic finite loop space at the prime 2. The centre of $K$ has been studied in detail in [6] and [7].

For $K$ an unstable algebra over the Steenrod algebra, a central element is a pair $(V, \phi)$, with $V \in \mathcal{V}^f$ and $\phi\colon K \to H^*(V)$ a morphism in $\mathcal{K}$ that satisfies some property that we do not wish to recall in full generality. We will only recall the easier characterization of a central element of $K$ from [3], in the case where $K$ is connected and noetherian.

**Definition 2.11.** Let $K$ be an unstable algebra, $K$ is connected if $K$ has an augmentation $\epsilon_K : K \to \mathbb{F}_p$ which induces an isomorphism $K^0 \xrightarrow{\cong} \mathbb{F}_p$.

*Notation 2.12.* We denote by $\epsilon_{K,V}$ or $\epsilon_V$, when there is no ambiguity on $K$, the composition of $\epsilon_K$ with the injection from $\mathbb{F}_p$ to $H^*(V)$.

The following propositions in Dwyer and Wilkerson's articles use less restrictive hypotheses. However, the hypothesis that $K$ is noetherian will be sufficient for this article.

**Proposition 2.13** ([2, Proof of Theorem 3.2])**.** *Let $K$ be a connected, noetherian, unstable algebra, then $(V, \epsilon_V)$ is central for all $V \in \mathcal{V}^f$.*

We recall the following results of [3].

**Proposition 2.14** ([3, Proposition 3.4])**.** *Let $K$ be a connected, noetherian, unstable algebra. Then, for $\phi \in \mathrm{Hom}_{\mathcal{K}}(K, H^*(V))$, $(V, \phi)$ is central if and only if there exists a morphism from $K$ to $K \otimes H^*(V)$ such that the following diagram commutes:*

$$\begin{array}{ccc} & & K \\ & \overset{\mathrm{id}}{\nearrow} & \uparrow{\scriptstyle \mathrm{id}\otimes\epsilon_{H^*(V)}} \\ K & \longrightarrow & K\otimes H^*(V) \\ & \underset{\phi}{\searrow} & \downarrow{\scriptstyle \epsilon_K\otimes\mathrm{id}} \\ & & H^*(V). \end{array}$$

*Notation 2.15.* We denote by $\mathbf{C}(K)$ the class of central elements of $K$.

**Corollary 2.16** ([7])**.** *Let $K$ be a connected, noetherian, unstable algebra. For $\phi \in \mathrm{Hom}_{\mathcal{K}}(K, H^*(V))$, $(V, \phi)$ is central if and only if $K$ has a structure of $H^*(V)$-comodule*

*$\kappa$ in $\mathcal{K}$, such that the following diagram commutes:*

$$\begin{array}{ccc} K & \xrightarrow{\ \kappa\ } & K \otimes H^*(V) \\ & \searrow^{\phi} & \downarrow{\scriptstyle \epsilon_K \otimes \mathrm{id}} \\ & & H^*(V). \end{array}$$

In particular, this implies:

**Proposition 2.17.** *Let $K$ be a connected, noetherian, unstable algebra, then for $\phi \in \mathbf{C}(K)$ and $\alpha\colon V \to E$ a morphism in $\mathcal{V}^f$, $(V, \alpha^* \circ \phi) \in \mathbf{C}(K)$.*

*Example 2.18.* For $W \in \mathcal{V}^f$, the addition in $W$, $\nabla_W$, induces on $H^*(W)$ a coalgebra structure in $\mathcal{K}$. Then, for every morphism of unstable modules $\phi$ from $H^*(W)$ to $H^*(V)$, one can take the composition of $\nabla_W^*$ with $\mathrm{id}_{H^*(W)} \otimes \phi$ to define a $H^*(V)$-comodule structure on $H^*(W)$ satisfying the hypothesis of Corollary 2.16. Therefore $(V, \phi)$ is central.

**Theorem 2.19** ([7])**.** *For $K$ noetherian and connected, there exists, up to isomorphism, a unique central element $(C, \gamma)$, such that $\gamma$ makes $H^*(C)$ into a finitely generated $K$-module and* $\dim(C)$ *is maximal. It is called the centre of $K$.*

*For $(C, \gamma)$ the centre of $K$, any central element of $K$ has the form $(V, \alpha^*\gamma)$ for some $V \in \mathcal{V}^f$ and $\alpha \in \mathrm{Hom}(V, C)$.*

We end this section by giving a characterization of central elements of a noetherian, connected, *nil*-closed unstable algebra $K$, using only properties of the functor $g(K)$.

*Notation 2.20.* When there is no ambiguity on $K$, we denote by $\epsilon_W$ the following composition

$$K \xrightarrow{\epsilon_K} \mathbb{F}_p \hookrightarrow H^*(W).$$

**Lemma 2.21.** *For $K \in \mathcal{K}$ connected, noetherian and for $\phi$ from $K$ to $H^*(V)$ central, there is a natural group action of* $(\mathrm{Hom}_{\mathbb{F}_p}(W, V), +)$ *on* $\mathrm{Hom}_{\mathcal{K}}(K, H^*(W))$ *that satisfies $\alpha \cdot \epsilon_W = \alpha^* \phi$.*

*Proof.* Since $K$ is connected, it admits a unit $1_K$. We have an isomorphism between $\mathrm{Hom}_{\mathcal{K}}(K, H^*(W)) \times \mathrm{Hom}_{\mathcal{K}}(H^*(V), H^*(W))$ and $\mathrm{Hom}_{\mathcal{K}}(K \otimes H^*(V), H^*(W))$ that maps $(\phi, \psi)$ to the unique morphism $\gamma$ in $\mathcal{K}$ such that $\gamma(1_K \otimes h) = \psi(h)$ and $\gamma(k \otimes 1_{H^*(W)}) = \phi(k)$ for all $k \in K$ and $h \in H^*(W)$. The result is a direct consequence of Corollary 2.16 and of the isomorphism $\mathrm{Hom}_{\mathcal{K}}(K \otimes H^*(V), H^*(W)) \cong \mathrm{Hom}_{\mathcal{K}}(K, H^*(W)) \times \mathrm{Hom}_{\mathcal{K}}(H^*(V), H^*(W)) \cong \mathrm{Hom}_{\mathcal{K}}(K, H^*(W)) \times \mathrm{Hom}_{\mathbb{F}_p}(W, V)$. □

**Proposition 2.22.** *If $K$ is connected, noetherian and nil-closed, $\phi \in \mathrm{Hom}_{\mathcal{K}}(K, H^*(V))$ is central if and only if, for any $W \in \mathcal{V}^f$ and any $\psi \in \mathrm{Hom}_{\mathcal{K}}(K, H^*(W))$, there is a unique element $\phi \boxplus \psi \in \mathrm{Hom}_{\mathcal{K}}(K, H^*(V \oplus W))$ such that*

$$\phi = \iota_V^*(\phi \boxplus \psi)\colon K \longrightarrow H^*(V \oplus W) \xrightarrow{\iota_V^*} H^*(V),$$

*and*

$$\psi = \iota_W^*(\phi \boxplus \psi)\colon K \longrightarrow H^*(V \oplus W) \xrightarrow{\iota_W^*} H^*(W).$$

*Proof.* If, for any $W \in \mathcal{V}^f$ and any $\psi \in \mathrm{Hom}_{\mathcal{K}}(K, H^*(W))$, there is a unique element $\phi \boxplus \psi$ that satisfies both conditions, one can define a morphism $\kappa$ from $\mathrm{Hom}_{\mathcal{K}}(K, H^*(W)) \times \mathrm{Hom}_{\mathbb{F}_p}(W, V)$ to $\mathrm{Hom}_{\mathcal{K}}(K, H^*(W))$ that maps $(\psi, \alpha)$ to $(\alpha \oplus \mathrm{id}_W)^*(\phi \boxplus \psi)$. It is natural and maps $(\psi, 0)$ to $\psi$ and $(\epsilon_W, \alpha)$ to $\alpha^*\phi$. By Proposition 2.5, since $K$ is $nil$-closed, $m_1(\mathrm{Hom}_{\mathcal{K}}(K, H^*(W)) \times \mathrm{Hom}_{\mathbb{F}_p}(W, V)) \cong K \otimes H^*(V)$. Therefore, $m_1(\kappa)$ is a morphism from $K$ to $K \otimes H^*(V)$ that satisfies the hypothesis of Proposition 2.14 and $(V, \phi)$ is central.

The converse is proven in [3, Lemma 4.6]. □

## 3. Formal categories of elements

In Section 5, we want to classify noetherian, $nil$-closed, unstable sub-algebras $K$ of $H^*(W)$, for some vector space $W$, and describe their central elements. To do so, we consider the category of elements of functors of the form $\mathrm{Hom}_{\mathcal{K}}(K, H^*(\_))$ for such sub-algebras $K$.

In this section, we start by describing such categories of elements and their properties in the case where $K$ is noetherian and a sub-algebra of $H^*(W)$. Then, we describe central elements of $K$ in terms of the category of elements of $\mathrm{Hom}_{\mathcal{K}}(K, H^*(\_))$.

### 3.1. Category of elements : an intrinsic characterisation

We recall that, for $S \in \mathcal{S}\mathrm{et}^{(\mathcal{V}^f)^{\mathrm{op}}}$, the category of elements of $S$ is the category $\mathfrak{S}_S$, whose objects are the pairs $(V, \phi)$ with $V \in \mathcal{V}^f$ and $\phi \in S(V)$ and whose morphisms from $(V, \phi)$ to $(W, \psi)$ are the linear morphisms $\alpha$ from $V$ to $W$, such that $\alpha^*\psi = \phi$. There exists a functor from $\mathfrak{S}_S$ to $\mathcal{V}^f$ that maps $(V, \phi)$ to $V$.

We give an intrinsic description of such categories.

**Definition 3.1.** A formal category of elements is a pair $(\mathcal{C}, \mathcal{S})$ where $\mathcal{C}$ is a category and $\mathcal{S}$ is a functor from $\mathcal{C}$ to $\mathcal{V}^f$, which satisfies:

(1) $\mathcal{S}$ is faithful,

(2) for all $V \in \mathcal{V}^f$, $\mathcal{S}^{-1}(\{V\})$ is a set,

(3) for $\alpha$ a linear morphism from $V$ to $W$ and for $c \in \mathcal{C}$ such that $\mathcal{S}(c) = W$, there exists a unique element $\alpha^* c \in \mathcal{C}$ and a unique morphism $\gamma$ from $\alpha^* c$ to $c$ in $\mathcal{C}$ such that $\mathcal{S}(\gamma) = \alpha$.

We denote by $\mathfrak{F}$ the category whose objects are the formal categories of elements and whose morphisms from $(\mathcal{C}, \mathcal{S})$ to $(\mathcal{C}', \mathcal{S}')$ are the functors $F$ from $\mathcal{C}$ to $\mathcal{C}'$ such that $\mathcal{S} = \mathcal{S}' \circ F$.

*Example 3.2.* For $S \in \mathcal{S}\mathrm{et}^{(\mathcal{V}^f)^{\mathrm{op}}}$ and $\mathcal{S}$ the functor from $\mathfrak{S}_S$ to $\mathcal{V}^f$ that maps $(V, \phi)$ onto $V$, $(\mathfrak{S}_S, \mathcal{S})$ is a formal category of element.

**Lemma 3.3.** *For $c \in \mathcal{C}$ and two composable morphisms $\alpha$ and $\beta$ in $\mathcal{V}^f$ with $\mathcal{S}(c)$ the codomain of $\beta$ we have:*

(1) $\mathrm{id}^*_{\mathcal{S}(c)}\, c = c$,

(2) $\alpha^*(\beta^* c) = (\beta \circ \alpha)^* c$.

*Proof.* For the first statement, we just have to notice that $\mathrm{id}_c$ is a morphism from $c$ to $c$ that satisfies $\mathcal{S}(\mathrm{id}_c) = \mathrm{id}_{\mathcal{S}(c)}$. For the second statement, we have morphisms $\gamma$ from $\beta^* c$ to $c$ and $\delta$ from $\alpha^*(\beta^* c)$ to $\beta^* c$ such that $\mathcal{S}(\gamma) = \beta$ and $\mathcal{S}(\delta) = \alpha$, therefore $\gamma \circ \delta$ is a morphism from $\alpha^*(\beta^* c)$ to $c$ that satisfies $\mathcal{S}(\gamma \circ \delta) = \beta \circ \alpha$. □

We take the opportunity to prove the following lemma, which we will use on many occasions in this article.

**Lemma 3.4.**

(1) *Given $a$, $b$, $c$ and $d$ objects in $\mathcal{C}$ and morphisms $\alpha$ from $a$ to $c$, $\beta$ from $b$ to $d$ and $\gamma$ from $c$ to $d$ and given a linear morphism $\lambda$ such that the following diagram is a commutative square in $\mathcal{V}^f$:*

$$\begin{array}{ccc} \mathcal{S}(a) & \xrightarrow{\lambda} & \mathcal{S}(b), \\ {\scriptstyle \mathcal{S}(\alpha)} \downarrow & & \downarrow {\scriptstyle \mathcal{S}(\beta)} \\ \mathcal{S}(c) & \xrightarrow[\mathcal{S}(\gamma)]{} & \mathcal{S}(d) \end{array}$$

*there exists a unique morphism $\widetilde{\lambda}$ from $a$ to $b$ such that $\mathcal{S}(\widetilde{\lambda}) = \lambda$ and the following diagram commutes:*

$$\begin{array}{ccc} a & \xrightarrow{\tilde{\lambda}} & b \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ c & \xrightarrow[\gamma]{} & d. \end{array}$$

(2) *Given $a$, $b$, $c$ and $d$ objects in $C$ and morphisms $\alpha$ from $a$ to $c$, $\beta$ from $b$ to $d$ and $\gamma$ from $a$ to $b$ with $\mathcal{S}(\alpha)$ surjective, and given a linear morphism $\lambda$ such that the following diagram is a commutative square in $\mathcal{V}^f$:*

$$\begin{array}{ccc} \mathcal{S}(a) & \xrightarrow{\mathcal{S}(\gamma)} & \mathcal{S}(b), \\ {\scriptstyle \mathcal{S}(\alpha)}\downarrow & & \downarrow{\scriptstyle \mathcal{S}(\beta)} \\ \mathcal{S}(c) & \xrightarrow[\lambda]{} & \mathcal{S}(d) \end{array}$$

*there exists a unique morphism $\widetilde{\lambda}$ from $c$ to $d$ such that $\mathcal{S}(\widetilde{\lambda}) = \lambda$ and the following diagram commutes:*

$$\begin{array}{ccc} a & \xrightarrow{\gamma} & b \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ c & \xrightarrow[\tilde{\lambda}]{} & d. \end{array}$$

*Proof.* For the first statement, we consider $\widetilde{\lambda}$ the map from $\lambda^*b$ to $b$, by construction, $\mathcal{S}(\beta \circ \widetilde{\lambda}) = \mathcal{S}(\gamma \circ \alpha)$, therefore, by Lemma 3.3, $a = \mathcal{S}(\gamma \circ \alpha)^*d = \lambda^*b$ and $\beta \circ \widetilde{\lambda} = \gamma \circ \alpha$.

For the second statement, since $\mathcal{S}(\alpha)$ is surjective, we can consider a right inverse $\iota$ from $\mathcal{S}(c)$ to $\mathcal{S}(a)$. Then, using Lemma 3.3, $\iota^*a = \iota^*\mathcal{S}(\alpha)^*c = \mathrm{id}^*_{\mathcal{S}(c)}\, c = c$. For $\widetilde{\iota}$ the induced morphism from $c$ to $a$, we have $\alpha \circ \iota = \mathrm{id}_c$ and the unique valid choice for $\widetilde{\lambda}$ is $\widetilde{\lambda} = \beta \circ \gamma \circ \widetilde{\iota}$. □

**Theorem 3.5.** *The categories $\mathcal{S}\mathrm{et}^{(\mathcal{V}^f)^{\mathrm{op}}}$ and $\mathfrak{F}$ are equivalent.*

*Proof.* We have a functor

$$\begin{aligned} \mathcal{S}\mathrm{et}^{(\mathcal{V}^f)^{\mathrm{op}}} &\longrightarrow \mathfrak{F} \\ S &\longmapsto (\mathfrak{S}_S, \mathcal{S}), \end{aligned}$$

with $\mathcal{S}$ defined as in Example 3.2. And a functor

$$\begin{aligned} \mathfrak{F} &\longrightarrow \mathcal{S}\mathrm{et}^{(\mathcal{V}^f)^{\mathrm{op}}} \\ (C, \mathcal{S}) &\longmapsto (V \mapsto \mathcal{S}^{-1}(\{V\})), \end{aligned}$$

that maps the morphism $\alpha$ from $V$ to $W$ to the application from $\mathcal{S}^{-1}(\{W\})$ to $\mathcal{S}^{-1}(\{V\})$ that maps $c \in \mathcal{S}^{-1}(W)$ to the unique $\alpha^* c$.

Those two functors are quasi-inverses. □

**Definition 3.6.** We say that $(\mathcal{C}, \mathcal{S})$ is connected if there is a unique element $\epsilon$ in $\mathcal{C}$ such that $\mathcal{S}(\epsilon) = 0$. In this case, we denote by $\epsilon_V$ the elements of the form $0^*\epsilon$ where 0 denotes the trivial morphism in $\mathcal{V}^f$ from $V$ to 0.

## 3.2. Noetherian formal categories of elements

In [8], the authors described the functors in $\mathcal{P}\mathrm{fin}^{(\mathcal{V}^f)^{\mathrm{op}}}$ that arise from a noetherian unstable algebra. Such functors have values in (discrete profinite) sets.

*Notation 3.7.* For $K$ a noetherian unstable algebra, we will denote by $\mathfrak{S}_K$ the category of elements of the functor $\mathrm{Hom}_{\mathcal{K}}(K, H^*(\_))$.

We describe the formal categories of elements of the form $\mathfrak{S}_K$, with $K$ noetherian.

**Proposition 3.8.** *Let $(\mathcal{C}, \mathcal{S}) \in \mathfrak{F}$ and $c \in \mathcal{C}$. Then, there exists a unique sub-vector space $U$ of $\mathcal{S}(c)$, denoted by* $\ker(c)$*, such that:*

(1) *For all $c' \in \mathcal{C}$ and all morphisms $\gamma\colon c \to c'$,* $\ker(\mathcal{S}(\gamma)) \subset U$.

(2) *There exists $c' \in \mathcal{C}$ and $\gamma\colon c \to c'$ such that* $\ker(\mathcal{S}(\gamma)) = U$.

(3) *There exists $c_0 \in \mathcal{C}$ and $\gamma_0\colon c \to c_0$ such that $\mathcal{S}(\gamma_0)$ is the projection from $\mathcal{S}(c)$ to $\mathcal{S}(c)/U$.*

*Proof.* This is a direct consequence of Theorem 3.5 and Proposition-Definition 5.1 in [8]. □

**Definition 3.9.** For $(\mathcal{C}, \mathcal{S}) \in \mathfrak{F}$ and $c \in \mathcal{C}$, we say that $c$ is regular if $\ker(c) = 0$.

We can now define a notion of a noetherian formal category of elements, such that $(\mathcal{C}, \mathcal{S})$ is noetherian if and only if there exists $K \in \mathcal{K}$ noetherian such that $(\mathcal{C}, \mathcal{S}) \cong \mathfrak{S}_K$.

**Definition 3.10.** A formal category of elements $(\mathcal{C}, \mathcal{S})$ is noetherian if the following conditions are satisfied:

(1) for all $V \in \mathcal{V}^f$, $\mathcal{S}^{-1}(V)$ is finite,

(2) there exists $d \in \mathbb{N}$ such that $\mathcal{C}$ contains no regular object $c$ such that $\dim(\mathcal{S}(c)) > d$,

(3) for all $\gamma\colon c \to c'$ in $\mathcal{C}$, $\ker(c) = \mathcal{S}(\gamma)^{-1}(\ker(c'))$.

*Notation 3.11.* For $(\mathcal{C}, \mathcal{S}) \in \mathfrak{F}$ which satisfies the two first conditions in Definition 3.10, $\mathcal{S}^{-1} \in \mathcal{P}\mathrm{fin}_{\omega}^{(\mathcal{V}^f)^{\mathrm{op}}}$ (see [8]). We denote by $\mathfrak{L}(\mathcal{C}, \mathcal{S})$ the image of $\mathcal{S}^{-1}$ by $m_1 : \mathcal{P}\mathrm{fin}_{\omega}^{(\mathcal{V}^f)^{\mathrm{op}}} \cong \mathcal{K}/\mathcal{N}\mathrm{il} \xrightarrow{m} \mathcal{K}$.

**Proposition 3.12.**

(1) *If $K \in \mathcal{K}$ is noetherian, $\mathfrak{S}_K$ is noetherian.*

(2) *If $(\mathcal{C}, \mathcal{S}) \in \mathfrak{F}$ is noetherian, then $\mathfrak{L}(\mathcal{C}, \mathcal{S}) \in \mathcal{K}$ is noetherian.*

*Proof.* It is a direct consequence of Theorem 3.5 and Theorem 7.1 in [8]. □

## 3.3. Rector's Category

**Definition 3.13.** For $(\mathcal{C}, \mathcal{S}) \in \mathfrak{F}$, $\mathfrak{R}_{\mathcal{C}}$ is the full subcategory of $\mathcal{C}$ of regular objects.

*Remark 3.14.* In the case where $K$ is a noetherian unstable algebra, $\mathfrak{R}_K := \mathfrak{R}_{\mathfrak{S}_K}$ is Rector's category of $K$. Rector's category of $K$ is defined in [14] as the full subcategory of $\mathfrak{S}_K$ whose objects are the pairs $(V, \phi)$ such that $H^*(V)$ is finitely generated as a $K$-module. It is a result from [8] that this condition is equivalent to $(V, \phi)$ being regular.

For $(\mathcal{C}, \mathcal{S})$ noetherian, the category $\mathfrak{R}_{\mathcal{C}}$ behaves nicely and, furthermore, one can "reconstruct" $(\mathcal{C}, \mathcal{S})$ from $\mathfrak{R}_{\mathcal{C}}$. This fact will be the main ingredient in the classification problem that we are addressing in this article.

*Remark 3.15.* For $(\mathcal{C}, \mathcal{S})$ in $\mathfrak{F}$, $\mathcal{S}|_{\mathfrak{R}_{\mathcal{C}}}$ is a faithful functor from $\mathfrak{R}_{\mathcal{C}}$ to $\mathcal{V}^f$ but it does not satisfy that, for any $\alpha$ a linear morphism from $V$ to $W$, and for $c \in \mathfrak{R}_{\mathcal{C}}$ such that $\mathcal{S}(c) = W$, there exists uniques $c' \in \mathfrak{R}_{\mathcal{C}}$ and $\gamma$ from $c'$ to $c$ such that $\mathcal{S}(\gamma) = \alpha$. Indeed, if $\alpha$ is not injective $\alpha^* c \in \mathcal{C}$ is the unique object that satisfies that condition and it is not regular. Yet, if $(\mathcal{C}, \mathcal{S})$ is noetherian, that condition is satisfied if and only if $\alpha$ is injective.

*Notation 3.16.* We denote by $\mathcal{VI}$ the wide subcategory of $\mathcal{V}^f$ that contains all injective morphisms.

**Definition 3.17.** A formal category of elements on $\mathcal{VI}$ is a pair $(\mathcal{R}, \mathcal{S})$ where $\mathcal{R}$ is a category and $\mathcal{S}$ is a functor from $\mathcal{R}$ to $\mathcal{VI}$, which satisfies:

(1) $\mathcal{S}$ is faithful,

(2) for all $V \in \mathcal{V}^f$, $\mathcal{S}^{-1}(\{V\})$ is a set

(3) for $\alpha$ an injective morphisms from $V$ to $W$ and for $c \in \mathcal{R}$ such that $\mathcal{S}(c) = W$, there exists a unique $\alpha^* c \in \mathcal{R}$ and a unique $\gamma$ from $\alpha^* c$ to $c$ in $\mathcal{R}$ such that $\mathcal{S}(\gamma) = \alpha$.

We denote by $\mathfrak{F}\mathfrak{I}$ the category whose objects are the formal categories of elements on $\mathcal{VI}$ and whose morphisms from $(\mathcal{R}, \mathcal{S})$ to $(\mathcal{R}', \mathcal{S}')$ are the functors $F$ from $\mathcal{R}$ to $\mathcal{R}'$ such that $F \circ \mathcal{S} = \mathcal{S}'$.

**Lemma 3.18.** *For $(C, \mathcal{S})$ in $\mathfrak{F}$ noetherian, $(\mathfrak{R}_C, \mathcal{S}|_{\mathfrak{R}_C})$ is formal on $\mathcal{VI}$.*

*Proof.* It is straightforward from the definition of a noetherian object in $\mathfrak{F}$. □

We explain now how to reconstruct a noetherian object $(C, \mathcal{S})$ in $\mathfrak{F}$ from $(\mathfrak{R}_C, \mathcal{S}|_{\mathfrak{R}_C})$.

**Definition 3.19.** For $(\mathcal{R}, \mathcal{S}) \in \mathfrak{F}\mathfrak{I}$, let $(\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}})$ be the following formal category of elements. The objects of $\widetilde{\mathcal{R}}$ are triples $(V, U, c)$ with $V \in \mathcal{V}^f$, $U$ a sub-vector space of $V$ and $c \in \mathcal{R}$ such that $\mathcal{S}(c) = V/U$. The morphisms from $(V', U', c')$ to $(V, U, c)$ are pairs $(\alpha, \gamma)$ with $\alpha$ a linear map from $V'$ to $V$ and $\gamma \in \mathcal{R}(c', c)$ that satisfies:

(1) $\alpha^{-1}(U) = U'$,

(2) $\mathcal{S}(\gamma)$ is the map induced by $\alpha$ from $V'/U'$ to $V/U$.

Finally, $\widetilde{\mathcal{S}}$ is the functor that maps $(V, U, c)$ to $V$ and $(\alpha, \gamma)$ to $\alpha$.

**Theorem 3.20.**

(1) *For $(C, \mathcal{S}) \in \mathfrak{F}$ noetherian, $(C, \mathcal{S}) \cong (\widetilde{\mathfrak{R}_C}, \widetilde{\mathcal{S}}|_{\mathfrak{R}_C})$.*

(2) *For $(\mathcal{R}, \mathcal{S}) \in \mathfrak{F}\mathfrak{I}$, $(\mathcal{R}, \mathcal{S}) \cong (\mathfrak{R}_{\widetilde{\mathcal{R}}}, \widetilde{\mathcal{S}}|_{\mathfrak{R}_{\widetilde{\mathcal{R}}}})$. Also $(\widetilde{\mathcal{R}}, \widetilde{\mathcal{S}})$ is noetherian if and only if $\mathcal{S}^{-1}(\{V\})$ is finite for every $V \in \mathcal{V}^f$ and there exists $d \in \mathbb{N}$ such that $\mathcal{S}^{-1}(\{V\})$ is empty for $\dim(V) > d$.*

*Proof.* For the first statement, the functor in the first direction maps $c$ to $(V, \ker(c), c_0)$, where $c_0$ is defined as in Proposition 3.8. Since $(C, \mathcal{S})$ is noetherian, $c_0$ is indeed regular. For $\beta$ from $c'$ to $c$ in $C$, by Proposition 3.8, we have the following diagram in $C$:

$$\begin{array}{ccc} c' & \xrightarrow{\beta} & c \\ \gamma_0' \downarrow & & \downarrow \gamma_0 \\ c_0' & & c_0, \end{array}$$

with $\mathcal{S}(c'_0) = \mathcal{S}(c')/\ker(c')$ and $\mathcal{S}(c_0) = \mathcal{S}(c)/\ker(c)$. $\ker(c') = \mathcal{S}(\beta)^{-1}(\ker(c))$ so $\mathcal{S}(\beta)$ induces a morphism from $\mathcal{S}(c'_0)$ to $\mathcal{S}(c_0)$. By Lemma 3.4, this morphism can be obtained in a unique way as a morphism $\mathcal{S}(\widetilde{\beta})$ with $\widetilde{\beta}$ from $c'_0$ to $c_0$. The morphism $\beta$ is mapped to $(\mathcal{S}(\beta), \widetilde{\beta})$.

The functor in the other direction is the one that maps $(V, U, c_0)$ to the unique $c$ for which there is a $\gamma_0$ from $c$ to $c_0$ such that $\mathcal{S}(\gamma_0)$ is the projection from $V$ to $V/U$. For $(\alpha, \beta)$ a morphism in $\widetilde{\mathfrak{R}_C}$, we have the following diagram:

$$\begin{array}{ccc} c' & & c \\ \gamma'_0 \downarrow & & \downarrow \gamma_0 \\ c'_0 & \xrightarrow{\beta} & c_0. \end{array}$$

By Lemma 3.4, there is a unique $\widetilde{\alpha}$, from $c'$ to $c$, such that $\mathcal{S}(\widetilde{\alpha}) = \alpha$, $(\alpha, \beta)$ is mapped to $\widetilde{\alpha}$.

It is easy to check that the descriptions above define morphisms in $\mathfrak{F}$ and that they are inverses.

For the second statement, it is enough to check that $(V, U, c)$ is regular in $\widetilde{\mathcal{R}}$ if and only if $U = 0$. On the one hand, for $U \neq 0$ there is the morphism $(\pi, \mathrm{id}_c)$ from $(V, U, c)$ to $(V/U, 0, c)$, with $\pi$ the projection from $V$ to $V/U$, therefore $(V, U, c)$ is not regular. On the other hand, morphisms $(\alpha, \gamma)$ in $\widetilde{\mathcal{R}}$ from $(V, 0, c)$ satisfy $\alpha = \mathcal{S}(\gamma)$. Since $\gamma$ is a morphism in $\mathcal{R}$, $\alpha$ is injective, therefore $(V, 0, c)$ is regular. □

## 3.4. **Some classification problems**

In Section 5, we will consider the following problem: can we classify the sub-unstable algebras $K$ of $H^*(W)$ that are *nil*-closed, noetherian and such that the injection $\phi\colon K \hookrightarrow H^*(W)$ is regular?

*Notation 3.21.* For $W \in \mathcal{V}^f$ and $K \in \mathcal{K}$, we denote by $\mathbf{W}$ the category of elements of the functor $\mathrm{Hom}_{\mathbb{F}_p}(\_, W) \cong \mathrm{Hom}_{\mathcal{K}}(H^*(W), H^*(\_))$.

An injection $\phi$ from $K$ to $H^*(W)$ induces a surjection of formal categories of elements from $\mathbf{W}$ to $\mathfrak{S}_K$ (by surjection, we mean a functor that is a surjection on objects but not on morphisms), this surjection maps $(W, \mathrm{id}_W)$ onto $(W, \phi)$.

We consider $(C, \mathcal{S})$ a noetherian formal category of elements, and $\phi$ a surjection from $\mathbf{W}$ to $(C, \mathcal{S})$. Since $\phi$ is a surjection, any object of $C$ has the form $\alpha^* c$ for some morphism $\alpha$ from some vector space $V$ to $W$, and since $(C, \mathcal{S})$ is noetherian, $\alpha^* c$ is regular if and only if $\alpha$ is injective. We get the following lemma.

**Lemma 3.22.** *For $(C, \mathcal{S}) \in \mathfrak{F}$ noetherian and $\phi\colon \mathbf{W} \twoheadrightarrow (C, \mathcal{S})$ such that $\phi(W, \mathrm{id}_W)$ is regular, $\phi$ is induced by a surjection of formal categories of elements on $\mathcal{VI}$ from $\mathfrak{R}_\mathbf{W}$ to $\mathfrak{R}_C$.*

In this subsection, we address the following problem: how to classify formal categories $(\mathcal{R}, \mathcal{S})$ on $\mathcal{VI}$ with a fixed surjection $\phi\colon \mathfrak{R}_\mathbf{W} \twoheadrightarrow (\mathcal{R}, \mathcal{S})$.

**Definition 3.23.** Let $\mathfrak{R}_\mathbf{W} \downarrow \mathfrak{F}\mathfrak{I}$ be the category whose objects are formal categories of elements $(\mathcal{R}, \mathcal{S})$ on $\mathcal{VI}$ with a fixed surjection $\phi\colon \mathfrak{R}_\mathbf{W} \twoheadrightarrow (\mathcal{R}, \mathcal{S})$, where a surjection means a map in $\mathfrak{F}\mathfrak{I}$ that is a surjection on objects, and whose morphisms are morphisms in $\mathfrak{F}\mathfrak{I}$ compatible with the surjections from $\mathfrak{R}_\mathbf{W}$.

The category $\mathfrak{R}_\mathbf{W}$ (where we forget the structure of formal category of elements on $\mathcal{VI}$) admits the following skeleton. The objects of $\mathcal{S}k$ are given by $(W, \mathrm{id}_W)$ and the pairs of the form $(U, \iota_U)$ with $U$ a sub-vector space of $W$ and $\iota_U$ the inclusion of $U$ in $W$, and the morphisms of $\mathcal{S}k$ are the identities and the inclusions of sub-spaces. Since a morphism from $(U, \iota_U)$ to $(R, \iota_R)$ correspond to a factorisation of $\iota_U$ by $\iota_R$, $\mathcal{S}k$ is full, and since for any objects $(V, \alpha)$ in $\mathfrak{R}_\mathbf{W}$ there is a unique isomorphism from $(V, \alpha)$ to an element of $\mathcal{S}k$, which is $(\mathrm{Im}(\alpha), \iota_{\mathrm{Im}(\alpha)})$, the inclusion of $\mathcal{S}k$ in $\mathfrak{R}_\mathbf{W}$ is an equivalence of categories.

For $\phi\colon \mathfrak{R}_\mathbf{W} \twoheadrightarrow (\mathcal{R}, \mathcal{S})$ an object in $\mathfrak{R}_\mathbf{W} \downarrow \mathfrak{F}\mathfrak{I}$, the image of $\mathcal{S}k$ by $\phi$ is not in general a skeleton of $\mathcal{R}$. Since $\phi$ is a surjection, it contains an element in each isomorphism class of object in $\mathcal{R}$, but this element might not be unique, and also it might not be a full subcategory of $\mathcal{R}$. Indeed, for $U$ and $U'$ two sub-vector spaces of $W$, there might be morphisms $\gamma$ from $U$ to $U'$ such that $\iota_U \neq \iota_{U'} \circ \gamma$ but $\phi(U, \iota_U) = \phi(U', \iota_{U'} \circ \gamma)$. We define a groupoid $\mathcal{G}_{(\mathcal{R}, \mathcal{S}, \phi)}$ (denoted only $\mathcal{G}_\mathcal{R}$ when there is no ambiguity) with objects the images of objects in $\mathcal{S}k$ that will capture all the informations of $(\mathcal{R}, \mathcal{S}, \phi)$.

**Definition 3.24.** For $(\mathcal{R}, \mathcal{S})$ a formal category of elements on $\mathcal{VI}$ and for $\phi$ a surjection from $\mathfrak{R}_\mathbf{W}$ to $(\mathcal{R}, \mathcal{S})$, let $\mathcal{G}_{(\mathcal{R}, \mathcal{S}, \phi)}$ be the groupoid whose objects are the sub-vector spaces $U$ of $W$, and such that $\mathcal{G}_{(\mathcal{R}, \mathcal{S}, \phi)}(U, U')$ is the set of isomorphisms $\alpha$ from $U$ to $U'$ such that there exists $\gamma$ from $\phi(U, \iota_U)$ to $\phi(U', \iota_{U'})$ in $\mathcal{R}$ with $\mathcal{S}(\gamma) = \alpha$.

**Lemma 3.25.** *Let $(\mathcal{R}, \mathcal{S}, \phi)$ be an object in $\mathfrak{R}_\mathbf{W} \downarrow \mathfrak{F}\mathfrak{I}$. Then, $\mathcal{G}_{(\mathcal{R}, \mathcal{S}, \phi)}$ satisfies the following property. For $\alpha \in \mathcal{G}_{(\mathcal{R}, \mathcal{S}, \phi)}$, for $M$ a sub-space of $U$, and for $\alpha_M : M \to \alpha(M)$ the restriction of $\alpha$ to $M$ corestricted to $\alpha(M)$, $\alpha_M \in \mathcal{G}_{(\mathcal{R}, \mathcal{S}, \phi)}(M, \alpha(M))$.*

*Proof.* If $\gamma$ from $\phi(U, \iota_U)$ to $\phi(U', \iota'_U)$ satisfies $\mathcal{S}(\gamma) = \alpha$, then, for $\iota_M^U$ the inclusion of $M$ in $U$, $\gamma \circ \phi(\iota_M^U)$ is a morphism from $\phi(M, \iota_M)$ to $\phi(U', \iota_{U'})$ in $\mathcal{R}$. One checks that it factorises as $\phi(\iota_{\alpha(M)}^{U'}) \circ \gamma'$ for some $\gamma'$ that satisfies $\mathcal{S}(\gamma') = \alpha_M$. □

**Definition 3.26.** For $\mathcal{G}$ a groupoid whose objects are the sub-vector spaces of $W$, and whose morphisms are isomorphisms of vector spaces, we say that $\mathcal{G}$ has the restriction property if, for all $U$, $U'$ and $\alpha \in \mathcal{G}(U, U')$, $\alpha_M$ is in $\mathcal{G}(M, \alpha(M))$.

A morphism $F$ in $\mathfrak{R}_{\mathbf{W}} \downarrow \mathfrak{F}\mathfrak{I}$ from $(\mathcal{R}, \mathcal{S}, \phi)$ to $(\mathcal{R}', \mathcal{S}', \phi')$ induces an inclusion of groupoids from $\mathcal{G}_{(\mathcal{R},\mathcal{S},\phi)}$ to $\mathcal{G}_{(\mathcal{R}',\mathcal{S},\phi')}$, indeed for all $U$ and $U'$ subspaces of $W$, and for all $\alpha \in \mathcal{G}_{(\mathcal{R},\mathcal{S},\phi)}(U, U')$, for $\gamma$ between $\phi(U, \iota_U)$ and $\phi(U', \iota_{U'})$ such that $\mathcal{S}(\gamma) = \alpha$, $F(\gamma)$ is an isomorphism from $\phi'(U, \iota_U)$ to $\phi'(U', \iota_{U'})$ in $\mathcal{R}'$ and $\mathcal{S}(\gamma) = \mathcal{S}'(F(\gamma)) = \alpha$.

**Definition 3.27.** Let *Groupoid*$(W)$ be the category whose objects are groupoids with the restriction property and with objects the subspaces of $W$, and whose morphisms are the inclusions of groupoids.

We want to prove that the categories $\mathfrak{R}_{\mathbf{W}} \downarrow \mathfrak{F}\mathfrak{I}$ and *Groupoid*$(W)$ are equivalent. Let us first explain how $\mathcal{G}_{(\mathcal{R},\mathcal{S},\phi)}$ captures all the information about morphisms in $\mathcal{R}$.

**Lemma 3.28.** *For* $(\mathcal{R}, \mathcal{S}, \phi) \in \mathfrak{R}_{\mathbf{W}} \downarrow \mathfrak{F}\mathfrak{I}$ *and for* $(U, \gamma)$ *and* $(V, \nu)$ *two objects in* $\mathfrak{R}_{\mathbf{W}}$*, we have*

$$\mathrm{Hom}_{\mathcal{R}}(\phi(U, \gamma), \phi(V, \nu)) \cong \bigsqcup_{U' \lhd V\ ;\ \dim(U')=\dim(U)} \mathcal{G}_{\mathcal{R}}(\gamma(U), \nu(U')).$$

*Proof.* Let $\alpha$ be a morphism from $\phi(U, \gamma)$ to $\phi(V, \nu)$ in $\mathcal{R}$. Then, $\mathcal{S}(\alpha) \in \mathcal{VI}$ factorises uniquely as $\iota \circ \widetilde{\alpha}$ with $\widetilde{\alpha}$ an isomorphism from $U$ to $U' = \mathcal{S}(\alpha)(U)$ and $\iota$ the inclusion of $U'$ in $V$. Then $\nu|^{U'} \circ \widetilde{\alpha} \circ (\gamma|^{\gamma(U)})^{-1}$ is an element of $\mathcal{G}_{\mathcal{R}}(\gamma(U), \nu(U'))$. It is easy to check that $\alpha \mapsto \nu|^{U'} \circ \widetilde{\alpha} \circ (\gamma|^{\gamma(U)})^{-1}$ defines a bijection. □

We construct a quasi-inverse to the functor from $\mathfrak{R}_{\mathbf{W}} \downarrow \mathfrak{F}\mathfrak{I}$ to *Groupoid*$(W)$ that maps $(\mathcal{R}, \mathcal{S}, \phi)$ to $\mathcal{G}_{(\mathcal{R},\mathcal{S},\phi)}$.

**Definition 3.29.** For $\mathcal{G} \in$ *Groupoid*$(W)$, $\sim_{\mathcal{G}}$ is the following equivalence relation on objects of $\mathfrak{R}_{\mathbf{W}}$. For $\beta$ and $\gamma$ from $V$ to $W$, $(V, \beta) \sim_{\mathcal{G}} (V, \gamma)$ if there is $\alpha$ in $\mathcal{G}(\beta(V), \gamma(V))$ such that $\widetilde{\gamma} = \alpha \circ \widetilde{\beta}$, for $\widetilde{\gamma}$ and $\widetilde{\beta}$ the corestrictions of $\gamma$ and $\beta$ to their images.

We denote by $[V, \beta]_{\mathcal{G}}$, or simply $[V, \beta]$ when there is no ambiguity, the equivalence class of $(V, \beta)$.

Since $\mathcal{G}$ has the restriction property, for $\beta$ and $\gamma$ from $V$ to $W$ with $(V, \beta) \sim_{\mathcal{G}} (V, \gamma)$, and for $\delta$ from some vector space $H$ to $V$, $(H, \beta \circ \delta) \sim_{\mathcal{G}} (H, \gamma \circ \delta)$. The following category is therefore well defined and it is in an obvious way an element of $\mathfrak{R}_{\mathbf{W}} \downarrow \mathfrak{F}\mathfrak{I}$.

**Definition 3.30.** We defined $\mathfrak{R}_{\mathbf{W}/\sim_{\mathcal{G}}} \in \mathfrak{R}_{\mathbf{W}} \downarrow \mathfrak{F}\mathfrak{I}$ as the category whose objects are the equivalence classes $[V, \beta]$ and whose morphisms from $[H, \eta]$ to $[V, \beta]$ is the set of morphisms $\delta$ from $H$ to $V$ such that $(H, \beta \circ \delta) \sim_{\mathcal{G}} (H, \eta)$. The functor from $\mathfrak{R}_{\mathbf{W}/\sim_{\mathcal{G}}}$ to

$\mathcal{VI}$ is the one that map $[V,\phi]$ to $V$ and the morphism $\alpha$ to itself and the surjection from $\mathfrak{R}_{\mathbf{W}}$ is given by $(V,\beta) \mapsto [V,\beta]$.

Finally, if $\mathcal{G}$ is included in $\mathcal{G}'$, the surjection from $\mathfrak{R}_{\mathbf{W}}$ to $\mathfrak{R}_{\mathbf{W}/\sim_{\mathcal{G}'}}$ factorises through $\mathfrak{R}_{\mathbf{W}} \twoheadrightarrow \mathfrak{R}_{\mathbf{W}/\sim_{\mathcal{G}}}$, $\mathcal{G} \mapsto \mathfrak{R}_{\mathbf{W}/\sim_{\mathcal{G}}}$ is therefore a functor from *Groupoid*$(W)$ to $\mathfrak{R}_{\mathbf{W}} \downarrow \mathfrak{F}\mathfrak{I}$.

*Example 3.31.* We consider $G$ a subgroup of $\mathrm{Gl}(W)$. We define $\mathfrak{g}(G) \in$ *Groupoid*$(W)$ by $\mathfrak{g}(G)(U,U')$ is the set of restriction to $U$ of morphisms in $G$ such that $g(U) = U'$. Then, $(V,\gamma) \sim_{\mathfrak{g}(G)} (V,\beta)$ if and only if there is $g \in G$ such that $\beta = g \circ \gamma$. Since $\mathrm{Hom}_{\mathcal{K}}(H^*(W)^G, H^*(U)) \cong \mathrm{Hom}(U,W)/\sim$ with $\alpha \sim \beta$ if and only if there is $g \in G$ such that $\beta = g \circ \alpha$, $\mathfrak{R}_{H^*(W)^G}$ and $\mathfrak{R}_{\mathbf{W}/\sim_{\mathfrak{g}(G)}}$ are isomorphic in $\mathfrak{R}_{\mathbf{W}} \downarrow \mathfrak{F}\mathfrak{I}$.

**Theorem 3.32.** *The categories* $\mathfrak{R}_{\mathbf{W}} \downarrow \mathfrak{F}\mathfrak{I}$ *and Groupoid*$(W)$ *are equivalent.*

*Proof.* The equivalence is given by the functors $\mathcal{G} \mapsto \mathfrak{R}_{\mathbf{W}/\sim_{\mathcal{G}}}$ and $(\mathcal{R},\mathcal{S},\phi) \mapsto \mathcal{G}_{\mathcal{R}}$. We have to prove that they are quasi-inverses. We consider $\mathcal{G}_{\mathfrak{R}_{\mathbf{W}/\sim_{\mathcal{G}}}}(U,U')$ for $U$ and $U'$ isomorphic subspaces of $W$. It is the set of isomorphisms $\alpha$ from $U$ to $U'$ such that $\iota_{U'} \circ \alpha \sim_{\mathcal{G}} \iota_U$ and, by definition, this is the case if and only if $\alpha \in \mathcal{G}(U,U')$. Therefore $\mathcal{G}_{\mathfrak{R}_{\mathbf{W}/\sim_{\mathcal{G}}}} = \mathcal{G}$.

In the other direction, for $(\mathcal{R},\mathcal{S},\phi) \in \mathfrak{R}_{\mathbf{W}} \downarrow \mathfrak{F}\mathfrak{I}$, if $(V,\beta) \sim_{\mathcal{G}_{\mathcal{R}}} (V,\gamma)$ there is $\alpha \in \mathcal{G}_{\mathcal{R}}(\beta(V),\gamma(V))$ such that $\widetilde{\gamma} = \alpha \circ \widetilde{\beta}$, for $\widetilde{\gamma}$ and $\widetilde{\beta}$ the corestriction of $\gamma$ and $\beta$ to their images. Then, $\phi(V,\gamma) = \widetilde{\gamma}^*\phi(\gamma(V),\iota_{\gamma(V)}) = \widetilde{\beta}^*(\alpha^*\phi(\gamma(V),\iota_{\gamma(V)}))$. But since $\alpha \in \mathcal{G}_{\mathcal{R}}(\beta(V),\gamma(V)), \alpha^*\phi(\gamma(V),\iota_{\gamma(V)}) = \phi(\beta(V),\iota_{\beta(V)})$. Therefore, $\phi(V,\gamma) = \phi(V,\beta)$. We can therefore define a surjective map $\Lambda$ from objects of $\mathfrak{R}_{\mathbf{W}/\sim_{\mathcal{G}_{\mathcal{R}}}}$ to objects of $\mathcal{R}$ defined by $[V,\gamma] \mapsto \phi(V,\gamma)$.

We prove that $\Lambda$ is injective. If $\phi(V,\gamma) = \phi(V,\beta)$, since $\gamma$ and $\beta$ are injective morphisms, there is a unique $\alpha$ from $\beta(V)$ to $\gamma(V)$ such that $\widetilde{\gamma} = \alpha \circ \widetilde{\beta}$. Therefore,

$$\widetilde{\beta}^*\phi(\beta(V),\iota_{\beta(V)}) = \widetilde{\gamma}^*\phi(\gamma(V),\iota_{\gamma(V)}) = \widetilde{\beta}^*(\alpha^*\phi(\gamma(V),\iota_{\gamma(V)})).$$

Since $\widetilde{\beta}$ is an isomorphism, we get $\phi(\beta(V),\iota_{\beta(V)}) = \alpha^*\phi(\gamma(V),\iota_{\gamma(V)})$. Hence, $\alpha \in \mathcal{G}_{\mathcal{R}}(\beta(V),\gamma(V))$ and therefore $(V,\gamma) \sim_{\mathcal{G}_{\mathcal{R}}} (V,\beta)$.

Finally, the functoriality of $\Lambda$ is straightforward and the fact that it is an isomorphism of categories is a consequence of Lemma 3.28. It is clear that $\Lambda$ defines an isomorphism in $\mathfrak{R}_{\mathbf{W}} \downarrow \mathfrak{F}\mathfrak{I}$. □

## 4. **Central elements**

In this section, we start by defining central elements of a formal category of elements in such a way that for $K \in \mathcal{K}$ noetherian, connected and *nil*-closed, central elements of $K$ coincide with central elements of $\mathfrak{S}_K$. Then, for $(C,\mathcal{S}) \in \mathfrak{F}$, we explain why it is enough to determine regular central elements of $(\mathcal{C}, \mathcal{S})$. Finally, we characterise central elements of $(\mathcal{R}, \mathcal{S}, \phi) \in \mathfrak{F}\mathfrak{I}$ using $\mathcal{G}_{(\mathcal{R},\mathcal{S},\phi)}$.

## 4.1. Central elements in a formal category of elements

We define central elements of a formal category of elements in such a way that for $K$ noetherian, $(V, \phi)$ is central for $K$ if and only if it is central as an element of $\mathfrak{S}_K$.

**Definition 4.1.** For $(\mathcal{C}, \mathcal{S}) \in \mathfrak{F}$ and $c \in \mathcal{C}$, we say that $c$ is central if for any $c' \in \mathcal{C}$, $c$ and $c'$ have a coproduct.

**Lemma 4.2.** *For $(\mathcal{C}, \mathcal{S})$ a formal category of elements, $c$ and $c'$ have a coproduct in $\mathcal{C}$, if and only if there exists a unique object $c \boxplus c'$ that satisfies $\mathcal{S}(c \boxplus c') = \mathcal{S}(c) \oplus \mathcal{S}(c')$ and such that there exists morphisms $\iota$ and $\gamma$ from $c$ and $c'$ to $c \boxplus c'$ with $\mathcal{S}(\iota)$ the inclusion of $\mathcal{S}(c)$ in $\mathcal{S}(c) \oplus \mathcal{S}(c')$ and $\mathcal{S}(\gamma)$ the inclusion of $\mathcal{S}(c')$. In this case, $c \boxplus c'$ is a coproduct.*

*Proof.* For any $d$ in $\mathcal{C}$ with maps $\iota$ and $\gamma$ from $c$ and $c'$ to $d$, $\mathcal{S}(\iota)$ and $\mathcal{S}(\gamma)$ factorises through $\mathcal{S}(\iota) \oplus \mathcal{S}(\gamma)$ from $\mathcal{S}(c) \oplus \mathcal{S}(c')$ to $\mathcal{S}(d)$. Therefore, since $(\mathcal{C}, \mathcal{S})$ is a formal category of elements, there exists $c'' \in \mathcal{C}$ such that $\mathcal{S}(c'') = \mathcal{S}(c) \oplus \mathcal{S}(c')$ and $\delta$ from $c''$ to $d$ such that $\iota$ and $\gamma$ factorises through $\delta$. Furthermore, the induced maps $\widetilde{\iota}$ and $\widetilde{\gamma}$ satisfy that $\mathcal{S}(\widetilde{\iota})$ is the inclusion of $\mathcal{S}(c)$ in $\mathcal{S}(c) \oplus \mathcal{S}(c')$ and $\mathcal{S}(\widetilde{\gamma})$ is the inclusion of $\mathcal{S}(c')$.

Therefore, if there is a unique $c \boxplus c'$ that satisfies the required conditions, it satisfies the universal property of the coproduct.

Conversely, if $c$ and $c'$ admit a coproduct $d$, the canonical injections $\iota$ and $\gamma$ factors through an object $c''$ as above, and by the universal property of the coproduct, $c''$ is isomorphic to $d$ and therefore a coproduct of $c$ and $c'$. Finally, for $e \in \mathcal{C}$ and $f$ and $f'$ from $c$ and $c'$ to $e$ such that $\mathcal{S}(e) = \mathcal{S}(c) \oplus \mathcal{S}(c')$, $\mathcal{S}(f)$ is the inclusion of $\mathcal{S}(c)$ and $\mathcal{S}(f')$ is the inclusion of $\mathcal{S}(c')$, we have $e = c''$. Indeed, $f$ and $f'$ factorises through $c''$ and $f''$, the induced morphism from $c''$ to $e$, satisfies $\mathcal{S}(f'') = \mathrm{id}_{\mathcal{S}(c)\oplus\mathcal{S}(c')}$. Therefore, $c'' = \mathrm{id}^*_{\mathcal{S}(e)}\, e = e$. ◻

*Remark 4.3.* By definition of a formal category of elements, there is a map $\gamma$ from $c$ to $c'$ such that $\mathcal{S}(\gamma) = \alpha$ if and only if $\alpha^* c' = c$. Therefore, when it is defined, $c \boxplus c'$ is the only element such that $\iota^*_{\mathcal{S}(c)}(c \boxplus c') = c$ and $\iota^*_{\mathcal{S}(c')}(c \boxplus c') = c'$, for $\iota_{\mathcal{S}(c)}$ and $\iota_{\mathcal{S}(c')}$ the inclusions of $\mathcal{S}(c)$ and $\mathcal{S}(c')$ in $\mathcal{S}(c) \oplus \mathcal{S}(c')$.

**Proposition 4.4.** *For $K$ a noetherian, nil-closed and connected unstable algebra, $(V, \phi)$ is central if and only if it is central as an element of $\mathfrak{S}_K$.*

*Proof.* This is a direct consequence of Lemma 4.2 and Proposition 2.22. ◻

## 4.2. Central elements of a noetherian and connected formal category of elements

The connectedness (see Definition 3.6) of $(C, \mathcal{S})$ plays an important role in describing its central elements.

**Proposition 4.5.** *If $(C, \mathcal{S})$ is not connected, $(C, \mathcal{S})$ admits no central elements.*

*Proof.* For $x \in C$ such that $\mathcal{S}(x) = 0$, we can consider $C_x$ the set of elements $c$ in $C$ such that $0^*c = x$. Then, for any map $\gamma\colon c \to c'$ in $C$, $c \in C_x$ if and only if $c' \in C_x$. For any $c \in C$, we can take $c'$ that is not in $C_{0^*c}$, in this case $c \boxplus c'$ should be both in $C_{0^*c}$ and in $C_{0^*c'}$ which is not possible, therefore $c \boxplus c'$ is not defined and $c$ is not central. □

**Proposition 4.6.** *For $(C, \mathcal{S})$ connected and noetherian, $\epsilon_V$ is central for any $V \in \mathcal{V}^f$.*

*Proof.* Let $c \in C$, we consider $\pi$ the projection from $\mathcal{S}(c) \oplus V$ to $\mathcal{S}(c)$. We show that, $\pi^*c$ is a coproduct of $c$ and $\epsilon_V$.

We have $\iota_V^*\pi^*c = 0^*c = \epsilon_V$ and $\iota_{\mathcal{S}(c)}^*\pi^*c = \mathrm{id}_{\mathcal{S}(c)}^*\, c = c$. We show that it is the unique element that satisfies both identities.

Let $c'$ such that $\mathcal{S}(c') = \mathcal{S}(c) \oplus V$ and $\iota_V^*c' = \epsilon_V$ and $\iota_{\mathcal{S}(c)}^*c' = c$. Since $(C, \mathcal{S})$ is noetherian, $\iota_V^{-1}(\ker(c')) = \ker(\epsilon_V) = V$. Therefore, $V \subset \ker(c')$. We get that $c' = \pi^*c_0'$, for some element such that $\mathcal{S}(c_0') = \mathcal{S}(c)$. Furthermore, $c = \iota_{\mathcal{S}(c)}^*c' = \iota_{\mathcal{S}(c)}^*\pi^*c_0' = c_0'$.

By Lemma 4.2, $\pi^*c$ is a coproduct of $c$ and $\epsilon_V$. □

**Proposition 4.7.** *For $(C, \mathcal{S})$ a noetherian and connected formal category of elements, if $c$ is central, for any morphism $\gamma$ from $c'$ to $c$ in $C$, $c'$ is central.*

*Proof.* We prove first the case where $\mathcal{S}(\gamma)$ is an injection. Up to isomorphism, we can suppose that $\mathcal{S}(\gamma)$ is the inclusion of a sub-space of $\mathcal{S}(c)$. We use the following notations $W = \mathcal{S}(c')$ and $S$ is a complementary sub-space of $W$ in $\mathcal{S}(c)$. Then, for any $d \in C$, and $H = \mathcal{S}(d)$, $(\iota_W \oplus \mathrm{id}_H)^*(c \boxplus d)$ satisfies $\mathcal{S}((\iota_W \oplus \mathrm{id}_H)^*(c \boxplus d)) = W \oplus H$ and $\iota_W^*(\iota_W \oplus \mathrm{id}_H)^*(c \boxplus d) = \iota_W^*c = c'$ and $\iota_H^*(\iota_W \oplus \mathrm{id}_H)^*(c \boxplus d) = d$.

We need to show that it is the only element that satisfies both identities. Let $c'' \in C$ such that $\mathcal{S}(c'') = W \oplus H$, $\iota_W^*c'' = c'$ and $\iota_H^*c'' = d$. Since $c$ is central, we can consider $c \boxplus c''$. Then, $\mathcal{S}(c \boxplus c'') = S \oplus W_1 \oplus W_2 \oplus H$, with $W_1$ and $W_2$ to copies of $W$, $S \oplus W_1$ corresponding to $\mathcal{S}(c)$ and $W_2 \oplus H$ to $\mathcal{S}(c'')$.

We want to prove that $\iota_{S\oplus W_2}^*(c \boxplus c'') = c$. Indeed, in this case we would have $\iota_{S\oplus W_2\oplus H}^*(c \boxplus c'') = c \boxplus d$ and therefore $(\iota_W \oplus \mathrm{id}_H)^* c \boxplus d = (\iota_{W_2} \oplus \mathrm{id}_H)^*\iota_{S\oplus W_2\oplus H}^*(c \boxplus c'')$ which is equal to $\iota_{W_2\oplus H}^*(c \boxplus c'') = c''$.

We consider $\iota_{S\oplus W_1\oplus W_2}^* c \boxplus c''$, it is equal to $c \boxplus c'$. But, if we consider $\delta$ from $S \oplus W_1 \oplus W_2$ to $S \oplus W$ defined by $\delta(s \oplus w_1 \oplus w_2) = s \oplus (w_1 + w_2)$, we have $\mathcal{S}(\delta^*c) = S \oplus W_1 \oplus W_2$

and $\iota^*_{S\oplus W_1}\delta^* c = c$ and $\iota^*_{W_2}\delta^* c = c'$, therefore $c \boxplus c' = \delta^* c$. We get that $\iota^*_{S\oplus W_2}(c \boxplus c'') = \iota^*_{S\oplus W_2}\delta^* c = \mathrm{id}^*_{S\oplus W}\, c = c$.

Finally, we consider the case where $\mathcal{S}(\gamma)$ is not an injection, we consider $S$ a complementary subspace of $\ker(\mathcal{S}(\gamma))$ in $\mathcal{S}(c')$ and $\alpha$ the restriction of $\mathcal{S}(\gamma)$ to $S$. Then, $c' = \alpha^* c \boxplus \epsilon_{\ker(\mathcal{S}(\gamma))}$, the centrality of $c'$ is, therefore, a direct consequence of the centrality of $\alpha^* c$ and $\epsilon_{\ker(\mathcal{S}(\gamma))}$. □

A direct consequence of Proposition 4.7 is that for $c \in C$ and $c_0$ as in Proposition 3.8, $c$ is central if and only if $c_0$ is central, where $c_0$ is regular. We introduce a characterization of regular central elements intrinsic to $\mathfrak{R}_C$.

**Lemma 4.8.** *For $(C, \mathcal{S})$ noetherian and $c \in C$ regular, $c$ is central if and only if it admits a coproduct in $\mathfrak{R}_C$ with any regular element $c'$.*

*Proof.* First, we suppose that $c$ is central. For $c' \in C$ regular and for $U = \ker(c \boxplus c')$, by Proposition 3.8 there exists $c''$ regular such that $\mathcal{S}(c'') = \mathcal{S}(c) \oplus \mathcal{S}(c')/U$ and such that $c \boxplus c' = \pi^* c''$, for $\pi$ the projection from $\mathcal{S}(c) \oplus \mathcal{S}(c')$ on $\mathcal{S}(c) \oplus \mathcal{S}(c')/U$. We prove that $c''$ is a coproduct of $c$ and $c'$ in $\mathfrak{R}_C$. We consider two maps $\iota$ and $\gamma$ from $c$ and $c'$ to a regular element $d$. Those factorises uniquely through the canonical injection from $c$ and $c'$ into $c \boxplus c'$, we denote by $\delta$ the induced morphism from $c \boxplus c'$ to $d$. Since $d$ is regular, we have $U = \ker(c \boxplus c') = \mathcal{S}(\delta)^{-1}(\ker(d)) = \ker(\mathcal{S}(\delta))$. By Lemma 3.4, we get that $\delta$ factorises uniquely through $c \boxplus c' \to c''$, therefore $c''$ is a coproduct of $c$ and $c'$ in $\mathfrak{R}_C$.

Conversely, if $c''$ is a coproduct of $c$ and $c'$ in $\mathfrak{R}_C$, we prove that $c$ and $c'$ admit a coproduct in $C$. For $\iota_c$ and $\iota_{c'}$ the canonical injections from $c$ and $c'$ to $c''$, we consider $(\iota_c \oplus \iota_{c'})^* c''$. Let $d \in C$ with $\iota$ and $\gamma$ morphisms from $c$ and $c'$ to $d$. For $d_0$ such that $\mathcal{S}(d_0) = \mathcal{S}(d)/\ker(d)$ as in Proposition 3.8, $\iota$ and $\gamma$ induce morphisms in $\mathfrak{R}_C$ from $c$ and $c'$ to $d_0$. Since $c''$ is a coproduct in $\mathfrak{R}_C$, those factorise through a unique morphism $\delta\colon c'' \to d_0$. We get the following diagram in $C$:

$$\begin{array}{ccc} (\iota_c \oplus \iota_{c'})^* c'' & & d \\ \downarrow & & \downarrow \\ c'' & \xrightarrow[\delta]{} & d_0. \end{array}$$

By construction, the following diagram is commutative in $\mathcal{V}^f$:

$$\begin{array}{ccc} \mathcal{S}(c) \oplus \mathcal{S}(c') & \xrightarrow{\mathcal{S}(\iota)\oplus\mathcal{S}(\gamma)} & \mathcal{S}(d) \\ \downarrow & & \downarrow \\ \mathcal{S}(c'') & \xrightarrow[\mathcal{S}(\delta)]{} & \mathcal{S}(d_0). \end{array}$$

By Lemma 3.4, we get a unique factorisation of $\iota$ and $\gamma$ through $(\iota_c \oplus \iota_{c'})^* c''$, it is therefore a coproduct of $c$ and $c'$ in $C$. □

This leads to the following definition.

**Definition 4.9.** For $(\mathcal{R}, \mathcal{S})$ a formal category of elements on $\mathcal{VI}$, $c \in \mathcal{R}$ is said to be central if it admits a coproduct in $\mathcal{R}$ with any element $c' \in \mathcal{R}$.

## 4.3. Central elements of objects in $\mathfrak{R}_{\mathbf{W}} \downarrow \mathfrak{F}\mathfrak{I}$

We end this section by describing the central elements of an object $(\mathcal{R}, \mathcal{S}, \phi)$ in $\mathfrak{R}_{\mathbf{W}} \downarrow \mathfrak{F}\mathfrak{I}$ using its associated groupoid.

Since any object in $\mathcal{R}$ is isomorphic to some object of the form $\phi(U, \iota_U)$ with $U$ a sub-vector space of $W$, it is enough to describe central elements of the form $\phi(V, \iota_V)$ with $V$ a sub-vector space of $W$.

**Lemma 4.10.** *For $(\mathcal{R}, \mathcal{S}, \phi) \in \mathfrak{R}_{\mathbf{W}} \downarrow \mathfrak{F}\mathfrak{I}$ and $V$ and $U$ two subspaces of $W$, if $\phi(V, \iota_V)$ and $\phi(U, \iota_U)$ admit a coproduct in $\mathcal{R}$ then $\phi(V + U, \iota_{V+U})$ is a coproduct of $\phi(V, \iota_V)$ and $\phi(U, \iota_U)$ in $\mathcal{R}$.*

*Proof.* The injections of $V$ and $U$ in $V + U$ induce morphisms in $\mathcal{R}$ from $\phi(V, \iota_V)$ and $\phi(U, \iota_U)$ to $\phi(V + U, \iota_{V+U})$. For $\phi(V, \iota_V) \sqcup \phi(U, \iota_U)$ a coproduct of $\phi(V, \iota_V)$ and $\phi(U, \iota_U)$, it induces a morphism $\gamma$ from $\phi(V, \iota_V) \sqcup \phi(U, \iota_U)$ to $\phi(V + U, \iota_{V+U})$. Since $\gamma$ is a morphism in $\mathcal{R}$, $\mathcal{S}(\gamma)$ is injective, but $\mathcal{S}(\gamma)$ factorises the inclusions of $U$ and $V$ in $V + U$, it is therefore surjective. We get that $\gamma$ is an isomorphism in $\mathcal{R}$ and $\phi(V + U, \iota_{V+U})$ is a coproduct in $\mathcal{R}$. □

**Theorem 4.11.** *For $(\mathcal{R}, \mathcal{S}, \phi) \in \mathfrak{R}_{\mathbf{W}} \downarrow \mathfrak{F}\mathfrak{I}$ and $V$ a sub-vector space of $W$, $\phi(V, \iota_V)$ is central in $(\mathcal{R}, \mathcal{S})$ if and only if:*

1. *for any $U$ and $U'$ subspaces of $W$, $\alpha \in \mathcal{G}_{\mathcal{R}}(U, U')$ and $v \in V \cap U$, we have $v \in U'$ and $\alpha(v) = v$,*
2. *for any $U$ and $U'$ subspaces of $W$ and $\alpha$ an isomorphism from $U$ to $U'$ such that $\alpha(v) = v$ for all $v \in V \cap U$, $\alpha \in \mathcal{G}_{\mathcal{R}}(U, U')$ if and only if $\bar{\alpha} \in \mathcal{G}_{\mathcal{R}}(V + U, V + U')$, where $\bar{\alpha}$ is the morphism that maps $v \in V$ to itself and $u \in U$ to $\alpha(u)$.*

*Proof.* We consider $V$ such that $\phi(V, \iota_V)$ is central in $(\mathcal{R}, \mathcal{S})$. Then, for any $U$ and $U'$ subspaces of $W$ and for $\alpha \in \mathcal{G}_{\mathcal{R}}(U, U')$, since $\mathcal{G}_{\mathcal{R}}$ satisfies the restriction property, we can consider $\alpha_{V \cap U} \in \mathcal{G}_{\mathcal{R}}(V \cap U, \alpha(V \cap U))$. By composing it with the inclusion of $U'$ in $V + U'$ we get a morphism $\alpha'$ from $\phi(V \cap U, \iota_{V \cap U})$ to $\phi(V + U', \iota_{V+U'})$ in $\mathcal{R}$. Also, the inclusion of $V$ in $V + U'$ induces a morphism $\iota$ from $\phi(V, \iota_V)$ to $\phi(V + U', \iota_{V+U'})$.

By Lemma 4.10 and since $V \cap U \subset V$, there is a unique morphism $\lambda$ from $\phi(V, \iota_V)$ to $\phi(V + U', \iota_{V+U'})$ that factorises both $\iota$ and $\alpha'$. Then, $\mathcal{S}(\lambda)$ factorises both the inclusion of $V$ in $V + U'$ and the restriction of $\alpha$ to $V \cap U$. We get that $\alpha(v) = v$ for $v \in V \cap U$. We have proven the necessity of condition (1).

We prove now the necessity of condition (2). For $\alpha$ an isomorphism from $U$ to $U'$ that satisfies $\alpha(v) = v$ for all $v \in V \cap U$, $\alpha$ is the restriction of $\bar{\alpha}$ to $U$. Since $\mathcal{G}_\mathcal{R}$ has the restriction property, $\bar{\alpha} \in \mathcal{G}_\mathcal{R}(V + U, V + U')$ implies that $\alpha \in \mathcal{G}_\mathcal{R}(U, U')$. Conversely, if $\alpha \in \mathcal{G}_\mathcal{R}(U, U')$, $\alpha$ and the inclusion of $V$ induces morphisms in $\mathcal{R}$ from $\phi(V, \iota_V)$ and $\phi(U, \iota_U)$ to $\phi(V + U', \iota_{V+U'})$. By Lemma 4.10, those factorises through a morphism $\gamma$ from $\phi(V+U, \iota_{V+U})$ to $\phi(V+U', \iota_{V+U'})$ and by construction we have $\mathcal{S}(\gamma) = \bar{\alpha}$, therefore $\bar{\alpha} \in \mathcal{G}_\mathcal{R}(V + U, V + U')$.

Finally, we prove that the conditions (1) and (2) are sufficient. We need to prove that all objects of the form $\phi(V + U, \iota_{V+U})$ are coproducts of $\phi(V, \iota_V)$ and $\phi(U, \iota_U)$, for $U$ a subspace of $W$. Since condition (1) is satisfied, and by Lemma 3.28, any morphism from $\phi(V, \iota_V)$ is an inclusion. Then, for any pair of morphisms $\iota$ and $\gamma$ from $\phi(V, \iota_V)$ and $\phi(U, \iota_U)$ to some $\phi(H, \iota_H) \in \mathcal{R}$, we have $V \subset H$ and $\mathcal{S}(\iota)$ is the inclusion of $V$ and $\alpha = \mathcal{S}(\gamma)$ is an element of $\mathcal{G}_\mathcal{R}(U, U')$ for $U' = \mathcal{S}(\gamma)(U)$. Then, by condition (2), $\bar{\alpha} \in \mathcal{G}_\mathcal{R}(V+U, V+U')$, therefore there is a unique $\bar{\gamma}$ such that $\mathcal{S}(\bar{\gamma}) = \bar{\alpha}$. The composition of $\bar{\gamma}$ with the inclusion of $V + U'$ in $H$ is the unique morphism from $\phi(V + U, \iota_{V+U})$ to $\phi(H, \iota_H)$ that factorises $\iota$ and $\gamma$. Therefore, $\phi(V + U, \iota_{V+U})$ is a coproduct of $\phi(V, \iota_V)$ and $\phi(U, \iota_U)$. □

## 5. The algebras $H^*(W)^\mathcal{G}$

In this section, we apply the results of Sections 3 and 4 to some classification problems about *nil*-closed, integral, noetherian, unstable algebras. Before we explain in more detail the focus of this section, let us recall the first theorem of Adams–Wilkerson.

**Definition 5.1** ([8, Part II.2])**.** For $K \in \mathcal{K}$, the transcendence degree of $K$ is $d \in \mathbb{N} \cup \{\infty\}$, the supremum of the cardinals of finite sets of homogeneous elements in $K$ which are algebraically independent.

*Remark 5.2.* If $K$ is noetherian, the transcendence degree of $K$ is finite.

Let us recall the theorem of Adams–Wilkerson.

**Theorem 5.3** ([8, Theorem 3])**.** *Let $K$ be an integral, unstable algebra of transcendence degree less or equal to* $\dim(W)$*, then there exists an injection $\phi$ from $K$ to $H^*(W)$. Furthermore, this injection is regular if and only if the transcendence degree of $K$ equals* $\dim(W)$.

Therefore, every integral, $nil$-closed, noetherian, unstable algebra is isomorphic to a $nil$-closed, noetherian sub-unstable algebra of some $H^*(W)$. In the first sub-section we define $H^*(W)^{\mathcal{G}}$ for $\mathcal{G} \in$ *Groupoid*$(W)$. Then, $\mathcal{G} \mapsto H^*(W)^{\mathcal{G}}$ defines an explicit one-to-one correspondence between the objects of *Groupoid*$(W)$ and the noetherian, $nil$-closed, unstable sub algebras of $H^*(W)$ of transcendence degree $\dim(W)$.

Let us now recall the definition of the primitive elements of a comodule.

**Definition 5.4.** For $K \in \mathcal{K}$ provided with a $H^*(V)$-comodule structure $\kappa$ in $\mathcal{K}$, the algebra of primitive elements of $K$ is the sub-algebra of $K$ whose elements are those satisfying that $\kappa(x) = x \otimes 1$, for 1 the unit of $H^*(V)$. We will denote by $P(K, \kappa)$ the algebra of primitive elements of $K$ for the $H^*(V)$-comodule structure $\kappa$.

*Remark 5.5.* By Corollary 2.16, for all $(V, \phi) \in \mathbf{C}(K)$, there is a unique structure $\kappa_\phi$ of $H^*(V)$-comodule on $K$ such that $(\epsilon_K \otimes \mathrm{id}_{H^*(V)}) \circ \kappa_\phi = \phi$.

*Notation 5.6.* We will also denote $P(K, \kappa_\phi)$ by $P(K, \phi)$.

The problem that we are interested in is the following. If we fix $V$ some finite dimensional vector space and $P$ some unstable algebra, can we classify, under suitable hypothesis, the connected, noetherian, integral, $nil$-closed unstable algebras $K$, satisfying that $K$ admit a $H^*(V)$-comodule structure $\kappa$ in $\mathcal{K}$, whose algebra of primitive elements is isomorphic to $P$. Since, every $nil$-closed, noetherian, integral, unstable algebra of transcendence degree $\dim(W)$ is isomorphic to some $H^*(W)^{\mathcal{G}}$, we need to be able to identify the primitive elements associated with a regular central element $(V, \phi)$ of $H^*(W)^{\mathcal{G}}$.

In the second subsection, we consider $H^*(W)^{\mathcal{G}}$ and an inclusion $\delta$ from some vector space $V$ to $W$, such that $(V, \delta^*\phi) \in \mathbf{C}(H^*(W)^{\mathcal{G}})$ for $\phi$ the inclusion of $H^*(W)^{\mathcal{G}}$ in $H^*(W)$. Then, we prove that $P(H^*(W)^{\mathcal{G}}, \delta^*\phi)$ is a $nil$-closed and noetherian sub-algebra of $\pi^*(H^*(W/\mathrm{Im}(\delta)))$ for $\pi$ the projection from $W$ to $W/\mathrm{Im}(\delta)$. Since $\pi^*$ is injective, there exists $H^*(W/\mathrm{Im}(\delta))^{\mathcal{G}'} \subset H^*(W/\mathrm{Im}(\delta))$ such that $P(H^*(W)^{\mathcal{G}}, \delta^*\phi) = \pi^*(H^*(W/\mathrm{Im}(\delta))^{\mathcal{G}'})$. We conclude this sub-section by explaining how to determine $\mathcal{G}'$ from $\mathcal{G}$.

We conclude this section, by giving some applications of those results.

## 5.1. Noetherian, $nil$-closed, unstable sub-algebras of $H^*(W)$

In this sub-section, we give an explicit one-to-one correspondence between *Groupoid*$(W)$ and the noetherian, $nil$-closed, unstable sub algebra of $H^*(W)$ of transcendence degree $\dim(W)$.

**Theorem 5.7.** *For all $W \in \mathcal{V}^f$, there is a one-to-one correspondence between the set of nil-closed and noetherian sub-algebras of $H^*(W)$ whose transcendence degree is* $\dim(W)$ *and Groupoid*$(W)$.

*Proof.* By Theorem 3.32, there is a one-to-one correspondence between isomorphism classes in $\mathfrak{R}_{\mathbf{W}} \downarrow \mathfrak{F}\mathfrak{I}$ and the set of objects in *Groupoid*$(W)$. Thus, we have to justify that the set of *nil*-closed and noetherian sub-algebras of $H^*(W)$ of transcendence degree $\dim(W)$ are in one-to-one correspondence with isomorphism classes in $\mathfrak{R}_{\mathbf{W}} \downarrow \mathfrak{F}\mathfrak{I}$. Let $K$ be a *nil*-closed, noetherian, sub-algebra of $H^*(W)$ whose transcendence degree is $\dim(W)$. Then, for $\phi_K$ the inclusion of $K$ in $H^*(W)$, since the transcendence degree of $K$ is $\dim(W)$, by the Theorem of Adams–Wilkerson, $\phi_K$ is regular. Then, since $K$ is noetherian, $\phi_K$ induces a surjection from $\mathfrak{R}_{\mathbf{W}}$ to $\mathfrak{R}_K$ that we also denote by $\phi_K$, by abuse of notation. This defines a map $h$ from the set of *nil*-closed and noetherian sub-algebras of $H^*(W)$ whose transcendence degree is $\dim(W)$ to the set of isomorphism classes in $\mathfrak{R}_{\mathbf{W}} \downarrow \mathfrak{F}\mathfrak{I}$.

For $K$ and $K'$ two such sub-algebras of $H^*(W)$, $(\mathfrak{R}_K, \phi_K)$ and $(\mathfrak{R}_{K'}, \phi_{K'})$ are not necessarily isomorphic in $\mathfrak{R}_{\mathbf{W}} \downarrow \mathfrak{F}\mathfrak{I}$ if $K$ and $K'$ are isomorphic (under an isomorphism $\eta$) in $\mathcal{K}$. We also need $\phi_K = \phi_{K'} \circ \eta$. This is the case if and only if $K = K'$ and $\eta$ is the identity. The theorem is therefore a consequence of Theorems 2.7 and 3.20 and Proposition 3.12. □

We recall that the notation $\mathfrak{L}$ has been defined in Notation 3.11.

**Definition 5.8.** For $\mathcal{G}$ an object in *Groupoid*$(W)$ and for $q_{\mathcal{G}}$ the canonical surjection from $\mathfrak{R}_{\mathbf{W}}$ to $\mathfrak{R}_{\mathbf{W}/\sim_{\mathcal{G}}}$, $H^*(W)^{\mathcal{G}}$ is the image of the map

$$\mathfrak{L}(\widetilde{q}_{\mathcal{G}})\colon \mathfrak{L}(\widetilde{\mathfrak{R}}_{\mathbf{W}/\sim_{\mathcal{G}}}) \hookrightarrow H^*(W).$$

*Remark 5.9.* The application, $\mathcal{G} \mapsto H^*(W)^{\mathcal{G}}$ defines a contravariant functor between *Groupoid*$(W)$ and the poset of *nil*-closed, noetherian, sub-algebras of $H^*(W)$, whose transcendence degrees are $\dim(W)$, ordered by inclusion.

**Corollary 5.10.** *Any nil-closed, integral, noetherian, unstable, algebra whose transcendence degree is equal to* $\dim(W)$ *is isomorphic to $H^*(W)^{\mathcal{G}}$ for some $\mathcal{G}$.*

*Proof.* It is a reformulation of the theorem of Adams–Wilkerson using Theorem 5.7. □

*Example 5.11.* For $G$ a sub-group of $\mathrm{Gl}(W)$, $H^*(W)^{\mathfrak{g}(G)} = H^*(W)^G$, for $H^*(W)^G$ the algebra of invariant element of $H^*(W)$ under the action of $G$.

Let us identify precisely the sub-algebra $H^*(W)^{\mathcal{G}}$ of $H^*(W)$.

**Proposition 5.12.** *Let $\mathcal{G} \in$ Groupoid$(W)$. Then,*

$$H^*(W)^{\mathcal{G}} = \{x \in H^*(W)\ ;\ \alpha^* \iota_{U'}^*(x) = \iota_U^*(x) \ \text{for all } \alpha \in \mathcal{G}(U, U')\}.$$

*Proof.* Let $\phi$ be the inclusion of $H^*(W)^{\mathcal{G}}$ in $H^*(W)$ and let $K(\mathcal{G}) = \{x \in H^*(W)\ ;\ \alpha^*\iota^*_{U'}(x) = \iota^*_U(x) \text{ for all } \alpha \in \mathcal{G}(U,U')\}$. By construction, $\alpha^*\iota^*_{U'}\phi = \iota^*_U\phi$ for all $\alpha \in \mathcal{G}(U,U')$ and for all sub-spaces $U$ and $U'$ of $W$. Then,

$$H^*(W)^{\mathcal{G}} \subset K(\mathcal{G}).$$

Furthermore, the inclusion from $K(\mathcal{G})$ to $H^*(W)$ induces a surjection from $\mathfrak{R}_{\mathbf{W}}$ to $\mathfrak{R}_{K(\mathcal{G})}$ which factorises through an isomorphism from $\mathfrak{R}_{\mathbf{W}/\sim_{\mathcal{G}}}$ to $\mathfrak{R}_{K(\mathcal{G})}$. The existence of this factorization is a direct consequence of the definitions of $K(\mathcal{G})$ and $\sim_{\mathcal{G}}$, and it is injective, since the inclusion of $H^*(W)^{\mathcal{G}}$ induces a right inverse $\mathfrak{R}_{K(\mathcal{G})} \to \mathfrak{R}_{\mathbf{W}/\sim_{\mathcal{G}}}$.

We get the following diagram:

$$\begin{array}{ccccc}
H^*(W)^{\mathcal{G}} & \hookrightarrow & K(\mathcal{G}) & \hookrightarrow & H^*(W) \\
\downarrow{\scriptstyle \eta_{H^*(W)^{\mathcal{G}}}} & & \downarrow{\scriptstyle \eta_{K(\mathcal{G})}} & & \downarrow{\scriptstyle \eta_{H^*(W)}} \\
l_1(H^*(W)^{\mathcal{G}}) & \xrightarrow{\cong} & l_1(K(\mathcal{G})) & \hookrightarrow & l_1(H^*(W)),
\end{array}$$

where $\eta$ denotes the unit of the adjunction between $f$ and $m$. Then, since $H^*(W)^{\mathcal{G}}$ and $H^*(W)$ are *nil*-closed, $\eta_{H^*(W)^{\mathcal{G}}}$ and $\eta_{H^*(W)}$ are isomorphisms. Furthermore, $K(\mathcal{G})$ is a sub unstable algebra of $H^*(W)$, hence it does not contain any nilpotent sub-module, and $\eta_{K(\mathcal{G})}$ is injective. Then, the commutativity of the diagram implies that $\eta_{K(\mathcal{G})}$ is an isomorphism, and therefore that $H^*(W)^{\mathcal{G}} = K(\mathcal{G})$. □

**Corollary 5.13.** *The correspondence of Theorem 5.7 is an isomorphism of posets, for the order on sub-algebras of $H^*(W)$ which is the reverse of the inclusion.*

*Proof.* Indeed, Proposition 5.12 implies that if $\mathcal{G}$ is a sub-groupoid of $\mathcal{G}'$, $H^*(W)^{\mathcal{G}'} \subset H^*(W)^{\mathcal{G}}$. □

**Definition 5.14.** For $g \in \mathrm{Gl}(W)$, and $\mathcal{G} \in \mathit{Groupoid}(W)$, $g \cdot \mathcal{G}$ is the groupoid in $\mathit{Groupoid}(W)$ defined by $\beta \in g \cdot \mathcal{G}(R,R')$, for $\beta$ an isomorphisms between subspaces $R$ and $R'$ of $W$, if there exist $\alpha \in \mathcal{G}(U,U')$ for $U = g^{-1}(R)$ and $U' = g^{-1}(R')$, such that the following diagram commutes:

$$\begin{array}{ccc}
U & \xrightarrow{g|_U^R} & R \\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\
U' & \xrightarrow[g|_{U'}^{R'}]{} & R'.
\end{array}$$

This defines a poset preserving action of $\mathrm{Gl}(W)$ on $\mathit{Groupoid}(W)$.

*Remark 5.15.* This action generalizes the action by conjugation on $Group(W)$. Indeed, for $G$ a subgroup of $\mathrm{Gl}(W)$ and $g \in \mathrm{Gl}(W)$, $g \cdot \mathfrak{g}(G) = \mathfrak{g}(gGg^{-1})$.

**Proposition 5.16.** *For $g \in \mathrm{Gl}(W)$ and $\mathcal{G} \in Groupoid(W)$,*

$$H^*(W)^{g \cdot \mathcal{G}} = (g^{-1})^*(H^*(W)^{\mathcal{G}}).$$

*Proof.* This is a direct consequence of Proposition 5.12. □

*Remark 5.17.* We want to notice that the $(H^*(W)^{\mathcal{G}})_{\mathcal{G} \in Groupoid(W)}$ does not constitute a minimal list for representing elements of isomorphism classes of $nil$-closed, integral and noetherian unstable algebras of transcendence degree $\dim(W)$. For $g \in \mathrm{Gl}(W)$ and $\mathcal{G} \in Groupoid(W)$, $g \cdot \mathcal{G}$ needs not to be equal to $\mathcal{G}$, but, by Proposition 5.16, $H^*(W)^{\mathcal{G}} \cong H^*(W)^{g \cdot \mathcal{G}}$.

Conversely, since the inclusion of $H^*(W)^{\mathcal{G}}$ in $H^*(W)$ induces a surjection from $\mathrm{Hom}_{\mathcal{K}}(H^*(W), H^*(W))$ to $\mathrm{Hom}_{\mathcal{K}}(H^*(W)^{\mathcal{G}}, H^*(W))$, and since $g \mapsto g^*$ induces an isomorphism between $\mathrm{Hom}_{\mathcal{K}}(H^*(W), H^*(W))$ and $\mathrm{Gl}(W)$, we have that if $H^*(W)^{\mathcal{G}} \cong H^*(W)^{\mathcal{H}}$, there exists $g \in \mathrm{Gl}(W)$ such that $H^*(W)^{\mathcal{H}} = (g^{-1})^*(H^*(W)^{\mathcal{G}})$. By Proposition 5.16, $\mathcal{H} = g \cdot \mathcal{G}$.

## 5.2. Centrality and primitive elements of $H^*(W)^{\mathcal{G}}$

Throughout this sub-section, we fix $V$ and $W$ two objects in $\mathcal{V}^f$, as well as an injection $\delta$ from $V$ to $W$.

We consider $K$ a $nil$-closed, noetherian unstable sub algebra of $H^*(W)$ of transcendence degree $\dim(W)$, such that $(V, \delta^*\phi) \in \mathbf{C}(K)$, for $\phi$ the inclusion of $K$ in $H^*(W)$. We start by explaining why the $H^*(V)$-comodule structure on $K$ induced by $\delta^*\phi$ is induced from the $H^*(V)$-comodule structure on $H^*(W)$ given by $(\mathrm{id}_W + \delta)^* \colon H^*(W) \to H^*(W) \otimes H^*(V)$.

Then, for $K = H^*(W)^{\mathcal{G}}$, we explain how to determine the primitive elements of this comodule structure from $\mathcal{G}$.

**Proposition 5.18.** *Let $K$ be a noetherian unstable sub algebra of $H^*(W)$ of finite transcendence degree $\dim(W)$ such that $(V, \delta^*\phi) \in \mathbf{C}(K)$, for $\phi$ the inclusion of $K$ in $H^*(W)$. The $H^*(V)$-comodule structure $\kappa$ on $K$, induced by $\delta^*\phi$ and Corollary 2.16, fits into the following commutative diagram:*

$$\begin{array}{ccc} K & \xrightarrow{\kappa} & K \otimes H^*(V) \\ \phi \downarrow & & \downarrow \phi \otimes \mathrm{id}_{H^*(V)} \\ H^*(W) & \xrightarrow[(\mathrm{id}_W + \delta)^*]{} & H^*(W) \otimes H^*(V). \end{array}$$

*Proof.* We consider the following diagram:

$$\begin{array}{ccc} K & \xrightarrow{\quad\kappa\quad} & K \otimes H^*(V) \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\phi\otimes \mathrm{id}_{H^*(V)}} \\ H^*(W) & & H^*(W) \otimes H^*(V). \end{array}$$

The existence of a morphism $\psi^*$ from $H^*(W)$ to $H^*(W) \otimes H^*(V)$ which turns it into a commutative diagram is a consequence of the surjectivity of $\phi^*$ from $\mathrm{Hom}_{\mathcal{K}}(H^*(W), H^*(W \oplus V))$ to $\mathrm{Hom}_{\mathcal{K}}(K, H^*(W \oplus V))$. We only have to justify why we can take $\psi = \mathrm{id}_W + \delta$. We have that the composition of $(\phi \otimes \mathrm{id}_{H^*(V)}) \circ \kappa$ with $\epsilon_K \otimes \mathrm{id}_{H^*(V)}$ is equal to $\delta^*\phi$ and that with $\mathrm{id}_{H^*(W)} \otimes \epsilon_{H^*(V)}$ is equal to $\phi$. Hence, since $\delta^*\phi$ is central, $(\phi \otimes \mathrm{id}_{H^*(V)}) \circ \kappa$ is the unique element in the inverse image of $\phi$ under $\rho_{\mathrm{Hom}_{\mathcal{K}}(K,H^*(\_)),(V,\delta^*\phi)}$. But $(\mathrm{id}_W + \delta)^*\phi$ is also in this inverse image of $\phi$, hence the diagram commutes. □

We consider $(\mathrm{id}_W + \delta)^* : H^*(W) \to H^*(W) \otimes H^*(V)$ which is the $H^*(V)$-comodule structure on $H^*(W)$ associated with $(V, \delta^*) \in \mathbf{C}(H^*(W))$.

**Proposition 5.19.** *Let $K$ be a noetherian unstable sub algebra of $H^*(W)$ of finite transcendence degree* $\dim(W)$ *such that $(V, \delta^*\phi) \in \mathbf{C}(K)$, for $\phi$ the inclusion of $K$ in $H^*(W)$. Then, we have a pullback diagram of the following form:*

$$\begin{array}{ccc} P(K, \delta^*\phi) & \hookrightarrow & K \\ \big\downarrow & & \big\downarrow{\scriptstyle\phi} \\ H^*(W/\mathrm{Im}(\delta)) & \underset{\pi^*}{\hookrightarrow} & H^*(W). \end{array}$$

*Proof.* Proposition 5.18 says that the following diagram commutes:

$$\begin{array}{ccc} K & \xrightarrow{\quad\kappa\quad} & K \otimes H^*(V) \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\phi\otimes \mathrm{id}_{H^*(V)}} \\ H^*(W) & \xrightarrow[(\mathrm{id}_W + \delta)^*]{} & H^*(W) \otimes H^*(V). \end{array}$$

This means that the $H^*(V)$-comodule structure on $K$ is induced by that on $H^*(W)$. Hence, the primitive elements of $K$ are simply the primitive elements of $H^*(W)$ that are in $K$. But the comodule structure on $H^*(W)$ is the morphism $(\mathrm{id}_W + \delta)^*$ whose algebra of primitive elements is the image of $H^*(W/\mathrm{Im}(\delta))$ under $\pi^*$, for $\pi$ the projection from $W$ to $W/\mathrm{Im}(\delta)$. □

**Corollary 5.20.** *Let $K$ be a noetherian unstable sub algebra of $H^*(W)$ of finite transcendence degree $\dim(W)$ such that $(V, \delta^*\phi) \in \mathbf{C}(K)$, for $\phi$ the inclusion of $K$ in $H^*(W)$. Then, the following is a pushout diagram in $\mathcal{S}\mathrm{et}^{(\mathcal{V}^f)^{\mathrm{op}}}$:*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{V}^f}(\_, W) & \twoheadrightarrow & \mathrm{Hom}_{\mathcal{K}}(K, H^*(\_)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{V}^f}(\_, W/\mathrm{Im}(\delta)) & \twoheadrightarrow & \mathrm{Hom}_{\mathcal{K}}(P(K, \delta^*\phi), H^*(\_)). \end{array}$$

*Proof.* It is a direct consequence of Lemma 2.10 and of Proposition 5.19. □

We can thus identify $\mathrm{Hom}_{\mathcal{K}}(P(K, \delta^*\phi), H^*(\_))$ in this context. In particular, we show that $P$ is always noetherian.

**Lemma 5.21.** *For $S$ a set, and $\sim_1$ and $\sim_2$ two equivalence relations on $S$, we denote by $\sim$ the smallest equivalence relation on $S$ (in the sense that $\{(a, b) \in S \times S\ ;\ a \sim b\} \subset S \times S$ is the smallest) such that, for all $a$ and $b$ in $S$ such that $a \sim_1 b$ or $a \sim_2 b$, $a \sim b$. Then, the following is a pushout in $\mathcal{S}\mathrm{et}$:*

$$\begin{array}{ccc} S & \twoheadrightarrow & S/{\sim_1} \\ \downarrow & & \downarrow \\ S/{\sim_2} & \twoheadrightarrow & S/{\sim}\ . \end{array}$$

*Proof.* Let $\Sigma$ denote the pushout of

$$\begin{array}{ccc} S & \twoheadrightarrow & S/{\sim_1} \\ \downarrow & & \\ S/{\sim_2} & & . \end{array}$$

Then, for $s\colon S \to \Sigma$ the composition of the projection from $S$ to $S/{\sim_1}$ with the surjective application $S/{\sim_1} \to \Sigma$, $s$ is surjective. We define $\sim'$ the equivalence relation on $S$ defined by $a \sim' b$ if and only if $s(a) = s(b)$. $\Sigma$ is isomorphic in $\mathcal{S}\mathrm{et}$ with $S/{\sim'}$ and we will show that $\sim' = \sim$.

By commutativity of the pushout diagram, for $a$ and $b$ in $S$ such that $a \sim_1 b$ or $a \sim_2 b$, $s(a) = s(b)$. Suppose that $\sim'$ is not the smallest such equivalence relation. Then, there exists $x$ and $y$ with $x \sim' y$ and an equivalence relation $\sim''$, satisfying that for $a$ and $b$ such that $a \sim_1 b$ or $a \sim_2 b$, $a \sim'' b$, and such that $x$ is not equivalent to $y$ for $\sim''$. Then, the following diagram is commutative:

$$\begin{array}{ccc} S & \twoheadrightarrow & S/{\sim_1} \\ \downarrow & & \downarrow \\ S/{\sim_2} & \twoheadrightarrow & S/{\sim''}, \end{array}$$

and factorise by a morphism $S/{\sim'} \to S/{\sim''}$. This is a contradiction, so $\sim' = \sim$. □

*Remark 5.22.* For $\sim_1$ and $\sim_2$ as in Lemma 5.21, and for $S$ finite, the smallest equivalence relation $\sim$ on $S$ such that, for all $a$ and $b$ in $S$ such that $a \sim_1 b$ or $a \sim_2 b$, $a \sim b$, is the equivalence relation defined by $a \sim b$ if there is a finite family $(s_i)_{i \in [\![1,n]\!]}$ of objects in $S$ such that:

(1) $s_1 = a$,

(2) $s_n = b$,

(3) for all $1 \leq i \leq n$, if $i$ is odd $s_i \sim_1 s_{i+1}$ and if $i$ is even $s_i \sim_2 s_{i+1}$.

We deduce the following proposition.

**Proposition 5.23.** *Let $K$ be a noetherian unstable sub algebra of $H^*(W)$ of finite transcendence degree $\dim(W)$ such that $(V, \delta^*\phi) \in \mathbf{C}(K)$, for $\phi$ the inclusion of $K$ in $H^*(W)$. Then, for $\zeta$ and $\gamma$ in $\mathrm{Hom}_{\mathcal{V}^{\mathcal{I}}}(U, V)$, $\gamma^*\phi|_{P(K,\delta^*\phi)} = \zeta^*\phi|_{P(K,\delta^*\phi)} \in \mathrm{Hom}_{\mathcal{K}}(P(K, \delta^*\phi), H^*(U))$ if and only if there exists a family $(\epsilon_i)_{i \in [\![1,n]\!]} \in \mathrm{Hom}_{\mathcal{V}^f}(U, W)^n$ with $n \in \mathbb{N}$ greater than 1, such that:*

(1) $\gamma = \epsilon_1$,

(2) $\zeta = \epsilon_n$,

(3) *for all $1 \leq i \leq n-1$, $\epsilon_i^*\phi = \epsilon_{i+1}^*\phi$ if $i$ is odd and $\pi \circ \epsilon_i = \pi \circ \epsilon_{i+1}$ if $i$ is even.*

*Proof.* By Corollary 5.20, the following is a pushout:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{V}^f}(\_, W) & \xrightarrow{k} & \mathrm{Hom}_{\mathcal{K}}(K, H^*(\_)) \\ \downarrow \pi_* & & \downarrow p \\ \mathrm{Hom}_{\mathcal{V}^f}(\_, W/\mathrm{Im}(\delta)) & \twoheadrightarrow & \mathrm{Hom}_{\mathcal{K}}(P(K, \delta^*\phi), H^*(\_)), \end{array}$$

where $k$ maps $\zeta\colon U \to W$ to $\zeta^*\phi\colon K \to H^*(U)$, $p$ maps $\psi\colon K \to H^*(U)$ to $\psi|_{P(K,\delta^*\phi)}$ and $\pi_*$ maps $\zeta\colon U \to W$ to $\pi \circ \zeta\colon U \to W/\mathrm{Im}(\delta)$. Then, by Lemma 5.21, $p \circ k(\zeta) =$

$p \circ k(\gamma)$ if and only if there exists a family $(\epsilon_i)_{i \in [\![1,n]\!]} \in \mathrm{Hom}_{\mathcal{V}^f}(U, W)^n$ with $n \in \mathbb{N}$ greater than 1, such that $\gamma = \epsilon_1$, $\zeta = \epsilon_n$ and for all $1 \leq i \leq n-1$, $k(\epsilon_i) = k(\epsilon_{i+1})$ if $i$ is odd and $\pi_*(\epsilon_i) = \pi_*(\epsilon_{i+1})$ if $i$ is even. □

**Corollary 5.24.** *Let $K$ be a noetherian unstable sub algebra of $H^*(W)$ of finite transcendence degree* $\dim(W)$ *such that $(V, \delta^*\phi) \in \mathbf{C}(K)$, for $\phi$ the inclusion of $K$ in $H^*(W)$. Then, for $\zeta \in \mathrm{Hom}_{\mathcal{V}^f}(U, W)$, $\ker(\zeta^*\phi|_{P(K,\delta^*\phi)}) = \ker(\pi \circ \zeta)$.*

*Proof.* Let $\zeta_0 \in \mathrm{Hom}_{\mathcal{V}^f}(U/\ker(\zeta^*\phi|_{P(K,\delta^*\phi)}), W)$ such that $\zeta^*\phi|_{P(K,\delta^*\phi)} = \pi_U^*\zeta_0^*\phi|_{P(K,\delta^*\phi)}$, with $\pi_U$ the projection from $U$ to $U/\ker(\zeta^*\phi|_{P(K,\delta^*\phi)})$. Let $\epsilon_1 = \zeta_0 \circ \pi_U$, $\epsilon_n = \zeta$ and for all $i$ $\epsilon_i^*\phi = \epsilon_{i+1}^*\phi$ if $i$ is odd and $\pi \circ \epsilon_i = \pi \circ \epsilon_{i+1}$ if $i$ is even. Then, since $\mathrm{Hom}_{\mathcal{K}}(K, H^*(\_))$ is noetherian, $\ker(\pi \circ \epsilon_i) = \ker(\pi \circ \epsilon_{i+1})$ for all $1 \leq i \leq n-1$. Hence, $\ker(\pi \circ \zeta) = \ker(\pi \circ \zeta_0 \circ \pi_U) = \ker(\zeta^*\phi|_{P(K,\delta^*\phi)})$. □

**Corollary 5.25.** *Let $K$ be a noetherian unstable sub algebra of $H^*(W)$ of finite transcendence degree* $\dim(W)$ *such that $(V, \delta^*\phi) \in \mathbf{C}(K)$, for $\phi$ the inclusion of $K$ in $H^*(W)$. Then, $\mathfrak{S}_{P(K,\delta^*\phi)}$ is noetherian.*

*Proof.* The two first conditions are straightforward. Let $\zeta^*\phi|_{P(K,\delta^*\phi)}$ in $\mathrm{Hom}_{\mathcal{K}}(P(K,\delta^*\phi), H^*(U))$ and let $\alpha$ be a morphism from a vector space $Y$ to $U$. Then, by Corollary 5.24

$$\ker(\alpha^*\zeta^*\phi|_{P(K,\delta^*\phi)}) = \ker(\pi \circ \zeta \circ \alpha).$$

This is equal to

$$\alpha^{-1}(\ker(\pi \circ \zeta)) = \alpha^{-1}(\ker(\zeta^*\phi|_{P(K,\delta^*\phi)})).$$ □

**Theorem 5.26.** *Let $K$ be a noetherian unstable sub algebra of $H^*(W)$ of finite transcendence degree* $\dim(W)$ *such that $(V, \delta^*\phi) \in \mathbf{C}(K)$, for $\phi$ the inclusion of $K$ in $H^*(W)$. Then, $P(K, \delta^*\phi)$ is nil-closed and noetherian.*

*Proof.* Since, $P(K, \delta^*\phi)$ is the kernel of $\kappa - \mathrm{id}_K \otimes 1$ from $K$ to $K \otimes H^*(V)$ which are $nil$-closed, for $\kappa$ the comodule structure of $K$ associated with $\delta^*\phi$, and since $f$ is exact and $m$ is left-exact, the following is an exact sequence:

$$0 \longrightarrow l_1(P(K, \delta^*\phi)) \longrightarrow l_1(K) \xrightarrow{l_1(\kappa - \mathrm{id}_K \otimes 1)} l_1(K \otimes H^*(V)).$$

Therefore, since $K$ is $nil$-closed, $P(K, \delta^*\phi)$ is also $nil$-closed. Then, the noetherianity of $P(K, \delta^*\phi)$ is a consequence of the noetherianity of $\mathfrak{S}_{P(K,\delta^*\phi)}$ and of Proposition 3.12. □

*Remark 5.27.* We have identified $P(K, \delta^*\phi)$ with a sub-algebra of $H^*(W/\mathrm{Im}(\delta))$. Furthermore, we proved that $P(K, \delta^*\phi)$ is $nil$-closed and noetherian, and (because we took $\delta$ to be an injection) Corollary 5.24 implies that the inclusion from $P(K, \delta^*\phi)$ into $H^*(W/\mathrm{Im}(\delta))$ is regular. Therefore, by Theorem 5.7, $P(K, \delta^*\phi)$ has the form $H^*(W/\mathrm{Im}(\delta))^{\mathcal{G}'}$, for some $\mathcal{G}' \in Groupoid(W/\mathrm{Im}(\delta))$.

This leads to the following question: for $W$ and $V$ in $\mathcal{V}^f$, for $\delta$ an inclusion from $V$ to $W$ and for $\mathcal{G}'$ a groupoid with the restriction property and whose objects are the sub-spaces of $(W/\mathrm{Im}(\delta))$, which are the groupoids $\mathcal{G} \in \mathit{Groupoid}(W)$, such that

(1) $H^*(W)^{\mathcal{G}}$ is a sub $H^*(V)$-comodule of $H^*(W)$ for the comodule structure induced by $\delta$,

(2) the intersection of $H^*(W)^{\mathcal{G}}$ with $\pi^*(H^*(W/\mathrm{Im}(\delta)))$ is the image under $\pi^*\colon H^*(W/\mathrm{Im}(\delta)) \to H^*(W)$ of $H^*(W/\mathrm{Im}(\delta))^{\mathcal{G}'}$.

*Remark 5.28.* $H^*(W)^{\mathcal{G}}$ is a sub $H^*(V)$-comodule of $H^*(W)$ for the comodule struture induced by $\delta$ if and only if $\mathcal{G}$ satisfies the two conditions of Theorem 4.11.

We recall that, from the beginning of this sub-section, $V$ and $W$ are fixed objects of $\mathcal{V}^f$ and $\delta$ a fixed injective morphism from $V$ to $W$.

**Theorem 5.29.** *Let $\mathcal{G}$ be a groupoid with the restriction property and whose objects are the sub-vector spaces of $W$, such that $H^*(W)^{\mathcal{G}}$ is a sub $H^*(V)$-comodule of $(H^*(W), (\mathrm{id}_W + \delta)^*)$. For $\mathcal{G}'$ the only object of $\mathit{Groupoid}(W/\mathrm{Im}(\delta))$ which satisfies that $\pi^*(H^*(W/\mathrm{Im}(\delta))^{\mathcal{G}'})$ is the algebra of primitive elements of $H^*(W)^{\mathcal{G}}$, the two following conditions are equivalent:*

(1) *$\alpha \in \mathcal{G}'(U, U')$, where $U$ and $U'$ are sub-vector spaces of $W/\mathrm{Im}(\delta)$ and $\alpha$ is an isomorphism from $U$ to $U'$,*

(2) *there exists $N$ and $N'$ sub spaces of $W$ such that $\pi$ induce isomorphisms from $N$ and $N'$ to $U$ and $U'$, as well as an element $\beta \in \mathcal{G}(N, N')$ such that $\alpha = \pi|_{N'}^{U'} \circ \beta \circ (\pi|_N^U)^{-1}$.*

*Proof.* We consider the pushout diagram of Corollary 5.20:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{V}^f}(\_, W) & \xrightarrow{\ k\ } & \mathrm{Hom}_{\mathcal{K}}(H^*(W)^{\mathcal{G}}, H^*(\_)) \\ {\scriptstyle \pi^*}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ \mathrm{Hom}_{\mathcal{V}^f}(\_, W/\mathrm{Im}(\delta)) & \xrightarrow[\ p\ ]{} & \mathrm{Hom}_{\mathcal{K}}(H^*(W/\mathrm{Im}(\delta))^{\mathcal{G}'}, H^*(\_)), \end{array}$$

where $\pi_*$ maps $\gamma\colon U \to W$ to $\pi \circ \gamma$, $k$ maps $\gamma$ to $\gamma^*\phi_{\mathcal{G}}$ for $\phi_{\mathcal{G}}$ the inclusion from $H^*(W)^{\mathcal{G}}$, $q$ maps $\psi\colon H^*(W)^{\mathcal{G}} \to H^*(U)$ to $\psi|_{\pi^*(H^*(W/\mathrm{Im}(\delta)))}$ the restriction of $\psi$ to $\pi^*(H^*(W/\mathrm{Im}(\delta)))$ and, finally, $p$ maps $\zeta$ from $U$ to $W/\mathrm{Im}(\delta)$ to $\zeta^*\phi_{\mathcal{G}'}$, for $\phi_{\mathcal{G}'}$ the inclusion of $H^*(W/\mathrm{Im}(\delta))^{\mathcal{G}'}$ into $H^*(W/\mathrm{Im}(\delta))$.

We fix a section $s$ from $W/\mathrm{Im}(\delta)$ to $W$. Since $\pi \circ s = \mathrm{id}_{W/\mathrm{Im}(\delta)}$, $\pi_*(s) = \mathrm{id}_{W/\mathrm{Im}(\delta)}$. Then, by commutativity of the pushout diagram, we have $\phi_{\mathcal{G}'} = q(s^*\phi_{\mathcal{G}}) = s^*\phi_{\mathcal{G}}|_{\pi^*(H^*(W/\mathrm{Im}(\delta)))}$.

By construction, there are natural isomorphisms $\mathfrak{R}_{H^*(W)^{\mathcal{G}}} \cong \mathfrak{R}_{\mathbf{W}/\sim_{\mathcal{G}}}$ and $\mathfrak{R}_{H^*(W/\mathrm{Im}(\delta))^{\mathcal{G}'}} \cong \mathfrak{R}_{\mathbf{W}/\mathbf{Im}(\delta)/\sim_{\mathcal{G}'}}$. These are the isomorphisms that map $(W, \phi_{\mathcal{G}})$ to $[W, \mathrm{id}_W]_{\mathcal{G}}$ and $(W/\mathrm{Im}(\delta), \phi_{\mathcal{G}'})$ to $[W/\mathrm{Im}(\delta), \mathrm{id}_{W/\mathrm{Im}(\delta)}]_{\mathcal{G}'}$ respectively.

Let us first prove (2) $\Rightarrow$ (1). We consider, $\beta \in \mathcal{G}(N, N')$ such that $\pi$ induces isomorphisms $\pi|_N^U$ and $\pi|_{N'}^{U'}$ between $N$ and $U$ and between $N'$ and $U'$. Let $\alpha$ be an isomorphism from $U$ to $U'$ such that $\alpha = \pi|_{N'}^{U'} \circ \beta \circ (\pi|_N^U)^{-1}$. Then, $(\pi|_{N'}^{U'})^{-1} \circ \alpha = \beta \circ (\pi|_N^U)^{-1}$. Therefore,

$$\begin{aligned}\alpha^*((\pi|_{N'}^{U'})^{-1})^*\iota_{N'}^*\phi_{\mathcal{G}} &= ((\pi|_N^U)^{-1})^*\beta^*\iota_{N'}^*\phi_{\mathcal{G}} \\ &= ((\pi|_N^U)^{-1})^*\iota_N^*\phi_{\mathcal{G}}.\end{aligned}$$

We can choose the section $s$ in such a way that $s \circ \iota_{U'} = \iota_{N'} \circ ((\pi|_{N'}^{U'})^{-1})$, then

$$\alpha^*\iota_{U'}^*s^*\phi_{\mathcal{G}} = ((\pi|_N^U)^{-1})^*\iota_N^*\phi_{\mathcal{G}}.$$

This implies that $\alpha^*\iota_{U'}^*s^*q(\phi_{\mathcal{G}}) = ((\pi|_N^U)^{-1})^*\iota_N^*q(\phi_{\mathcal{G}})$. Furthermore, $\pi \circ s \circ \iota_U = \pi \circ \iota_N \circ (\pi|_N^U)^{-1}$. Hence, we also have, by Proposition 5.23, that $\iota_U^*s^*q(\phi_{\mathcal{G}}) = ((\pi|_N^U)^{-1})^*\iota_N^*q(\phi_{\mathcal{G}})$. Hence,

$$\alpha^*\iota_{U'}^*s^*q(\phi_{\mathcal{G}}) = \iota_U^*s^*q(\phi_{\mathcal{G}}).$$

Since $s^*q(\phi_{\mathcal{G}}) = \phi_{\mathcal{G}'}$, this implies that $\alpha \in \mathcal{G}'(U, U')$, as required.

Now, let us prove the far more challenging (1) $\Rightarrow$ (2).

We consider $\alpha \in \mathcal{G}'(U, U')$ where $U$ and $U'$ are two sub-spaces of $W/\mathrm{Im}(\delta)$. Then,

$$\alpha^*\iota_{U'}^*\phi_{\mathcal{G}'} = \iota_U^*\phi_{\mathcal{G}'},$$

or, equivalently,

$$[U, \iota_{U'} \circ \alpha]_{\mathcal{G}'} = [U, \iota_U]_{\mathcal{G}'}.$$

By Proposition 5.23, since $\phi_{\mathcal{G}'} = s^*q(\phi_{\mathcal{G}})$, we have that $[H, \zeta]_{\mathcal{G}'} = [H, \gamma]_{\mathcal{G}'}$, for some $H \in \mathcal{V}^f$ and $\zeta$ and $\gamma$ injectives from $H$ to $W/\mathrm{Im}(\delta)$, if and only if there exists a family $(\epsilon_i)_{i \in [\![1,n]\!]} \in \mathrm{Hom}_{\mathcal{V}^f}(U, W)^n$ with $n \in \mathbb{N}$ greater than 1, such that $s \circ \gamma = \epsilon_1$, $s \circ \zeta = \epsilon_n$ and, for all $1 \le i \le n-1$, $\epsilon_i^*\phi_{\mathcal{G}} = \epsilon_{i+1}^*\phi_{\mathcal{G}}$ if $i$ is odd and $\pi \circ \epsilon_i = \pi \circ \epsilon_{i+1}$ if $i$ is even.

So let $(\epsilon_i)_{i \in [\![1,n]\!]} \in \mathrm{Hom}_{\mathcal{V}^f}(U, W)^n$ be such that $\epsilon_1 = s \circ \iota_U$, $\epsilon_n = s \circ \iota_{U'} \circ \alpha$ and for all $1 \le i \le n-1$, $\epsilon_i^*\phi_{\mathcal{G}} = \epsilon_{i+1}^*\phi_{\mathcal{G}}$ if $i$ is odd and $\pi \circ \epsilon_i = \pi \circ \epsilon_{i+1}$ if $i$ is even.

By induction, for all $i \in [\![1, n]\!]$, $\epsilon_i^*\phi_{\mathcal{G}}$ and $\pi \circ \epsilon_i$ are regular elements respectively of $\mathfrak{S}_{H^*(W)^{\mathcal{G}}}$ and $\mathfrak{S}_{\mathrm{Hom}_{\mathcal{V}^f}(U, W/\mathrm{Im}(\delta))}$. Hence, $\epsilon_i$ and $\pi \circ \epsilon_i$ are injections. For all $i$, let $N_i$ denote the image of $\epsilon_i$ in $W$, we denote also by $\widetilde{\epsilon_i}$ the corestriction of $\epsilon_i$ to $N_i$. Then, for $i$ odd, $\epsilon_i^*\phi_{\mathcal{G}} = \epsilon_{i+1}^*\phi_{\mathcal{G}}$ implies that there exists $\beta_i$ in $\mathcal{G}(N_i, N_{i+1})$ such that $\widetilde{\epsilon}_{i+1} = \beta_i \circ \widetilde{\epsilon}_i$.

We take some moment to explain a subtlety in the proof. We would like, for $i$ even, to have $\epsilon_i = \epsilon_{i+1}$. Then, the composition of the $\beta_i$ with $i$ odd would give an isomorphism $\beta$ between $N_1 = s(U)$ and $N_n = s(U')$ such that $\beta \in \mathcal{G}(N_1, N_n)$, since $\mathcal{G}$ is a groupoid and we would have $(s \circ \iota_{U'})|^{N_n} \circ \alpha = \beta \circ (s \circ \iota_U)|^{N_1}$. Since $(s \circ \iota_U)|^{N_1}$ and $(s \circ \iota_{U'})|^{N_n}$ are inverse isomorphisms of $\pi|^U_{N_1}$ and $\pi|^{U'}_{N_n}$, we would have $\alpha = \pi|^{U'}_{N_n} \circ \beta \circ (\pi|^U_{N_1})^{-1}$. If this were the case, we would have found a $\beta$ for any $N$ and $N'$ such that $\pi$ induces isomorphisms between $U$ and $N$ and between $U'$ and $N'$, and we would have done so without using the assumption that $\delta^*\phi_{\mathcal{G}}$ is central. Unfortunately, this naive approach fails, and $N$ and $N'$ must be chosen carefully. The hypothesis on the $\epsilon_i$ for $i$ even indicates how to modify our original $N_1$ and $N_n$ to make it work, using the centrality of $\delta^*\phi_{\mathcal{G}}$.

First notice that, since $\pi \circ \epsilon_i$ is injective for all $i$, we always have $N_i \cap \mathrm{Im}(\delta) = \{0\}$. Then, the assumption that, for $i$ even, $\pi \circ \epsilon_i = \pi \circ \epsilon_{i+1}$ implies that there exists $\rho_i$ from $U$ to $W$ whose image is inside $\mathrm{Im}(\delta)$ and such that $\epsilon_{i+1} = \epsilon_i + \rho_i$. Now, since $H^*(W)^{\mathcal{G}}$ is a sub $H^*(V)$-comodule of $(H^*(W), (\mathrm{id}_W + \delta)^*)$ we know that $[V, \delta]_{\mathcal{G}}$ is a central element of $\mathfrak{R}_{\mathbf{W}/\sim_{\mathcal{G}}}$. Then, by Theorem 4.11, for $i$ odd, we know that the isomorphisms $\bar{\beta}_i$ from $N_i \oplus \mathrm{Im}(\delta)$ to $N_{i+1} \oplus \mathrm{Im}(\delta)$ defined by $\bar{\beta}_i(n) = \beta_i(n)$ for $n \in N_i$ and $\bar{\beta}_i(v) = v$ for $v \in \mathrm{Im}(\delta)$ satisfy $\bar{\beta}_i \in \mathcal{G}(N_i \oplus \mathrm{Im}(\delta), N_{i+1} \oplus \mathrm{Im}(\delta))$. Moreover, for $i$ even, $\pi \circ \epsilon_i = \pi \circ \epsilon_{i+1}$ implies that $N_i \oplus \mathrm{Im}(\delta) = N_{i+1} \oplus \mathrm{Im}(\delta)$. Then, at each even step $i$, we can "correct" $\epsilon_{i-1}$ to get $\beta_{i-1} \circ \widetilde{\epsilon}_{i-1} = \widetilde{\epsilon}_{i+1}$ instead of $\widetilde{\epsilon}_i$.

For each $i \in [\![1, n]\!]$, we define $\epsilon'_i$ (the "corrected" $\epsilon_i$) by

$$\epsilon'_i := \iota^W_{N_i \oplus \mathrm{Im}(\delta)} \circ \left( \widetilde{\epsilon}_i \oplus \sum_{\{j \text{ even}\,;\, i \le j < n\}} \beta^{-1}_{i \to j} \circ \rho_j|^{N_j \oplus \mathrm{Im}(\delta)} \right),$$

where $\beta_{i \to j}$ is the composition of all the $\bar{\beta}_k$ with $k$ odd and $i \le k < j$. The family $(\epsilon'_i)_{i \in [\![1,n]\!]}$ satisfies the following:

(1) $\pi \circ \epsilon'_1 = \iota_U$, $\pi \circ \epsilon'_n = \iota_{U'} \circ \alpha$,

(2) for all $i$, if we denote by $N'_i$ the image of $\epsilon'_i$, then $N'_i \oplus \mathrm{Im}(\delta) = N_i \oplus \mathrm{Im}(\delta)$,

(3) for $i$ odd, if we denote by $\beta'_i$ the restriction of $\bar{\beta}_i$ to $N'_i$ corestricted to $N'_{i+1}$, $\widetilde{\epsilon}'_{i+1} = \beta'_i \circ \widetilde{\epsilon}'_i$, with $\beta'_i \in \mathcal{G}(N'_i, N'_{i+1})$), since $\bar{\beta}_i \in \mathcal{G}(N_i \oplus \mathrm{Im}(\delta), N_{i+1} \oplus \mathrm{Im}(\delta))$ and $\mathcal{G}$ has the restriction property,

(4) for $i$ even, $\epsilon'_i = \epsilon'_{i+1}$.

Then, let $N = N'_1$, $N' = N'_n$ and $\beta = (\beta'_k \circ \cdots \circ \beta'_3 \circ \beta'_1)$, where $k = n - 2$ if $n$ is odd, $k = n - 1$ otherwise. Then, $\beta \in \mathcal{G}(N, N')$ and $\beta \circ \widetilde{\epsilon}'_1 = \widetilde{\epsilon}'_n$. Finally, $\pi \circ \epsilon'_1 = \iota_U$ implies that $\widetilde{\epsilon}'_1 = (\pi|^U_N)^{-1}$ and $\pi \circ \epsilon'_n = \iota_{U'} \circ \alpha$ implies that $\widetilde{\epsilon}'_n = (\pi|^{U'}_{N'})^{-1} \circ \alpha$. Hence, $\alpha = \pi|^{U'}_{N'} \circ \beta \circ (\pi|^U_N)^{-1}$. □

## 5.3. **Applications**

We end this section by presenting some applications of Theorem 5.29. We consider some algebras $H^*(W)^{\mathcal{G}}$ that satisfy some conditions on their centre and associated sub-algebras of primitive elements.

We consider first the case where the centre of $H^*(W)^{\mathcal{G}}$ has dimension $\dim(W)$. Since the centre is a regular element of $H^*(W)^{\mathcal{G}}$, we can take it to be $(W, \phi_{\mathcal{G}})$ for $\phi_{\mathcal{G}}$ the inclusion in $H^*(W)$. By Theorem 4.11, $W$ is invariant under any morphism in $\mathcal{G}$. Therefore, $\mathcal{G}$ is the groupoid in *Groupoid*$(W)$ that contains only trivial morphisms and $H^*(W)^{\mathcal{G}} = H^*(W)$. We get the following Proposition (that was already known).

**Proposition 5.30.** *Let $K$ be a noetherian, nil-closed, integral, unstable algebra of transcendence degree $d$. We assume that the centre of $K$ is of dimension $d$. Then, $K \cong H^*(W)$ with* $\dim(W) = d$.

Let us now consider the case where the centre is of dimension $\dim(W) - 1$. Up to isomorphism, the centre is induced by the inclusion in $W$ of a sub-vector space $C$. Then, the elements of $C$ are invariant under any morphism in $\mathcal{G}$. From condition (2) in Theorem 4.11 and since $\mathcal{G}$ satisfies the restriction property, any morphism in $\mathcal{G}$ is the restriction of a morphism in $\mathcal{G}(C \oplus S, C \oplus S')$ with $S$ and $S'$ complementary sub-spaces of $C$ in $W$, hence, they are the restriction of a morphism in $\mathcal{G}(W, W)$. Therefore, $\mathcal{G} = \mathfrak{g}(\mathcal{G}(W, W))$. We get the following Theorem.

**Theorem 5.31.** *Let $K$ be a noetherian, nil-closed, integral, unstable algebra of transcendence degree $d$. Then, the centre of $K$ has dimension $d - 1$ if and only if, for $W$ such that* $\dim(W) = d$*, there exists $G$ a sub-group of* $\mathrm{Gl}(W)$ *such that $K$ is isomorphic to $H^*(W)^G$ and such that the sub-vector space of $W$ of invariant elements under $G$ has dimension* $d - 1$.

A *nil*-closed, noetherian, integral unstable algebra that is not an algebra of invariant elements of the form $H^*(W)^G$, must have a centre of dimension at most $d - 2$, with $d$ the transcendence degree of $K$. We give such examples for $p = 2$, using Theorems 5.7 and 5.29.

**Proposition 5.32.** *There are, up to isomorphism,* 5 *nil-closed, noetherian, unstable algebras $K$ of transcendence degree* 3*, whose centre has dimension* 1 *and such that the algebra of primitive elements of $K$ is isomorphic to $H^*(V_2)$ for $V_k$ the vector space of dimension $k$. They can be realised as sub-algebras of $H^*(V_3) \cong \mathbb{F}_2[x, y, z]$ as:*

(1) $\mathbb{F}_2[y, z, x(x+y)(x+z)(x+y+z)]$,

(2) $\mathbb{F}_2[y, z, x(x+y)(x+z)(x+y+z)] + \mathbb{F}_2[y, z, x(x+y)]y$,

(3) $\mathbb{F}_2[x, y, z](y + z) \oplus \mathbb{F}_2[z, x(x + z)]$,

(4) $\mathbb{F}_2[z, x(x + z)] \oplus \mathbb{F}_2[x, y, z](y + z)y \oplus \mathbb{F}_2[y, x(x + y)]y$,

(5) $\mathbb{F}_2[z, x(x+z)]z \oplus \mathbb{F}_2[y, x(x+y)]y \oplus \mathbb{F}_2[y, x(x+y)](y+z)y \oplus \mathbb{F}_2[x, y, z](y+z)yz$.

*Among them, only the first one is an algebra of invariant elements.*

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