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An extension of the standard multifractional Brownian motion

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Abstract

In this paper, firstly, we generalize the definition of the bifractional Brownian motion $B^{H,K} := \big(B^{H,K}\ ;\ t \geq 0\big)$, with parameters $H \in (0,1)$ and $K \in (0,1]$, to the case where H is no longer a constant, but a function $H(\cdot)$ of the time index t of the process. We denote this new process by $B^{H(\cdot),K}$. Secondly, we study its time regularities, the local asymptotic self-similarity and the long-range dependence properties.

1. Introduction

In recent years, the famous fractional Brownian motion $B^H := \big(B_t^H\ ;\ t \geq 0\big)$, (fBm for short), with Hurst parameter $H \in (0,1)$, has considerable interest due to its applications in various scientific areas including: telecommunications, finance, turbulence and image processing, (see for examples: Addison and Ndumu [1], Cheridito [9], Comegna et al. [13], Samorodnitsky and Taqqu [29] and Taqqu [30]). The fBm was firstly introduced by Kolmogorov [20], and was later made popular by Mandelbrot and Van Ness [26]. It is the only centered and self-similar Gaussian process with stationary increments and covariance function:

$$R^H(t,s) := \mathbb{E}\big(B_t^H B_s^H\big) = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big), \quad \forall\ s,t \geq 0.$$

The case $H = \frac{1}{2}$ correspond to the standard Brownian motion.

For small increments, in models such as turbulence, fBm seems a good model but it is inadequate for large increments. For this reason, Houdré and Villa [18] have explored the existence of a Gaussian process which preserve some of the properties of the fBm such as self-similarity and stationarity of small increments, and can enlarge modeling tool kit. This process, denoted by $B^{H,K} := \big(B_t^{H,K}\ ;\ t \geq 0\big)$, is called the bifractional Brownian

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motion, (bfBm for short), with parameters $H \in (0,1)$ and $K \in (0,1]$ and covariance function:

$$R^{H,K}(t,s) := \mathbb{E}\big(B_t^{H,K} B_s^{H,K}\big) = \frac{1}{2^K}\left[\Big(t^{2H} + s^{2H}\Big)^K - |t-s|^{2HK}\right], \quad \forall\, t,s \geq 0.$$

For large details about bfBm, we refer to [14, 18, 22, 28].

The increments of $B^{H,K}$ are only independents in the case of the standard Brownian motion: ($H = \frac{1}{2}, K = 1$), and they are not stationary for any $K \in]0,1[$, except the case of the fBm: ($K = 1$), however, $B^{H,K}$ is quasi-helix in the sense of Kahane [19]:

$$2^{-K}|t-s|^{2HK} \leq \mathbb{E}\big(B_t^{H,K} - B_s^{H,K}\big)^2 \leq 2^{1-K}|t-s|^{2HK}, \quad \forall\, s,t \geq 0. \tag{1.1}$$

Moreover, according to [18], if we put: $\sigma_\varepsilon^2(t) := \mathbb{E}\big(B_{t+\varepsilon}^{H,K} - B_t^{H,K}\big)^2$, then,

$$\lim_{\varepsilon \to 0} \frac{\sigma_\varepsilon^2(t)}{\varepsilon^{2HK}} = 2^{1-K}, \quad t > 0.$$

Therefore, the small increments of $B^{H,K}$ are approximately stationary. For the large increments, Maejima and Tudor [25] have proved that, when $h \to +\infty$, the sequence of increments process:

$$\Big(B_{t+h}^{H,K} - B_h^{H,K}\ ;\ t \geq 0\Big)$$

converges modulo a constant, in the sense of the finite dimensional distributions, to the fBm $\big(B_t^{HK}\ ;\ t \geq 0\big)$ with Hurst parameter HK. This result can be interpreted like the bfBm has stationary increments for large increments. The key ingredient used in [25] is a decomposition in law of the bfBm presented by Lei and Nualart [22] as follows:

Let $W := \big(W_\theta\ ;\ \theta \geq 0\big)$ be a standard Brownian motion independent of $B^{H,K}$. For any $K \in (0,1)$, let $X^K := \big(X_t^K\ ;\ t \geq 0\big)$ be the centered Gaussian process defined by:

$$X_t^K := \int_0^{+\infty} \big(1 - e^{-\theta t}\big)\theta^{-\frac{(1+K)}{2}}\, \mathrm{d}W_\theta,$$

with the covariance function:

$$\mathbb{E}\big(X_t^K X_s^K\big) = \frac{\Gamma(1-K)}{K}\Big[t^K + s^K - (t+s)^K\Big], \quad \forall\, t,s \geq 0.$$

Γ is the well known Gamma function.

The authors in [22] showed by setting: $X_t^{H,K} := X_{t^{2H}}^K$, that:

$$\Big(C_1(K) X_t^{H,K} + B_t^{H,K}\ ;\ t \geq 0\Big) \stackrel{d}{=} \big(C_2(K) B_t^{HK}\ ;\ t \geq 0\big), \tag{1.2}$$

where $C_1(K) = \sqrt{\frac{2^{-K}K}{\Gamma(1-K)}}$, $C_2(K) = 2^{\frac{(1-K)}{2}}$, $\stackrel{d}{=}$ means equality of all finite dimensional distributions and $X_t^{H,K}$ and $B_t^{H,K}$ are independent. The second application of (1.2) given in [25] is that the long-range dependence, (LRD for short), of the process $B^{H,K}$ depends on the value of the product HK:

- Long-memory: for every $a \in \mathbb{N}^*$: $\sum_{n\geq 0} \mathrm{cor}_{B^{H,K}}(a, a+n) = +\infty$, if $2HK > 1$,
- Short-memory: for every $a \in \mathbb{N}^*$: $\sum_{n\geq 0} \mathrm{cor}_{B^{H,K}}(a, a+n) < +\infty$, if $2HK \leq 1$,

where

$$\mathrm{cor}_{B^{H,K}}(a, a+n) := \mathbb{E}\Big[\big(B^{H,K}_{a+1} - B^{H,K}_{a}\big)\big(B^{H,K}_{a+n+1} - B^{H,K}_{a+n}\big)\Big].$$

This result was appeared also in Remark 7 in [28]. Cioczek-Georges and Mandelbrot [10] used a sum of micropulses to obtain limit processes with interesting properties, like Brownian motion, fractional Brownian motion and bifractional Brownian motion. Recently, Marouby [27] used this model in the case of bifractional Brownian sheet. The bfBm was introduced in [18] with the aim of enriching the catalogue of the processes likely to be used within the framework of modeling. Bardina and Es-sebaiy [5] enlarged the zone of existence of bfBm. This extension was used by Lifshits et al. [24] for proving new probabilistic inequalities.

Now, we are ready to introduce our new process: since the model of the fBm B^H may be restrictive for different phenomena due to the fact that all its interesting properties are governed by the Hurst parameter H, this gave the motivation to Benassi et al. [6] and Lévy-Véhel and Peltier [23] to introduce, independently, a new model to generalize the fBm: It's the multifractional Brownian motion, (mBm for short). Contrarily to the fBm, the almost sure Hölder exponent of the mBm is allowed to vary along the trajectory, a useful feature when one needs to model processes whose regularity evolves in time, such as Internet traffic or images. The definition of the mBm in [23] is based on the moving average representation of the fBm, where the constant Hurst parameter H is substituted by a functional $H(\,\cdot\,)$ as follows:

$$\begin{aligned}\widetilde{B}^{H(t)}_t = \frac{1}{\Gamma\big(H(t)+\frac{1}{2}\big)}\Bigg(\int_{-\infty}^{0}\Big[(t-u)^{H(t)-\frac{1}{2}} - (-u)^{H(t)-\frac{1}{2}}\Big]W(\mathrm{d}u) \\ + \int_0^t (t-u)^{H(t)-\frac{1}{2}} W(\mathrm{d}u)\Bigg), \quad t \geq 0,\end{aligned}$$

where $H(\,\cdot\,) : [0, +\infty) \mapsto [\mu, \nu] \subset (0, 1)$ is a Hölder continuous function of exponent $\beta > 0$, and W is a standard Brownian motion on $\mathbb{R}$.

The authors in [6] defined the mBm by means of the harmonisable representation of the fBm as follows:

$$\widehat{B}^{H(t)}_t = \int_{\mathbb{R}} \frac{e^{it\xi}-1}{|\xi|^{H(t)+\frac{1}{2}}}\widehat{W}(\mathrm{d}\xi), \quad t \geq 0,$$

where $\widehat{W}(\xi)$ is the Fourier transform of the series representation of white noise with respect to an orthonormal basis of $L^2(\mathbb{R})$. From these definitions, it's easy to see that the mBm is a zero mean Gaussian processes whose increments are in general neither independents nor stationary. It is proved by Cohen [12] that the two representations of mBm are equivalent, up to a multiplicative deterministic function. This function is explicitly given by Boufoussi et al. [8]. Moreover, in Ayache et al. [4], the covariance function of the standard mBm $B^{H(\cdot)}$: (i.e. the variance at time 1 is 1), has been deduced from its harmonisable representation as follows:

$$\mathbb{E}\big(B_t^{H(t)}B_s^{H(s)}\big) = D\big(H(t),H(s)\big)\Big[t^{H(t)+H(s)} + s^{H(t)+H(s)} - |t-s|^{H(t)+H(s)}\Big],$$

where,

$$D(x,y) := \frac{\sqrt{\Gamma(2x+1)\Gamma(2y+1)\sin(\pi x)\sin(\pi y)}}{2\Gamma(x+y+1)\sin\big(\frac{\pi(x+y)}{2}\big)}.$$

Clearly, if $H(\cdot)\equiv H$ a constant in $(0,1)$, $D(H,H)=\frac{1}{2}$, and we find the covariance function of the fBm B^H: the zero mean Gaussian process with stationary increments.

In the same spirit as [6] and [23], since all the properties of the bfBm $B^{H,K}$ is governed by the unique number HK, we introduce in this note a generalization of $B^{H,K}$, by substituting to the parameter H in the covariance function $R^{H,K}$, a Hölder function $H(\cdot) : [0,+\infty) \mapsto [\mu,\nu] \subset (0,1)$ with exponent $\beta > 0$. More precisely:

Definition 1.1. We define a new centered Gaussian process, starting from zero and denoted by $B^{H(\cdot),K} := \big(B_t^{H(t),K}\ ;\ t\geq 0\big)$, by the covariance function:

$$R^{H(\cdot),K}(t,s) := \big(D(H(t),H(s))\big)^K\Big[\big(t^{H(t)+H(s)} + s^{H(t)+H(s)}\big)^K - |t-s|^{(H(t)+H(s))K}\Big].$$

Remark 1.2. Clearly, when $K=1$, $B^{H(\cdot),K}$ is a standard mBm. When $H(\cdot)\equiv H$ a constant in $(0,1)$, $B^{H(\cdot),K}$ is a bfBm with parameters $H\in(0,1)$ and $K\in(0,1]$. However, if $K\neq 1$, $B^{H(\cdot),K}$ does not have stationary increments and the fBm is the only Gaussian self-similar process with stationary increments, (see for example [29]).

2. The existence of $B^{H(\cdot),K}$

In this section we prove the existence of our process by using the same argument used in [18] for the bfBm.

Proposition 2.1. *For any $K\in(0,1]$ and $H(\cdot) : [0,+\infty) \mapsto [\mu,\nu] \subset (0,1)$ a Hölder continuous function, the covariance function $R^{H(\cdot),K}$ appeared in Definition 1.1 is positive-definite.*

Proof. We assume that $K \in (0,1)$ since the special case $K = 1$ is evident. We use the following identity:

$$\vartheta^K = \frac{K}{\Gamma(1-K)} \int_0^{+\infty} (1 - e^{-\vartheta x}) x^{-1-K} \mathrm{d}x, \quad \forall\, \vartheta \geq 0,$$

where Γ is the gamma function. For any $c_1, \ldots, c_n \in \mathbb{R}$, we have:

$$\begin{aligned}
&\sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j R^{H(\cdot),K}(t_i, t_j) \\
&\quad = \frac{K}{\Gamma(1-K)} \int_0^{+\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \Big[- e^{-xD\left(H(t_i),H(t_j)\right)\left(t_i^{H(t_i)+H(t_j)} + t_j^{H(t_i)+H(t_j)}\right)} \\
&\qquad\quad + e^{-xD\left(H(t_i),H(t_j)\right)\left(|t_i - t_j|^{H(t_i)+H(t_j)}\right)} \Big] x^{-1-K} \mathrm{d}x \\
&\quad = \frac{K}{\Gamma(1-K)} \int_0^{+\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j e^{-xD\left(H(t_i),H(t_j)\right)\left(t_i^{H(t_i)+H(t_j)} + t_j^{H(t_i)+H(t_j)}\right)} \\
&\qquad\quad \times \Big[e^{xD(H(t_i),H(t_j))\left(t_i^{H(t_i)+H(t_j)} + t_j^{H(t_i)+H(t_j)} - |t_i - t_j|^{H(t_i)+H(t_j)}\right)} - 1 \Big] x^{-1-K} \mathrm{d}x.
\end{aligned}$$

We know by [4] that $D\big(H(t), H(s)\big)\left(t^{H(t)+H(s)} + s^{H(t)+H(s)} - |t-s|^{H(t)+H(s)}\right)$ is positive-definite, then so is:

$$e^{xD\left(H(t),H(s)\right)\left(t^{H(t)+H(s)} + s^{H(t)+H(s)} - |t-s|^{H(t)+H(s)}\right)} - 1, \quad \forall\, x \geq 0,$$

which gives the proof of the proposition. □

3. Regularities of the trajectories of $B^{H(\cdot),K}$

In this section, we deal with the regularities of the trajectories of $B^{H(\cdot),K}$. We follows the same method used in the case of the mBm, (see [8]). For this, we need the following regularity of the bfBm $B^{H,K}$ with respect to the constant parameter H, we use (1.2) the decomposition in law of the bfBm $B^{H,K}$.

Proposition 3.1. *Let* $[a,b] \subset [0,+\infty)$ *and* $[\alpha,\gamma] \subset (0,1]$, *and consider* $B^{H,K}$ *a bfBm with parameters* $H \in [\alpha,\gamma]$ *and* $K \in]0,1]$. *Then, there exists a finite positive constant* $C(\alpha,\gamma,K)$ *such that, for all* $H, H' \in [\alpha,\gamma]$, *we have:*

$$\sup_{t\in[a,b]} \mathbb{E}\big(B_t^{H,K} - B_t^{H',K}\big)^2 \leq C(\alpha,\gamma,K)|H - H'|^2.$$

Proof. Using (1.2) and the elementary inequality: $(a-b)^2 \leq 2a^2 + 2b^2$, we obtain:

$$\mathbb{E}\big(B_t^{H,K} - B_t^{H',K}\big)^2 \leq 2C_2^2(K)\mathbb{E}\big(B_t^{HK} - B_t^{H'K}\big)^2 + 2C_1^2(K)\mathbb{E}\big(X_t^{H,K} - X_t^{H',K}\big)^2.$$

In view of Lemma 3.1 in [8], (see also [23]), we know that:

$$\mathbb{E}\big(B_t^{HK} - B_t^{H'K}\big)^2 \leq C_1(\alpha,\gamma,K)|H-H'|^2, \tag{3.1}$$

where,

$$C_1(\alpha,\gamma,K) = 4K^2 \sup_{t\in[a,b]}\left(\int_0^1 \frac{1-\cos(t\theta)}{\theta^{2\gamma+1}}(\log(\theta))^2 \mathrm{d}\theta + \int_1^{+\infty} \frac{1}{\theta^{2\alpha+1}}(\log(\theta))^2 \mathrm{d}\theta\right) < +\infty.$$

Now, let us deal with the process $X^{H,K}$. We have by the Itô's isometry:

$$\mathbb{E}\big(X_t^{H,K} - X_t^{H',K}\big)^2 = \int_0^{+\infty} \big(e^{-\theta t^{2H'}} - e^{-\theta t^{2H}}\big)^2 \theta^{-(1+K)} \mathrm{d}\theta.$$

Without loss of generality, we suppose that $H < H'$.

Making use of the theorem on finite increments, (see for example [11]), for the function $x \mapsto e^{-\theta t^{2x}}$ for $x \in (H,H')$, there exists $\xi \in (H,H')$ such that:

$$\mathbb{E}\big(X_t^{H,K} - X_t^{H',K}\big)^2 = 4|H-H'|^2 t^{4\xi}\log^2(t)\int_0^{+\infty} e^{-2\theta t^{2\xi}}\theta^{1-K}\mathrm{d}\theta.$$

- If $t \leq 1$, then $|t\log(t)| \leq e^{-1}$, and:

  $$\mathbb{E}\big(X_t^{H,K} - X_t^{H',K}\big)^2 \leq \frac{1}{(e\alpha)^2}|H-H'|^2\int_0^{+\infty} e^{-2\theta t^{2\gamma}}\theta^{1-K}\mathrm{d}\theta.$$

  Then,

  $$\mathbb{E}\big(X_t^{H,K} - X_t^{H',K}\big)^2 \leq C_2(\alpha,\gamma,K)|H-H'|^2,$$

  where,

  $$C_2(\alpha,\gamma,K) = \frac{1}{(e\alpha)^2}\sup_{t\in[a,b]}\left(\int_0^1 e^{-2\theta t^{2\gamma}}\theta^{1-K}\mathrm{d}\theta + \int_1^{+\infty} e^{-2\theta t^{2\gamma}}\theta^{1-K}\mathrm{d}\theta\right) < +\infty.$$

- If $t \geq 1$, we obtain:

  $$\mathbb{E}\big(X_t^{H,K} - X_t^{H',K}\big)^2 \leq C_3(\alpha,\gamma,K)|H-H'|^2.$$

  where,

  $$C_3(\alpha,\gamma,K) = \left[\sup_{t\in[a,b]}\big(4t^{4\gamma}\log^2(t)\big)\right]\left[\sup_{t\in[a,b]}\left(\int_0^{+\infty} e^{-2\theta t^{2\alpha}}\theta^{1-K}\mathrm{d}\theta\right)\right] < +\infty.$$

Finally,

$$\mathbb{E}\big(X_t^{H,K} - X_t^{H',K}\big)^2 \leq C(\alpha,\gamma,K)|H-H'|^2, \tag{3.2}$$

where

$$C(\alpha,\gamma,K) = \max\big(C_2(\alpha,\gamma,K)\,;\, C_3(\alpha,\gamma,K)\big).$$

Consequently, by combining (3.1) and (3.2), we conclude the lemma. □

Remark 3.2.

(1) A similar result is obtained by Ait Ouahra and Sghir [3, Lemma 3.2], for the sub-fractional Brownian motion S^H with parameter $H \in (0,1)$. It's a continuous centered Gaussian process, starting from zero, with covariance function:

$$\mathbb{E}(S_t^H S_s^H) = t^H + s^H - \frac{1}{2}\Big[(t+s)^H + |t-s|^H\Big].$$

(2) In the case of fBm B^H, (i.e. $K = 1$), a similar result is given, independently, in [8], by using the moving average representation of fBm, and in [23], by using the harmonisable representation of fBm.

We turn now our interest to the study of the time regularities of our process.

Theorem 3.3. *Let $H(\cdot) : [0,+\infty) \mapsto [\mu,\nu] \subset (0,1)$ be a Hölder continuous function with exponent $\beta > 0$ and $\sup_{t\geq 0} H(t) < \beta$. Then, there exists a finite positive constant $C(\mu,\nu,K)$ such that:*

$$\mathbb{E}\big(B_t^{H(t),K} - B_s^{H(s),K}\big)^2 \leq C(\mu,\nu,K)|t-s|^{2\left(H(t)\vee H(s)\right)K}, \quad \textit{for all } t,s \in [0,1].$$

Proof. By the elementary inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we have:

$$\begin{aligned}\mathbb{E}\big(B_t^{H(t),K} - B_s^{H(s),K}\big)^2 &= \mathbb{E}\big(B_t^{H(t),K} - B_s^{H(t),K} + B_s^{H(t),K} - B_s^{H(s),K}\big)^2\\ &\leq 2\mathbb{E}\big(B_t^{H(t),K} - B_s^{H(t),K}\big)^2 + 2\mathbb{E}\big(B_s^{H(t),K} - B_s^{H(s),K}\big)^2,\end{aligned}$$

where $B_s^{H(t),K}$ is bfBm with parameters $H(t)$ and K.

By virtue of (1.1) and Proposition 3.1 and the fact that $H(\cdot) : [0,+\infty[\to [\mu,\nu] \subset (0,1)$, we get:

$$\begin{aligned}\mathbb{E}\big(B_t^{H(t),K} - B_s^{H(s),K}\big)^2 &\leq 2^{2-K}|t-s|^{2H(t)K} + 2C(\mu,\nu,K)|H(t)-H(s)|^2.\\ &\leq 2^{2-K}|t-s|^{2H(t)K} + 2C'(\mu,\nu,K)|t-s|^{2\beta}.\end{aligned}$$

Since $\sup_{t\geq 0} H(t) < \beta$ and $KH(t) \in (0,1)$, we deduce that: $|t-s|^{2\beta} \leq |t-s|^{2KH(t)}$. Thus,

$$\mathbb{E}\big(B_t^{H(t),K} - B_s^{H(s),K}\big)^2 \leq C_4(\mu,\nu,K)|t-s|^{2KH(t)},$$

where $C_4(\mu,\nu,K) = 2^{2-K} + 2C'(\mu,\nu,K)$.

Since the roles of t and s are symmetric, we obtain the desired result. □

To prove the next result: Theorem 3.5, we need the following classical lemma.

Lemma 3.4. *Let Y be a real centered Gaussian random variable. Then for all real $\alpha > 0$, we have:*

$$\mathbb{E}|Y|^{\alpha} = c(\alpha)\big(\mathbb{E}|Y|^2\big)^{\frac{\alpha}{2}}, \quad \textit{where} \quad c(\alpha) = \frac{2^{\frac{\alpha}{2}}\Gamma(\frac{\alpha+1}{2})}{\Gamma(\frac{1}{2})}$$

Theorem 3.5. *Let $H(\cdot) : [0, +\infty) \mapsto [\mu, \nu] \subset (0, 1)$ be a Hölder continuous function with exponent $\beta > 0$ and $\sup_{t\geq 0} H(t) < \beta$. Then, there exists $\delta > 0$, and for any integer $m \geq 1$, there exist $M_m > 0$, such that:*

$$\mathbb{E}\Big(B_t^{H(t),K} - B_s^{H(s),K}\Big)^m \geq M_m|t-s|^{m(H(t)\wedge H(s))K},$$

for all $s, t \geq 0$ such that $|t - s| < \delta$.

Proof. Using the elementary inequality: $(a + b)^2 \geq \frac{1}{2}a^2 - b^2$, we obtain:

$$\begin{aligned}\mathbb{E}\Big(B_t^{H(t),K} - B_s^{H(s),K}\Big)^2 &= \mathbb{E}\Big(B_t^{H(t),K} - B_s^{H(t),K} + B_s^{H(t),K} - B_s^{H(s),K}\Big)^2 \\ &\geq \frac{1}{2}\mathbb{E}\Big(B_t^{H(t),K} - B_s^{H(t),K}\Big)^2 - \mathbb{E}\Big(B_s^{H(t),K} - B_s^{H(s),K}\Big)^2.\end{aligned}$$

Moreover, by using (1.1) and Proposition 3.1, we obtain:

$$\begin{aligned}\mathbb{E}\Big(B_t^{H(t),K} - B_s^{H(s),K}\Big)^2 &\geq \frac{1}{2^{1+K}}|t-s|^{2H(t)K} - \mathbb{E}\Big(B_s^{H(t),K} - B_s^{H(s),K}\Big)^2 \\ &\geq \frac{1}{2^{1+K}}|t-s|^{2H(t)K} - C(\mu,\nu,K)|H(t)-H(s)|^2.\end{aligned}$$

Since the function $H(\cdot)$ is Hölder continuous with exponent β, we get:

$$\begin{aligned}\mathbb{E}\Big(B_t^{H(t),K} - B_s^{H(s),K}\Big)^2 &\geq \frac{1}{2^{1+K}}|t-s|^{2H(t)K} - C(\mu,\nu,K)|t-s|^{2\beta} \\ &= |t-s|^{2H(t)K}\bigg(\frac{1}{2^{1+K}} - C(\mu,\nu,K)|t-s|^{2(\beta-H(t)K)}\bigg).\end{aligned}$$

Since $KH(t) < \beta$, we can choose δ small enough such that for all $s, t \geq 0$, and $|t - s| < \delta$, we have:

$$\frac{1}{2^{1+K}} - C(\mu,\nu,K)|t-s|^{2(\beta-H(t)K)} > 0.$$

Indeed, it suffices to choose $\delta < \Big(\frac{1}{2^{1+K}C(\mu,\nu,K)} \wedge 1\Big)^{\eta}$, where: $\eta = \frac{1}{2}\big(\beta - K\sup_{t\geq 0} H(t)\big)^{-1}$.

Finally, we get:

$$\mathbb{E}\Big(B_t^{H(t),K} - B_s^{H(s),K}\Big)^2 \geq M|t-s|^{2H(t)K}, \quad \text{for all } |t-s| < \delta,$$

where $M = \Big(\frac{1}{2^{1+K}} - C''(\mu,\nu,K)\delta^{\gamma}\Big)$ and $\gamma = 2(\beta - K\nu)$.

Since $B^{H(\cdot),K}$ is a Gaussian process, then by Lemma 3.4 and the fact that the roles of t and s are symmetric, we obtain the desired result. □

Remark 3.6. It is well known by Berman [7] that, for a zero-mean Gaussian process $X := \big(X(t)\,;\, t \in [0,T]\big)$ with bounded variance, the variance condition:

$$\int_0^T \int_0^T \big(\mathbb{E}|X_t - X_s|^2\big)^{-\frac{1}{2}} \mathrm{d}s\mathrm{d}t < +\infty,$$

is sufficient for the local time $L(t,x)$ of X to exist on $[0,T]$ almost surely and to be square integrable as a function of x:

$$\int_{\mathbb{R}} L^2\big([a,b],x\big)\mathrm{d}x < +\infty, \quad \big([a,b] \subset [0,+\infty[\big).$$

For more information on local time, the reader is referred to [8, 17, 32] and the references therein. In our case, we obtain for any interval I with $|I| < \delta$:

$$\int_I \int_I \big(\mathbb{E}\big|B_t^{H(t),K} - B_s^{H(s),K}\big|^2\big)^{-\frac{1}{2}} \mathrm{d}s\mathrm{d}t \le C'(\mu,\gamma,K) \int_I \int_I |t-s|^{-K \sup_{r\in I} H(r)} \mathrm{d}s\mathrm{d}t < +\infty,$$

because $K \sup_{r\in I} H(r) < 1$. Then $B^{H(\cdot),K}$ possesses, on any interval $I \subset [a,b]$ with length $|I| < \delta$, a local time which is square integrable as function of x. Finally, since $[a,b]$ is a finite interval, we can obtain the local time on $[a,b]$ by a patch-up procedure, i.e., we partition $[a,b]$ into $\bigcup_{i=1}^n [a_{i-1},a_i]$, such that $|a_i - a_{i-1}| < \delta$, and define: $L([a,b],x) := \sum_{i=1}^n L([a_i,a_{i-1}],x)$, where $a_0 = a$ and $a_n = b$.

4. Local asymptotic self similarity property

The dependence of $H(\cdot)$ with respect to the time t destroys all the invariance properties that we had for the fBm. For example the mBm is no more self-similar, nor with stationary increments. However, the authors in [22] showed that with the condition that $H(\cdot)$ is β−Hölder continuous with exponent $\beta > 0$ and $\sup_{t\in\mathbb{R}^+} H(t) < \beta$, the mBm is locally asymptotically self-similar, (LASS for short), in the following sense:

$$\lim_{\rho\to 0^+} \left(\frac{B_{t+\rho u}^{H(t+\rho u)} - B_t^{H(t)}}{\rho^{H(t)}}\,;\, u \ge 0 \right) \overset{d'}{=} \big(B_u^{H(t)}\,;\, u \ge 0\big),$$

where $B_u^{H(t)}$ is a fBm with Hurst parameter $H(t)$, and $\overset{d'}{=}$ stands for the convergence of finite dimensional distributions. Some authors use the term localizability for locally asymptotically self-similarity, (see Falconer [15, 16]).

Our process is an other example of Gaussian process who loses the self similarity property when H depend on t. However, we show in the following result that it is LASS.

Before we deal with the proof of our result, we need the following lemma proved by Ait Ouahra et al. [2, see Theorem 2.6].

Lemma 4.1. *Let $B^{H,K}$ a bfBm with parameters $K \in (0,1)$ and $H \in (0,1)$. Then,*

$$\mathbb{E}\left(\frac{B^{H,K}_{t+\rho u} - B^{H,K}_{t}}{\rho^{HK}}\right)^2 \xrightarrow[\rho\to 0]{} 2^{1-K}u^{2HK}.$$

Now, we are ready to state and prove our result.

Proposition 4.2. *Consider $H(\cdot)$ a β-Hölder continuous function with exponent $\beta > 0$ such that $\sup_{t\geq 0} H(t) < \beta$, then $B^{H(\cdot),K}$ is LASS:*

$$\lim_{\rho\to 0^+}\left(\frac{B^{H(t+\rho u),K}_{t+\rho u} - B^{H(t),K}_{t}}{\rho^{H(t)K}}\ ;\ u \geq 0\right) \overset{d'}{=} \left(2^{1-K}B^{H(t)K}_{u}\ ;\ u\geq 0\right),$$

where $B^{H(t)K}$ is a fBm with the Hurst parameter $H(t)K$.

Proof. We use the same arguments used in [23] in the case of the mBm, (see [23, Proposition 5]). We prove the convergence in distribution by showing the following two statements:

$$\mathbb{E}\left(\frac{B^{H(t+\rho u),K}_{t+\rho u} - B^{H(t),K}_{t}}{\rho^{H(t)K}}\right) \xrightarrow[\rho\to 0]{} 0, \tag{4.1}$$

$$\mathbb{E}\left(\frac{B^{H(t+\rho u),K}_{t+\rho u} - B^{H(t),K}_{t}}{\rho^{H(t)K}}\right)^2 \xrightarrow[\rho\to 0]{} \sigma_t^2, \tag{4.2}$$

where,

$$\sigma_t^2 = 2^{1-K}\operatorname{Var}\left(\frac{B^{H(t)K}_{t+\rho u} - B^{H(t)K}_{t}}{\rho^{H(t)K}}\right) = 2^{1-K}u^{2H(t)K},$$

and $B^{H(t)K}$ is a fBm with the Hurst parameter $H(t)K$.

We deal with (4.2) since (4.1) is obvious. We have:

$$\begin{aligned}
&\mathbb{E}\left(\frac{B^{H(t+\rho u),K}_{t+\rho u} - B^{H(t),K}_{t}}{\rho^{H(t)K}}\right)^2\\
&\qquad = \mathbb{E}\left(\frac{B^{H(t+\rho u),K}_{t+\rho u} - B^{H(t),K}_{t+\rho u}}{\rho^{H(t)K}}\right)^2 + \mathbb{E}\left(\frac{B^{H(t),K}_{t+\rho u} - B^{H(t),K}_{t}}{\rho^{H(t)K}}\right)^2\\
&\qquad\qquad + 2\mathbb{E}\left[\frac{\left(B^{H(t+\rho u),K}_{t+\rho u} - B^{H(t),K}_{t+\rho u}\right)\left(B^{H(t),K}_{t+\rho u} - B^{H(t),K}_{t}\right)}{\rho^{2H(t)K}}\right].
\end{aligned}$$

In view of Proposition 3.1, and the fact that $H(\cdot)$ is $\beta-$Hölder continuous function, we have:

$$\begin{aligned}\mathbb{E}\left(\frac{B_{t+\rho u}^{H(t+\rho u),K}-B_{t+\rho u}^{H(t),K}}{\rho^{H(t)K}}\right)^2 &\leq C(K)\frac{|H(t+\rho u)-H(t)|^2}{\rho^{2KH(t)}}\\ &\leq C'(K)(\rho u)^{2\left(\beta-KH(t)\right)}.\end{aligned}$$

Since $K\sup_t H(t)<\beta$, we get:

$$\mathbb{E}\left(\frac{B_{t+\rho u}^{H(t+\rho u),K}-B_{t+\rho u}^{H(t),K}}{\rho^{H(t)K}}\right)^2 \longrightarrow 0 \quad \text{as} \quad \rho\longrightarrow 0.$$

In view of Lemma 4.1, we know that:

$$\mathbb{E}\left(\frac{B_{t+\rho u}^{H(t),K}-B_{t}^{H(t),K}}{\rho^{H(t)K}}\right)^2 \underset{\rho\to 0}{\longrightarrow} 2^{1-K}u^{2H(t)K}.$$

Now, by Schwartz's inequality, (1.1) and Proposition 3.1, we have:

$$\begin{aligned}&\mathbb{E}\left[\frac{\left(B_{t+\rho u}^{H(t+\rho u),K}-B_{t+\rho u}^{H(t),K}\right)\left(B_{t+\rho u}^{H(t),K}-B_{t}^{H(t),K}\right)}{\rho^{2H(t)K}}\right]\\ &\qquad\leq\left[\mathbb{E}\left(\frac{B_{t+\rho u}^{H(t+\rho u),K}-B_{t+\rho u}^{H(t),K}}{\rho^{H(t)K}}\right)^2\right]^{\frac{1}{2}}\left[\mathbb{E}\left(\frac{B_{t+\rho u}^{H(t),K}-B_{t}^{H(t),K}}{\rho^{H(t)K}}\right)^2\right]^{\frac{1}{2}}\\ &\qquad\leq C\rho^{\beta-H(t)K}u^{\beta+H(t)K}\longrightarrow 0,\quad \text{for } \rho\longrightarrow 0,\ \text{since } K\sup_t H(t)<\beta.\end{aligned}$$

Hence, we deduce that:

$$\mathbb{E}\left(\frac{B_{t+\rho u}^{H(t+\rho u),K}-B_{t}^{H(t),K}}{\rho^{H(t)K}}\right)^2 \underset{\rho\to 0}{\longrightarrow} 2^{1-K}u^{2H(t)K}.$$

Consequently the LASS property is proved. □

5. Long range dependence

The long range dependence, (LRD for short), and long memory are synonymous notions. LRD measures long-term correlated processes. LRD is a characteristic of phenomena whose autocorrelation functions decay rather slowly. The presence and the extent of LRD is usually measured by the parameters of the process. Most of the definitions of LRD appearing in literature for stationary process are based on the second-order properties of a stochastic process. Such properties include asymptotic behavior of covariances, spectral density, and variances of partial sums. The specialness of LRD is a connection between long memory and stationarity, (see for example [29] in case of fBm and [25] and [28] in case of bfBm). For our process $B^{H(\cdot),K}$, we use the same arguments used in [4] in case of standard mBm. Of course, the definitions must be adapted in our case since mBm and our extension does not have stationary increments, (see for example [4]):

Definition 5.1.

(a) Let Y be a second order process: i.e., $\mathbb{E}\big(Y^2(t)\big) < +\infty$, for all $t \geq 0$. Y is said to have a LRD if there exists a function $\alpha(s)$ taking values in $(-1, 0)$ such that:

$$\forall\ s \geq 0, \quad \mathrm{cor}_Y(s, s+h) \approx h^{\alpha(s)} \quad \text{as } h \text{ tends to } +\infty,$$

where $\mathrm{cor}_Y(s,t) := \frac{\mathrm{cov}_Y(s,t)}{\sqrt{\mathbb{E}(Y^2(s))\mathbb{E}(Y^2(t))}}$.

(b) Let Y be a second-order process. Y is said to have a LRD if:

$$\forall\ s \geq 0,\ \forall\ \delta \geq 0, \quad \sum_{k=0}^{+\infty} |\mathrm{cor}_Y(s, s+k\delta)| = +\infty.$$

In the next propositions, we prove some results about covariances and correlations of our process and its increments. In the sequel, we denote $f(t) \approx g(t)$ if there exist $0 < c < d < +\infty$ such that for all sufficiently large t: $c \leq \frac{f(t)}{g(t)} \leq d$. We put: $\mathrm{cov}(t,s) := R^{H(\cdot),K}(t,s)$ the covariance function of our process $B^{H(\cdot),K}$ and $\mathrm{cor}(t,s)$ its correlation function.

5.1. Asymptotic behavior of the covariance and the correlation of $B^{H(\cdot),K}$

Proposition 5.2. *When t tends to infinity, and for all fixed $s \geq 0$, we have:*

(i) $K(H(t)+H(s)) < 1 \Rightarrow \mathrm{cov}(t,s) \approx t^{(H(t)+H(s))(K-1)}$.

(ii) $K(H(t)+H(s)) > 1 \Rightarrow \mathrm{cov}(t,s) \approx t^{K(H(t)+H(s))-1}$.

(iii) $K(H(t)+H(s)) < 1 \Rightarrow \mathrm{cor}(t,s) \approx t^{-H(t)}$.

(iv) $K(H(t)+H(s)) > 1 \Rightarrow \mathrm{cor}(t,s) \approx t^{KH(s)-1}$.

Proof.

(i) *and* (ii). Firstly, Recall that $H(t)+H(s)$ and $\big(D(H(t),H(s)\big)^K$ are bounded. We have:

$$\mathrm{cov}(t,s) = \big(D\big(H(t),H(s)\big)\big)^k t^{(H(t)+H(s))K}\left[\left(1+\Big(\frac{s}{t}\Big)^{H(t)+H(s)}\right)^K - \left|1-\frac{s}{t}\right|^{(H(t)+H(s))K}\right].$$

Consequently, the Taylor expansion as $t \longrightarrow +\infty$, gives:

$$\mathrm{cov}(t,s) \approx KD\big(H(t),H(s)\big)^K \Big[s^{H(t)+H(s)}t^{(H(t)+H(s))(K-1)} \\ + (H(t)+H(s))st^{K(H(t)+H(s))-1}\Big],$$

where the leading term is:

$$K\big(D(H(t),H(s))\big)^K s^{H(t)+H(s)}t^{(H(t)+H(s))(K-1)} \quad \text{if } K(H(t)+H(s)) < 1,$$

and,

$$K\big(D(H(t),H(s))\big)^K (H(t)+H(s))st^{K(H(t)+H(s))-1} \quad \text{if } K(H(t)+H(s)) > 1.$$

(iii) *and* (iv). Using once again a Taylor expansion of:

$$\mathrm{cor}(t,s) := \big(D(H(t),H(s))\big)^K \frac{\big(t^{H(t)+H(s)} + s^{H(t)+H(s)}\big)^K - |t-s|^{(H(t)+H(s))K}}{s^{KH(s)}t^{KH(t)}},$$

where the leading term in this case is:

$$K\big(D(H(t),H(s))\big)^K s^{H(t)+(1-K)H(s)}t^{-H(t)+(K-1)H(s)} \quad \text{if } K(H(t)+H(s)) < 1,$$

and,

$$K\big(D(H(t),H(s))\big)^K (H(t)+H(s))t^{KH(s)-1}s^{1-KH(s)} \quad \text{if } K(H(t)+H(s)) > 1. \quad \square$$

Since both $-H(t)$ and $KH(s)-1$ belong to $(-1,0)$ for all t,s, we have the following result:

Corollary 5.3. *For all admissible $H(t)$, our process $B^{H(\cdot),K}$ has LRD in the sense of Definition 5.1(b). If for all s $K(H(t)+H(s)) > 1$ for all sufficiently large t, then $B^{H(\cdot),K}$ has LRD in the sense of Definition 5.1(a), with functional LRD exponent: $\alpha(s) = KH(s)-1$.*

5.2. Asymptotic behavior of the covariance and the correlation of the increments of $B^{H(\cdot),K}$

In the following results, to simplify the notation, let us denote:

$$L(s,t) := \max\big(H(t)+H(s), H(t+1)+H(s), H(t)+H(s+1), H(t+1)+H(s+1)\big).$$

Proposition 5.4. *Let Y be the unit time increments of $B^{H(\cdot),K}$: $Y(t) = B_{t+1}^{H(t+1),K} - B_t^{H(t),K}$. Then, when t tends to infinity, and for all fixed $s \geq 0$ such that the four quantities: $H(t)+H(s)$, $H(t+1)+H(s)$, $H(t)+H(s+1)$, and $H(t+1)+H(s+1)$ are all different, we have:*

(i) $KL(s,t) < 1 \Rightarrow \mathrm{cov}_Y(t,s) \approx t^{L(s,t)(K-1)}$.

(ii) $KL(s,t) > 1 \Rightarrow \mathrm{cov}_Y(t,s) \approx t^{KL(s,t)-1}$.

(iii) $KL(s,t) < 1 \Rightarrow \mathrm{cor}_Y(t,s) \approx t^{-K\max(H(t),H(t+1))}$.

(iv) $KL(s,t) > 1 \Rightarrow \mathrm{cor}_Y(t,s) \approx t^{K\max(H(s),H(s+1))-1}$.

Proof.

(i) *and* (ii). By definition, we have:

$$\mathrm{cov}_Y(t,s) = \mathrm{cov}(t+1,s+1) - \mathrm{cov}(t+1,s) - \mathrm{cov}(t,s+1) + \mathrm{cov}(t,s).$$

Applying the Taylor expansion to each covariance, we obtain:

- if $KL(s,t) < 1$, from Proposition 5.2, it follows that:

$$\mathrm{cov}_Y(t,s) \approx t^{L(s,t)(K-1)}.$$

- if at least one of $K\big(H(t)+H(s)\big)$; $K\big(H(t+1)+H(s)\big)$; $K\big(H(t)+H(s+1)\big)$ and $K\big(H(t+1)+H(s+1)\big)\big)$ is greater than one, the order of $\mathrm{cov}_Y(t,s)$ will be the maximum of these value, since they all differ. More precisely, denoting (t',s') the couple where the maximum of $H(t)+H(s)$; $H(t+1)+H(s)$; $H(t)+H(s+1)$ and $H(t+1)+H(s+1)$ is attained, we get:

$$\mathrm{cov}_Y(t,s) = K\big(D(H(t),H(s))\big)^K \big(H(t')+H(s')\big)s't'^{K(H(t')+H(s'))-1} + o\big(t'^{K(H(t')+H(s'))-1}\big).$$

(iii) *and* (iv). Again, this is simply obtained using Proposition 5.2 and the fact that $E(Y^2(t)) = O\big(t^{2K\max(H(t),H(t+1))}\big)$ if $H(t)$ and $H(t+1)$ differ (otherwise cancellation occur and the leading term is different). The exponent in the case where $KL(s,t) > 1$ results from the identity :

$$\max\big(H(t)+H(s), H(t+1)+H(s), H(t)+H(s+1), H(t+1)+H(s+1)\big) \\ -\max\big(H(t);H(t+1)\big) = \max\big(H(s),H(s+1)\big). \quad \square$$

Corollary 5.5. *For all admissible $H(t)$, our process $B^{H(\cdot),K}$ has LRD in the sense of Definition 5.1 (b). If, for all s, $KL(s,t) > 1$, for all sufficiently large t, the increments of $B^{H(\cdot),K}$ have LRD in the sense of Definition 5.1. As well as in the sense of Definition 5.1 (a), with functional long range dependence exponent $\alpha(s) = K\max\big(H(s),H(s+1)\big) - 1$.*

Proof. Obviously, both $K\max\big(H(s),H(s+1)\big) - 1$ and $-K\max\big(H(t),H(t+1)\big)$ belong to $(-1;0)$. $\square$

5.3. Conclusion and Outlook

(i) It will be interesting to study a general case of Gaussian process of the form $B^{H(\cdot),K(\cdot)}$ where both the parameters H and K depend on the time t.

(ii) A response of the problem of the decomposition in law of our process will be useful to generalize a popular results for the bfBm like the existence and the Hölder regularities of its local time, (see [2] in case of bfBm and [8] in case of mBm).

(iii) For future work, we plan to study the natural question of the local non-determinism property of local time, (LND for short). This property is useful to prove the joint continuity and the Höder regularity of local time. It is introduced by Berman [7], and based on the relative conditioning error:

$$V_p = \frac{\mathrm{Var}\left\{X_{t_p} - X_{t_{p-1}} | X_{t_1}, \ldots X_{t_{p-1}}\right\}}{\mathrm{Var}\left\{X_{t_p} - X_{t_{p-1}}\right\}}$$

where for $p \geq 2,\ t_1 < \ldots < t_p$ are arbitrary ordered points in an open interval J. To estimate V_p in the case of mBm, Boufoussi et al. [8] have used an integral representation with respect to the Brownian motion. In case of bfBm, by using the Lamperti's transform [21] based on the self-similarity property, Tudor and Xiao [31] have proved a strong property then LND, called the strongly locally non-deterministic. Therefore they obtained the joint continuities of local times. For our process, we propose study the LND property by using the LASS property.

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References

[1] P. S. Addison and A. S. Ndumu. Engineering applications of fractional Brownian motion: self-affine and self-similar random processes. *Fractals*, 7(2):151–157, 1999.

[2] Mohamed Ait Ouahra, Hanae Ouahhabi, and Aissa Sghir. Continuity in law of some additive functionals of bifractional Brownian motion. *Stochastics*, 91(4):613–628, 2019.

[3] Mohamed Ait Ouahra and Aissa Sghir. Continuity in law with respect to the hurst index of some additive functionals of sub-fractional brownian motion. *Stochastic Anal. Appl.*, 35(4):677–690, 2017.

[4] Antoine Ayache, Serge Cohen, and Jacques Lévy Véhel. The covariance structure of multifractional Brownian motion, with application to long range dependence. In *ICASSP 2000 (Istanbul, Turkey, 2000)*. IEEE, 2002.

[5] Xavier Bardina and Khalifa Es-Sebaiy. An extension of bifractional Brownian motion. *Commun. Stoch. Anal.*, 5(2):333–340, 2011.

[6] Albert Benassi, Stéphane Jaffard, and Daniel Roux. Elliptic Gaussian random processes. *Rev. Mat. Iberoam.*, 13:19–90, 1997.

[7] Simeon M. Berman. Local times and sample function properties of stationary Gaussian processes. *Trans. Am. Math. Soc.*, 137:277–299, 1969.

[8] Brahim Boufoussi, Marco Dozzi, and Raby Guerbaz. On the local time of multifractional Brownian motion. *Stochastics*, 78(1):33–49, 2006.

[9] Patrick Cheridito. Mixed fractional Brownian motion. *Bernoulli*, 7(6):913–934, 2001.

[10] Renata Cioczek-Georges and Benoît B. Mandelbrot. Alternative micropulses and fractional Brownian motion. *Stochastic Processes Appl.*, 64(2):143–152, 1996.

[11] Michael J. Cloud and Byron C. Drachman. *Inequalities: With Applications to Engineering*. Springer, 1998.

[12] Serge Cohen. From self-similarity to local self-similarity: the estimation problem. In M. Dekking, J. Lévy-Véhel, E. Lutton, and C. Tricot, editors, *Fractals: Theory and Applications in Engineering*, pages 3–16. Springer, 1999.

[13] A. Comegna, A. Coppola, V. Comegna, A. Sommella, and C. D. Vitale. Use of a fractional Brownian motion model to mimic spatial horizontal variation of soil physical and hydraulic properties displaying a power-law variogram. *Procedia Environ. Sci.*, 19:416–425, 2013.

[14] Khalifa Es-Sebaiy and Ciprian A. Tudor. Multidimensional bifractional Brownian Motion. Itô and Tanaka formulas. *Stoch. Dyn.*, 7(3):365–388, 2007.

[15] Kenneth J. Falconer. Tangent fields and the local structure of random fields. *J. Theor. Probab.*, 15(3):731–750, 2002.

[16] Kenneth J. Falconer. The local structure of random processes. *J. Lond. Math. Soc.*, 67(3):657–672, 2003.

[17] Donald Geman and Joseph Horowitz. Occupation densities. *Ann. Probab.*, 8(1):1–67, 1980.

[18] Christian Houdré and José Villa. An example of infinite dimensional quasi-helix. In *Stochastic models. Seventh symposium on probability and stochastic processes*, volume 336 of *Contemporary Mathematics*, pages 195–201. American Mathematical Society, 2003.

[19] Jean-Pierre Kahane. Hélices et quasi-hélices. *Adv. Math.*, 7B:417–433, 1981.

[20] Andrei N. Kolmogorov. The Wiener spiral and some other interesting curves in Hilbert space. *Dokl. Akad. Nauk SSSR*, 26(2):115–118, 1940.

[21] John Lamperti. Semi-stable stochastic processes. *Trans. Am. Math. Soc.*, 104:64–78, 1962.

[22] Pedro Lei and David Nualart. A decomposition of the bifractional Brownian motion and some applications. *Stat. Probab. Lett.*, 79(5):619–624, 2009.

[23] Jacques Lévy-Vehel and Romain F. Peltier. Multifractional Brownian motion: definition and preliminary results. Technical Report RR-2645, INRIA, 1996.

[24] Mikhail Lifshits, René L. Schilling, and Ilya Tyurin. A Probabilistic Inequality Related to Negative Definite Functions. In *High Dimensional Probability VI*, volume 66 of *Progress in Probability*, pages 73–80. Birkhäuser, 2013.

[25] Makoto Maejima and Ciprian A. Tudor. Limits of bifractional Brownian noises. *Commun. Stoch. Anal.*, 2(3):369–383, 2008.

[26] Benoît B. Mandelbrot and John W. Van Ness. Fractional Brownian Motions, Fractional Noises and Applications. *SIAM Rev.*, 10:422–437, 1968.

[27] Matthieu Marouby. Micropulses and different types of Brownian motion. *J. Appl. Probab.*, 48(3):792–810, 2011.

[28] Francesco Russo and Ciprian A. Tudor. On bifractional Brownian motion. *Stochastic Processes Appl.*, 116(5):830–856, 2006.

[29] Gennady Samorodnitsky and Murad S. Taqqu. *Stable non-Gaussian random processes: Stochastic models with infinite variance*. Stochastic Modeling. Chapman & Hall, 1994.

[30] Murad S. Taqqu. The modelling of ethernet data and of signals that are heavy-tailed with infinite variance. *Scand. J. Stat.*, 29(2):273–295, 2002.

[31] Ciprian A. Tudor and Yimin Xiao. Sample path properties of bifractional Brownian motion. *Bernoulli*, 13(4):1023–1052, 2007.

[32] Yimin Xiao. Hölder conditions for the local times and the Hausdorff measure of the level sets of Gaussian random fields. *Probab. Theory Relat. Fields*, 109(1):129–157, 1997.

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