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Well-posedness of a non local ocean-atmosphere coupling model: study of a 1D Ekman boundary layer problem with non-local KPP-type turbulent viscosity profile

SOPHIE THERY

Abstract

This paper addresses the mathematical analysis of the ocean-atmosphere coupling problem, including Coriolis force, non-local turbulent closure and realistic nonlinear interface conditions. We introduce a 1D vertical model corresponding to a coupled Ekman boundary layer problem with non-local turbulent viscosities. The interest of this model lies in its proximity to realistic ones by considering the numerical strategies employed to take into account the turbulent scale. Well-posedness is first studied in stationary and non-stationary states considering generalized parameterized turbulent viscosities. We establish sufficient criteria on the viscosity profiles for the uniqueness of solution and find that they are not met for parameters in the order of magnitude used in ocean and atmosphere models. To identify precisely the conditions of well-posedness, we therefor establish a necessary and sufficient criterion for the stationary state. We show that there is non-uniqueness of the solution when considering typical viscosity profiles from ocean and atmosphere models. Eventually, we illustrate that non-uniqueness is produced by an inconsistency between the viscosity profile and the boundary layer parametrisation.

1. Introduction

Ocean-atmosphere (OA) interactions play a critical role for several applications, like forecasting the trajectories of tropical cyclones, seasonal weather forecasting, or climate studies. Therefore numerical modeling systems for such applications generally couple an oceanic model with an atmospheric model, with complex interface conditions (referred to as a "bulk closure") that model these interactions. However ocean and atmosphere models have originally been constructed separately, by two distinct communities. Thus the question of the mathematical coherence of such a coupled system naturally arises, since there is no guarantee that all possible associations of an atmospheric model, an oceanic model and interface "bulk" conditions will lead to a well-posed problem.

The translation of such an OA coupled model into a single global mathematical model is challenging and gives rise to specific difficulties. A first global OA coupled model has been presented and studied by [16] as a coupling of the so-called primitive equations with nonlinear interface conditions. Many studies on the well-posedness of the primitive equations (without coupling) can be found in the context of ocean or atmosphere modeling (see for example [6] or [14, 15]). OA coupled models mainly differ by the

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strategies they use to take into account the turbulent scales (turbulent closure) and the interactions between the two domains (bulk closure). The turbulent closure scheme considered hereafter relies on the Boussinesq hypothesis which requires the definition of a turbulent viscosity profile. The resulting model is known as the coupled Ekman layer problem [9]. The present study addresses the well-posedness of such an Ekman layer coupled model that takes into account the specificities brought by the bulk interface conditions. Mathematical results can be obtained for this model, which is representative of the physics and numerics of realistic models. The combination of the turbulent closure schemes with specific interface conditions from the bulk formulation leads to a non-local coupled problem with nonlinear interface conditions, that is, the entire viscosity profile depends directly on the solution at the interface. This non-locality is the main difficulty in tackling the mathematical analysis. A first study of this model was proposed by [22]. In the present work, we present a first synthetic step in the analysis on this non-local problem in the context of OA coupling, and investigate the well-posedness of this coupled problem by searching for constraints on the turbulent viscosity profiles.

As a global approach to establish well-posedness, we will adapt a method from existing work in the fluid-fluid interaction community. A very simplified problem of what would illustrate the non-local nature of our model would be the system :

$$-\partial_z \left(\nu(z, |\mathbf{u}(0)|)\partial_z \mathbf{u}(z)\right) = \mathbf{g} \qquad z \in]0, \mathrm{Z}^\infty[\tag{1.1a}$$

$$\mathbf{u}(\mathrm{Z}^\infty) = 0 \tag{1.1b}$$

$$\nu(0, |\mathbf{u}(0)|)\partial_z \mathbf{u}(0) = |\mathbf{u}(0)|\, \mathbf{u}(0) \tag{1.1c}$$

for a given source term $\mathbf{g}$ and a parameterized viscosity profile ν. The type of non-local character we are dealing with here has rarely been considered in the literature. Here we will refer to a stationary model proposed by [3] which considers a model close to (1.1) on a two-fluid interaction with a turbulent viscosity profile different from the one considered here. The authors prove the existence of solutions and show that the uniqueness depends strongly on the viscosity profile and the regularity of the solution itself. More precisely, a fixed-point method is used to study the uniqueness of solutions and it appears that it depends on the ratio between the H^1-norm on the solution, the variation of the viscosity profile and its minimum. In our case, we bring some additional elements to be more representative of the realistic model by considering the coupling and bulk formulations at the interface. On the other hand, we simplify the study by considering the viscosity as a parametrisation $\nu(z, u^*)$ with u^* depending on the trace terms. This assumption will allow us to simplify the study of the well-posedness and give conditions on the parametrized profile of ν. Therefore, even if our model is simplified, the criteria that we establish contain the same ingredients and follow the same behavior as in [3]. These criteria for the uniqueness of the solution are sufficient criteria. These are not satisfy for viscosity profile with parameters of the order of magnitude used in OA models and the uniqueness of solution is therefore not guaranteed. To further investigate this issue, we then introduce a sufficient and necessary criterion that ensures the well-posedness of the stationary problem. However, this criterion can only be computed for viscosity profiles for which the main equation can be solved explicitly. Adapting this strategy to viscosity profiles representative to those used in OA models, shows to non-uniqueness of the solution. As will be highlighted, non-uniqueness stems from the combination between these specific viscosity profiles and interface conditions.

The remainder of this paper is organized as follows. We first construct the OA model in Section 2. Starting from the primitive equations, we briefly describe the usual assumptions considered to obtain the model in each domain and the interface conditions (Section 2.1). Then we describe the viscosity profiles considered, which are representative of OA coupled models (Section 2.2). We also provide representative values and viscosity profiles for numerical illustration. The global problem under investigation is summarized in Definition 2.4. The study of the well-posedness is discussed in Section 3 for general viscosity profile. Our global strategy is to solve the non-local problem using a fixed-point method. We first focus on the stationary state of the problem in Section 3.1. In line with the work of [3] we show the existence of stationary solution and give uniqueness criteria on the viscosity profiles. The well-posedness of the non-stationary state is studied in Section 3.2. In the line of the work of [7], we give a criterion to have the existence of a non stationary solution in the neighborhood of an existing stationary solution. We finally give well-posedness criteria on the non-stationary state following the same procedure than for the stationary state. In Section 4.1 we study the well-posedness of the problem for viscosity profiles specific with parameters in the OA order of magnitude. We illustrate in paragraph Section 4.2 that the uniqueness criteria given in Section 3 are too restrictive for the OA order of magnitude. To fully answer the question of uniqueness, we give a necessary and sufficient criterion for the well-posedness of the stationary problem (Section 4.3), case without Coriolis effect is quickly discuss. We finally apply this necessary and sufficient criterion to viscosity profiles that are representative of those used in the OA model (Section 4.4) and the non-uniqueness of solution is confirmed for these viscosity profiles. We conclude by giving some conditions on interface conditions that would guarantee uniqueness.

2. Construction of the ocean-atmosphere coupled model

In this section, we build our ocean-atmosphere model by justifying the main steps. Readers wishing to dispense with the justification of the model construction can refer directly to Definition 2.4 where the final model is recalled.

2.1. A simplified 1D coupled model

To establish a coupled model, we start from the Navier Stokes equations on which we will make simplifying assumptions, while keeping the most important ingredients to obtain a relevant coupled model. The steps involved in building our model are the following.

In each domain:

(1) Starting from the Navier–Stokes equations with density stratification in both oceanic and atmospheric domains, we make the following classical hypotheses to obtain the so-called primitive equations :

- Hydrostatic hypothesis and Boussinesq approximation (the variations of the fluid density are weak)
- We consider the earth rotation, represented by the Coriolis force

This leads to

$$\begin{aligned}
\partial_t \mathbf{u}_h + f\mathbf{k}\times\mathbf{u}_h - \nu_m \Delta \mathbf{u}_h &= \frac{\nabla_h p}{\rho_0} - \nabla\cdot(\mathbf{U}\otimes\mathbf{U}_h) \\
\partial_z p &= -g\rho' \qquad \text{(hydrostatic balance)} \\
\nu_m \nabla\cdot\mathbf{u}_h &= 0 \qquad \text{(incompressibility)} \\
\partial_t \boldsymbol{\Phi} &= \mathbf{F}_{\boldsymbol{\Phi}} - \nabla_h\cdot(\mathbf{u}_h\boldsymbol{\Phi}) \\
\rho &= \rho_{\text{eos}}(\boldsymbol{\Phi}, z)
\end{aligned} \tag{2.1}$$

where $\mathbf{U} = (\mathbf{u}_h, w)$ represents the speed (wind or current), p is the pressure, ρ the density provided by an equation of state ρ_{eos}, μ^m the molecular viscosity and $\boldsymbol{\Phi}$ is a general symbol for tracers (salinity, temperature...). The well-posedness of these primitive equations are widely studied, some first studies related to ocean and atmosphere model can be found in [5], or [14, 15], and we refer to [23] for a recent review of existing results.

(2) In order to take into account the fine-scale dynamics that is not resolved by the numerical grid, equations (2.1) must be supplemented by "sub-grid" parameterization schemes. The classical approach to introduce these parameterizations

consists in using the so-called Reynolds decomposition of each variable ϕ into a “resolved” averaged component $\overline{\phi}$ and an “unresolved” component ϕ', with $\overline{\phi'} = 0$. Using this decomposition in (2.1) gives for the first line

$$\partial_t \overline{\mathbf{u}_h} + f\mathbf{e}_z \times \overline{\mathbf{u}_h} - \nu_m \Delta \overline{\mathbf{u}_h} - \nabla_h \cdot \overline{\mathbf{u}'_h \mathbf{u}'_h} - \partial_z \overline{w' \mathbf{u}'_h} = \frac{\nabla_h p}{\rho_0} \tag{2.2}$$

where terms of the form $\overline{\phi' \varphi'}$ represent the effect of unresolved scales on the resolved scales. To close the system, we use a turbulent closure considering the Boussinesq hypothesis which gives $\overline{w' \varphi'}$ in terms of the known resolved-scale variables as:

$$\overline{\mathbf{u}' \varphi'} = -\left(\nu_{t,x} \partial_x \overline{\varphi}, \nu_{t,y} \partial_y \overline{\varphi}, \nu_{t,z} \partial_z \overline{\varphi}\right)^T$$

where ν_t are turbulent viscosities (a.k.a. eddy-viscosity) depending on space and time and potentially other parameters. These turbulent viscosities are parameterized using different closure schemes that we will discuss in Section 2.2. Note ν_t is strongly non-isotropic between horizontal direction and vertical direction. From fluid mechanics notations, we introduce the constraint tensor:

$$\boldsymbol{\sigma} = -\overline{p}\mathbf{I}_3 + \rho_0 \begin{pmatrix} (\nu_x^t + \nu_m)\partial_x \overline{u} & (\nu_x^t + \nu_m)\partial_x \overline{v} & 0 \\ (\nu_y^t + \nu_m)\partial_y \overline{u} & (\nu_y^t + \nu_m)\partial_y \overline{v} & 0 \\ (\nu_z^t + \nu_m)\partial_z \overline{u} & (\nu_z^t + \nu_m)\partial_z \overline{v} & 0 \end{pmatrix} \tag{2.3}$$

with $\mathbf{I}_3$ the identity matrix.

For more details on approximations and closure assumptions taken so far, we can refer to [25]. These assumptions are common in oceanic and atmospheric models used for climate simulations (see for example [17]).

At the interface. The interface conditions between the ocean and the atmosphere are complicated, due to the complexity of the natural phenomena they describe. Very close to the interface, dedicated parameterizations are applied and superposed to the numerical models. These interface parameterizations rely on the [19] (MO) theory that assumes constant vertical fluxes and a wall-law in this near-interface zone. The numerical counterparts to MO theory are the so-called bulk formulations, see [22] for more details. In order to formulate interface conditions consistently, we separate the near-interface zone whose flow is governed by MO theory from the rest of the domain where the primitive equations are considered. The altitude that limit the near-interface to the boundary layer is defined by δ_o in the ocean and δ_a with $\delta_o < 0 < \delta_a$. We assume $(p_\alpha, \mathbf{u}_\alpha)$ are parameterized in $[\delta_o, \delta_a]$ for $\alpha \in \{o, a\}$. The ocean and atmosphere domain are defined as $\Omega_o = [Z_o^\infty, \delta_o[$ and $\Omega_a =]\delta_a, Z_a^\infty]$. In the following, we consider the interface between the ocean and the atmosphere as a buffer zone $[\delta_o, \delta_a]$ where MO theory applies. The interface condition at the ocean surface are given by the continuity of the constraint $\boldsymbol{\sigma}$ and the continuity of $\mathbf{u}$. Applying the MO theory in $[\delta_o, \delta_a]$ (which takes the form of a non-linear friction law see [22] for a detailled description), leads to the following conditions :

$$\rho_o(\nu_o^m + \nu_{t,o}(\delta_o, t))\, \partial_z \overline{\mathbf{u}}_{h,o}(\delta_o, t) = \rho_a(\nu_a^m + \nu_{t,a}(\delta_a, t))\, \partial_z \overline{\mathbf{u}}_{h,a}(\delta_a, t) \tag{2.4a}$$

$$(\nu_a^m + \nu_{t,a}(\delta_a, t))\, \partial_z \overline{\mathbf{u}}_{h,a}(\delta_a, t) = C_D\,(u^*) \left\| \overline{\mathbf{u}}_{h,a}(\delta_a, t) - \overline{\mathbf{u}}_{h,o}(\delta_o, t) \right\| \times \left(\overline{\mathbf{u}}_{h,a}(\delta_a, t) - \overline{\mathbf{u}}_{h,o}(\delta_o, t) \right) \tag{2.4b}$$

$$u^* = \sqrt{C_D(u^*)} \left\| \overline{\mathbf{u}}_{h,o}(\delta_a, t) - \overline{\mathbf{u}}_{h,o}(\delta_o, t) \right\| \tag{2.4c}$$

with ν_a^m (resp. ν_o^m) the molecular viscosity in the atmosphere (resp. in the ocean) at and u^* the friction velocity. The coefficient C_D is given by the MO theory. This type of non-linear interface condition based on friction laws is widely studied in fluid-structure interactions theory (see for example [4]).

Simplifying assumptions. To reduce the complexity of the problem, we make the following assumptions:

- In the buffer zone $]\delta_o, \delta_a[$, equations are parameterized. These parameterizations are taken into account in $\sqrt{C_D}$, that depends on u^* itself. However the role of C_D is minor in our context and can be considered as constant. According to [13], we set $C_D = 1.2 \times 10^{-3}$.

- We make an assumption of horizontal homogeneity, justified by the fact that in this study we are focusing on exchanges that are predominantly in the vertical direction. Therefore the terms in $\partial_x \bullet$ and $\partial_y \bullet$ are neglected, with the exception of the horizontal pressure gradient.

- It is assumed that the geostrophic winds/currents, noted $\mathbf{g} := (u^g, v^g)$, are known and are defined by the equilibrium

$$-f u_\alpha^g = \frac{1}{\rho_{0,\alpha}} \partial_y p_\alpha \qquad f v_\alpha^g = \frac{1}{\rho_{0,\alpha}} \partial_x p_\alpha$$

  with $\alpha \in \{o, a\}$ ($\alpha = o$ in the ocean, $\alpha = a$ in the atmosphere). This assumption allows us to decouple the different variables and to consider a condition at the outer edge of the media.

$$\overline{\mathbf{u}}_\alpha(\mathrm{Z}_\alpha^\infty) = \mathbf{g}(\mathrm{Z}_\alpha^\infty) \tag{2.5}$$

Finally our model in each domain can be written as:

$$\partial_t \overline{\mathbf{u}_h} + f\mathbf{e}_z \times \overline{\mathbf{u}_h} - \nu_m \Delta \overline{\mathbf{u}_h} - \partial_z \left(\nu_t(z, \mathbf{u}_h, \dots)\partial_z \overline{\mathbf{u}_h}\right) = f\mathbf{e}_z \times \mathbf{g} \qquad \text{on } (\delta_\alpha, \mathrm{Z}_\alpha^\infty) \quad (2.6)$$

In [11] a rigorous derivation of the Ekman layer equation (2.6) is obtained from multiple scales asymptotic technique. Their derivation indicates that such model is relevant to describe the evolution of atmospheric horizontal velocities at large scale. From now on, the overbar notation $\overline{\cdot}$ and $\cdot_h$ are neglected. Equation (2.6) is considered both in the oceanic and the atmospheric domains.

2.2. Viscosity profiles and reference values for ocean-atmosphere coupling

In this study, we consider viscosity profiles based on the parameterizations of [27] and [12] commonly used in ocean-atmosphere models and adapted to the Ekman layer problem by [18]. In the following we will refer to the corresponding viscosity as K-Profile Parameterization (KPP) viscosity. There exists different turbulent closure schemes with different degrees of complexity [7]. We focus here on a closure scheme based on a so-called zeroth-order closure i.e ν is directly diagnosed from u^* and z and does not involve additional evolution equations as is the case for parameterizations based on the turbulent kinetic energy (TKE) via the Prandtl–Kolmogorov relation. In the stationary case a TKE-based viscosity profile would depend locally on the wind shear. A mathematical analysis of a model close to (2.1)–(2.4) is made in the stationary case by [3] and [24], with a TKE viscosity profile. They prove the existence of a solution and highlight issues that occur to prove the uniqueness of such solution. In this study we have a similar objective but with a KPP viscosity profile and show that the same global uniqueness issues are encountered.

Definition 2.1 (The KPP viscosity profile)**.** The KPP viscosity profile is built to be consistent with the MO theory near the interface and to connect continuously with the constant molecular viscosity outside the boundary layer [20]. $\nu \in \mathrm{C}^1(\delta_\alpha, h_\alpha)$ only depends on u^* and z, and is such that

$$\nu_\alpha(u^*, z) = \nu(u^*_\alpha, z) = \begin{cases} \nu_\alpha^m & \text{on } (h_\alpha, \mathrm{Z}_\alpha^\infty) \\ D_\alpha(u^*, z)\, H\left(1 - \dfrac{z}{h_\alpha}\right) + \nu_\alpha^m & \text{on } (\delta_\alpha, h_\alpha) \end{cases} \quad (2.7)$$

with $D_\alpha \geq 0$ for all $z \in (\delta_\alpha, \mathrm{Z}_\alpha^\infty)$, H is the Heaviside function, h_α depending on u^* and ν_α^m the molecular viscosity. It must also satisfy

- consistency with the MO theory : $D_\alpha(\delta) \approx \kappa u_\alpha^* \delta \gg \nu_\alpha^m$ with κ the Von Karman constant (≈ 0.4) and $\partial_z D_\alpha(\delta) \approx \kappa u_\alpha^*$. with $u_o^* = \lambda u_a^* = \lambda u^*$ [1].

- order of magnitude assumptions : $|\mathrm{Z}_\alpha^\infty| \gg |\delta_\alpha|$ and $D_\alpha(z) \gg \nu_\alpha^m$ for all $z \in (\delta_\alpha, h_\alpha(1-\epsilon))$ for an $\epsilon \ll 1$. Also
  - $\nu_o^m = \lambda \nu_a^m$ with $\lambda = \sqrt{\rho_o/\rho_a} \approx 0.03$
  - $h_\alpha = c_\alpha u^*$ with $c_\alpha = \widetilde{c}_\alpha/|f|$ and $\widetilde{c}_\alpha$ taken from $\widetilde{c}_a = 0.2$ ([1]) and $\widetilde{c}_o = -0.7\lambda$ [12].
  - $|f| \approx 5 \times 10^{-5}\ \mathrm{s}^{-1}$
  - $u^* \in [10^{-3}, 1[\ \mathrm{ms}^{-1}$

  The range of values for u^* is the one considered in [22] and corresponds to "classic" values for this parameter in OA models.

Definition 2.2 (Reference values for the ocean-atmosphere context)**.** We consider a KPP viscosity profile specific to the application of OA coupling as given by [20]:

$$D_\alpha(u_\alpha^*, z) = \kappa u_\alpha^* |z| \left(1 - \frac{z}{h_\alpha}\right)^2 \tag{2.8}$$

To apply our results to the specific OA framework, we choose a number of fixed parameters, which we will call *reference values for ocean-atmosphere coupling* :

- $\mathrm{Z}_a^\infty = 3000\mathrm{m}$ and $\mathrm{Z}_o^\infty = -500\mathrm{m}$

- $\delta_a = 10\mathrm{m}$ and $\delta_o = 1\mathrm{m}$

- $\nu_a^m = 15.6 \times 10^{-6}\mathrm{m}^2\mathrm{s}^{-1}$ and $\nu_o^m = 5 \times 10^{-7}\mathrm{m}^2\mathrm{s}^{-1}$

- u^* is taked such that $\delta_\alpha < h_\alpha < \mathrm{Z}_\alpha^\infty$ for $\alpha \in \{o, a\}$ i.e $u^* \in]3.24 \times 10^{-3}, 9.75 \times 10^{-1}[\ \mathrm{ms}^{-1}$.

2.3. Model problem

Notation 2.3. Oceanic and the atmospheric domains are denoted as Ω_α with

$$\Omega_a =]\delta_a, \mathrm{Z}_a^\infty[\qquad \Omega_o =]\mathrm{Z}_o^\infty, \delta_o[\qquad \text{with } \mathrm{Z}_o^\infty < \delta_0 < 0 < \delta_a < \mathrm{Z}_a^\infty$$

We also define a notation for the "jump" of a variable in the vicinity of the interface:

$$\left[\mathbf{u}\right]_o^a := \mathbf{u}_a(\delta_a, t) - \mathbf{u}_o(\delta_o, t)$$

and $|\mathbf{u}|$ denote the norm of the vector $\mathbf{u}$.

[1]This choice of u^* can be justified by the construction of u^* in the interface buffer zone, and is detailed in [22]

Definition 2.4 (The OA coupled model)**.** The coupled OA model studied here is the one-dimensional problem:

$$\partial_t \mathbf{u}_\alpha + f\mathbf{u}_\alpha^\perp - \partial_z\left(\nu_\alpha(z,u^*(t))\mathbf{u}_\alpha\right) = f\mathbf{g}_\alpha^\perp \qquad \text{on } \Omega_\alpha \times]0,T[\tag{2.9a}$$

$$\mathbf{u}_\alpha(\mathrm{Z}_\alpha^\infty, t) = \mathbf{g}_\alpha(\mathrm{Z}_\alpha^\infty, t) \qquad \text{on }]0,T[\tag{2.9b}$$

$$\mathbf{u}_\alpha(z, t=0) = \mathbf{u}_\alpha^0(z) \qquad \text{on } \Omega_o \cup \Omega_a \tag{2.9c}$$

$$\nu_o\, \partial_z \mathbf{u}_o(\delta_o, t) = \lambda^2 \nu_a\, \partial_z \mathbf{u}_a(\delta_a, t) \qquad \text{on }]0,T[\tag{2.9d}$$

$$\nu_a\, \partial_z \mathbf{u}_a(\delta_a, t) = C_D \left|\left[\mathbf{u}(t)\right]_o^a\right| \left[\mathbf{u}(t)\right]_o^a \qquad \text{on }]0,T[\tag{2.9e}$$

$$u^*(t) = \sqrt{C_D}\left[\mathbf{u}(t)\right]_o^a \qquad \text{on }]0,T[\tag{2.9f}$$

with f, λ and C_D are known constants $\mathbf{g}_\alpha, \mathbf{u}_0$ are source terms and $\mathbf{u} = (u,v)^T$ and the scalar u^* are the unknowns, with notation $\alpha \in \{o,a\}$ and $\mathbf{u}^\perp = (-v,u)^\mathrm{T}$, bold notation denotes 2D vectors.

Non-local aspects come from the interface condition that depends on u^* which itself depends on the jump of the solution around the interface. The parameter u^* thus makes it possible to group together all the non-local aspects of the problem. To study well-posedness of such problem, we will first rewrite it using a fixed point fomulation in Section 3. A more specific study will be made on the spacial case of KPP viscosity profiles in Section 4.1.

3. Well-posedness criteria for general viscosity profile

In this section we study the well-posedness of problem (2.9) for a general viscosity profile $\nu(u^*, z)$. The main strategy to handle the non-locality is to rewrite the model as a fixed point formulation on u^*. This strategy has already been developed by [3] on a close problem in the stationary state and without Coriolis effect. Authors of this paper prove the existence of solutions for TKE viscosity profiles, but the uniqueness of the solution is obtained under some restrictive conditions on the viscosity profile and its variations. The conclusions we reach on the uniqueness of solutions in this section are similar.

Definition 3.1 (Local problem)**.** We define the general problem $\mathcal{P}$ describe by (2.9) and $\mathcal{P}^e$ its stationary version. For a fixed u^* the local problem given by (2.9a)–(2.9e) is called $\mathcal{L}(u^*)$ and $\mathcal{L}^e(u^*)$ for the stationary state.

Notation 3.2. In all this section, we suppose there exist $\nu_\alpha^m > 0$ such that $\nu_\alpha^m \leq \nu(z,u^*)$ for all $(z,u^*) \in \overline{\Omega_\alpha} \times \mathbb{R}_+$, with $\alpha \in \{o,a\}$. We define the scalar product on the domain

$\Omega := \Omega_o \cup \Omega_a$ as:

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathrm{OA}} := \int_{\Omega_a} \mathbf{u}_a \cdot \mathbf{v}_a + \lambda^{-2} \int_{\Omega_o} \mathbf{u}_o \cdot \mathbf{v}_o$$

and we define the associated norm:

$$\|\mathbf{u}\|_{\mathrm{OA}}^2 := \langle \mathbf{u}, \mathbf{u} \rangle_{\mathrm{OA}} = \|\mathbf{u}_a\|_{\mathrm{L}^2(\Omega_a)}^2 + \lambda^{-2} \|\mathbf{u}_o\|_{\mathrm{L}^2(\Omega_o)}^2 .$$

Since $\lambda > 0$, $\|\cdot\|_{\mathrm{OA}}$ is a norm equivalent to $\mathrm{L}^2(\Omega)$. In this paragraph we will use space $\mathbb{V} := \mathbb{V}_a \cup \mathbb{V}_o$ such that $\mathbb{V}_\alpha := \left\{\mathbf{v} \in \mathrm{H}^1(\Omega_\alpha), \mathbf{v}(\mathrm{Z}_\alpha^\infty) = 0\right\}$.

3.1. Existence and uniqueness for the stationary state

In this paragraph we gives the a necessary conditions that ensure the well-posedness of the stationary state problem $\mathcal{P}^e$.

$$f\mathbf{u}_\alpha^\perp - \partial_z \left(\nu_\alpha(u^*)\partial_z \mathbf{u}_\alpha\right) = f\mathbf{g}_\alpha^\perp \qquad \text{on } \Omega_\alpha \tag{3.1a}$$

$$\mathbf{u}_\alpha(\mathrm{Z}_\alpha^\infty) = \mathbf{u}_\alpha^g(\mathrm{Z}_\alpha^\infty) \tag{3.1b}$$

$$\nu_o\, \partial_z \mathbf{u}_o(\delta_o) = \lambda^2 \nu_a\, \partial_z \mathbf{u}_a(\delta_a) \tag{3.1c}$$

$$\nu_a\, \partial_z \mathbf{u}_a(\delta_a) = C_D \left|\left[\mathbf{u}\right]_o^a\right| \left[\mathbf{u}\right]_o^a \tag{3.1d}$$

$$u^* = \sqrt{C_D}\, \left|\left[\mathbf{u}\right]_o^a\right| \tag{3.1e}$$

We first study the well-posedness of the local problem $\mathcal{L}^e(u^*)$ given by (3.1a)–(3.1d) and we then proof the existence of solution of full stationary problem $\mathcal{P}^e$. The uniqueness of solution is given by criteria on the viscosity profile using a Banach fixed point method.

Proposition 3.3 (Well-posedness of the stationary local problem $\mathcal{L}^e(u^*)$)**.** *Suppose that* ν_α *is bounded and* $\mathbf{u}^g, \in \mathrm{H}^1(\Omega)$. *Then we have a unique weak solution* $\mathbf{u} \in \mathrm{H}^1(\Omega)$ *of the stationary local problem* (3.1a)–(3.1d). *Moreover if* $\nu_\alpha \in \mathrm{C}^1\left(\overline{\Omega}\right)$ *and* $\mathbf{u}^g, \in \mathrm{H}^2(\Omega)$ *then* $\mathbf{u} \in \mathrm{H}^2(\Omega)$ *that will be used in Theorem 3.10.*

Proof. Model (3.1a)–(3.1d) involves a non homogeneous external boundary conditions in Z_α^∞. To write the problem in his weak formulation, we use a lifting that will deletes the external boundary condition. To simplify the writing we suppose $\mathbf{u}^g \in \mathrm{H}^2(\Omega)$ and use it as the lifting. The problem we consider in this section is given by :

$$f\widetilde{\mathbf{u}}_\alpha^\perp - \partial_z \left(\nu_\alpha(u^*)\partial_z \widetilde{\mathbf{u}}_\alpha\right) = \partial_z \left(\nu_\alpha(u^*)\partial_z \mathbf{g}_\alpha\right) \qquad \text{on } \Omega_\alpha$$

$$\widetilde{\mathbf{u}}_\alpha(\mathrm{Z}_\alpha^\infty) = 0$$

$$\nu_o\, \left(\partial_z \widetilde{\mathbf{u}}_o + \partial_z \mathbf{g}\right)(\delta_o) = \lambda^2 \nu_a\, \left(\partial_z \widetilde{\mathbf{u}}_a + \partial_z \mathbf{g}_a\right)(\delta_a)$$

$$\nu_a\, \left(\partial_z \widetilde{\mathbf{u}}_a + \partial_z \mathbf{g}_a\right)(\delta_a) = C_D \left|\left[\widetilde{\mathbf{u}} + \mathbf{g}\right]_o^a\right| \left[\widetilde{\mathbf{u}} + \mathbf{g}\right]_o^a$$

We have chosen a non-classical lifting[2] here to gain explicit control on the solution by the source term $\mathbf{u}^g$ and simplify the computation that will be used in Proposition 3.5. Consider the first equation on the domain Ω_a. Multiplying by a test function $\mathbf{v}_a^T \in \mathbb{V}$ and integrating on Ω_a leads to the formulation:

$$\int_{\delta_a}^{Z_a^\infty} \mathbf{v}_a \cdot f\widetilde{\mathbf{u}}_a^\perp + \int_{\delta_a}^{Z_a^\infty} \partial_z \mathbf{v}_a \cdot \nu_a \partial_z \widetilde{\mathbf{u}}_a + \mathbf{v}_a(\delta_a) \cdot \nu_a \left(\partial_z \widetilde{\mathbf{u}}_a(\delta_a) + \partial_z \mathbf{g}_a(\delta_a)\right) = -\int_{\delta_a}^{Z_a^\infty} \nu_a \partial_z \mathbf{v}_a \cdot \partial_z \mathbf{g}_a$$

The same stands for the ocean domain. Then multiplying the formulation from the oceanic part by λ^{-2} and adding the formulation from the atmospheric part, we get the weak formulation of problem :

Find a unique solution $\widetilde{\mathbf{u}} \in \mathbb{V}$ such that

$$f\left\langle \widetilde{\mathbf{u}}^\perp, \mathbf{v}\right\rangle_{\mathrm{OA}} + \left\langle \sqrt{\nu}\partial_z \widetilde{\mathbf{u}}, \sqrt{\nu}\partial_z \mathbf{v}\right\rangle_{\mathrm{OA}} + C_D \left|\left[\widetilde{\mathbf{u}} + \mathbf{g}\right]_o^a\right| \left[\widetilde{\mathbf{u}} + \mathbf{g}\right]_o^a \cdot \left[\mathbf{v}\right]_o^a = -\left\langle \partial_z \mathbf{g}, \nu \partial_z \mathbf{v}\right\rangle_{\mathrm{OA}} \tag{3.2}$$

for all $\mathbf{v} \in \mathbb{V}$ with $\langle \cdot, \cdot \rangle_{\mathrm{OA}}$ is define in Notation 3.2. We define $(\mathbb{V}_m)_{m\geq 0}$ the increasing sequence of finite-dimensional Hilbert subspaces such that $\mathbb{V} = \bigcup_{m\geq 0} \mathbb{V}_m$ Applying the Brouwer's fixed point,we show in Appendix A.1 that there exit $\widetilde{\mathbf{u}}_m \in \mathbb{V}_m$ solution of the weak formulation (3.2). Taking $\mathbf{v} = \widetilde{\mathbf{u}}_m$ in (3.2) and using Young inequality $|x^2 y| \leq 2\epsilon |x|^3/3 + |y|^3/(3\epsilon^2)$ we have :

$$\left\|\sqrt{\nu}\partial_z \widetilde{\mathbf{u}}_m\right\|_{\mathrm{OA}}^2 + \frac{2C_D}{3}\left|\left[\widetilde{\mathbf{u}}_m + \mathbf{g}\right]_o^a\right|^3 \leq \overline{\nu}\left\|\partial_z \mathbf{g}\right\|_{\mathrm{OA}}^2 + \frac{2C_D}{3}\left|\left[\mathbf{g}\right]_o^a\right|^3 \tag{3.3}$$

with $\overline{\nu}_\alpha = \|\nu_\alpha\|_{\mathrm{L}^\infty(\overline{\Omega}\times\mathbb{R}_+)}$. There exist a sub-sequence $(\mathbf{u}_{m_l})_{l\geq 0}$ and $\widetilde{\mathbf{u}} \in \mathbb{V}$, such that

- $(\widetilde{\mathbf{u}}_{m_l})$ converge weakly to $\widetilde{\mathbf{u}}$ in $\mathbb{V}$
- $(\widetilde{\mathbf{u}}_{m_l})$ converge to $\widetilde{\mathbf{u}}$ in $\mathrm{C}^0(\overline{\Omega})$ (Morrey's inequality)

Previous convergence properties gives $\widetilde{\mathbf{u}}_{\alpha,m}(\delta_\alpha)$ converge to $\widetilde{\mathbf{u}}(\delta_\alpha)$. So $\left|\left[\widetilde{\mathbf{u}}_{m_l} + \mathbf{g}\right]_o^a\right| \times \left[\widetilde{\mathbf{u}}_{m_l} + \mathbf{g}\right]_o^a$ converge to the boundary term in $\left|\left[\widetilde{\mathbf{u}} + \mathbf{g}\right]_o^a\right| \left[\widetilde{\mathbf{u}} + \mathbf{g}\right]_o^a$ and $\widetilde{\mathbf{u}}$ is a solution of (3.2). The uniqueness is proved by showing that if we have two solutions $\widetilde{\mathbf{u}}, \widetilde{\mathbf{u}}' \in \mathbb{V}$ of (3.2), and choosing $\mathbf{v} = \widetilde{\mathbf{u}} - \widetilde{\mathbf{u}}'$ they satisfy:

$$\left\|\sqrt{\nu}\partial_z\left(\widetilde{\mathbf{u}} - \widetilde{\mathbf{u}}'\right)\right\|_{\mathrm{OA}} + \left(\left|\left[\widetilde{\mathbf{u}} + \mathbf{g}\right]_o^a\right| \left[\widetilde{\mathbf{u}} + \mathbf{g}\right]_o^a - \left|\left[\widetilde{\mathbf{u}}' + \mathbf{g}\right]_o^a\right| \left[\widetilde{\mathbf{u}}' + \mathbf{g}\right]_o^a\right)\left(\left[\widetilde{\mathbf{u}}\right]_o^a - \left[\widetilde{\mathbf{u}}'\right]_o^a\right) = 0 \tag{3.4}$$

[2]For example choosing the lifting $\mathbf{g}_\alpha = \mathbf{u}_\alpha^g(Z_\alpha^\infty)(z - \delta_\alpha)$ would allow to have $\mathbf{u}^g \in \mathrm{L}^2(\Omega)$

Remark that $\left(\left[\widetilde{\mathbf{u}}\right]_o^a - \left[\widetilde{\mathbf{u}}'\right]_o^a\right) = \left(\left[\widetilde{\mathbf{u}}+\mathbf{g}\right]_o^a - \left[\widetilde{\mathbf{u}}'+\mathbf{g}\right]_o^a\right)$ And since $(\|\mathbf{x}\|\,\mathbf{x} - \|\mathbf{y}\|\,\mathbf{y})\,(\mathbf{x}-\mathbf{y}) \geq 0$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, we have uniqueness in $\mathbb{V}$. To obtain more regularity, since we have $\mathbf{g} \in \mathrm{H}^2(\Omega)$ and $\nu_\alpha \in \mathrm{C}^1(\overline{\Omega}_\alpha)$ and using classical method (see for example [10]), we obtain the weak solution $\widetilde{\mathbf{u}}$ in $\mathrm{H}^2(\Omega)$ and consequently $\mathbf{u} = \widetilde{\mathbf{u}} + \mathbf{g} \in \mathrm{H}^2\,(\Omega)$. □

Theorem 3.4 (Existence of solution for the stationary problem $\mathcal{P}^e$)**.** *Let* $\mathbf{g} \in \mathrm{H}^1(\Omega)$ *and* $\nu_\alpha : (z, u^*)$ *bounded on* $\left(\overline{\Omega}, \mathbb{R}_+\right)$. *Then there exists at least one couple* $(\mathbf{u}^e, u_e^*) \in \mathrm{H}^1(\Omega) \times \mathbb{R}_+$ *solution of* (3.1) *such that*

$$\|\mathbf{u}-\mathbf{g}\|_{\mathrm{H}^1(\Omega)} \leq \|\mathbf{g}\|_{\mathrm{H}^1(\Omega)} \qquad \text{and} \qquad u^* \leq \sqrt{C_D}\left|\left[\mathbf{g}\right]_o^a\right| \tag{3.5}$$

with $\overline{\nu} = \|\nu\|_{\mathrm{L}^\infty}$. *Moreover if* $\mathbf{g} \in \mathrm{H}^2(\Omega)$ *and if* $\nu_\alpha \in \mathrm{C}^1\overline{\Omega} \times \mathbb{R}_+$ *then the couple solution* $(\mathbf{u}^e, u_e^*)$ *is in* $\in \mathrm{H}^2(\Omega) \times \mathbb{R}_+$ *that will be used for Theorem 3.7.*

Proof. Weak formulation of the non local problem $\mathcal{P}^e$ is : Find $(\widetilde{\mathbf{u}}, u^*) \in \mathbb{V} \times \mathbb{R}_+$ such that

$$\begin{aligned} f\left\langle \widetilde{\mathbf{u}}^\perp, \mathbf{v}\right\rangle_{\mathrm{OA}} + \left\langle \sqrt{\nu}\partial_z \widetilde{\mathbf{u}}, \sqrt{\nu}\partial_z \mathbf{v}\right\rangle_{\mathrm{OA}} + C_D\left|\left[\widetilde{\mathbf{u}}+\mathbf{g}\right]_o^a\right|\left[\widetilde{\mathbf{u}}+\mathbf{g}\right]_o^a \cdot \left[\mathbf{v}\right]_o^a + u^* v^* \\ = -\left\langle \partial_z \mathbf{g}, \nu \partial_z \mathbf{v}\right\rangle_{\mathrm{OA}} + \sqrt{C_D}\left|\left[\widetilde{\mathbf{u}}+\mathbf{g}\right]_o^a\right| v^* \end{aligned} \tag{3.6}$$

for all $(\mathbf{v}, v^*) \in \mathbb{V} \times \mathbb{R}_+$. Using the Brouwer's fixed point method, we show there exist a solution of the weak formulation $(\widetilde{\mathbf{u}}_m, u_m^*) \in \mathbb{V}_m \times \mathbb{R}_+$ satisfying (3.5), with $\|\widetilde{\mathbf{u}}_m\|_{\mathrm{H}^1(\Omega)} \leq \|\mathbf{g}\|_{\mathrm{H}^1(\Omega)}$ and $u_m^* \leq \sqrt{C_D}\left|\left[\mathbf{g}\right]_o^a\right|$. See Appendix A.1 for details. Same arguments than in the local problem hold for the convergence of a sub-sequence $\widetilde{\mathbf{u}}_{m,l}$ to $\widetilde{\mathbf{u}} \in \mathbb{V}$ and $u_{m,l}^*$ to u^* such that $(\widetilde{\mathbf{u}}, u^*)$ is solution of (3.6). Then using $\nu_\alpha(u^*) \in \mathrm{C}^1(\overline{\Omega})$ and $\mathbf{g} \in \mathrm{H}^2(\Omega)$ in the weak formulation gives the a solution of the non local problem $\mathcal{P}^e$ in $\mathrm{H}^2(\Omega)$. □

Let us now proceed by analyzing the uniqueness of the solution. The general strategy to ensure uniqueness of the solution of $\mathcal{P}^e$ is to consider the model as a fixed-point formulation for a stationary problem. Using a Banach fixed-point theorem on u^*, we give criteria on ν_α and source terms $\mathbf{g}$ to obtain the uniqueness of a couple solution $(\mathbf{u}, u^*)$. We first introduce results derived from Proposition 3.3 that will be used to apply the fixed-point method in Theorem 3.6.

Proposition 3.5 (Existence of bound global bound on the local solution)**.** *Let define*

$$\mathcal{M}^e := \sup_{u^* \in \mathbb{R}_+} \left\|\sqrt{\nu}\partial_z \mathbf{u}\right\|_{\mathrm{OA}}^2 \qquad \text{with } \mathbf{u} \text{ the unique solution of } \mathcal{L}^e(u^*) \tag{3.7}$$

If $\nu(z, u^*)$ *is bounded and if* $\mathbf{u}^g \in \mathrm{H}^2(\Omega)$ *then we can give an upper bound to* $\mathcal{M}$ *depending on source term:*

$$\mathcal{M}^e \leq \overline{\nu}\,\|\partial_z \mathbf{u}^g\|_{\mathrm{OA}}^2 + \frac{2C_D}{3}\left|\left[\mathbf{u}^g\right]_o^a\right|^3 := \mathcal{G}^e \tag{3.8}$$

with $\overline{\nu} = \max_{\alpha\in\{o,a\}} \|\nu_\alpha\|_{\mathrm{L}^\infty\left(\overline{\Omega}\times\mathbb{R}_+\right)}$. *Moreover we define* $u^*_{\max} = \left(3\sqrt{C_D}\mathcal{G}^e/2\right)^{1/3}$ *then for every solution* $\mathbf{u}$ *of* $\mathcal{L}^e(u^*)$ *we have* $\sqrt{C_D}\left|[\mathbf{u}]_o^a\right| \leq u^*_{\max}$.

Proof. If $\mathbf{u}^g \in \mathrm{H}^2(\Omega)$ then we can take $\mathbf{g} = \mathbf{u}^g$ and $\mathbf{r} = 0$. From the weak formulation (3.2) taking $\mathbf{v} = \widetilde{\mathbf{u}}$ we have $\left\langle \sqrt{\nu}\partial_z\mathbf{u}, \sqrt{\nu}\partial_z\widetilde{\mathbf{u}}\right\rangle_{\mathrm{OA}} + C_D\left|[\mathbf{u}]_o^a\right|[\mathbf{u}]_o^a\cdot[\widetilde{\mathbf{u}}]_o^a = 0$. Rewrite $\widetilde{\mathbf{u}} = \mathbf{u} - \mathbf{g}$ and apply Cauchy–Schwarz and Young's inequality gives.

$$\frac{1}{2}\left\|\sqrt{\nu}\partial_z\mathbf{u}\right\|^2_{\mathrm{OA}} + \frac{C_D}{3}\left|[\mathbf{u}]_o^a\right|^3 \leq \frac{1}{2}\left\|\sqrt{\nu}\partial_z\mathbf{g}\right\|^2_{\mathrm{OA}} + \frac{C_D}{3}\left|[\mathbf{g}]_o^a\right|^3$$

If ν is bounded it imply (3.8). Moreover taking $u^*_{\max}$ such that $\mathcal{G}^e = 2(u^*_{\max})^3/(3\sqrt{C_D})$ we would have $\sqrt{C_D}\left|[\mathbf{u}]_o^a\right| \leq u^*_{\max}$. □

Theorem 3.6 (Uniqueness criteria for the stationary solution)**.** *Let* $\nu_\alpha(z,u^*)$ *be bounded,* $0 < \nu^m_\alpha \leq \nu_\alpha(z,u^*)$ *and source term* $\mathbf{g} \in \mathrm{H}^1(\Omega)$. *Assume viscosity profile* ν_α *satisfy:*

$$\left\|\frac{\nu(z,u^*) - \nu(z,v^*)}{\sqrt{\nu(z,u^*)\nu(z,v^*)}}\right\|_{\mathrm{L}^\infty\left(\overline{\Omega}\right)} \leq \left(C\sqrt{C_D\mathcal{M}^e}\right)^{-1}|u^* - v^*| \qquad \forall\, u^*, v^* \geq 0 \tag{3.9}$$

with

$$C^2 = \max\left(\frac{2\sqrt{2}|\Omega_a|}{\nu^m_a}, \lambda^2\frac{2\sqrt{2}|\Omega_o|}{\nu^m_o}\right).$$

Then there exists a unique solution of $\mathcal{P}^e$ *such that* $(\mathbf{u}^e, u^*_e) \in \mathrm{H}^1(\Omega)\times\mathbb{R}^+$. *The condition* (3.9) *is hard to check in practice, a Lipschitz condition that implies* (3.9) *is :*

$$\|\nu(u^*) - \nu(v^*)\|_{\mathrm{L}^\infty\left(\overline{\Omega}\right)} \leq \frac{\nu^m_\alpha}{C\sqrt{C_D\mathcal{G}^e}}|u^* - v^*| \qquad \forall\, u^*, v^* \geq 0 \tag{3.10}$$

with $\mathcal{G}^e$ *define in* (3.8) *depending on source term. Moreover, for a* $u^*_{\max} = \left(3\sqrt{C_D}\mathcal{G}^e/2\right)^{1/3}$, *we can ensure that* $u^* \leq u^*_{\max}$ *and condition* (3.9) *and* (3.10) *can be restricted to* $u^*, v^* \in [0, u^*_{\max}]$.

Proof. We use a fixed point approach, we define u^*_e the fixed point of the map

$$F^e := \begin{cases} \mathbb{R}_+ \longrightarrow \mathbb{R}_+ & \text{where } \mathbf{u} \text{ is the unique solution of} \\ u^* \longrightarrow \sqrt{C_D}\left|[\mathbf{u}]_o^a\right| & \mathcal{L}^e(u^*) \text{ given by Definition 3.1} \end{cases}$$

From Proposition 3.3 the local problem $\mathcal{L}^e(u^*)$ is well-posed in $\mathrm{H}^1(\Omega)$ for a fixed u^*. We pose $u^*, v^* \geq 0$ and $\mathbf{u}$ the unique solution of $\mathcal{L}^e(u^*)$ and $\mathbf{v}$ the unique solution of $\mathcal{L}^e(v^*)$. We define $\mathbf{w} := \mathbf{u} - \mathbf{v} \in \mathbb{V}$. We want to show that F^e is a contracting mapping, which means $\sqrt{C_D}\left|\left|[\mathbf{u}]_o^a\right| - \left|[\mathbf{v}]_o^a\right|\right| < |u^* - v^*|$. Using inverse triangular inequality we have $\left||[\mathbf{u}]_o^a| - |[\mathbf{v}]_o^a|\right| \leq \left|[\mathbf{u}]_o^a - [\mathbf{v}]_o^a\right| = \left|[\mathbf{w}]_o^a\right|$. Applying trace theorem and Poincare inequality we can bound $|\mathbf{w}(\delta_\alpha)|^2 = \left|2\left\langle\partial_z \mathbf{w}, \mathbf{w}\right\rangle_{\Omega_\alpha}\right| \leq \frac{\sqrt{2}|\Omega_\alpha|}{\nu_\alpha^m}\left\|\sqrt{\nu_\alpha}\partial_z \mathbf{w}\right\|_2^2$, and finally

$$\left||[\mathbf{u}]_o^a| - |[\mathbf{v}]_o^a|\right|^2 \leq C^2\left\|\sqrt{\nu(u^*)}\partial_z \mathbf{w}\right\|_{\mathrm{OA}}^2 \qquad \text{with } C^2 = \max\left(\frac{2\sqrt{2}|\Omega_a|}{\nu_a^m}, \lambda^2\frac{2\sqrt{2}|\Omega_o|}{\nu_o^m}\right) \tag{3.11}$$

It now remains to control $\left\|\sqrt{\nu(u^*)}\partial_z \mathbf{w}\right\|_{\mathrm{OA}}^2$. Subtracting equations from $\mathcal{L}^e(u^*)$ and $\mathcal{L}^e(v^*)$ gives

$$\begin{aligned}
f\mathbf{w}_\alpha^\perp - \partial_z\left(\nu_\alpha(u^*)\partial_z \mathbf{w}_\alpha + (\nu(u^*) - \nu(u^*))\partial_z \mathbf{v}_\alpha\right) &= 0 \qquad \text{in } \Omega_\alpha \\
\mathbf{w}(\mathrm{Z}_\alpha^\infty) &= 0 \\
\nu_o(u^*)\partial_z \mathbf{w}_o(\delta_o) + \left(\nu_o(u^*) - \nu_o(v^*)\right)\partial_z \mathbf{v}_o(\delta_o) &= \lambda^2\nu_a\partial_z \mathbf{w}_a(\delta_a) \\
&\quad + \lambda^2\left(\nu_a(u^*) - \nu_a(v^*)\right)\partial_z \mathbf{v}_a(\delta_a) \\
\nu(u^*)\partial_z \mathbf{w}_a(\delta_a) + \left(\nu_a(u^*) - \nu_a(v^*)\right)\partial_z \mathbf{v}_a(\delta_a) &= C_D\left(|[\mathbf{u}]_o^a|\,[\mathbf{u}]_o^a - |[\mathbf{v}]_o^a|\,[\mathbf{v}]_o^a\right)
\end{aligned}$$

and energy estimate yields:

$$\begin{aligned}
\left\|\sqrt{\nu(u^*)}\partial_z \mathbf{w}\right\|_{\mathrm{OA}}^2 &+ C_D\left(|[\mathbf{u}]_o^a|\,[\mathbf{u}]_o^a - |[\mathbf{v}]_o^a|\,[\mathbf{v}]_o^a\right)\cdot[\mathbf{w}]_o^a \\
&= -\left\langle\partial_z \mathbf{w}_\alpha, \left(\nu_\alpha(u^*) - \nu_\alpha(v^*)\right)\partial_z \mathbf{v}_\alpha\right\rangle_\alpha .
\end{aligned}$$

Second term in the l.h.s is positive and r.h.s is bounded using :

$$\begin{aligned}
\left|\left\langle\partial_z \mathbf{w}_\alpha, \left(\nu_\alpha(u^*) - \nu_\alpha(v^*)\right)\partial_z \mathbf{v}_\alpha\right\rangle_{\mathrm{OA}}\right| &= \left|\left\langle\sqrt{\nu(u^*)}\partial_z \mathbf{w}, \eta\sqrt{\nu(v^*)}\partial_z \mathbf{v}\right\rangle_{\mathrm{OA}}\right| \\
&\leq \frac{1}{2}\left\|\sqrt{\nu(u^*)}\partial_z \mathbf{w}\right\|_{\mathrm{OA}}^2 + \frac{1}{2}\max_{\alpha\in\{o,a\}}\|\eta_\alpha\|_{\mathrm{L}^\infty(\Omega_\alpha)}^2\left\|\sqrt{\nu(v^*)}\partial_z \mathbf{v}\right\|_{\mathrm{OA}}^2
\end{aligned}$$

with $\eta_\alpha = \frac{\nu_\alpha(u^*) - \nu_\alpha(v^*)}{\sqrt{\nu_\alpha(u^*)\nu_\alpha(v^*)}}$ for $\alpha \in \{a, o\}$. From Proposition 3.5 we have $\left\|\sqrt{\nu(v^*)}\partial_z \mathbf{v}\right\|_{\mathrm{OA}}^2 \leq \mathcal{M}^e$. If ν satisfy (3.9) there exits a $L \in \left[0, 2\left(\sqrt{C_D}\sqrt{\mathcal{M}^e}C\right)^{-1}\right]$ such that $\eta_\alpha \leq L|u^* - v^*|$ and using the definition of $\mathcal{M}^e$ we have :

$$C_D C^2\left\|\sqrt{\nu(u^*)}\partial_z \mathbf{w}\right\|_{\mathrm{OA}}^2 \leq C_D C^2 L^2\mathcal{M}^e|u^* - v^*|^2$$

and it conclude the proof. The condition (3.10) is obtained using $\|\eta_\alpha(z)\|_{\mathrm{L}^\infty(\Omega_\alpha)} \leq \|\nu_\alpha(z, u^*) - \nu_\alpha(z, v^*)\|_{\mathrm{L}^\infty(\Omega_\alpha)}/\nu_\alpha^m$. Also, taking $u_{\max}^*$ such that $3\sqrt{C_D}G^e \leq (u_{\max}^*)^3$ then by Proposition 3.5 we have $\sqrt{C_D}\left|[\mathbf{u}]_o^a\right| \leq u_{\max}^*$. □

In sum, the condition for the uniqueness of solutions depends on the bounds and the variations of ν and H^1 norm on $\mathbf{u}$. Generally speaking, if the product between $\|\partial_{u^*}\nu\|_{\mathrm{L}^\infty}$ and $\|\partial_z \mathbf{u}\|_{\mathrm{L}^\infty}$ is small compared to the minimum of ν, we can ensure the uniqueness of the stationary solution $\mathbf{u}$. The control of the H^1 norm on $\mathbf{u}$ by the source terms is given by Proposition 3.5. Uniqueness criteria linking the viscosity profile and the source term are given in Theorem 3.6. These criteria state that the product between $\|\partial_{u^*}\nu\|_{\mathrm{L}^\infty}$ and norm on source term $\mathbf{g}$ must be small compared to the minimum of ν. A practical application of this condition will be made in paragraph Section 4.2, where we analyze the well-posedness of viscosity profiles used in the OA framework.

3.2. **Existence and uniqueness of the non-stationary state**

In this paragraph we study the well-posedness on the non local problem $\mathcal{P}$ given by system (2.9). We first study the existence of non stationary solutions in a neighborhood of the stationary state using the method proposed by [7]. A similar case have been studied by [8] to the local coupled problem with nonlinear interface conditions but with constant viscosity. In Theorem 3.7, we extend these results to the non-local coupled problem and the non locality is treated using a Banach fixed point method.

Theorem 3.7 (Existence in the neighborhood of a stationary state)**.** *If there exists a solution* $(\mathbf{u}^e, u_e^*) \in \mathrm{H}^2(\Omega) \times \mathbb{R}_+$ *to the stationary state* $\mathcal{P}^e$ *and if the viscosity* $\nu(z, u_e^*)$ *satisfies*

$$\sqrt{C_D} \left\| \frac{\partial_{u^*}\nu(u_e^*)}{\sqrt{\nu(u_e^*)}} \partial_z \mathbf{u}^e \right\|_{\mathrm{OA}}^2 < 2u_e^* \tag{3.12}$$

Then there exists $(\mathbf{u}, u^*) \in \mathrm{L}^2\left(0, T; \mathrm{H}^2(\Omega)\right) \cap \mathrm{H}^1\left(0, T; \mathrm{L}^2(\Omega)\right) \times \mathrm{L}^2(0, T)$ *a solution of* $\mathcal{P}$ *in an neighborhood of* $(\mathbf{u}^e, u_e^*)$*. Moreover, since* $(\mathbf{u}^e, u_e^*)$ *is solution of the stationary problem, a condition that would imply* (3.12) *is :*

$$\left\| \frac{\partial_{u^*}\nu(u_e^*)}{\nu(u_e^*)} \right\|_{\mathrm{OA}}^2 < \frac{2u_e^*}{\sqrt{C_D}\mathcal{G}^e} \tag{3.13}$$

with $\mathcal{G}^e$ *depend on* $\mathbf{g}$ *is defined in* (3.8).

Proof. The proof is based on the method developed by [7], where the authors consider TKE-type viscosity profiles which bring non-linearity in the main equations. Step are recall in Appendix A.1. After linearisation around the stationary state, we have to prove that the linear model

$$\begin{cases} \partial_t \mathbf{v}_\alpha + f\mathbf{v}_\alpha^\perp - \partial_z\left(\nu_{\alpha,e}\partial_z \mathbf{v}\right) = \partial_z\left(v^*\nu'_{\alpha,e}\partial_z \mathbf{u}^e\right) + \mathbf{\Phi} & \text{on } \Omega_\alpha \times]0,T[\\ \mathbf{v}_\alpha(z,t=0) = \mathbf{\Phi}_0 & \text{on } \Omega_o \cup \Omega_a \\ \mathbf{v}_\alpha(\mathrm{Z}_\alpha^\infty, t) = \mathbf{\Phi}_\infty & \text{on }]0,T[\\ \nu_{o,e}\,\partial_z \mathbf{v}_o(\delta_o,t) + v^*(t)\nu'_{o,e}\partial_z \mathbf{u}_o^e(\delta_o) & \\ \qquad = \lambda^2\nu_{a,e}\,\partial_z\mathbf{v}_a(\delta_a) + \lambda^2 v^*(t)\nu'_{a,e}\partial_z\mathbf{u}_a^e(\delta_a) + \mathbf{\Phi}_{I,1} & \text{on }]0,T[\\ \nu_{a,e}\,\partial_z\mathbf{v}_a(\delta_a,t) + v^*(t)\nu'_{a,e}\partial_z\mathbf{u}_a^e(\delta_a) & \\ \qquad = \sqrt{C_D}u_e^*\left(\left(\left[\mathbf{v}\right]_o^a\cdot\mathbf{e}_\tau\right)\mathbf{e}_\tau + \left[\mathbf{v}\right]_o^a\right) + \mathbf{\Phi}_{I,2} & \text{on }]0,T[\end{cases} \tag{3.14a}$$

$$v^* = \sqrt{C_D}\left[\mathbf{v}\right]_o^a\cdot\mathbf{e}_\tau + \mathbf{\Phi}_* \tag{3.14b}$$

with notation $\nu_{\alpha,e} = \nu_\alpha(z,u_e^*)$, $\mathbf{e}_\tau = \left[\mathbf{u}^e\right]_o^a / \left|\left[\mathbf{u}^e\right]_o^a\right|$ and $\nu'_{\alpha,e} = \partial_{u^*}\nu(z,u_e^*)$, is well-posed on $\mathcal{X} := \mathrm{L}^2\left(0,T;\mathrm{H}^2(\Omega)\right) \cap \mathrm{H}^1\left(0,T;\mathrm{L}^2(\Omega)\right) \times \mathrm{L}^2(0,T)$ for all $\mathbf{Y} = (\mathbf{\Phi},\mathbf{\Phi}_0,\mathbf{\Phi}_\infty,\mathbf{\Phi}_{I,1},\mathbf{\Phi}_{I,2},\mathbf{\Phi}_*) \in \mathcal{Y} := \mathrm{L}^2\left(0,T,\mathrm{L}^2(\Omega)\right), \times \mathrm{H}^1(\Omega) \times \mathrm{L}^2(0,T)^4$. First note that for any $\mathbf{Y} \in \mathcal{Y}$, if $\nu(z,u_e^*) \in \mathrm{C}^1(\overline{\Omega}\times\mathbb{R})$ and $\mathbf{u}^e \in \mathrm{H}^2(\Omega)$ there exist a unique solution of (3.14a) in $\mathcal{X}$. For a given stationary solution $(u_e^*,\mathbf{u}^e)$ of $\mathcal{P}^e$, we define for every $\mathbf{Y} \in \mathcal{Y}$:

$$F_{\mathbf{Y}} := \begin{cases} \mathrm{L}^2(0,T) \longrightarrow \mathrm{L}^2(0,T) \\ \quad v^* \longrightarrow \sqrt{C_D}\left[\mathbf{v}\right]_o^a\cdot\mathbf{e}_\tau + \mathbf{\Phi}_* \quad \text{where } \mathbf{v} \text{ is the unique solution of (3.14a)} \end{cases}$$

We want to show that $F_{\mathbf{Y}}$ is a contracting mapping for all $\mathbf{Y} \in \mathcal{Y}$. Consider $u^*, v^* \in \mathrm{L}^2(0,T)$ and $\mathbf{u},\mathbf{v}$ the corresponding solution of (3.14a), we pose $w^* = u^* - v^*$ and $\mathbf{w} = \mathbf{u} - \mathbf{v}$. Then, by linearity, $(w^*,\mathbf{w})$ is solution of (3.14a) with $\mathbf{Y} = 0$. Thus an a priori estimate can be computed:

$$\partial_t \left\|\mathbf{w}\right\|_{\mathrm{OA}}^2 + \left\|\sqrt{\nu}\partial_z\mathbf{w}\right\|_{\mathrm{OA}}^2 + \sqrt{C_D}u_e^*\left|\left[\mathbf{w}\right]_o^a\right|^2\left(\cos(\theta_\tau)^2 + 1\right) = -w^*\left\langle\partial_z\mathbf{w},\nu'\partial_z\mathbf{u}^e\right\rangle_{\mathrm{OA}}$$

Using the following bound :

$$\left|w^*\left\langle\partial_z\mathbf{w},\nu'\partial_z\mathbf{u}^e\right\rangle_{\mathrm{OA}}\right| \leq \left\|\sqrt{\nu}\partial_z\mathbf{w}\right\|_{\mathrm{OA}}^2 + \frac{|w^*|^2}{4}\left\|\frac{\nu'_\alpha}{\sqrt{\nu_\alpha}}\partial_z\mathbf{u}^e\right\|_{\mathrm{OA}}^2$$

and integrating on $(0,t)$ gives the apriori estimate:

$$\left\|\mathbf{w}\right\|_{\mathrm{OA}}^2(t) + \left(\cos(\theta_\tau)^2 + \frac{1}{2}\right)\int_0^t \sqrt{C_D}u_e^*\left|\left[\mathbf{w}\right]_o^a\right|^2 \leq \frac{1}{4}\left\|\frac{\nu'_\alpha}{\sqrt{\nu_\alpha}}\partial_z\mathbf{u}^e\right\|_{\mathrm{OA}}^2\int_0^t |w^*|^2$$

Therefore, if $\nu(u_e^*)$ and $(\mathbf{u}^e,u_e^*)$ satisfy (3.12) then $F_{\mathbf{Y}}$ is a contracting mapping in $\mathrm{L}^2(0,T)$ for all $\mathbf{Y} \in \mathcal{Y}$. $\square$

The existence of a non-stationary solution, as shown by criterion (3.12), requires the product between $\partial_{u^*} \nu/\sqrt{\nu}$ and $\partial_z \mathbf{u}$ to be small enough. Product of the same nature has been found in Theorem 3.6 when analyzing the uniqueness of a stationary solution. It highlighting the key role of this product when we use the fixed point method.

Similarly, the general non-local problem $\mathcal{P}$ is considered as a fixed point problem on u^* considering local problem $\mathcal{L}(u^*)$ defined in Definition 3.1. The main difficulty encountered in the time dependency of ν. First remark that, by definition, the time regularity of ν is the time regularity of u^*. Then, to apply the Banach's fixed point as in Theorems 3.6 and 3.7, the time regularity on ν has to be chosen such that it ensures the same regularity for the trace of the solution $\mathbf{u}_\alpha(\delta_\alpha, t)$. Moreover, as it appears in the stationary state, the criterion will depend on the $\mathrm{L}^\infty(0,T;\mathbb{V})$ norm of $\mathbf{u}$ a solution of the local problem $\mathcal{L}(u^*)$. We show in Proposition 3.8 that taking $\nu \in \mathrm{C}^1\left([0,T]\times\overline{\Omega}\right)$ guarantees $\mathbf{u} \in \mathrm{L}^\infty(0,T;\mathbb{V})$ and $\mathbf{u}_\alpha(\delta) \in \mathrm{C}^1[0,T]$. This regularity is then used in the Banach's fixed point method in Theorem 3.10 to give a criterion on the well-posedness.

Proposition 3.8 (Well-posedness of the local system $\mathcal{L}(u^*)$)**.** *We suppose a given* $\nu_\alpha \in \mathrm{C}^1\left([0,T]\times\overline{\Omega_\alpha}\right)$*. Then, for* $\mathbf{g} \in \mathrm{H}^2\left(0,T;\mathrm{H}^1(\Omega)\right)$ *and* $\mathbf{u}^0 \in \mathrm{H}^2(\Omega)$ *we have a unique solution* $\mathbf{u} \in \mathrm{L}^2(0,T;\mathrm{H}^2(\Omega)) \cap \mathrm{H}^1(0,T;\mathbb{V})$ *of the local problem* $\mathcal{L}(u^*)$ *given by* (2.9a)–(2.9e)*. Especially we have* $\mathbf{u} \in \mathrm{L}^\infty(0,T;\mathbb{V})$ *and* $\partial_t \left|\left[\mathbf{u}\right]_o^a\right| \in \mathrm{C}^0[0,T]$ *that will be useful for Theorem 3.10.*

Proof. Model (2.9) involves a non homogeneous external boundary conditions in Z_α^∞. To write the problem in his weak formulation, we use a lifting that will deletes the external boundary condition. Taking $\widetilde{\mathbf{u}} = \mathbf{u} - \mathbf{g}$, the problem we consider in this section is given by:

$$
\begin{aligned}
\partial_t\widetilde{\mathbf{u}}_\alpha + f\widetilde{\mathbf{u}}_\alpha^\perp - \partial_z(\nu_\alpha(z,u^*(t))\partial_z\widetilde{\mathbf{u}}_\alpha) &= -\partial_t\mathbf{g}_\alpha + \partial_z(\nu_\alpha(z,u^*(t))\partial_z\mathbf{g}_\alpha) && \text{on } \Omega_\alpha\times]0,T[\\
\widetilde{\mathbf{u}}_\alpha(\mathrm{Z}_\alpha^\infty,t) &= 0 && \text{on }]0,T[\\
\widetilde{\mathbf{u}}_\alpha(z,t=0) &= \widetilde{\mathbf{u}}_0 && \text{on } \Omega_o\cup\Omega_a\\
\nu_o\,\partial_z\widetilde{\mathbf{u}}_o(\delta_o,t) + \nu_o\,\partial_z\mathbf{g}_o(\delta_o,t) &= \lambda^2\nu_a\,\partial_z\widetilde{\mathbf{u}}_a(\delta_a,t) + \nu_a\,\partial_z\mathbf{g}_a && \text{on }]0,T[\\
\nu_a\,\partial_z\widetilde{\mathbf{u}}_a(\delta_a,t) + \nu_a\,\partial_z\widetilde{\mathbf{u}}_a^g &= C_D\left|\left[\widetilde{\mathbf{u}}+\mathbf{g}\right]_o^a\right|\left[\widetilde{\mathbf{u}}+\mathbf{g}\right]_o^a && \text{on }]0,T[
\end{aligned}
$$

The weak formulation of the local problem $\mathcal{L}(u^*)$ is : for a given $u^* \in \mathrm{H}^1(0,T)$ find $\widetilde{\mathbf{u}}$ solution of

$$
\begin{aligned}
\langle\partial_t\widetilde{\mathbf{u}},\mathbf{v}\rangle_{\mathrm{OA}} + f\left\langle\widetilde{\mathbf{u}}^\perp,\mathbf{v}\right\rangle_{\mathrm{OA}} + \langle\nu(u^*)\partial_z\widetilde{\mathbf{u}},\partial_z\mathbf{v}\rangle_{\mathrm{OA}} + C_D\left|\left[\mathbf{u}\right]_o^a\right|\left[\mathbf{u}\right]_o^a\cdot\left[\mathbf{v}\right]_o^a \qquad &(3.16)\\
= -\langle\nu\partial_z\mathbf{g},\partial_z\mathbf{v}\rangle_{\mathrm{OA}} - \langle\partial_t\mathbf{g},\widetilde{\mathbf{v}}\rangle_{\mathrm{OA}} \qquad &(3.17)
\end{aligned}
$$

for all $\mathbf{v} \in \mathbb{V}$ with $\mathbf{u} = \widetilde{\mathbf{u}} + \mathbf{g}$. Proof is given in Appendix A.2. □

For any solution of $\mathcal{L}(u^*)$, the norm in $\mathrm{L}^\infty(0,T;\mathbb{V})$ would depends on $\partial_t u^*$. In the application of the fixed point method, u^* will be obtained from the trace of a previous solution on the local problem, so that, we need to control the norm of the trace of the solution for all u^* in a given space. We prove in Proposition 3.9 that this would be possible by controlling $\|\partial_t u^*\|_{\mathrm{L}^2}$ and we give a necessary condition to apply the fixed point method used in Theorem 3.10.

Proposition 3.9 (Existence of bound global bound in $\mathrm{L}^\infty(0,T;\mathbb{V})$)**.** *We define* $\mathbb{V}_B^* := \{u^* \in \mathrm{C}^1(0,T), \|\partial_t u^*\|_{\mathrm{L}^2(0,T)} \leq B\}$ *for given* $B \geq 0$. *Suppose* $\mathbf{g} \in \mathrm{H}^2(0,T;\mathrm{H}^1(\Omega))$ *and* $\mathbf{u}^0 \in \mathrm{H}^2(\Omega)$ *a solution a solution of the stationary state* $\mathcal{P}^e$. *We define*

$$\mathcal{M} := \sup_{u^*\in\mathbb{V}_B^*} \sup_{t\in]0,T[} \left\|\sqrt{\nu}\partial_z \mathbf{u}\right\|_{\mathrm{OA}}^2 \qquad \textit{with } \mathbf{u} \textit{ the unique solution of } \mathcal{L}(u^*) \tag{3.18}$$

If $\nu \in \mathrm{C}^1(\overline{\Omega}\times\mathbb{R})$, *there exist a constant D depending on T and bound on ν such that*

$$\mathcal{M} \leq D(T,\|\nu\|_{\mathrm{L}^\infty})\left(\|\mathbf{g}\|_{\mathrm{H}^2(0,T;\mathrm{H}^1(\Omega))}^2 + \left\|\mathbf{u}^0\right\|_{\mathrm{H}^2(\Omega))}^2\right) e^{N^2B^2} := \mathcal{G} \tag{3.19}$$

with $N = \max_{\alpha\in\{o,a\}} \|\partial_{u^*}\nu_\alpha/\nu_\alpha\|_{\mathrm{L}^\infty(\overline{\Omega}\times\mathbb{R})}^2$. *Moreover, there exist condition linking source term* $\mathbf{g}$, $\mathbf{u}^0$, *viscosity profile* ν, *B and T such that for all* $u^* \in \mathbb{V}_B^*$, *the unique solution* $\mathbf{u}$ *of* $\mathcal{L}(u^*)$ *verify* $\sqrt{C_D}\left[\mathbf{u}\right]_o^a \in \mathbb{V}_B^*$. *This condition can take the form as a upper bound on norm on* $\mathbf{g}$ *and* $\mathbf{u}^0$ *like*

$$D(T,\|\nu\|_{\mathrm{L}^\infty})\left(\|\mathbf{g}\|_{\mathrm{H}^2(0,T;\mathrm{H}^1(\Omega))}^2 + \left\|\mathbf{u}^0\right\|_{\mathrm{H}^2(\Omega))}^2\right) \leq \frac{B^2}{1+2B^2N^2e^{B^2N^2}} \tag{3.20}$$

with $C^2 = \max_{\alpha\in\{o,a\}} \frac{\sqrt{2}|\Omega_\alpha|}{\nu_\alpha^m}$.

Proof. Suppose $u^* \in \mathbb{V}_B^*$ for given B. From weak formulation (3.16), taking $\mathbf{v} = \mathbf{u} - \mathbf{g}$ it gives

$$\begin{aligned}\|\partial_t\mathbf{u}\|_{\mathrm{OA}}^2 + f\,\langle\widetilde{\mathbf{u}}_m,\partial_t\widetilde{\mathbf{u}}_m\rangle_{\mathrm{OA}} + \left\langle\sqrt{\nu}\partial_z\mathbf{u},\partial_{z,t}\mathbf{u}\right\rangle_{\mathrm{OA}} + C_D\left|\left[\mathbf{u}\right]_o^a\right|\left[\mathbf{u}\right]_o^a\cdot\left[\partial_t\mathbf{u}\right]_o^a \\ = C_D\left|\left[\mathbf{u}\right]_o^a\right|\left[\mathbf{u}_m\right]_o^a\cdot\left[\partial_t\mathbf{g}\right]_o^a + \left\langle\nu\partial_{z,t}\mathbf{g},\partial_z\mathbf{u}\right\rangle_{\mathrm{OA}} + \langle\partial_t\mathbf{g},\partial_t\mathbf{u}\rangle_{\mathrm{OA}}\end{aligned}$$

Integrating in time and using same strategy than in Appendix A.2 to obtain (A.3), would gives

$$\left\|\sqrt{\nu}\partial_z\mathbf{u}\right\|_{\mathrm{OA}}^2 \leq B_1 + \int_0^t \|\mu\|_{\mathrm{L}^\infty(\Omega)}^2 \left\|\sqrt{\nu}\partial_z\mathbf{u}\right\|_{\mathrm{OA}}^2$$

Using Gronwall theorem we have $\left\|\sqrt{\nu}\partial_z\widetilde{\mathbf{u}}\right\|_{\mathrm{OA}}^2 \leq B_1\exp\left(\int_0^t \|\mu\|_{\mathrm{L}^\infty(\Omega)}^2\right)$. By hypothesis we have $\int_0^t \|\mu\|_{\mathrm{L}^\infty(\Omega)}^2 \leq B^2N^2$ and then $\mathcal{M} \leq B_1 e^{N^2B^2}$. Now using equation (A.4) on $\mathbf{u}$

gives :

$$\int_0^t \left\|\sqrt{\nu}\partial_{z,t}\widetilde{\mathbf{u}}\right\|_{\mathrm{OA}}^2 \leq B_2 + \int_0^t \|\mu\|_{\mathrm{L}^\infty(\Omega)}^2 \left(\left\|\sqrt{\nu}\partial_z\mathbf{u}\right\|_{\mathrm{OA}}^2\right) \leq B_2 + N^2B^2\mathcal{M}$$

with B_2 depending on $\mathbf{g}$ and $\mathbf{u}^0$ and bound on ν. Then we can bound $\left|\left[\partial_t\mathbf{u}\right]_o^a\right|^2 \leq 2\left|\left[\partial_t\widetilde{\mathbf{u}}\right]_o^a\right|^2 + 2\left|\left[\partial_t\mathbf{g}\right]_o^a\right|^2 \leq 2C^2\left\|\sqrt{\nu}\partial_z\widetilde{\mathbf{u}}\right\|_{\mathrm{OA}}^2 + 2\left|\left[\partial_t\mathbf{g}\right]_o^a\right|^2$ that gives

$$\int_0^t C_D\left|\left[\partial_t\mathbf{u}\right]_o^a\right|^2 \leq B_2 + 2\int_0^t \|\mu\|_{\mathrm{L}^\infty(\Omega)}^2 \left(\left\|\sqrt{\nu}\partial_z\mathbf{u}\right\|_{\mathrm{OA}}^2\right) \leq B_3 + 2N^2B^2\mathcal{M}$$

with $B_3 = 2C^2B_2 + 2\left|\left[\partial_t\mathbf{g}\right]_o^a\right|^2$.Taking the expression of B_2 given in (A.4), there exist $D_i > 0$ depending on T and bound on ν such that $B_i \leq D_i\left(\|\mathbf{g}\|_{\mathrm{H}^2(0,T;\mathrm{H}^1(\Omega))}^2 + \left\|\mathbf{u}^0\right\|_{\mathrm{H}^2(\Omega))}^2\right)$ for $i \in \{1,2,3\}$ and

$$B_3 + 2N^2B^2\mathcal{M} \leq D_3\left(\|\mathbf{g}\|_{\mathrm{H}^2(0,T;\mathrm{H}^1(\Omega))}^2 + \left\|\mathbf{u}^0\right\|_{\mathrm{H}^2(\Omega)}^2\right)\left(1 + 2B^2N^2e^{N^2B^2}\right)$$

Taking $D = \max_{i\in\{1,2\}} D_i$, if we have (3.20) then $\left(B_3 + 2N^2B^2\left(\mathcal{M}^2\right)\right) \leq B^2$ and $\sqrt{C_D\int_0^t\left|\left[\partial_t\mathbf{u}\right]_o^a\right|^2} \leq B$. □

Remark the condition (3.20) is not optimal and can be improved for specific $\mathbf{g}$. Generally speaking the product between the variation on ν and global norm on the source term have to be small enough. This condition is of same nature as uniqueness criteria previously established in the stationary state. Consequently, the condition (3.20) is an additional criterion to the well-posedness criteria that we will be establish in Proposition 3.9. We now have all the tools necessary to apply the fixed point method on the non-stationary and non-local problem.

Theorem 3.10 (Existence and uniqueness of the non local problem)**.** *Let* $\mathbf{g} \in \mathrm{H}^2(0,T;\mathrm{H}^1(\Omega)) \cap \mathrm{L}^\infty(0,T;\mathrm{H}^2(\Omega))$ *and* $\mathbf{u}^0 \in \mathrm{H}^2(\Omega)$ *a solution of the stationary state* $\mathcal{P}^e$ *with source term* $\mathbf{g}(t=0)$*. We suppose* $\nu(z,u^*) \in \mathrm{C}^1\left(\overline{\Omega}\times\mathbb{R}_+\right)$*. We suppose* $\mathbf{g}$*, and* ν *such that there exist* $B \geq 0$ *satisfying condition* (3.20)*. If* ν_α *also satisfy*

$$\left\|\frac{\nu(z,a)-\nu(z,b)}{\sqrt{\nu(z,a)\nu(z,b)}}\right\|_{\mathrm{L}^\infty\left(\overline{\Omega}\right)} \leq \left(C\sqrt{C_D\mathcal{M}}\right)^{-1}|a-b| \qquad \forall\, a,b \geq 0 \tag{3.21}$$

with $\mathcal{M}$ *define in Proposition 3.9, then there exist a unique solution of* $\mathcal{P}$ *given by* (2.9) *such that* $(\mathbf{u},u^*) \in \mathrm{L}^2(0,T;\mathrm{H}^2(\Omega)) \cap \mathrm{H}^1(0,T;\mathrm{H}^1(\Omega)\times\mathrm{C}^1[0,T])$*, and* $\|\partial_t u^*\|_{\mathrm{L}^2(0,T)} \leq B$*. A Lipschitz condition depending on source term that implies* (3.21) *is*

$$\|\nu(z,a)-\nu(z,b)\|_{\mathrm{L}^\infty\left(\overline{\Omega}\right)} \leq \frac{\nu_\alpha^m}{C\sqrt{C_D\mathcal{G}}}|a-b| \qquad \forall\, a,b \geq 0 \tag{3.22}$$

with $\mathcal{G}$ given by (3.19). *The condition* (3.20) *imply that $\mathcal{G}$ is decreasing when* $\|\mathbf{g}\|_{\mathrm{H}^2(0,T;\mathrm{H}^1(\Omega))\cap\mathrm{H}^1(0,T;\mathrm{H}^2(\Omega))}$, *and B decrease.*

Proof. The proof follow the same step than in the stationary state. We consider the map

$$F := \begin{cases} \mathbb{V}_B^* \longrightarrow \mathbb{V}_B^* & \text{where } \mathbf{u} \text{ is the solution of} \\ u^* \longrightarrow \sqrt{C_D}\left|\left[\mathbf{u}\right]_o^a\right| & \mathcal{L}(u^*) \text{ given by Definition 3.1} \end{cases}$$

According to Theorem 3.4, there exist solution of the stationary problem $(\mathbf{u}^0, u_e^*) \in \mathrm{H}^2(\Omega)\times\mathbb{R}_+$with source term $\mathbf{g}^0 \in \mathrm{H}^2(\Omega)$. If $\mathbf{g} \in \mathrm{H}^1(0,T;\mathrm{H}^2(\Omega))$ then $\mathbf{g}(t=0) \in \mathrm{H}^2(\Omega))$ and $(\mathbf{u}^0, u_e^*)$ satisfy the compatibility solution. According to Proposition 3.9 if $u^* \in \mathbb{V}_{M,B}^*$ with $u^*(0) = u_e^*$ then the unique solution $\mathbf{u}$ of $\mathcal{L}(u^*)$ is such that $\sqrt{C_D}\left|\left[\mathbf{u}+\mathbf{g}\right]_o^a\right| \in \mathbb{V}_{M,B}^*$. We want to show that F is a contracting mapping and the proof follow the step for the stationary state. We pose $u^*, v^* \in \mathrm{C}^1[0,T]$ and $\mathbf{u}$ the unique solution of $\mathcal{P}^e(u^*)$ and $\mathbf{v}$ the unique solution of $\mathcal{P}^e(v^*)$. We define $\mathbf{w} := \mathbf{u}-\mathbf{v}$. We want to find a condition to ensure that F is a contracting mapping, which means $\sqrt{C_D}\left\|\left|\left[\mathbf{u}\right]_o^a\right| - \left|\left[\mathbf{v}\right]_o^a\right|\right\| < |u^*-v^*|$. Subtracting equation on $\mathbf{u}$ and $\mathbf{v}$ gives:

$$\begin{aligned} \|\mathbf{u}-\mathbf{v}\|_{\mathrm{OA}}^2(t) + \int_0^t \left\|\sqrt{\nu(u^*)}\partial_z(\mathbf{u}-\mathbf{v})\right\|_{\mathrm{OA}}^2 + \int_0^t A_1 \\ = -\int_0^t \left\langle \sqrt{\nu(u^*)}(\mathbf{u}-\mathbf{v}), \eta\sqrt{\nu(v^*)}\mathbf{v}\right\rangle_{\mathrm{OA}} \end{aligned}$$

$$A_1 = C_D\left(\left[\widetilde{\mathbf{u}}\right]_o^a - \left[\widetilde{\mathbf{v}}\right]_o^a\right)\cdot\left(\left|\left[\widetilde{\mathbf{u}}\right]_o^a\right|\left[\widetilde{\mathbf{u}}\right]_o^a - \left|\left[\widetilde{\mathbf{v}}\right]_o^a\right|\left[\widetilde{\mathbf{v}}\right]_o^a\right)$$

with $\eta = (\nu(u^*)-\nu(v^*))/\sqrt{\nu(u^*)\nu(v^*)}$. Remark that $C_D|\left|\left[\widetilde{\mathbf{u}}\right]_o^a\right| - \left|\left[\widetilde{\mathbf{v}}\right]_o^a\right||^3 \le A_1$. Using (3.11) we have $\int_0^t \left\|\left|\left[\widetilde{\mathbf{u}}\right]_o^a\right| - \left|\left[\widetilde{\mathbf{v}}\right]_o^a\right|\right\|^2 \le C^2\int_0^t \left\|\sqrt{\nu(u^*)}\partial_z(\mathbf{u}-\mathbf{v})\right\|_{\mathrm{OA}}^2$ and then

$$\int_0^t \left\|\sqrt{\nu(u^*)}\partial_z(\mathbf{u}-\mathbf{v})\right\|_{\mathrm{OA}}^2 \le \mathcal{M}\max_{\alpha\in\{o,a\}}\int_0^t \|\eta_\alpha\|_{\mathrm{L}^\infty(\Omega_\alpha)}^2$$

Injecting (3.21) it conclude the proof. Criterion (3.22) is obtained using $\nu_\alpha(z,u^*) \ge \nu_\alpha^m$ for all z, u^* and definition on $\mathcal{G}$. □

As the Banach's fixed point method is applied in the same way in the stationary and the non stationary state, well-posedness criteria for the non stationary state contain the same ingredients as uniqueness criteria for the stationary state. Generally speaking, the product between $\|\partial_{u^*}\nu\|_{\mathrm{L}^\infty}$ and norm on source term $\mathbf{g}$ have to be small compare to the minimum of ν. In the non-stationary state, an additional condition of the same nature have to be verified to apply the fixed-point method. Note that uniqueness criteria given Propositions 3.3 and 3.8 are sufficient criteria. Then for viscosity profile with large variations these conditions can be too restrictive and it not properly answer the question of the uniqueness of solution. We will see in Section 4.1 that this is the case with viscosity profile specific to the OA framework and will give another method to answer the question of the uniqueness of solution in the stationary state for this specific viscosity profile.

4. **Well-posedness for KPP viscosty profiles**

4.1. **No Uniqueness for OA order of magnitude**

In Section 3.1, the existence of a solution in the stationary state has been proved for a general parametrized viscosity profile $\nu(z, u^*)$. However the uniqueness of the solution is proved only for very smooth profiles of ν. In this section, we discuss the uniqueness of a solution for viscosity profiles used in realistic OA coupled models as given by Definition 2.1. We will see in Section 4.2 that the criteria given in the general case cannot ensure the uniqueness of solution for parameters with orders of magnitude of OA coupling models. To answer the question of uniqueness, we give a necessary and sufficient condition in the stationary state in Section 4.3. This condition can be computed by solving the stationary problem analytically. This is done in Section 4.4 for approximated KPP viscosity profiles. We will see that there is non uniqueness of solutions because the size of the buffer zone (delimited by $[\delta_o, \delta_a]$) is fixed. Throughout this section, we define values of u^* that are considered as physically acceptable in the context of OA coupling.

Definition 4.1 (Interval of u^*)**.** Since u^* is related to the thickness of the turbulent layer $|h_\alpha|$, values of u^* are limited by their physical definitions and by the constraint $h_\alpha \in (\delta_\alpha, \mathrm{Z}_\alpha^\infty)$. Under these constraints, we define I^* as the interval of u^* corresponding to physically acceptable values

$$I^* := \left]\max\left(\frac{\delta_a}{c_a}, \frac{\delta_o}{c_o}\right), \min\left(\frac{\mathrm{Z}_a^\infty}{c_a}, \frac{\mathrm{Z}_o^\infty}{c_o}\right)\right[= \left]u^*_{\min}, u^*_{\max}\right[\tag{4.1}$$

Depending on the reference values given by Definition 2.2, this gives $I^* \approx]0.001, 0.7[$.

4.2. **Non-uniqueness for OA order of magnitude**

In Section 3 we have established criteria to ensure the well-posedness of the non-local problem in a general framework. In this paragraph, we wish to apply these results considering parameters of the order of magnitude of realistic OA coupling. As a first approximation, we suppose $\mathbf{g}$ to be constant in each domain.

Remark 4.2 (Bound on the viscosity profile). Before applying the different criteria, note that KPP vicosites are not bounded from their parametrisation, ($\|\nu\|_{\mathrm{L}^\infty(\Omega)}$ can tend to infinity when u^* goes to 0 or infinity) but since we assume a bounded interval of $u^* \in I^*$, we can define ν outside on I^* as constant and ensure that ν and $\partial_u^* \nu$ are bounded.

Here we want to test the existence criterion given in Theorem 3.4 and the uniqueness criterion given in Theorem 3.6 for a specific KPP viscosity profile. It depend on quantity $\mathcal{G}^e$ which can be express in terms of the source terms $\mathbf{g}$.

Proposition 4.3 (Criteria of uniqueness for constant source terms)**.** *We suppose $\mathbf{g}_\alpha$ to be constant and $u^*_{\max}$ is given by Definition 4.1.*

$$\mathcal{M}^e \leq \frac{1}{3} C_D \left| \left[\mathbf{u}^{\mathbf{g}} \right]_o^a \right|^3 := \mathcal{G}^e \tag{4.2}$$

If

$$\mathcal{G}^e \leq \min \left(\frac{(u^*_{\max})^3}{3\sqrt{C_D}} ; \min_{\alpha \in \{o,a\}} \frac{(\nu^m_\alpha)^2}{C_D C^2 \, \|\partial_{u^*} \nu_\alpha\|^2_{\mathrm{L}^\infty(\Omega)\times[0,u^*_{\max}]}} \right) \tag{4.3}$$

then we have a unique solution of the non local problem $\mathcal{P}^e$ satisfying $u^* \leq u^*_{\max}$.

Proof. Applying weak formulation (3.2) to $\partial_z \mathbf{g} = 0$ would provides (4.2). Then the first minimizer in (4.3) ensure existence of solution such that $u^* \leq u^*_{\max}$, moreover it would provide $\sqrt{C_D} \left| [\mathbf{u}]_o^a \right| \leq u^*_{\max}$. The second minimizer would imply the uniqueness criterion (3.10) for $u^* \leq u^*_{\max}$. □

Existence of solution in OA framework. According to Theorem 3.4, there exist (at least one) solution of the stationary problem such that $u^* \leq \sqrt{C_D} \left| [\mathbf{g}]_o^a \right|$. Considering the order of magnitude of realistic OA coupling we have $\left| [\mathbf{g}]_o^a \right| \approx 10\,\mathrm{ms}^{-1}$, that gives $u^* \leq 0.3\,\mathrm{ms}^{-1}$ which is consistent the value expected by Definition 4.1. We can now test the existence of non-stationary solution in the neighbour of stationary solution given by the Theorem 3.7. Considering KPP viscosity profiles given by Definition 2.2, we compute numerically $N = \|\partial_u \nu_\alpha(u^*)/(\nu_\alpha(u^*))\|_{\mathbf{L}^\infty(\Omega)}$ which turns out to be of the order of, at least, 10^4 (see Figure 4.2). The criterion (3.13) would be equivalent to $\left| [\mathbf{g}]_o^a \right| \leq \left(6u^*_e / (C_D^{3/2} N^2) \right)^{1/3} \approx 0.1 (u^*_e)^{1/3} \in \left] 4 \times 10^{-5}, 7 \times 10^{-2} \right[\, \mathrm{ms}^{-1}$ that would be much more smaller than the value expected in the OA framework.

Application of uniqueness criteria in OA framework. From the Definition 4.1 we have $u^*_{\max} \approx 0.7$. The first bound in (4.3) enquires $\left| [\mathbf{g}]_o^a \right| \leq u^*_{\max}/\sqrt{C_D}$ that is consistent with the OA order of magnitude. However, the second bound depend on variation on ν. Taking O'Brien viscosity profile given by (2.2) we have $\|\partial_u \nu_\alpha\|_\infty = 2\kappa |c_\alpha| u^*/3\sqrt{3}$, its values for the atmosphere domain are draw in Figure 4.2 and turn out to be of the order of, at least, 10^5. For example in the atmosphere domain, criterion (2.2) can be rewrite as

$$\max_{\alpha \in \{o,a\}} \|\partial_u \nu_\alpha(u^*)\|_{\mathbf{L}^\infty(\Omega)} \frac{\sqrt{C_D \mathcal{G}^e} C}{\nu^m_\alpha} \approx 3.6 \times 10^5 \left| [\mathbf{g}]_o^a \right|^{3/2} u^* \leq 1. \tag{4.4}$$

Then order of magnitude for $\left|\left[\mathbf{g}\right]_o^a\right|$ of for u^* such that we have uniqueness of solution have to be much more smaller than the value expected in the OA framework.

Non-uniqueness in the OA framework. In general, the viscosity profile conditions required to guarantee solution uniqueness are very restrictive and cannot be expected in the context of OA coupling. Here the criteria are sufficient but not necessary, and the bounds on the $\mathbb{V}$ norms to illustrate the purpose are taken broadly as a function of the source term. Therefore the fixed-point method as proposed here does not seem to be well adapted to fully answer the question of the well-posedness of the coupled problem. Either the bounds are too large, or the problem arises from the nature of the viscosity profiles with the constraints given by the model (see Section 2.2). To answer this question, we solve the non-local problem in the stationary state for some specific viscosity profiles.

4.3. A necessary and sufficient condition for the stationary problem

Section 3 was considering the coupled system as a fixed-point problem, and the obtained well-posedness criteria are too restrictive to be used in the OA context. Let assume that, in the stationary case, the problem can be solved with some given viscosity profiles that allow us to compute explicit solution of (2.9), and thus lead to a necessary and sufficient well-posedness criterion for existence and uniqueness of solutions. We seek to compute the explicit stationary solution considering constant in time and space geostrophic currents. Considering KPP viscosities leads us to consider values of $u^* \in I^*$ for which viscosity profiles have a physical meaning. Note that we will rewrite the interface condition as $\nu_a(\delta_a, u^*)\partial_z \mathbf{u}_a(\delta, t) = \sqrt{C_D} u^* \left[\mathbf{u}\right]_o^a$. The obtained system is equivalent to (2.9) and the following results will also hold to the original system (2.9) since the criterion given here is a necessary and sufficient condition for the well-posedness.

In the following, we will use the following change of variable to reduce the system (2.9) to a complex-variable system. Let us pose

$$\Phi_\alpha := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} u_\alpha \\ v_\alpha \end{pmatrix} =: \begin{pmatrix} \phi_{\alpha,1} \\ \phi_{\alpha,2} \end{pmatrix} \qquad \text{and} \qquad \Phi_\alpha^g := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \mathbf{g}_\alpha$$

Using this change of variable in (2.9) gives

$$\begin{pmatrix} if & 0 \\ 0 & -if \end{pmatrix} \Phi_\alpha - \partial_z \left(\nu_\alpha(z, u^*)\, \partial_z \Phi_\alpha\right) = \begin{pmatrix} if & 0 \\ 0 & -if \end{pmatrix} \Phi_\alpha^g \qquad \text{on } \Omega_\alpha, \tag{4.5a}$$

$$\Phi_\alpha(\mathrm{Z}_\alpha^\infty) = \Phi_\alpha^g(\mathrm{Z}_\alpha^\infty) \tag{4.5b}$$

$$\nu_o\, \partial_z \Phi_o(\delta_o, t) = \lambda^2 \nu_a\, \Phi_a(\delta_a, t) \tag{4.5c}$$

$$\nu_a^\delta\,\partial_z\Phi_a(\delta_a,t) = u^*\sqrt{C_D}\,(\Phi_a(\delta_a,t) - \Phi_o(\delta_o,t)) \tag{4.5d}$$

$$u^* = \sqrt{C_D}\,\left\|\Phi_a^\delta - \Phi_o^\delta\right\| \tag{4.5e}$$

with $\alpha \in \{o,a\}$ and $\|\Phi\| = \sqrt{\overline{\Phi}^T\Phi}$. Note that the change of variable conserves the norm i.e. $\|\mathbf{u}\|^2 = \|\Phi\|^2$. We have a pair of equation systems that can be translated as $\phi_{\alpha,1} = \overline{\phi_{\alpha,2}}$ and $\|\mathbf{u}\| = \|\Phi\| = \sqrt{2}|\phi_{\alpha,1}| = \sqrt{2}|\phi_{\alpha,2}|$. Finally system (4.5) can be split in two independent system, one on $\phi_{\alpha,1}$ and one $\phi_{\alpha,2}$. Considering $f > 0$ and $f < 0$ to taking into account the two systems of equations on $\phi_{\alpha,1}$ and $\phi_{\alpha,2}$ we obtain the system

$$if\phi_\alpha - \partial_z(\nu_\alpha(u^*,z)\partial_z\phi_\alpha(z)) = if\phi_\alpha^g \qquad \text{on } (\delta_\alpha, \mathrm{Z}_\alpha^\infty) \tag{4.6a}$$

$$\phi_\alpha(\mathrm{Z}_\alpha^\infty) = \phi_\alpha^g(\mathrm{Z}_\alpha^\infty) \tag{4.6b}$$

$$\nu_o(u^*,\delta_o)\,\partial_z\phi_o(\delta_o) = \lambda^2\nu_a(u^*,\delta_a)\,\partial_z\phi_a(\delta_a) \tag{4.6c}$$

$$\nu_a(u^*,\delta_a)\,\partial_z\phi_a(\delta_a) = \sqrt{C_D}u^*\,(\phi_a(\delta_a) - \phi_o(\delta_o)) \tag{4.6d}$$

$$u^* = \sqrt{2C_D}\,|\phi_a(\delta_a) - \phi_o(\delta_o)| \tag{4.6e}$$

Definition 4.4 (Definition of $\mathcal{S}_\alpha$)**.** For a given $\nu(u^*,z)$, solutions of

$$\begin{cases} if\phi_\alpha - \partial_z(\nu_\alpha(u^*,z)\partial_z\phi_\alpha(z)) = 0 \quad \text{on } (\delta_\alpha, \mathrm{Z}_\alpha^\infty) \\ \phi_\alpha(\mathrm{Z}_\alpha^\infty) = 0 \end{cases} \tag{4.7}$$

are given by $A_\alpha(u^*)\psi_\alpha(z,u^*)$, where A_α does not depend on z and ψ is composed by generating solution of the first line of (4.7). We define

$$\mathcal{S}_\alpha(u^*) := \frac{\psi(\delta_\alpha,u^*)}{\nu_\alpha(\delta_\alpha,u^*)\partial_z\psi(\delta_\alpha,u^*)} \tag{4.8}$$

Some profiles of $\mathcal{S}_\alpha$ have been given in [26] for different types of viscosity profiles.

Theorem 4.5 (Well-posedness criterion)**.** *Let define*

$$F : \begin{cases} I^* \times [2\pi[\longrightarrow \mathbb{C} \\ (u,\theta) \longrightarrow \dfrac{ue^{i\theta}}{\sqrt{2C_D}}\left(1 - u\sqrt{C_D}\mathcal{S}_a(u) + \lambda^2 u\sqrt{C_D}\mathcal{S}_o(u)\right) \end{cases} \tag{4.9}$$

Then problem (4.6) *is well-posed on $I \subset I^*$ if and only if F is injective on $I \times [2\pi[$. This amounts to determine the largest interval $I \subset I^*$ such that*

$$\left|\partial_u\left(\left|u\sqrt{C_D^{-1}} - u^2\mathcal{S}_a(u) + \lambda^2u^2\mathcal{S}_o(u)\right|\right)\right| > 0 \qquad \forall\, u \in I \subset I^* \tag{4.10}$$

If ϕ_α^g are constant we have the equality:

$$\phi_a^g - \phi_o^g = F(u^*, \arg(\phi_a(\delta_a) - \phi_o(\delta_o))) \tag{4.11}$$

Proof. Let pose ψ_α^p a particular solution of the main equation (4.6a)–(4.6b) and solutions in each domain can be written as $\phi_\alpha(z) = A_\alpha\psi_\alpha(z) + \psi_\alpha^p(z)$. Using interface conditions and definitions of $\mathcal{S}_\alpha$, we have:

$$\begin{aligned}\phi_a(\delta_a) - \phi_o(\delta_o) &= \psi_a^p(\delta_a) - \psi_o^p(\delta_o) + A_a\psi_a(\delta_a)(1 - \lambda^2\mathcal{S}_a^{-1}\mathcal{S}_o)\\ &= \psi_a^p(\delta_a) - \psi_o^p(\delta_o) + u^*\sqrt{C_D}\left(\phi_a^\delta - \phi_o^\delta\right)\mathcal{S}_a(1 - \lambda^2\mathcal{S}_a^{-1}\mathcal{S}_o)\\ \psi_a^p(\delta_a) - \psi_o^p(\delta_o) &= (\phi_a^\delta - \phi_o^\delta)(1 - \sqrt{C_D}u^*\mathcal{S}_a + \lambda^2\sqrt{C_D}u^*\mathcal{S}_o) \qquad (4.12)\end{aligned}$$

Let $u^* = \sqrt{2C_D}|\phi_a^\delta - \phi_o^\delta|$ and $\theta = \arg(\phi_a^\delta - \phi_o^\delta)$, which gives (4.11). Define G the application from $\mathbb{R}^2$ into $\mathbb{R}^2$ as $G : (u^*, \theta) \to (|F(u^*, \theta)|, \arg(F(u^*, \theta)))$. Then G is a diffeomorphism from $I \subset I^* \times [0, 2\pi[$ to $F(I, [0, 2\pi[)$ if $\det(J_G(u^*, \theta)) \neq 0$ for all $(u^*, \theta) \in I \times [0, 2\pi[$ with J_G the Jacobian matrix of G. Then

$$J_G(u, \theta) = \begin{pmatrix} \partial_u|F(u,\theta)| & \partial_\theta|F(u,\theta)| \\ \partial_u(\arg(F(u,\theta))) & \partial_\theta(\arg(F(u,\theta))) \end{pmatrix} = \begin{pmatrix} \partial_u|F(u,\theta)| & 0 \\ \partial_u(\arg(F(u,\theta))) & 1 \end{pmatrix}$$

Then $\det(J_G)(u, \theta) = \partial_u|F(u, \theta)|$. If ϕ_α^g is constant, it is a particular solution of (4.6a)–(4.6b) and we obtain (4.11) from (4.12). □

Proposition 4.6 (Particular case with no Coriolis force ($f = 0$))**.** *Let consider the simpler case where the Coriolis force is neglected ($f = 0$) in the system* (4.6)*. Then* (4.6a)–(4.6b) *can be solved*[3] *for every ν with*

$$\mathcal{S}_\alpha = -\mathcal{X}_\alpha \int_{\Omega_\alpha} \nu_\alpha^{-1}\mathrm{d}z \qquad \textit{where } \mathcal{X}_a = 1 \textit{ and } \mathcal{X}_o = -1 \qquad (4.13)$$

and the well-posedness criterion (4.10) *is an analytic expression on u^*. Under the hypothesis of KPP viscosity profiles (Definition 2.1), we find that the problem* (4.6) *is well-posed on the interval*

$$I = \left]u^*_{\min}, \min\left(u^*_{\max}, \frac{2}{3}w^*\right)\right[\qquad (4.14)$$

with $w^* \approx \frac{\mathrm{Z}_a^\infty - \lambda \mathrm{Z}_o^\infty}{c_a - \lambda c_o}$, and in the context of OA coupling $w^* \approx \frac{\mathrm{Z}_a^\infty}{c_a} = u^*_{\max}$.

Proof. If $f = 0$ the resolution of (4.6a)–(4.6b) gives $\nu_\alpha(z, u^*)\partial_z\mathbf{u}_\alpha = C_\alpha$ with C_α a constant in z. The interface condition can be written as $\lambda^{-2}C_o = C_a$. Then $\widetilde{\phi}_\alpha(z) = -\mathcal{X}_\alpha \int_{(z, \mathrm{Z}_\alpha^\infty)} C_\alpha\nu_\alpha^{-1}(z')dz'$ where $\mathcal{X}_a = 1$ and $\mathcal{X}_o = -1$, which leads to (4.13). Under the

[3] Remark that we consider $f = 0$ in the equation, but h_α still depending on f and we use a realistic value of f for h_α definition

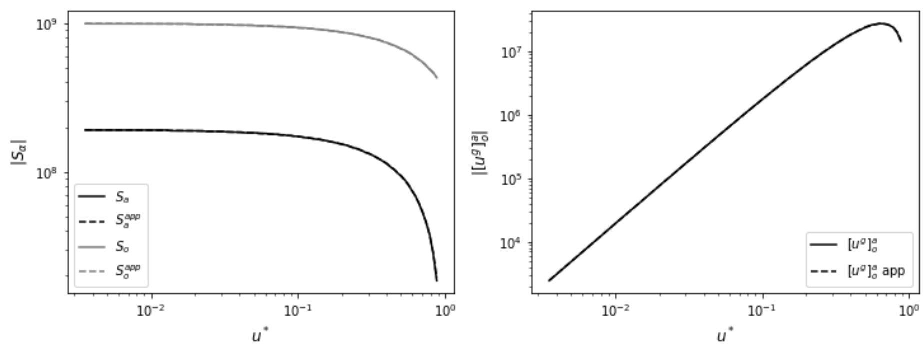

FIGURE 4.1. Left panel: computed $\mathcal{S}_\alpha$ (plain lines) for O'Brien KPP viscosity profile (Definition 2.2) and their approximations given by (4.13) (dashed lines). Curves are superimposed. Right panel: corresponding profile of $\left|\left[\phi^g\right]_o^a\right| = |F(u^*)|$

hypotheses of Definition 2.1, and especially $\nu_\alpha^m \ll D_\alpha(u^*, z)$ for all $z \in (\delta_\alpha, h_\alpha(1-\epsilon))$, and using $\nu_o^m = \lambda \nu_a^m$, we can approximate:

$$\mathcal{S}_\alpha \approx -\mathcal{X}_a \frac{|\mathrm{Z}_\alpha^\infty - h_\alpha|}{\nu_\alpha^m} \qquad -\mathcal{S}_a + \lambda^2 \mathcal{S}_o \approx \frac{\mathrm{Z}_a^\infty - \lambda \mathrm{Z}_o^\infty - (h_a - \lambda h_o)}{\nu_a^m}$$

$$\partial_u \left(-\mathcal{S}_a + \lambda^2 \mathcal{S}_o\right) \approx -\frac{c_a - \lambda c_o}{\nu_a^m}$$

Note $\mathcal{S} = -\mathcal{S}_a + \lambda^2 \mathcal{S}_o > 0$ and condition (4.10) can be rewritten as

$$\left|\sqrt{C_D}^{-1} + \partial_u \left((u^*)^2 \mathcal{S}(u^*)\right)\right| > 0.$$

Then the problem is well-posed if $\partial_u \left((u^*)^2 \mathcal{S}(u^*)\right) > 0$ or $\left|\partial_u \left((u^*)^2 \mathcal{S}(u^*)\right)\right| < \sqrt{C_D}^{-1}$. Since $\nu_a^m \ll u^* \sqrt{C_D}$, we can consider only $\partial_u \left((u^*)^2 \mathcal{S}(u^*)\right) > 0$. The expression of $\mathcal{S}$ gives us the root $\frac{2}{3} w^*$. □

This result illustrates the fact that there is non-uniqueness of solutions, and this non-uniqueness is inherent in the OA order of magnitude (with a global shape such that $\nu_\alpha \gg \nu_\alpha^m$ on $(\delta_\alpha, h_\alpha)$). For viscosity profiles given by Definition 2.1, Figure 4.1 presents the computed $\mathcal{S}_\alpha$, the profile of $\left|\left[\phi^g\right]_o^a\right| = |F(u^*)|$ for $u^* \in I^*$, and their respective approximations given by (4.13).

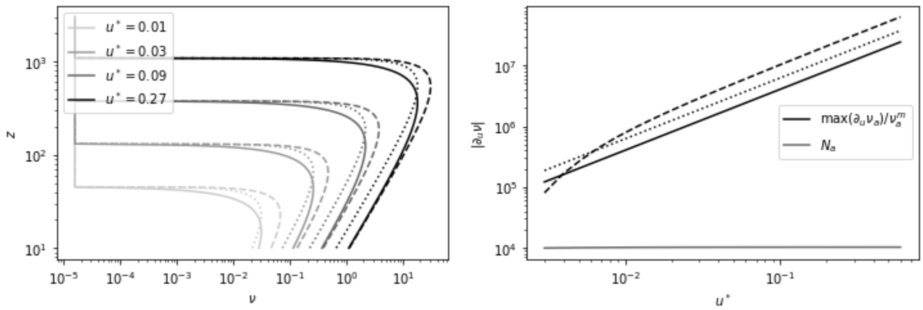

FIGURE 4.2. Left panel: O'Brien KPP viscotity profile ν_a^{Obrien} given by (2.8) (plain line), $\nu_{1,a}$ (dashed line) and $\nu_{2,a}$ (dotted line) given by Definition 4.7 for different values of u^*. Right panel: the corresponding values of $\|\partial_u \nu\|_{L^\infty(\Omega_a)} / \nu_a^m$ (black lines) and $N_a(u^*) = \|\partial_u \nu(u^*)/\nu(u^*)\|_{L^\infty(\Omega_a)}$ (grey line) for ν_a^{Obrien} only.

4.4. **Application to KPP viscosity profiles**

The resolution of (4.6a) cannot be explicitly computed for a general viscosity profile if $f \neq 0$. Here we propose to consider KPP viscosity profiles as given by Definition 2.2. The parameterization of ν in the context of OA coupling is generally given by a third-order polynomial in z to suit the hypotheses of the KPP viscosity profile. In order to simplify the resolution of (4.6a), we propose to approximate the viscosity by a second-order polynomial. This approximation can be justified by the concave profile of the viscosity profiles in the turbulent zone $(\delta_\alpha, h_\alpha)$. However, a third-order viscosity profile being necessary to ensure C^1 regularity of ν in h_α, we will assume here that considering only C^0 regularity on ν to compute $\mathcal{S}$ is not significant (indeed ν is not C^1 but $\partial_u \nu$ is bounded). To remain consistent with the OA context, we build a viscosity profile that verifies the KPP viscosity profile hypotheses (see Definition 2.1) except the C^1 continuity in h_α.

Definition 4.7 (Definition and hypotheses on approximation of ν)**.** To guarantee that the assumptions of Definition 2.1 are satisfied, and to ensure a relevant approximation, we will make the approximation that $\nu_\alpha^{\text{approx}}(h_\alpha) = \nu_\alpha(h_\alpha) = \nu_\alpha^{\text{approx}}(z = 0) = \nu_\alpha(z = 0) = \nu_\alpha^m$. One degree of freedom remains to approximate ν, which must guarantee the concave shape of ν^{approx}, and thus we add the condition $\nu^{\text{approx}}(\delta_\alpha) \approx \nu(\delta_\alpha)$. We denote this last degree of freedom A_α such that:

$$\nu_\alpha^{\text{approx}} = \begin{cases} \nu_\alpha = \nu_\alpha^m & \text{on } (h_\alpha, \mathrm{Z}_\alpha^\infty) \\ D_\alpha(z)\,\text{Heaviside}\left(1 - \dfrac{z}{h_\alpha}\right) + \nu_\alpha^m \in P^2(\mathbb{R}) & \text{on } (0, h_\alpha) \end{cases} \tag{4.15a}$$

$$\text{with } D_\alpha(z) = K_\alpha z(z - h_\alpha) > 0,\ D_\alpha(\delta) \text{ in the order of magnitude of } \kappa u^* \delta_\alpha \tag{4.15b}$$

Two examples of such approximations of ν^{Obrien} (given by (2.8)) are proposed here:

(1) to ensure $\nu_{1,\alpha}(\delta_\alpha) = \nu_\alpha^{\text{Obrien}}(\delta_\alpha)$ for all u^*, we take $K_{1,\alpha} = -\kappa u^* |\delta_\alpha - h_\alpha| / h_\alpha^2$.

(2) to ensure $\left\|\nu_{2,\alpha}(u^*)\right\|_{\mathrm{L}^\infty(\Omega)} = \left\|\nu^{\text{Obrien}}(u^*)\right\|_{\mathrm{L}^\infty(\Omega)}$ for all u^* we take $K_{2,\alpha} = -16\kappa/(27|c_\alpha|)$

According to Figure 4.2, $\left\|\partial_u \nu_{1,\alpha}\right\|_{\mathrm{L}^\infty(\Omega)}$ and $\left\|\partial_u \nu_{2,\alpha}\right\|_{\mathrm{L}^\infty(\Omega)}$ is of the same order of magnitude than $\left\|\partial_u \nu_\alpha^{\text{Obrien}}\right\|_{\mathrm{L}^\infty(\Omega)}$.

Proposition 4.8 (Resolution of the equations on each subdomain)**.** *Supposing $f \neq 0$, then for KPP viscosity profiles given by Definition 4.7 with $D_\alpha = K_\alpha z(z - h_\alpha)$ we have :*

$$\nu_\alpha(\delta_\alpha) S_\alpha \approx \delta_\alpha \left(\ln\left(\frac{\delta_\alpha}{h_\alpha}\right) - (1 + \xi_\alpha)^{-1} \right) \qquad \xi = -\frac{1}{2}\left(1 + \sqrt{1 + if/K_\alpha}\right) \tag{4.16}$$

This result is an asymptotic approximation of an exact computation of $\mathcal{S}$ given in Appendix A.3 by (A.5).

Proof. Using approximation (4.15), we can now solve the system (4.7). For each u^*, we separate each domain into two parts:

- the free zones, where interface turbulence has no impact, $(z \in (h_\alpha, \mathrm{Z}_\alpha^\infty))$:

  $$\widehat{\varphi}_\alpha(z) = B_{\alpha,l}\left(e^{\varsigma_\alpha z} - e^{\varsigma_\alpha(2\mathrm{Z}_\alpha^\infty - z)}\right) \qquad \partial_z \widehat{\varphi}\alpha = \varsigma_\alpha B_{\alpha,l}\left(e^{\varsigma_\alpha z} + e^{\varsigma_\alpha(2\mathrm{Z}_\alpha^\infty - z)}\right)$$

  with $\varsigma_\alpha = \sqrt{\frac{if}{\nu_\alpha^m}}$

- the turbulent zone, close to the interface $(z \in (\delta_\alpha, h_\alpha^*))$:

  $$\varphi_\alpha(z) = C_{\alpha,t} P_\alpha(r_\alpha(z)) + B_{\alpha,t} P_\alpha(-r_\alpha(z))$$

  $$\partial_z \varphi_\alpha(z) = \frac{\mathcal{X}_\alpha \xi_\alpha(\xi_\alpha + 1)}{h_\alpha \sqrt{1 + 4\mu_\alpha}} \left(C_{\alpha,t} G_\alpha(r_\alpha(z)) - B_{\alpha,t} G_\alpha(-r_\alpha(z))\right)$$

  with $\mathcal{X}_o = -1, \mathcal{X}_a = 1, r_\alpha(z) = \mathcal{X}_\alpha(1 - 2z/h_\alpha)/\sqrt{1 + 4\mu_\alpha}, \mu_\alpha = \nu_\alpha^m/(|K_\alpha| h_\alpha^2)$. P and G are Legendre polynomials which can also be written in terms of hypergeometric function $P_\alpha(r(z)) = {}_2F_1(\xi_\alpha + 1, -\xi_\alpha, 1, (1 - r(z))/2)$, and

$G_\alpha(\eta(z)) = {}_2F_1(\xi_\alpha + 2, 1 - \xi_\alpha, 2, (1 - r(z))/2)$. For a justification of this result see [26].

Results in Proposition 4.8 are then obtained by considering a solution in $\mathrm{C}^1(\Omega_\alpha)$ and using some asymptotic expansions mostly based on the fact that $\mu \ll 1$. See Appendix A.3 for the full computation to obtain (A.5) and (4.16). □

Theorem 4.9 (Non-uniqueness for KPP viscosity profiles)**.** *The well-posedness criterion* (4.10) *applied to the problem* (4.6)*, gives the following well-posedness properties for every viscosity profiles as given in Definition 4.7:*

- *There exists a unique solution to the problem* (4.6) *on* $]2u^*_{\min}, u^*_{\max}]$ *if*

$$1 - \frac{\kappa}{2\sqrt{C_D}} \le \frac{u^* \partial_u \nu_\alpha(\delta_\alpha, u^*)}{\nu_\alpha(\delta_\alpha, u^*)} \le 2 + \frac{\kappa}{2\sqrt{C_D}} \left| \ln\left(\frac{\delta_\alpha}{H_\alpha}\right) + 1 \right|^{-1} \tag{4.18}$$

  In OA order of magnitude we have $\frac{\kappa}{2\sqrt{C_D}} \approx 5.8$ *and* $\frac{\kappa}{2\sqrt{C_D}} \left| \ln\left(\frac{\delta_\alpha}{H_\alpha}\right)\right|^{-1} \approx 2$.

- *There exists a root* $w^* \approx 2u^*_{\min}$ *to* (4.10) *therefore problem* (4.6) *has least two solutions on any interval containing* w^*.

Proof. The proof is given in Appendix A.4 and use result (4.16). It is based on the study of the sign of $\partial_u |F|$ given by (4.9). It shows that assuming (4.18) for $u^* \in I$ allows to ensure that $\partial_u |F| > 0$ whatever the sign of f. Also we show that the sign of $\partial_u |F|$ changes for a $w^* \approx 2u^*_{\min}$ which gives non-uniqueness of solution. □

Theorem 4.9 shows that, for KPP viscosity profiles, we have the non-uniqueness of solution for physically relevant values of u^*. Even if an approximation is made on viscosity profiles for the computations, it seems that the well-posedness issues are inherent to the global KPP viscosity profiles and the OA orders of magnitude. For the reference OA values from Definition 2.2, we plot in Figure 4.3 the $\mathcal{S}_\alpha$ as given by (A.5) and their respective approximations given by (4.16). The corresponding profile of $\left|\left[\phi^g\right]_o^a\right| = |F(u^*)|$ for $u^* \in I^*$ drawn black in the right panel of Figure 4.3, shows an inflexion point of $|F(u^*)|$ close to $2u^*_{\min}$. Note that, according to the computation in Appendix A.3, the inflection point w^* goes to 0 if $|\delta_\alpha|$ goes to 0, and then the equivalent problem considering an interface with zero thickness would be well-posed. Theorem 4.9 highlights the role of boundary layer parameterization and uniqueness issues appears for low values of u^*. Indeed, by asymptotics, if $u^* \to u^*_{\min}$ the parametrization at the interface which gives $\nu = \kappa u^* z + \nu^m_\alpha$ is not consistent with the parametrization in the ocean or atmosphere domains $\nu \to \nu^m_\alpha$ and the boundary layer becomes more important than the turbulent layer. Non-uniqueness issues is consequently inherent to the nature of the KPP viscosity

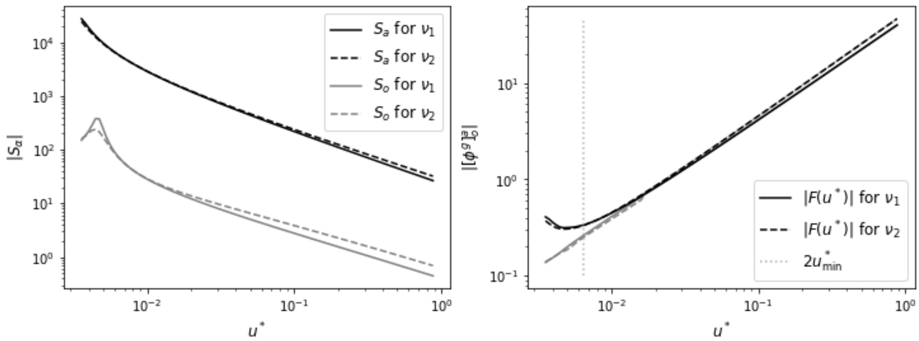

FIGURE 4.3. For KPP viscosity profiles with reference values (see Definition 2.2). Left panel: $\mathcal{S}_\alpha$ given for, $\nu_{\alpha,1}$ in continuous line, $\nu_{\alpha,2}$ in dashed line. Black lines correspond to the atmospheric part and grey lines to the oceanic part. Right panel: the corresponding profiles of $\left|\left[\phi^g\right]_o^a\right| = |F(u^*)|$ with the same line style. Dot-dashed line and doted line are superimposed. Black lines correspond to the case of fixed δ_α and grey lines to the case $\delta_\alpha = h_\alpha/4$.

profile with fixed boundary layer thickness. Therefore, from a mathematical point of view, we can solve the problem by assuming that the buffer zone varies as a function of u^*. Under these conditions, we can show that the problem is generally well posed for viscosity profiles compatible with KPP viscosity profiles.

Theorem 4.10 (Uniqueness for non-fixed buffer zone)**.** *Let consider* (4.6) *and suppose* $\delta_\alpha = \beta h_\alpha$ *for a constant* $\beta < e^{-1} \approx 0.3$. *Then, there exists an unique solution to* (4.6) *in* $u^* \in [u^*_{\min}, u^*_{\max}]$ *if*

$$\frac{u^* \partial_u \nu_\alpha(\delta_\alpha, u^*)}{\nu_\alpha(\delta_\alpha, u^*)} \leq 2 + \frac{\kappa}{2\sqrt{C_D}} |\ln(\beta) + 1|^{-1} \tag{4.19}$$

As an example with $\beta = 1/4$ *we have* $\frac{\kappa}{2\sqrt{C_D}} |\ln(\beta) + 1|^{-1} \approx 12$

Proof. Replacing δ_α by βh_α in the equation (A.5) then asymptotic (4.16) remains true. The proof given in Appendix A.4 shows that $\partial_u |F| > 0$ for all u^* if ν statisfy (4.19). □

To illustrate Theorem 4.10, the profile of $\left|\left[\phi^g\right]_o^a\right| = |F(u^*)|$ considering $\delta_a = \min(h_a/4, 10)$ is drawn in grey in Figure 4.3 right panel for the two examples $\nu_{\alpha,1}$ and $\nu_{\alpha,2}$. Graphs illustrate that, indeed, $|F(u^*)|$ is monotoneous on I^*. Adapting the thickness of the buffer zone to the thickness of the turbulent layer is thus a solution to guaranty the well-posedness of the stationary problem. However, in numerical models the buffer zone size in fixed by vertical grid and this assumption could not be directly applied in practice. Nevertheless this example shows that buffer zone parameterizations have a significant impact on the well-posedness of the problem for low values of u^* and that adapting these parameterizations could solve the uniqueness issue encountered with KPP viscosity profiles.

5. Conclusion

To summarize, we constructed here a global OA coupled model considering realistic boundary conditions and taking into account the numerical strategy used by the actual implemented models. Our coupled model can be described as a non-local Ekman boundary layer problem with parameterized turbulent viscosity profiles and nonlinear interface conditions. The non-local property lies in the dependency of the turbulent viscosity profiles to the jump of the solution around the interface. The interface is described by a buffer zone where solutions are parametrized as it is done in realistic OA numerical models. The existence of solutions has been proved on a close problem in the stationary case, and it was shown that the uniqueness of the solution is possible only for viscosity profiles with low variations. We adapted this method, based on a fixed-point problem, to our model and discussed on the application in the OA context. Criteria on viscosity profiles that ensure well-posedness in the stationary and non-stationary cases was given. These criteria imply that the uniqueness of the solution can be guarantee for viscosity profiles with slow variations, which it is not relevant in the OA framework. For the stationary problem, when it is possible to solve the main equation, we gave a sufficient and necessary well-posedness criterion that ensures existence and uniqueness of a solution. We first applied this criterion to the problem without Coriolis force leading to a well-posedness criterion for every parameterized turbulent viscosities. Finally, we applied this well-posedness criterion considering the Coriolis effect and KPP viscosity profiles. We show that there is non-uniqueness of the solution for an interval of physically relevant solutions. This non-uniqueness is valid not only for a specific viscosity profile but for a general viscosity profile which follows the hypothesis imposed in OA context. This uniqueness issue was solved by adapting the thickness of the interface buffer zone.

This paper is a synthetic work on the well-posedness properties on a simplified but somewhat realistic OA coupled problem. It confirms that, even in a simplified model, the regularity issues involved by the non-local behavior of the problem remain valid, both for the stationary and the non-stationary cases. By giving a sufficient condition on the viscosity profiles to ensure the well-posedness, it highlights that the naive resolution based on a fixed-point problem is not adapted to the OA framework. Indeed the regularity of the viscosity profiles considered in this framework does not satisfy the given necessary condition. Also, on the stationary problem where some more precise computation can be made, results show that the consideration of the Coriolis effect is indispensable and can change the nature of the solution. The non uniqueness issues are also relevant for viscosity profiles derived from oceanic and atmospheric models. The non-uniqueness of solution is related to the combination between the viscosity profiles, the interface condition and the parametrisation in the interface zone, as used in the realistic OA coupled models. Generally speaking, the non-uniqueness issues appear because of a incompatibility between the boundary layer parametrisation and the parametrised viscosity profile when the turbulent layer is small. In this paper, we have tried to stay within the hypotheses considered close to the realistic models, but improve the compatibility between the viscosity profile and the interface parametrisations would be a key to ensure the well-posedness of the non-local OA coupling model.

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Appendix

A.1. **Existence of solution of the stationary state**

Well-posedness of the stationary weak formulation of the local problem, proof of Proposition 3.3. We define $\mathbb{V} := \mathbb{V}_a \cup \mathbb{V}_o$ such that $\mathbb{V}_\alpha := \left\{\mathbf{v} \in \mathrm{H}^1(\Omega_\alpha), \mathbf{v}(\mathrm{Z}_\alpha^\infty) = 0\right\}$ and we define $(\mathbb{V}_m)_{m\geq 0}$ the increasing sequence of finite-dimensional Hilbert subspaces such that $\mathbb{V} = \bigcup_{m\geq 0} \mathbb{V}_m$ and the continuous mapping $\mathbf{\Phi}_m : \mathbb{V}_m \to \mathbb{V}_m$ such that

$$\begin{aligned}\langle \Phi_m(\widetilde{\mathbf{u}}_m), \mathbf{v}_m\rangle = \left\langle \sqrt{\nu}\widetilde{\mathbf{u}}_m, \sqrt{\nu}\mathbf{v}_m\right\rangle_{\mathrm{OA}} + f\left\langle \widetilde{\mathbf{u}}_m^{\perp}, \mathbf{v}_m\right\rangle_{\mathrm{OA}} \\ + C_D \left|\left[\mathbf{u}_m\right]_o^a\right| \left[\mathbf{u}_m\right]_o^a \cdot \left[\mathbf{v}_m\right]_o^a + \langle \partial_z \mathbf{g}, \partial_z \mathbf{v}_m\rangle_{\mathrm{OA}} \quad \text{(A.1)}\end{aligned}$$

with $\mathbf{u}_m = \widetilde{\mathbf{u}}_m + \mathbf{g}$ The existence of solution $\widetilde{\mathbf{u}}_m$ of $\Phi_m(\widetilde{\mathbf{u}}_m) = 0$ is proved using a monotonicity method, see [10] chapter 9 for more details. We need to prove that there exists $r > 0$ such that $\langle\Phi_m(\widetilde{\mathbf{u}}_m), \widetilde{\mathbf{u}}_m)\rangle \geq 0$ for $\|\widetilde{\mathbf{u}}_m\| = r$. We first minimize $\langle\Phi_m(\widetilde{\mathbf{u}}_m), \widetilde{\mathbf{u}}_m\rangle$

trace theorem we have

$$\begin{aligned}
&\Phi_m(\widetilde{\mathbf{u}}_m, \widetilde{\mathbf{u}}_m) \\
&= \left\|\sqrt{\nu}\partial_z\widetilde{\mathbf{u}}_m\right\|_{\mathrm{OA}}^2 + C_D \left|\left[\mathbf{u}_m\right]_o^a\right|^3 + \langle \nu\partial_z\mathbf{g}, \partial_z\widetilde{\mathbf{u}}_m\rangle_{\mathrm{OA}} - C_D \left|\left[\mathbf{u}_m\right]_o^a\right| \left[\mathbf{u}_m\right]_o^a \cdot \left[\mathbf{g}\right]_o^a \\
&\geq \left\|\sqrt{\nu}\partial_z\widetilde{\mathbf{u}}_m\right\|_{\mathrm{OA}} \left(\left\|\sqrt{\nu}\partial_z\widetilde{\mathbf{u}}_m\right\|_{\mathrm{OA}} - \left\|\sqrt{\nu}\partial_z\mathbf{g}\right\|_{\mathrm{OA}}\right) + C_D \left|\left[\mathbf{u}\right]_o^a\right|^2 \left(\left|\left[\widetilde{\mathbf{u}}+\mathbf{g}\right]_o^a\right| - \left|\left[\mathbf{g}_m\right]_o^a\right|\right)
\end{aligned}$$

Taking $\widetilde{\mathbf{u}}_m = \mathbf{g}$ then $\left|\left[\widetilde{\mathbf{u}}\right]_o^a\right| = 2\left|\left[\mathbf{g}\right]_o^a\right|$ and $\langle\Phi_m(\mathbf{g}), \mathbf{g})\rangle \geq 0$. Applying the Brouwer's fixed point, there exists a solution in $\widetilde{\mathbf{u}}_m \in \mathbb{V}_m$ such that $\|\widetilde{\mathbf{u}}_m\|_{\mathrm{OA}} + \|\partial_z\widetilde{\mathbf{u}}_m\|_{\mathrm{OA}} \leq \|\mathbf{g}\|_{\mathrm{OA}} + \|\partial_z\mathbf{g}\|_{\mathrm{OA}}$.

Existence of solution of the non local stationary problem, proof of Theorem 3.4. We use the same step that the proof of the well-posedness of the local problem above, with the additional constraint $u^* = \sqrt{C_D}\left|\left[\widetilde{\mathbf{u}}\right]_o^a\right|$. Let us note $\Psi_m : \mathbb{V}_m \times \mathbb{R} \to \mathbb{V}_m \times \mathbb{R}$ with $\mathbb{V}_m \subset \mathbb{V}$ and given by $\Psi_m(\widetilde{\mathbf{u}}_m, u^*) = \left(\Phi_{m,u^*}(\widetilde{\mathbf{u}}_m), u^* - \sqrt{C_D}\left|\left[\widetilde{\mathbf{u}}_m\right]_o^a\right|\right)$ and such that $\Psi_m(\widetilde{\mathbf{u}}_m, u^*) \cdot (\mathbf{v}_m, v^*) = \left\langle\Phi_{m,u^*}(\widetilde{\mathbf{u}}_m), \mathbf{v}_m\right\rangle + v^*(u^* - \sqrt{C_D}\left|\left[\widetilde{\mathbf{u}}_m\right]_o^a\right|))$ with Φ_{m,u^*} is Φ_m by (A.1) with $\nu_\alpha = \nu_\alpha(u^*)$. Then using same bounding than previous proof we have:

$$\begin{aligned}
&\Psi_m(\widetilde{\mathbf{u}}_m, u^*) \cdot (\widetilde{\mathbf{u}}_m, u^*) \\
&\qquad \geq \left\|\sqrt{\nu}\partial_z\widetilde{\mathbf{u}}_m\right\|_{\mathrm{OA}} \left(\left\|\sqrt{\nu}\partial_z\widetilde{\mathbf{u}}_m\right\|_{\mathrm{OA}} - \left\|\sqrt{\nu}\partial_z\mathbf{g}\right\|_{\mathrm{OA}}\right) \\
&\qquad\qquad + C_D \left|\left[\mathbf{u}\right]_o^a\right|^2 \left(\left|\left[\widetilde{\mathbf{u}}+\mathbf{g}\right]_o^a\right| - \left|\left[\mathbf{g}\right]_o^a\right|\right) + u^* \left(u^* - \sqrt{C_D}\left|\left[\widetilde{\mathbf{u}}_m+\mathbf{g}\right]_o^a\right|\right)
\end{aligned}$$

with $\overline{\nu}_\alpha = \|\nu_\alpha\|_\infty$ Taking $\widetilde{\mathbf{u}} = \mathbf{g}$ then first term in the r.h.s is zero and so second term in the r.h.s is positive. Also taking $u^*_{\max} = \sqrt{C_D}\left|\left[\mathbf{g}\right]_o^a\right|$. then $\Psi_m(\mathbf{g}, u^*_{\max}) \cdot (\mathbf{g}, u^*_{\max}) \geq 0$. Thus there exists a solution $\Psi_m(\widetilde{\mathbf{u}}_m, u^*_m) = 0$ with $\mathbb{V}_m \times [0, u^*_{\max}]$ and such that $\|\widetilde{\mathbf{u}}_m\|_{\mathrm{H}^1} \leq \|\mathbf{g}_m\|_{\mathrm{H}^1}$.

Recall step to prove existence in the neighborhood of stationary state, proof of Theorem 3.7. We adapt from [7] the method to our non-local problem and briefly recall the steps:

(1) Suppose a stationary solution $(\mathbf{u}^e, u^*_e) \in \mathrm{H}^1(\Omega) \times \mathbb{R}^+$ for a source term $\mathbf{u}^{e,g}$

(2) Define

$$\Psi(\mathbf{u}, u^*) = \left\{\begin{array}{l}
\partial_t\mathbf{u}_\alpha + f\mathbf{u}_\alpha^\perp - \partial_z(\nu(z)\partial_z\mathbf{u}_\alpha),\ \mathbf{u}_\alpha(t=0) - \mathbf{u}^e_\alpha,\ \mathbf{u}_\alpha(\mathrm{Z}^\infty_\alpha) - \mathbf{u}^e_\alpha(\mathrm{Z}^\infty_\alpha) \\
\lambda^2\nu_a\partial_z\mathbf{u}_a(\delta_a) - \nu_o\partial_z\mathbf{u}_o(\delta_o), \\
\nu_a(\delta_a)\partial_z\mathbf{u}_a(\delta_a) - C_D\left|\left[\mathbf{u}\right]_o^a\right|\left[\mathbf{u}\right]_o^a,\ u^* - \sqrt{C_D}\left|\left[\mathbf{u}\right]_o^a\right|
\end{array}\right\}$$

then Ψ is continuous from $\mathcal{X} := \mathrm{L}^2\left(0,T;\mathrm{H}^2(\Omega)\right) \cap \mathrm{H}^1\left(0,T;\mathrm{L}^2(\Omega)\right) \times \mathrm{L}^2(0,T)$ to $\mathcal{Y} := \mathrm{L}^2\left(0,T,\mathrm{L}^2(\Omega)\right) \times \mathrm{H}^1(\Omega) \times \mathrm{L}^2\left(0,T\right)^4$ and differentiable, noted $\boldsymbol{D}\Psi(\mathbf{u}^e, u_e^*)$.

(3) Prove that

$$\boldsymbol{D}\Psi(\mathbf{u}_e, u_e^*)(\mathbf{v}, v^*) = \left\{ \begin{array}{l} \partial_t \mathbf{v}_\alpha + f\mathbf{v}_\alpha^\perp - \partial_z(\nu_\alpha(z,u_e^*)\partial_z \mathbf{v}_\alpha + v^*\nu'_\alpha(z,u_e^*)\partial_z \mathbf{u}_\alpha^e), \\ \mathbf{v}_\alpha(t=0), \mathbf{v}_\alpha(\mathrm{Z}_\alpha^\infty), \lambda^2\nu_a\partial_z\mathbf{v}_a(\delta_a) - \nu_o\partial_z\mathbf{v}_o(\delta_o) \\ \;+ v^*(\nu'_a(u_e^*)\partial_z\mathbf{u}_a^e(\delta_a) - \lambda^2\nu'_o(u_e^*)\partial_z\mathbf{v}_o^e(\delta_o)), \nu_a(\delta_a)\partial_z\mathbf{u}_a(\delta_a) \\ \;+ v^*\nu'_a(u_e^*)\partial_z\mathbf{u}_a^e(\delta_a) - \sqrt{C_D}u_e^*\left(\left(\left[\mathbf{v}\right]_o^a\cdot\mathbf{e}_\tau\right)\mathbf{e}_\tau + \left[\mathbf{v}\right]_o^a\right), \\ v^* - \sqrt{C_D}\left|\left[\mathbf{v}\right]_o^a\right|\cdot\mathbf{e}_\tau \end{array} \right\}$$

with $\mathbf{u}_\tau = \left[\mathbf{u}^e\right]_o^a / \left|\left[\mathbf{u}^e\right]_o^a\right|$ and $\nu' = \partial_{u^*}\nu(u^*)$, is continuous and invertible from $\mathcal{X}$ to $\mathcal{Y}$

(4) Use the inverse theorem to obtain the existence of the solution around $(\mathbf{u}^e, u_e^*)$.

Finally, we have to show that the differential of Ψ is continuous and invertible from $\mathcal{X}$ to $\mathcal{Y}$. To ensure the continuity of $\boldsymbol{D}\Psi$ we have to suppose $\nu'_\alpha/\nu_\alpha \in \mathrm{L}^\infty(\Omega)$ that is given by hypothesis. To show the invertibility of $\boldsymbol{D}\Psi$, we have to prove that the linear model given by (3.14a) is well-posed.

A.2. Well-posedness of the local weak formulation

We prove here the Proposition 3.8. Suppose first $\mathbf{g} \in \mathrm{H}^1(0,T;\mathrm{H}^1(\Omega)$, and $\widetilde{\mathbf{u}}^0 \in \mathrm{H}^1(\Omega)$ and $\nu \in \mathrm{C}^1(\overline{\Omega}\times[0,T])$. Because we suppose Ω is one-dimensional space, during all the proof we will use that $\mathbf{g} \in \mathrm{C}(\overline{\Omega}\times[0,T])$ and $\mathbf{g}_\alpha(\delta_\alpha) \in \mathrm{H}^1(0,T)$ for $\alpha \in \{o,a\}$. We prove the existence of solution for the weak formulation:

$$\langle \partial_t \widetilde{\mathbf{u}}, \mathbf{v}\rangle_{\mathrm{OA}} + f\left\langle \widetilde{\mathbf{u}}^\perp, \mathbf{v}\right\rangle_{\mathrm{OA}} + \langle \nu\partial_z\widetilde{\mathbf{u}}, \partial_z\mathbf{v}\rangle_{\mathrm{OA}} + C_D\left|\left[\mathbf{u}\right]_o^a\right|\left[\mathbf{u}\right]_o^a\cdot\left[\mathbf{v}\right]_o^a$$
$$= -\langle \nu\partial_z\mathbf{g}, \partial_z\mathbf{v}\rangle_{\mathrm{OA}} - \langle \partial_t\mathbf{g}, \mathbf{v}\rangle_{\mathrm{OA}}$$

with $\mathbf{u} = \widetilde{\mathbf{u}} + \mathbf{g}$. Using Galerkin method, we can prove that there exists a unique solution of the weak formulation $\widetilde{\mathbf{u}} \in \mathrm{C}([0,T];\mathrm{L}^2(\Omega)) \cap \mathrm{L}^2(0,T,\mathbb{V})$ and $\widetilde{\mathbf{u}}(t=0) = \widetilde{\mathbf{u}}_0$. We define $(\mathbb{V}_m)_{m\geq 0}$ the increasing sequence of finite-dimensional Hilbert subspaces such that $\mathbb{V} = \bigcup_{m\geq 0}\mathbb{V}_m$. Suppose $\widetilde{\mathbf{u}}_m = (\sum c_k(t)e_k, \sum d_k(t)e_k)^T \in \mathbb{V}_m$ with $\mathbf{e}_k$ and orthogonal basis of $\mathbb{V}_m$. Then the weak formulation becomes a nonlinear ODE on c_k and d_k. Since all terms are continuous, by Cauchy–Lipschitz theorem, there exists a unique set of solutions with $c_k, d_k \in \mathrm{C}[0,T]$. Let us suppose $\widetilde{\mathbf{u}}_m \in \mathbb{V}_m$ is a solution of the weak formulation.

Taking $\mathbf{v} = \widetilde{\mathbf{u}}_m$, integrating on $]0,t[$, and applying Cauchy–Schwarz, Young inequality and Poincaré inquality on each term on the r.h.s, we finally have the apriori estimate:

$$\begin{aligned} \|\widetilde{\mathbf{u}}\|^2_{\mathrm{OA}}(t) + \frac{1}{2}\int_0^t \|\sqrt{\nu}\partial_z\widetilde{\mathbf{u}}\|^2_{\mathrm{OA}} + \frac{C_D}{3}\int_0^t \left|[\mathbf{u}]_o^a\right|^3 \\ \leq \frac{C_D}{3}\int_0^t \left|[\mathbf{g}]_o^a\right|^3 + \int_0^t \|\sqrt{\nu}\partial_r\mathbf{g}\|^2_{\mathrm{OA}} + 4\int_0^t \|\partial_t\mathbf{g}\|^2_{\mathrm{OA}} + \left\|\widetilde{\mathbf{u}}^0\right\|^2_{\mathrm{OA}} \quad \text{(A.2)} \end{aligned}$$

Since $\mathbf{g} \in \mathrm{H}^1(0,T;\mathrm{H}^1(\Omega))$ using Morrey inequality we have $\mathbf{g}_\alpha(\delta_\alpha) \in \mathrm{C}[0,T]$. According to (A.2), $\widetilde{\mathbf{u}}_m \in \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)) \cap \mathrm{L}^2(0,T;\mathbb{V})$ and $\partial_t\widetilde{\mathbf{u}}_m \in \mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega))$. Using Sobolev embedding convergence results, there exists $\widetilde{\mathbf{u}} \in \mathrm{L}^2(0,T;\mathbb{V}) \cap \mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))$ with $\partial_t\widetilde{\mathbf{u}} \in \mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega))$ and a $\mathcal{X} \in \mathrm{L}^{3/2}(0,T)$ such that

- $\widetilde{\mathbf{u}}_m$ converges weakly to $\widetilde{\mathbf{u}}$ in $\mathrm{L}^2(0,T,\mathbb{V})$
- $\partial_t\widetilde{\mathbf{u}}_m$ converges weakly to $\partial_t\widetilde{\mathbf{u}}$ in $\mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega))$
- $\left|[\widetilde{\mathbf{u}}_m]_o^a\right| [\widetilde{\mathbf{u}}_m]_o^a$ converges weakly to $\mathcal{X}$ in $\mathrm{L}^{3/2}(0,T)$
- $\widetilde{\mathbf{u}}_m$ converges to $\widetilde{\mathbf{u}}$ in $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$.

To prove that $\widetilde{\mathbf{u}}$ is the solution of the weak formulation, we have to prove that $\mathcal{X}$ is the term $C_D\left|[\widetilde{\mathbf{u}}]_o^a\right| [\widetilde{\mathbf{u}}]_o^a$. Because we are in 1D, $\widetilde{\mathbf{u}} \in \mathrm{L}^2(0,T,\mathrm{C}^0(\overline{\Omega}))$ and then $\widetilde{\mathbf{u}}_m(\delta)$ converges to $\widetilde{\mathbf{u}}(\delta)$ in $\mathrm{L}^2([0,T])$. It implies that $\left|[\mathbf{u}_m]_o^a\right| [\mathbf{u}_m]_o^a$ converges weakly to $\left|[\mathbf{u}]_o^a\right| [\mathbf{u}]_o^a$ in $\mathrm{L}^{3/2}([0,T])$. Finally, since $\nu(t)$ is continuous and bounded $\widetilde{\mathbf{u}}$ is a solution of the weak formulation with $\widetilde{\mathbf{u}}(t=0) = \widetilde{\mathbf{u}}_0$. The proof of the uniqueness of the solution is based on the same argument than in the stationary case to treate the non-linear terms (see (3.4)). Now, we want a bound for $\left\|\sqrt{\nu}\partial_z\widetilde{\mathbf{u}}\right\|^2_{\mathrm{OA}}$ in $\mathrm{L}^\infty(0,T)$ that will be necessary for Theorem 3.10. Taking $\mathbf{v} = \partial_t\widetilde{\mathbf{u}}_m$ in the weak formulation:

$$\begin{aligned} \|\partial_t\widetilde{\mathbf{u}}_m\|^2_{\mathrm{OA}} + f\left\langle \widetilde{\mathbf{u}}_m, \partial_t\widetilde{\mathbf{u}}_m\right\rangle_{\mathrm{OA}} + \left\langle \sqrt{\nu}\partial_z\widetilde{\mathbf{u}}_m, \partial_{z,t}\widetilde{\mathbf{u}}_m\right\rangle_{\mathrm{OA}} + C_D\left|[\mathbf{u}_m]_o^a\right| [\mathbf{u}_m]_o^a \cdot [\partial_t\mathbf{u}_m]_o^a \\ = C_D\left|[\mathbf{u}_m]_o^a\right| [\mathbf{u}_m]_o^a \cdot [\partial_t\mathbf{g}_m]_o^a - \left\langle \nu\partial_z\mathbf{g}, \partial_{z,t}\widetilde{\mathbf{u}}_m\right\rangle_{\mathrm{OA}} - \left\langle \partial_t\mathbf{g}, \partial_t\widetilde{\mathbf{u}}_m\right\rangle_{\mathrm{OA}} \end{aligned}$$

The term in f will disappear after integrating in time, indeed $\int_0^t \left\langle \widetilde{\mathbf{u}}_m^\perp, \partial_t\widetilde{\mathbf{u}}_m\right\rangle_{\mathrm{OA}} = \left\langle \widetilde{\mathbf{u}}_m^\perp, \widetilde{\mathbf{u}}_m\right\rangle_{\mathrm{OA}}(t) - \left\langle \widetilde{\mathbf{u}}_m^\perp, \widetilde{\mathbf{u}}_m\right\rangle_{\mathrm{OA}}(t=0) - \int_0^t \left\langle \partial_t\widetilde{\mathbf{u}}_m^\perp, \widetilde{\mathbf{u}}_m\right\rangle_{\mathrm{OA}} = -\int_0^t \left\langle \partial_t\widetilde{\mathbf{u}}_m^\perp, \widetilde{\mathbf{u}}_m\right\rangle_{\mathrm{OA}}$ and by symmetry $\int_0^t \left\langle \widetilde{\mathbf{u}}_m^\perp, \partial_t\widetilde{\mathbf{u}}_m\right\rangle_{\mathrm{OA}} = \int_0^t \left\langle \partial_t\widetilde{\mathbf{u}}_m^\perp, \widetilde{\mathbf{u}}_m\right\rangle_{\mathrm{OA}}$ so the term is null. Also using

$$\int_0^t \left(\partial_t [\mathbf{u}_m]_o^a\right) \cdot [\mathbf{u}_m]_o^a \left|[\mathbf{u}_m]_o^a\right| = \left[\frac{1}{3}\left|[\mathbf{u}_m]_o^a\right|^3\right]_0^t$$

$$2\int_0^t \left\langle \partial_t\partial_z\widetilde{\mathbf{u}}_m, \nu\partial_z\widetilde{\mathbf{u}}_m\right\rangle = \left[\left\|\sqrt{\nu}\partial_z\widetilde{\mathbf{u}}_m\right\|_2^2\right]_0^t - \int_0^t \left\langle \partial_z\widetilde{\mathbf{u}}_m, \partial_t\nu\partial_z\widetilde{\mathbf{u}}_m\right\rangle$$

it gives:

$$\int_0^t \|\partial_t \widetilde{\mathbf{u}}_m\|_{\mathrm{OA}}^2 + \frac{1}{2}\left\|\sqrt{\nu}\partial_z \widetilde{\mathbf{u}}_m\right\|_{\mathrm{OA}}^2(t) + \frac{C_D}{3}\left|\left[\mathbf{u}_m\right]_o^a\right|^3(t)$$
$$= A_0 + \int_0^t \left\langle \nu\partial_{t,z}\mathbf{g}, \partial_z \widetilde{\mathbf{u}}_m\right\rangle_{\mathrm{OA}} + \int_0^t \langle \partial_z \widetilde{\mathbf{u}}_m + \partial_z \mathbf{g}, \partial_t \nu \partial_z \widetilde{\mathbf{u}}_m\rangle_{\mathrm{OA}}$$
$$+ C_D \int_0^t \left|\left[\widetilde{\mathbf{u}}_m\right]_o^a\right| \left[\widetilde{\mathbf{u}}_m\right]_o^a \cdot \left[\partial_t \mathbf{g}\right]_o^a - \langle \partial_t \mathbf{g}, \partial_t \widetilde{\mathbf{u}}_m\rangle_{\mathrm{OA}} - \langle \partial_z \mathbf{g}, \nu\partial_z \widetilde{\mathbf{u}}_m\rangle_{\mathrm{OA}}$$

with $A_0 = \frac{1}{2}\left\|\sqrt{\nu}\partial_z \widetilde{\mathbf{u}}_0\right\|_{\mathrm{OA}}^2 + \frac{C_D}{3}\left|\left[\mathbf{u}_0\right]_o^a\right|^3$. We introduce the notation $\mu = \partial_t \nu / \nu$. Using Cauchy–Schwarz inequality and Young inequality on terms on the right hand side, there exist $C_2 > 0$ such that :

$$\int_0^t \|\partial_t \widetilde{\mathbf{u}}_m\|_{\mathrm{OA}}^2 + \left\|\sqrt{\nu}\partial_z \widetilde{\mathbf{u}}_m\right\|_{\mathrm{OA}}^2 + \frac{C_D}{3}\left|\left[\widetilde{\mathbf{u}}_m\right]_o^a\right|^3$$
$$\leq \int_0^t \|\mu\|_{\mathrm{L}^\infty(\Omega)}^2 \left\|\sqrt{\nu}\partial_z \widetilde{\mathbf{u}}_m\right\|_{\mathrm{OA}}^2$$
$$+ C_2\left(\|\mathbf{g}\|_{\mathrm{OA}}^2 + \int_0^t \left\|\sqrt{\nu}\partial_{z,t}\mathbf{g}\right\|_{\mathrm{OA}}^2 + C_D\int_0^t \left|\left[\partial_t \mathbf{g}\right]_o^a\right|^3 + A_0 + A_1\right) \quad \text{(A.3)}$$

with A_1 the r.h.s of (A.2). By hypothesis $\nu \in \mathrm{C}^1(\overline{\Omega} \times [0,T])$ then $\|\mu\|_{\mathrm{L}^\infty(\Omega)} \in \mathrm{C}^0([0,T])$ and $\mathbf{g} \in \mathrm{H}^1(0,T;\mathrm{H}^1(\Omega))$, if we had the hypothesis $\partial_t \mathbf{g} \in \mathrm{L}^3(0,T;\mathrm{H}^1(\Omega))$ so all term in the r.h.s exist. By Gronwall Theorem, we have $\left\|\sqrt{\nu}\partial_z \widetilde{\mathbf{u}}\right\|_{\mathrm{OA}}^2 \in \mathrm{C}^0[0,T]$ and $\widetilde{\mathbf{u}} \in \mathrm{L}^2(0,T;\mathrm{H}^2(\Omega)) \cap \mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega)) \cap \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega))$. To obtain better regularity on $\partial_t \widetilde{\mathbf{u}}$, we suppose $\widetilde{\mathbf{u}}^0 \in \mathrm{H}^2(\Omega)$ and $\mathbf{g} \in \mathrm{H}^2(0,T,\mathrm{H}^1(\Omega))$. We proceed in a similar way by first deriving the equation w.r.t. t, then potentially problematic terms are the term in ν and the boundary term. Using $\partial_t \partial_z u \partial_t(\nu\partial_z u) = \nu(\partial_t \partial_z u)^2 + \frac{1}{2}\partial_t \nu \partial_{tz} u \partial_z u$ and $\partial_t u \partial_t(|u|u) = \partial_t |u| |u| \frac{1}{2}\partial_t(u^2) + |u|(\partial_t u)^2 = (\partial_t |u|)^2 |u| + |u|(\partial_t u)^2$ we have:

$$\frac{1}{2}\partial_t \|\partial_t \widetilde{\mathbf{u}}\|_{\mathrm{OA}}^2 + \left\|\sqrt{\nu}\partial_t \partial_z \widetilde{\mathbf{u}}\right\|_{\mathrm{OA}}^2 + \left|\left[\mathbf{u}\right]_o^a\right|\left(\left(\partial_t \left|\left[\mathbf{u}\right]_o^a\right|\right)^2 + \left|\left[\partial_t \mathbf{u}\right]_o^a\right|^2\right)$$
$$= \left\langle \partial_t^2 \mathbf{g}, \partial_t \widetilde{\mathbf{u}}\right\rangle_{\mathrm{OA}} + \left\langle \sqrt{\nu}\partial_z \widetilde{\mathbf{u}}, \mu\sqrt{\nu}\partial_t \partial_z \widetilde{\mathbf{u}}\right\rangle_{\mathrm{OA}} + \left[\partial_t \mathbf{g}\right]_o^a \left(\partial_t \left[\mathbf{u}\right]_o^a \left|\left[\mathbf{u}\right]_o^a\right|\right)$$
$$- \left\langle \sqrt{\nu}\partial_z \mathbf{g}, \mu\sqrt{\nu}\partial_t \partial_z \widetilde{\mathbf{u}}\right\rangle_{\mathrm{OA}} - \left\langle \sqrt{\nu}\partial_z \partial_t \mathbf{g}, \sqrt{\nu}\partial_t \partial_z \widetilde{\mathbf{u}}\right\rangle_{\mathrm{OA}}$$

We integrate on $[0,t]$ and apply Young's inequality on all r.h.s terms, especially

$$\begin{aligned}
&\int_0^t \left|[\partial_t \mathbf{g}]_o^a \partial_t \left([\mathbf{u}]_o^a \left|[\mathbf{u}]_o^a\right|\right)\right| \\
&\quad = \int_0^t \left|[\partial_t \mathbf{g}]_o^a \sqrt{\left|[\mathbf{u}]_o^a\right|}\right| \left|\sqrt{\left|[\mathbf{u}]_o^a\right|}\left(\partial_t \left|[\mathbf{u}]_o^a\right| + \left|[\partial_t \mathbf{u}]_o^a\right|\right)\right| \\
&\quad \le \int_0^t \left|[\partial_t \mathbf{g}]_o^a\right|^2 \left|[\mathbf{u}]_o^a\right| + \frac{1}{2}\int_0^t \left|[\mathbf{u}]_o^a\right| \left(\left(\partial_t \left|[\mathbf{u}]_o^a\right|\right)^2 + \left|[\partial_t \mathbf{u}]_o^a\right|^2\right) \\
&\quad \le \int_0^t \frac{2}{3}\left|[\partial_t \mathbf{g}]_o^a\right|^3 + \int_0^t \frac{1}{3}\left|[\mathbf{u}]_o^a\right|^3 + \frac{1}{2}\int_0^t \left|[\mathbf{u}]_o^a\right| \left(\left(\partial_t \left|[\mathbf{u}]_o^a\right|\right)^2 + \left|[\partial_t \mathbf{u}]_o^a\right|^2\right)
\end{aligned}$$

We conclude there exist constant C_3 such that

$$\begin{aligned}
\|\partial_t \widetilde{\mathbf{u}}\|_{\mathrm{OA}}^2(t) &+ \int_0^t \left\|\sqrt{\nu}\partial_t\partial_z \widetilde{\mathbf{u}}\right\|_{\mathrm{OA}}^2 + \int_0^t \left|[\widetilde{\mathbf{u}}]_o^a\right| \left(\left(\partial_t \left|[\mathbf{u}]_o^a\right|\right)^2 + \left|[\partial_t \mathbf{u}]_o^a\right|^2\right) \\
&\le \int_0^t \|\mu\|_{\mathrm{L}^\infty(\Omega)}^2 \left(\left\|\sqrt{\nu}\partial_z \widetilde{\mathbf{u}}\right\|_{\mathrm{OA}}^2 + \left\|\sqrt{\nu}\partial_z \mathbf{g}\right\|_{\mathrm{OA}}^2\right) + A_2 \\
&\quad + C_3\left(\int_0^t \left\|\partial_t^2 \mathbf{g}\right\|_{\mathrm{OA}}^2 + \int_0^t \left|[\partial_t \mathbf{g}]_o^a\right|^3 + \int_0^t \left\|\sqrt{\nu}\partial_{z,t}\mathbf{g}\right\|_{\mathrm{OA}}^2 \right. \\
&\qquad \left. + \|\partial_t \widetilde{\mathbf{u}}\|_{\mathrm{OA}}^2(t=0) + A_0 + A_1\right) \qquad \text{(A.4)}
\end{aligned}$$

with A_2 the r.h.s of (A.3). Since ν is bound there exist a $C_4 > 0$ such that $\|\partial_t \widetilde{\mathbf{u}}\|_{\mathrm{OA}}^2(t = 0) \le C_4\left(\left\|\partial_{t,z}\mathbf{g}\right\|_{\mathrm{OA}}^2 + \left\|\widetilde{\mathbf{u}}^0\right\|_{\mathrm{H}^2(\Omega)}\right)$ (see [10, Chapter 7] for details). We obtain (using bound (A.4)) in particular $\partial_t \widetilde{\mathbf{u}} \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))$ and $\partial_t^2 \widetilde{\mathbf{u}} \in \mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega))$ that gives $\partial_t [\mathbf{u}]_o^a \in \mathrm{C}^0[0,T]$.

A.3. Computations for the stationary problem with KPP viscosity profiles, proof of Proposition 4.8

Resolution of the equation on each subdomain. Considering the equation in the proof of Proposition 4.8, to have $\mathrm{C}^1(\Omega_\alpha)$ regularity, especially in h_α^*, we have to add the constraints:

$$\begin{aligned}
C_{\alpha,l} &= \frac{A_{\alpha,t}P_\alpha(r_\alpha(h_\alpha)) + B_{\alpha,t}P_\alpha(-r_\alpha(h_\alpha))}{e^{\varsigma_\alpha h_\alpha} - e^{\varsigma_\alpha(2\mathrm{Z}_\alpha^\infty - h_\alpha)}} \quad \text{with } B_{\alpha,t} = \Pi_\alpha C_{\alpha,t} \\
\Pi_\alpha &= \frac{G_\alpha(r_\alpha(h_\alpha)) - \beta_\alpha P_\alpha(r_\alpha(h_\alpha))}{G_\alpha(-r_\alpha(h_\alpha)) + \beta_\alpha P_\alpha(-r_\alpha(h_\alpha))} \\
\beta_\alpha &= \frac{\varsigma_\alpha}{\tanh(\varsigma_\alpha(\mathrm{Z}_\alpha^\infty - h_\alpha))}\frac{h_\alpha\sqrt{1+4\mu_\alpha}}{\mathcal{X}_\alpha \xi_\alpha(\xi_\alpha + 1)}
\end{aligned}$$

Finally:

$$\varphi_\alpha(z) = C_{\alpha,t}\,[P_\alpha(r_\alpha(z)) + \Pi_\alpha P_\alpha(-r_\alpha(z))]$$

$$\partial_z\widehat{\varphi}_\alpha(z) = \frac{\mathcal{X}_\alpha\xi_\alpha(\xi_\alpha+1)}{h^*_\alpha\sqrt{1+4\mu_\alpha}}\,[G_\alpha(r_\alpha(z)) - \Pi_\alpha G_\alpha(-r_\alpha(z))]$$

noting $\eta = r(h) = -r(0)$, $P^+ = P(\eta)$ and $P^- = P(-\eta)$:

$$\nu_\alpha(\delta_\alpha)\mathcal{S}_\alpha = \mathcal{X}_\alpha\frac{h_\alpha\sqrt{1+4\mu_\alpha}}{\xi_\alpha(1+\xi_\alpha)}\frac{(G^-_\alpha+\beta_\alpha P^-_\alpha)P^\delta_\alpha + (G^+_\alpha-\beta_\alpha P^+_\alpha)P^{-\delta}_\alpha}{(G^-_\alpha+\beta_\alpha P^-_\alpha)G^\delta_\alpha - (G^+_\alpha-\beta_\alpha P^+_\alpha)G^{-\delta}_\alpha} \tag{A.5}$$

with $\mathcal{X}_a = 1$, $\mathcal{X}_o = -1$, and

$$\eta_\alpha = \mathcal{X}_\alpha\left(\sqrt{1+\frac{4\nu^m_\alpha}{h^2_\alpha|K_\alpha|}}\right)^{-1} \qquad \xi = -\frac{1}{2}\left(1+\sqrt{1+if/K_\alpha}\right)$$

$$\beta_\alpha = \frac{\varsigma_\alpha}{\tanh(\varsigma_\alpha(\mathrm{Z}^\infty_\alpha - h^*_\alpha))}\frac{h^*_\alpha}{\eta_\alpha\xi_\alpha(\xi_\alpha+1)} \qquad \zeta = \sqrt{if/\nu^m_\alpha}$$

$$P^\pm_\alpha = {}_2F_1(\xi+1,-\xi,1,(1-\pm\eta)/2) \qquad G^\pm_\alpha = {}_2F_1(\xi+2,1-\xi,2,(1-\pm\eta)/2)$$

where ${}_2F_1$ is the hypergeometric function (see [21]).

Asymptotics on hypergeometric function. Considering KPP viscosity profiles as in Definition 2.1 and approximation (4.15), we get the order of magnitude:

- $4\mu = \frac{4\nu^m_\alpha}{h^2_\alpha|K_\alpha|} \approx \frac{4\nu^m_\alpha|h-\delta|}{h^2_\alpha\kappa u^*} \ll 1.$
- $\frac{|f|}{|K_\alpha|} \approx \frac{f|h_\alpha-\delta_\alpha|}{\kappa u^*} < 4$ for reasonable value of $f < 7\times10^{-5}\ \mathrm{s}^{-1}$ or u^* not too close from $u^*_{\min}$. Then we can pose $\xi_\alpha \approx -1+\epsilon_\alpha$ with $\epsilon_\alpha \in \mathbb{C}$ and $|\epsilon_\alpha| < 1$.

The first inequality allows to write $\eta_\alpha \approx -\mathcal{X}_\alpha\,(1-2\mu_\alpha)$ and $r_\alpha(\delta_\alpha) = -\eta_\alpha(1-\frac{2\delta_\alpha}{h_\alpha})$. We focus on the computation on the oceanic part; by symmetry the atmospheric part follows the same principle. In the rest of the proof, indices $_o$ are omitted. We can use the following asymptotic taken from [2] and [21]:

$$P^+ = F(1+\xi,-\xi,1,\mu) \approx 1-\xi(1+\xi)\mu$$

$$P^- = F(1+\xi,-\xi,1,1-\mu) \approx \frac{\sigma-\ln(\mu)}{\Gamma(\xi+1)\Gamma(-\xi)}$$

$$P^\delta = F(1+\xi,-\xi,1,1-\delta/h) \approx \frac{\sigma-\ln(\delta/h)}{\Gamma(\xi+1)\Gamma(\xi)}$$

$$P^{-\delta} = F(1+\xi,-\xi,1,\delta/h) \approx 1-\xi(1+\xi)\frac{\delta}{h}$$

$$\begin{aligned} G^+ &= F(2+\xi, 1-\xi, 2, \mu) \approx 1 + (1-\xi)(2+\xi)\frac{\mu}{2} \\ G^- &= F(2+\xi, 1-\xi, 2, 1-\mu) \approx \frac{\Gamma(2)}{\mu\Gamma(\xi+2)\Gamma(1-\xi)} \\ G^\delta &= F(2+\xi, 1-\xi, 2, 1-\delta/h) \approx \frac{h\Gamma(2)}{\delta\Gamma(\xi+2)\Gamma(1-\xi)} \\ G^{-\delta} &= F(2+\xi, 1-\xi, 2, \delta/h^*) \approx 1 + (1-\xi)(2+\xi)\frac{\delta}{2h} \end{aligned}$$

with $\sigma = 2\Psi(1) - \Psi(1+\xi) - \Psi(-\xi)$ and Ψ the digamma function. The second inequality implies $|\xi| \ll 2$ thus term $\Gamma(\xi+2)$, $\Gamma(1-\xi)$ and σ in the order of magnitude of 1. The term $\left(\mu^{-1} + \beta\xi(\xi+1)(\ln(\mu) - \sigma)\right)(\sigma - \ln(\delta/h))$ is much more bigger than $(1-\beta)\left(1/(\xi(1+\xi)) - \delta/h\right)$, so terms with $(G^- + \beta P^-)$ are much more bigger than terms with $(G^+ - \beta P^+)$. Finally

$$\frac{\varphi(z)}{\partial_z \varphi(z)} \approx h\sqrt{1+4\mu}\frac{\delta}{h}\left(\ln(\delta/h) - \sigma\right) \approx \delta\left(\ln(\delta/h) - \sigma\right)$$

Using the properties from [21] and $\Psi(1+\xi) = \Psi(\epsilon) = \Psi(1+\epsilon) - \epsilon^{-1}$, $\Psi(-\xi) = \Psi(1-\epsilon)$ and $\Psi(1+\epsilon) = \sum_{k=2}^{\infty}(-1)^k\zeta(k)(\epsilon)^{k-1}$ we can simplify σ as:

$$\sigma = 2\Psi(1) + \epsilon^{-1} - \Psi(1+\epsilon) - \Psi(1-\epsilon) = \epsilon^{-1} + 2\sum_{k=1}^{\infty}\zeta(2k+1)\epsilon^{2k}$$

which gives the asymptotic $\sigma \approx (1+\xi)^{-1}$.

A.4. Well-posedness criteria for KPP viscosty profile

Research of an inflexion point, proof of Theorem 4.9. We pose $Y_\alpha = \frac{4K_\alpha}{f}$ then $\xi_\alpha \approx -1 - iY_\alpha^{-1} - Y_\alpha^{-2}$, with and $\sigma_\alpha \approx -1 + iY_\alpha$. According to the well-posedness criteria (4.10), we are searching for a solution of

$$\left|\partial_u\left(u^2\left(\sqrt{C_D}^{-1} - N_aX_a - N_oX_o\right)^2 + u^2\left(N_aY_a + N_oY_o\right)^2\right)\right| = 0 \qquad \text{(A.6)}$$

with $X_\alpha = \ln(\delta_\alpha/h_\alpha) + 1$, $N_a = u^*|\delta_a|/\nu_a^\delta$ and $N_o = \lambda^2 u^*|\delta_o|/\nu_o^\delta$ where $\nu_\alpha^\delta = \nu_\alpha(\delta_\alpha)$. By definition of K_α we have $Y_\alpha = 4\left(\nu_\alpha^\delta - \nu_\alpha^m\right)/(f\delta(\delta - h_\alpha))$. By hypothesys $\nu_\alpha^m \ll \nu_\alpha^\delta$ and we neglect ν_α^m from the computation. Then $N_aY_a = 4u^*/(f(\delta_a - h_a))$ and $N_oY_o = -4\lambda^2u^*/(f(\delta_o - h_o))$. The derivative on u^* gives $u^*N'_\alpha = N_\alpha R_\alpha$ with $R_\alpha = 1 - u^*\left(\nu_\alpha^\delta\right)'/\nu_\alpha^\delta$, $X'_\alpha = -1/u^*$ and $u^*\left(N_\alpha Y_\alpha\right)' = N_\alpha Y_\alpha\left(1 + h_\alpha/(\delta - \alpha - h_\alpha)\right)$. Equation (A.6) can be rewritten as $\partial_{u^*}\left((u^*)^2\mathbf{X}^2 + (u^*)^2\mathbf{Y}^2\right) = 0$ with $\mathbf{X} = \sqrt{C_D^{-1}} - N_aX_a -$

$N_o X_o$, and $\mathbf{Y} = N_a Y_a + N_o Y_o$. Then we look for a solution of $\mathbf{X}\left(\mathbf{X} + u\mathbf{X}'\right) + \mathbf{Y}\left(\mathbf{Y} + u\mathbf{Y}'\right) = 0$. We have

$$\mathbf{X} + u^*\mathbf{X}' = \sqrt{C_D^{-1}} - N_a X_a + N_a\left(1 - X_a R_a\right) - N_o X_o + N_o\left(1 - X_o R_o\right)$$

$$\text{and } \mathbf{Y} + u^*\mathbf{Y}' = N_a Y_a\left(2 - \frac{h_a}{h_a - \delta_a}\right) + N_o Y_o\left(2 - \frac{h_o}{h_o - \delta_o}\right)$$

First remark that the sign of f does not impact the sign of $\mathbf{Y}\left(\mathbf{Y} + u\mathbf{Y}\right)$. It comes $\mathbf{Y}\left(\mathbf{Y} + u\mathbf{Y}\right) > 0$ if $h_\alpha > 2\delta_\alpha$ and negative if $h_\alpha < 2\delta_\alpha$ for $\alpha \in \{o, a\}$. Because of the scale $\lambda^2 \ll 1$ on N_o, the root of $\mathbf{Y} + u\mathbf{Y}$ is close to $2\delta_a/c_a = 2u^*_{\min}$. We will show that, under assumption that $u^*$is not too large, the order of magnitude of $\mathbf{X}\left(\mathbf{X} + u^*\mathbf{X}'\right)$ is much more small compare to the order of magnitude of $\mathbf{Y}\left(\mathbf{Y} + u^*\mathbf{Y}'\right)$. And so there exist a root close to the root of $\mathbf{Y} + u^*\mathbf{Y}'$ that is in the order of magnitude of $2u^*_{\min}$.

- The order of magnitude of $|N_a Y_a| = 4u^*|h_a - \delta_a|^{-1}|f|^{-1} \in [24, \infty[$. Especially, when u^* is not too large, the order of magnitude of $|N_a Y_a|$ (for example, $h_a < \mathrm{Z}_a^\infty/2$ gives $|N_a Y_a| > 10^3$).

- We have $X_\alpha \in [1 + \ln(\delta/\mathrm{Z}^\infty), 1] \approx] - 5, 1[$ and $N_\alpha \approx \kappa^{-1}$ for reasonable value of u^* so $\mathbf{X} > 0$.

- we have $\mathbf{X} + u^*\mathbf{X}' = \sqrt{C_D^{-1}} + N_a\left(1 - X_a(1 + R_a)\right) + N_o\left(1 - X_o(1 + R_o)\right)$ and $X_\alpha(1+R_\alpha) < 0$ then $\mathbf{X}+u^*\mathbf{X}' > 0$. If $X_\alpha < 0$ (i.e $h_\alpha > 3\delta_\alpha$) and $1+R_a < 0$ then a condition to have positivity would be $1 + R_\alpha \geq \left(2\sqrt{C_D}N_\alpha\right)^{-1}\ln(\delta_\alpha/H_\alpha)^{-1} \approx -1.7$. If $0 < X_\alpha < 1$ and $1 + R_\alpha > 0$, a condition to have positivity would be $1 - X_\alpha(1+R_\alpha) \geq -R_\alpha \geq -\left(2\sqrt{C_D}N_\alpha\right)^{-1} \approx -5.8$. These gives conditions (4.18)

No inflexion point for buffer zone with variating thinckness, proof of Theorem 4.10. Suppose $\delta_\alpha = \beta h_\alpha$ with $\beta < e^{-1}$ a constant. Then using previous notation we would have $X_\alpha = \ln(\beta) + 1 < 0$ that is constant, $u^* N'_\alpha = N_\alpha\left(R_\alpha + 1\right)$, and $N_\alpha Y_\alpha = 4/(f c_\alpha(\beta - 1))$ is constant. Thus $\partial_u\left(u^2\mathbf{X}^2 + (u^*)^2\mathbf{Y}^2\right) = 0$ has an inflexion point for roots of $\mathbf{X}\left(\mathbf{X} + u^*\mathbf{X}'\right) + \mathbf{Y}^2$. In the order of magnitude of OA framework, order of magnitude of $\mathbf{Y}^2$ compare to the order of magnitude of $\mathbf{X}$ and $\mathbf{X}'$ and it would be enough to obtain $\mathbf{X}\left(\mathbf{X} + u^*\mathbf{X}'\right) + \mathbf{Y}^2 > 0$ for a large possibility of viscosity profile. Generally speaking, we have $\mathbf{X} > 0$ because $X_\alpha < 0$. Using the same king of argument than previously, $\mathbf{X} + u^*\mathbf{X}' = \sqrt{C_D^{-1}} - X_a N_a(2 + R_a) - X_o N_o(2 + R_o) > 0$ if $2 + R_\alpha > \left(2\sqrt{C_D}N_\alpha X_\alpha\right)^{-1}$ and imposing this will guaranty the uniqueness.

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Sophie Thery
Institute of Mathematics
University of Augsburg
86150 Augsburg
Germany
sophie.thery@uni-a.de