ANNALES MATHÉMATIQUES

BLAISE PASCAL

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Volume 32, nº 1 (2025), p. 77-121.

https://doi.org/10.5802/ambp.434

Publication éditée par le laboratoire de mathématiques Blaise Pascal de l'université Clermont Auvergne, UMR 6620 du CNRS Clermont-Ferrand — France

Publication membre du centre Mersenne pour l'édition scientifique ouverte
http://www.centre-mersenne.org/
e-ISSN : 2118-7436

Annales mathématiques Blaise Pascal **32**, 77-121 (2025)

Topologies on split Kac–Moody groups over valued fields

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Abstract

Let G be a minimal split Kac–Moody group over a valued field $\mathcal{K}$. Motivated by the representation theory of G, we define two topologies of topological group on G, which take into account the topology on $\mathcal{K}$.

Topologies sur les groupes de Kac–Moody déployés sur les corps valués

Résumé

Soit G un groupe de Kac–Moody déployé sur un corps local $\mathcal{K}$. Motivés par la théorie des représentation de G, nous introduisons deux topologies de groupe topologique sur G.

1. Introduction

1.1. Motivation from representation theory

Let G be a reductive group over a nonArchimedean local field $\mathcal{K}$. As G is finite dimensional over $\mathcal{K}$, G is naturally equipped with a topological group structure. Its admits a basis of neighbourhood of the identity consisting of open compact subgroups. A complex representation V of G is called smooth if for every $v \in V$, the fixator of v in G is open. To every compact open subgroup K of G is associated a Hecke algebra $\mathcal{H}_K$, which is the space of K-bi-invariant functions from G to $\mathbb{C}$ which have compact support. Let V be a smooth representation of G. Then the space of K-invariant vectors V^K is naturally equipped with the structure of an $\mathcal{H}_K$-module, and we can prove that this assignment induces a bijection between the irreducible smooth representations of G admitting a non zero K-invariant vector and the irreducible representations of $\mathcal{H}_K$.

Kac–Moody groups are infinite dimensional generalizations of reductive groups. For example, if $\mathring{\mathfrak{G}}$ is a split reductive group and $\mathcal{F}$ is a field, then the associated affine Kac–Moody group is a central extension of $\mathring{\mathfrak{G}}(\mathcal{F}[u, u^{-1}]) \rtimes \mathcal{F}^*$, where u is an indeterminate. Let now $G = \mathfrak{G}(\mathcal{K})$ be a split Kac–Moody group over $\mathcal{K}$. Recently, Hecke algebras were associated to G. In [5] and [11], Braverman and Kazhdan (in the affine case) and Gaussent and Rousseau (in the general case) associated a spherical Hecke algebra $\mathcal{H}_s$ to G, i.e. an algebra associated to the spherical subgroup $\mathfrak{G}(\mathcal{O})$ of G, where $\mathcal{O}$ is the ring of integers of $\mathcal{K}$. In [6] and [2], Braverman, Kazhdan and Patnaik and Bardy-Panse,

Keywords: Kac–Moody groups, valued fields, masures.

Gaussent and Rousseau defined the Iwahori–Hecke algebra $\mathcal{H}_I$ of G (associated to the Iwahori subgroup K_I of G). In [1], together with Abdellatif, we associated Hecke algebras to certain parahoric subgroups of G, which generalizes the construction of the Iwahori–Hecke algebra of G. In [17], [20] and [19], we associated and studied principal series representations of $\mathcal{H}_I$.

For the moment, there is no link between the representations of G and the representations of its Hecke algebras. It seems natural to try to attach an irreducible representation of G to each irreducible representation of $\mathcal{H}_I$. A more modest task would be to associate to each principal series representation I_τ of $\mathcal{H}_I$ a principal series representation $I(\tau)$ of G, which is irreducible when I_τ is.

Let T be a maximal split torus of G and Y be the cocharacter lattice of (G,T). Let B be a Borel subgroup of G containing T. Let $T_\mathbb{C} = \mathrm{Hom}_{\mathrm{Gr}}(Y,\mathbb{C}^*)$ and $\tau \in T_\mathbb{C}$. Then τ can be extended to a character $\tau : B \to \mathbb{C}^*$. Assume that G is reductive. Then the principal series representation $I(\tau)$ of G is the induction of $\tau\delta^{1/2}$ from B to G, where $\delta : B \to \mathbb{R}_+^*$ is the modulus character of B. More explicitly, this is the space of locally constant functions $f : G \to \mathbb{C}$ such that $f(bg) = \tau\delta^{1/2}(b)f(g)$ for every $g \in G$ and $b \in B$. Then G acts on $I(\tau)$ by right translation. Then $I_\tau := I(\tau)^{K_I}$ is a representation of $\mathcal{H}_I$. Assume now that G is a Kac–Moody group. Then we do not know what "locally constant" mean, but we can define the representation $\widehat{I(\tau)}$ of G as the set of functions $f : G \to \mathbb{C}$ such that $f(bg) = \tau\delta^{1/2}(b)f(g)$ for every $g \in G$ and $b \in B$. Let $\mathscr{T}_G$ be a topology of topological group on G such that K_I is open. Then

$$I(\tau)_{\mathscr{T}_G} := \{f \in \widehat{I(\tau)} \mid f \text{ is locally constant for } \mathscr{T}_G\} \tag{1.1}$$

is a subrepresentation of G containing $\widehat{I(\tau)}^{K_I}$. Thus if we look for an irreducible representation containing $\widehat{I(\tau)}^{K_I}$ it is natural to search it inside $I(\tau)_{\mathscr{T}_G}$. Moreover, the more $\mathscr{T}_G$ is coarse, the smaller $I(\tau)_{\mathscr{T}_G}$ is. We thus look for the coarsest topology of topological group on G for which K_I is open.

1.2. Topology on G, masure and main results

We now assume that $\mathcal{K}$ is any field equipped with a valuation $\omega : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ such that $\omega(\mathcal{K}^*) \supset \mathbb{Z}$. We no longer assume $\mathcal{K}$ to be local, and $\omega(\mathcal{K}^*)$ can be dense in $\mathbb{R}$. Let $\mathcal{O}$ be its ring of integers. Let $\mathfrak{G}$ be a split Kac–Moody group (à la Tits, as defined in [31]) and $G = \mathfrak{G}(\mathcal{K})$. In [10] and [30], Gaussent and Rousseau associated to G a kind of Bruhat–Tits building, called a masure, on which G acts (when G is reductive, $\mathcal{I}$ is the usual Bruhat–Tits building). They defined the spherical subgroup K_s as the fixator of some vertex 0 in the masure (we prove in Proposition 3.1 that $K_s = \mathfrak{G}^{\min}(\mathcal{O})$, where $\mathfrak{G}^{\min}$ is the minimal Kac–Moody group defined by Marquis in [24]). They also define the

Iwahori subgroup K_I as the fixator of some alcove C_0^+ of $\mathcal{I}$. Then we define the topology $\mathscr{T}_{\mathrm{Fix}}$ on G as follows. A subset V of G is open if for every $g \in V$, there exists a finite subset F of $\mathcal{I}$ such that $G_F.g \subset V$, where G_F is the fixator of F in G. Then we prove that $\mathscr{T}_{\mathrm{Fix}}$ is the coarsest topology of topological group on G for which K_I is open (see Proposition 4.14). However, it is not Hausdorff in general. Indeed, let $\mathcal{Z} \subset T$ be the center of G and $\mathcal{Z}_O = \mathcal{Z} \cap \mathfrak{T}(O)$. Then $\mathcal{Z}_O$ is the fixator of $\mathcal{I}$ in G and when $\mathcal{Z}_O$ is nontrivial (which already happens for $\mathrm{SL}_2(\mathcal{K})$), $\mathscr{T}_{\mathrm{Fix}}$ is not Hausdorff.

To address this issue, we define an other topology, $\mathscr{T}$, finer than $\mathscr{T}_{\mathrm{Fix}}$ and Hausdorff. Let $\mathbb{A} = Y \otimes \mathbb{R}$ be the standard apartment of $\mathcal{I}$ and $\Phi \subset \mathbb{A}^*$ be the set of roots of (G, T). Then $\mathcal{I} = \bigcup_{g \in G} g.\mathbb{A}$. Let us begin with the case where $G = \mathrm{SL}_2(\mathcal{K})$. Let $\varpi \in O$ be such that $\omega(\varpi) = 1$. For $n \in \mathbb{N}^*$, let $\pi_n : \mathrm{SL}_2(O) \to \mathrm{SL}_2(O/\varpi^n O)$ be the natural projection. Then a basis of the neighbourhood of the identity is given by the $(\ker \pi_n)_{n \in \mathbb{N}^*}$. Let $U^+ = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ and $U^- = \left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$. Then one can prove that $\ker \pi_n = (U^+ \cap \ker \pi_n).(U^- \cap \ker \pi_n).(T \cap \ker \pi_n)$. Let $\alpha, -\alpha$ be the two roots of (G, T). Let $x_\alpha : a \mapsto \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ and $x_{-\alpha} : a \mapsto \left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right)$. Then $x_\alpha(\varpi^n O)$ fixes $\{a \in \mathbb{A} \mid \alpha(a) \geq -n\}$ and $x_{-\alpha}(\varpi^n O)$ fixes $\{a \in \mathbb{A} \mid \alpha(a) \leq n\}$. Therefore if $\lambda \in \mathbb{A}$ is such that $\alpha(\lambda) = 1$ and $[-n\lambda, n\lambda] = \alpha^{-1}([-n, n])$, we have

$$\ker \pi_n = \left(U^+ \cap \mathrm{Fix}_G([-n\lambda, n\lambda])\right).\left(U^- \cap \mathrm{Fix}_G([-n\lambda, n\lambda])\right).(T \cap \ker \pi_n).$$

We now return to the general case for G. We prove that the topology associated to $(\ker \pi_n)_{n \in \mathbb{N}^*}$ is not a topology of topological group if G is not reductive (see Lemma 3.3). Let $(\alpha_i)_{i \in I}$ be the set of simple roots of (G, T) and $C_f^v = \{x \in \mathbb{A} \mid \alpha_i(x) > 0, \forall\, i \in I\}$. Let W^v be the Weyl group of (G, T) and $\lambda \in Y \cap \bigsqcup_{w \in W^v} w.C_f^v$. We define the following subset $\mathcal{V}_{n\lambda}$ of G, for $n \in \mathbb{N}^*$:

$$\mathcal{V}_{n\lambda} = \left(U^+ \cap \mathrm{Fix}_G([-n\lambda, n\lambda])\right).\left(U^- \cap \mathrm{Fix}_G([-n\lambda, n\lambda])\right).(T \cap \ker \pi_{2N(\lambda)}),$$

where $N(\lambda) = \min\{|\alpha(\lambda)| \mid \alpha \in \Phi_+\}$. We prove the following theorem:

Theorem 1.1 (see Theorem 4.8, Lemma 4.2, Proposition 4.21 and Proposition 5.13)**.**

(1) *For $n \in \mathbb{N}^*$ and $\lambda \in Y \cap \bigsqcup_{w \in W^v} w.C_f^v$, $\mathcal{V}_{n\lambda}$ is a subgroup of $\mathfrak{G}^{\min}(O)$.*

(2) *The topology $\mathscr{T}$ associated with $(\mathcal{V}_{n\lambda})_{n \in \mathbb{N}^*}$ is Hausdorff, independent of the choice of λ and equips G with the structure of a topological group.*

(3) *The topology $\mathscr{T}$ is finer than $\mathscr{T}_{\mathrm{Fix}}$ and if $\mathcal{K}$ is Henselian, we have $\mathscr{T} = \mathscr{T}_{\mathrm{Fix}}$ if and only if $\mathcal{Z}_O = \{1\}$.*

(4) *Every compact subset of G has empty interior (for $\mathscr{T}$).*

Note that $\mathscr{T}$ and $\mathscr{T}_{\text{Fix}}$ induce the same topologies on U^+ and U^-. The main difference comes from what happens in T. As the elements of $I(\tau)_{\mathscr{T}}$ and $I(\tau)_{\mathscr{T}_{\text{Fix}}}$ are left $\mathfrak{T}(\mathcal{O})$-invariant, these two spaces are actually equal (see Remark 4.22).

In [14], based on works of Kac and Peterson on the topology of $\mathfrak{G}(\mathbb{C})$, Hartnick, Köhl and Mars defined a Kac–Peterson topology on $\mathfrak{G}(\mathcal{F})$, for any local field $\mathcal{F}$ (Archimedean or not). Assume that $\mathcal{K}$ is local and let $\mathscr{T}_{KP}$ be the Kac–Peterson topology on G. We prove that when G is not reductive, then $\mathscr{T}$ is strictly coarser than $\mathscr{T}_{KP}$ (see Proposition 5.4) and thus $\mathscr{T}$ seems more adapted for our purpose.

Assume that $\mathfrak{G}$ is affine SL_2 (with a nonfree set of simple coroots). Then $G = \mathrm{SL}_2(\mathcal{K}[u,u^{-1}]) \rtimes \mathcal{K}^*$. Up to the assumption that $\ker \pi_n \subset \begin{pmatrix} 1+\varpi^n\mathcal{O}[u,u^{-1}] & \varpi^n\mathcal{O}[u,u^{-1}] \\ \varpi^n\mathcal{O}[u,u^{-1}] & 1+\varpi^n\mathcal{O}[u,u^{-1}] \end{pmatrix} \rtimes (1+\varpi^n\mathcal{O})$, for $n \in \mathbb{N}^*$, we prove that the topology $\mathscr{T}$ on G is associated to the filtration $(H_n)_{n\in\mathbb{N}^*}$, where $H_n = \ker(\pi_n) \cap \left(\begin{pmatrix} \mathcal{O}[(\varpi u)^n,(\varpi u^{-1})^n] & \mathcal{O}[(\varpi u)^n,(\varpi u^{-1})^n] \\ \mathcal{O}[(\varpi u)^n,(\varpi u^{-1})^n] & \mathcal{O}[(\varpi u)^n,(\varpi u^{-1})^n] \end{pmatrix}, \mathcal{K}^* \right)$.

The paper is organized as follows. In Section 2, we define Kac–Moody groups (as defined by Tits, Mathieu and Marquis) and the masures.

In Section 3, we define and study the subgroups $\ker \pi_n$ of $\mathfrak{G}^{\min}(\mathcal{O})$.

In Section 4, we define the topologies $\mathscr{T}$ and $\mathscr{T}_{\text{Fix}}$, and compare them.

In Section 5, we study the properties of $\mathscr{T}$ and $\mathscr{T}_{\text{Fix}}$: we prove that $\mathscr{T}_{KP}$ is strictly finer than $\mathscr{T}$, we describe the topology in the case of affine SL_2, and we prove that usual subgroups of G (i.e. T, N, B, etc.) are closed for $\mathscr{T}$.

Acknowledgements

I would like to thank Nicole Bardy-Panse and Guy Rousseau for the very helpful discussions we had on the subject. I also thank Stéphane Gaussent, Timothée Marquis and Dinakar Muthiah for useful conversations and suggestions and the anonymous referee for their very careful reading and useful comments.

2. Kac–Moody groups and masures

In this section, we define Kac–Moody groups and masures. Let $\mathcal{K}$ be a field. There are several possible definitions of Kac–Moody groups and we are interested in the minimal one $\mathfrak{G}(\mathcal{K})$, as defined by Tits in [31]. However, because of the lack of commutation relations in $\mathfrak{G}(\mathcal{K})$, it is convenient to embed it in its Mathieu's positive and negative completions $\mathfrak{G}^{\text{pma}}(\mathcal{K})$ and $\mathfrak{G}^{\text{nma}}(\mathcal{K})$. Then one define certain subgroups of $\mathfrak{G}(\mathcal{K})$ as the intersection of a subgroup of $\mathfrak{G}^{\text{pma}}(\mathcal{K})$ and $\mathfrak{G}(\mathcal{K})$. For example if $\mathfrak{G}$ is affine SL_2 (with a nonfree set of simple roots and coroots), then $\mathfrak{G}(\mathcal{K}) = \mathrm{SL}_2(\mathcal{K}[u,u^{-1}])$, $\mathfrak{G}^{\text{pma}}(\mathcal{K}) = \mathrm{SL}_2(\mathcal{K}(\!(u)\!))$ and $\mathfrak{G}^{\text{nma}}(\mathcal{K}) = \mathrm{SL}_2\left(\mathcal{K}(\!(u^{-1})\!)\right)$.

As we want to define congruence subgroups in our framework, we also need to work with Kac–Moody groups over rings: if $\mathcal{K}$ is equipped with a valuation ω and ϖ is such that $\omega(\varpi) = 1$, then we want to define $\ker \pi_n \subset \mathfrak{G}(\mathcal{O})$, where $\pi_n : \mathfrak{G}(\mathcal{O}) \to \mathfrak{G}(\mathcal{O}/\varpi^n\mathcal{O})$ is the natural projection. The functor defined by Tits in [31] goes from the category of rings to the category of groups. However the fact that it satisfies the axioms defined by Tits is proved only for fields (see [31, 3.9 Theorem 1]) and we do not know if it is "well-behaved" on rings, so we will consider it only as a functor from the category of fields to the category of groups. In [24, 8.8], Marquis introduces a functor $\mathfrak{G}^{\min}$ which goes from the category of rings to the category of groups and he proves that it has nice properties (see [24, Proposition 8.128]), especially on Bézout domains. We will use its functor $\mathfrak{G}^{\min}$. We have $\mathfrak{G}^{\min}(\mathcal{F}) \simeq \mathfrak{G}(\mathcal{F})$ for any field $\mathcal{F}$. This functor is defined as a subfunctor of $\mathfrak{G}^{\mathrm{pma}}$, so we first define Tits's functor, then Mathieu's functors and then Marquis's functor.

2.1. Standard apartment of a masure

2.1.1. Root generating system

A *Kac–Moody matrix* (or generalized Cartan matrix) is a square matrix $A = (a_{i,j})_{i,j\in I}$ indexed by a finite set I, with integral coefficients, and such that :

(i) $\forall\, i \in I,\ a_{i,i} = 2$;

(ii) $\forall\, (i, j) \in I^2, (i \neq j) \Rightarrow (a_{i,j} \leq 0)$;

(iii) $\forall\, (i, j) \in I^2,\ (a_{i,j} = 0) \Leftrightarrow (a_{j,i} = 0)$.

A *root generating system* is a 5-tuple $\mathcal{S} = (A, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ made of a Kac–Moody matrix A indexed by the finite set I, of two dual free $\mathbb{Z}$-modules X and Y of finite rank, and of a family $(\alpha_i)_{i\in I}$ (respectively $(\alpha_i^\vee)_{i\in I}$) of elements in X (resp. Y) called *simple roots* (resp. *simple coroots*) that satisfy $a_{i,j} = \alpha_j(\alpha_i^\vee)$ for all i, j in I. Elements of X (respectively of Y) are called *characters* (resp. *cocharacters*).

Fix such a root generating system $\mathcal{S} = (A, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ and set $\mathbb{A} := Y \otimes \mathbb{R}$. Each element of X induces a linear form on $\mathbb{A}$, hence X can be seen as a subset of the dual $\mathbb{A}^*$. In particular, the α_i's (with $i \in I$) will be seen as linear forms on $\mathbb{A}$. This allows us to define, for any $i \in I$, a *simple reflection* r_i of $\mathbb{A}$ by setting $r_i.v := v - \alpha_i(v)\alpha_i^\vee$ for any $v \in \mathbb{A}$. One defines the *Weyl group of* $\mathcal{S}$ as the subgroup W^v of $\mathrm{GL}(\mathbb{A})$ generated by $\{r_i \mid i \in I\}$. The pair $(W^v, \{r_i \mid i \in I\})$ is a Coxeter system, hence we can consider the length $\ell(w)$ with respect to $\{r_i \mid i \in I\}$ of any element w of W^v.

The following formula defines an action of the Weyl group W^v on $\mathbb{A}^*$:

$$\forall\, x \in \mathbb{A}, w \in W^v, \alpha \in \mathbb{A}^*,\ (w.\alpha)(x) := \alpha(w^{-1}.x).$$

Let $\Phi := \{w.\alpha_i \mid (w,i) \in W^v \times I\}$ (resp. $\Phi^\vee = \{w.\alpha_i^\vee \mid (w,i) \in W^v \times I\}$) be the set of *real roots* (resp. *real coroots*): then Φ (resp. $\Phi^\vee$) is a subset of the *root lattice* $Q := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ (resp. *coroot lattice* $Q^\vee = \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee$). By [23, 1.2.2(2)], one has $\mathbb{R}\alpha^\vee \cap \Phi^\vee = \{\pm\alpha^\vee\}$ and $\mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$ for all $\alpha^\vee \in \Phi^\vee$ and $\alpha \in \Phi$.

We define the *height* $\mathrm{ht} : Q \to \mathbb{Z}$ by $\mathrm{ht}(\sum_{i\in I} n_i\alpha_i) = \sum_{i\in I} n_i$, for $(n_i) \in \mathbb{Z}^I$.

2.1.2. **Vectorial apartment**

As in the reductive case, define the *fundamental chamber* as $C_f^v := \{v \in \mathbb{A} \mid \forall\ i \in I, \alpha_i(v) > 0\}$.

Let $\mathcal{T} := \bigcup_{w\in W^v} w.\overline{C_f^v}$ be the *Tits cone*. This is a convex cone (see [23, 1.4]).

For $J \subset I$, set $F^v(J) = \{x \in \mathbb{A} \mid \alpha_j(x) = 0,\ \forall\ j \in J \text{ and } \alpha_j(x) > 0,\ \forall\ j \in I \setminus J\}$. A *positive vectorial face* (resp. *negative*) is a set of the form $w.F^v(J)$ $(-w.F^v(J))$ for some $w \in W^v$ and $J \subset I$. Then by [26, 5.1 Théorème (ii)], the family of positive vectorial faces of $\mathbb{A}$ is a partition of $\mathcal{T}$ and the stabilizer of $F^v(J)$ is $W_J = \langle J \rangle$.

One sets $Y^{++} = Y \cap \overline{C_f^v}$ and $Y^+ = Y \cap \mathcal{T}$. An element of Y^+ is called *regular* if it does not belong to any wall, i.e. if it belongs to $\bigsqcup_{w\in W^v} w.C_f^v$.

Remark 2.1. By [21, Section 4.9] and [21, Section 5.8] the following conditions are equivalent:

(1) the Kac–Moody matrix A is of finite type (i.e. is a Cartan matrix),

(2) $\mathbb{A} = \mathcal{T}$

(3) W^v is finite.

2.2. **Split Kac–Moody groups over fields**

2.2.1. **Minimal Kac–Moody groups over fields**

Let $\mathfrak{G} = \mathfrak{G}_\mathcal{S}$ be the group functor associated in [31] with the root generating system $\mathcal{S}$, see also [26, 8]. Let $\mathcal{K}$ be a field. Let $G = \mathfrak{G}(\mathcal{K})$ be the *split Kac–Moody group over* $\mathcal{K}$ *associated with* $\mathcal{S}$. The group G is generated by the following subgroups:

- the fundamental torus $T = \mathfrak{T}(\mathcal{K})$, where $\mathfrak{T} = \mathrm{Spec}(\mathbb{Z}[X])$,
- the root subgroups $U_\alpha = \mathfrak{U}_\alpha(\mathcal{K})$, for $\alpha \in \Phi$, each isomorphic to $(\mathcal{K}, +)$ by an isomorphism x_α.

The groups X and Y correspond to the character lattice $\mathrm{Hom}(\mathfrak{T}, \mathbb{G}_m)$ and cocharacter lattice $\mathrm{Hom}(\mathbb{G}_m, \mathfrak{T})$ of $\mathfrak{T}$ respectively. One writes $\mathfrak{U}^{\pm}$ the subgroup of $\mathfrak{G}$ generated by the $\mathfrak{U}_\alpha$, for $\alpha \in \Phi^{\pm}$ and $U^{\pm} = \mathfrak{U}^{\pm}(\mathcal{K})$.

By a simple computation in SL_2, we have for $\alpha \in \Phi$ and $a, b \in \mathcal{K}$ such that $ab \neq -1$:

$$\begin{aligned} x_{-\alpha}(b)x_\alpha(a) &= x_\alpha(a(1+ab)^{-1})\alpha^\vee(1+ab)x_{-\alpha}(b(1+ab)^{-1}) \\ &= x_\alpha(a(1+ab)^{-1})x_{-\alpha}(b(1+ab))\alpha^\vee(1+ab), \end{aligned} \tag{2.1}$$

where $\alpha^\vee = w.\alpha_i^\vee$ if $\alpha = w.\alpha_i$, for $i \in I$ and $w \in W^v$.

Let $\mathfrak{N}$ be the group functor on rings such that if $\mathscr{R}'$ is a ring, $\mathfrak{N}(\mathscr{R}')$ is the subgroup of $\mathfrak{G}(\mathscr{R}')$ generated by $\mathfrak{T}(\mathscr{R}')$ and the $\widetilde{r}_i$, for $i \in I$, where

$$\widetilde{r}_i = x_{\alpha_i}(1)x_{-\alpha_i}(-1)x_{\alpha_i}(1). \tag{2.2}$$

Then if $\mathscr{R}'$ is a field with at least 4 elements, $\mathfrak{N}(\mathscr{R}')$ is the normalizer of $\mathfrak{T}(\mathscr{R}')$ in $\mathfrak{G}(\mathscr{R}')$.

Let $N = \mathfrak{N}(\mathcal{K})$ and $\mathrm{Aut}(\mathbb{A})$ be the group of affine automorphisms of $\mathbb{A}$. Then by [28, 1.4 Lemme], there exists a group morphism $\nu^v : N \to \mathrm{GL}(\mathbb{A})$ such that:

(1) for $i \in I$, $\nu^v(\widetilde{r}_i)$ is the simple reflection $r_i \in W^v$,

(2) $\ker \nu^v = T$.

The aim of the next two subsubsections is to define Mathieu's Kac–Moody group. This group is defined by assembling three ingredients: the group $\mathfrak{U}^{\mathrm{pma}}$, which corresponds to a maximal positive unipotent subgroup of $\mathfrak{G}^{\mathrm{pma}}$, the torus $\mathfrak{T}$ and copies of SL_2, one for each simple root α_i, $i \in I$.

2.2.2. **The affine group scheme $\mathfrak{U}^{\mathrm{pma}}$**

In this subsubsection, we define $\mathfrak{U}^{\mathrm{pma}}$. Let $\mathfrak{g}$ be the Kac–Moody Lie algebra over $\mathbb{C}$ associated with $\mathcal{S}$ (see [23, 1.2]) and $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ be its enveloping algebra. The group $\mathfrak{U}^{\mathrm{pma}}(\mathbb{C})$, will be defined as a subgroup of a completion of $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$. As we want to define $\mathfrak{U}^{\mathrm{pma}}(\mathscr{R})$, for any ring $\mathscr{R}$, we will also consider $\mathbb{Z}$-forms of $\mathfrak{g}$ and $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$.

The Lie algebra $\mathfrak{g}$ decomposes as $\mathfrak{g} = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, where $\Delta \subset Q$ is the *set of roots* and $\mathfrak{g}_\alpha$ is the proper space associated with α, for $\alpha \in \Delta$ (see [23, 1.2]). We have $\Delta = \Delta_+ \sqcup \Delta_-$, where $\Delta_+ = \Delta \cap Q_+$ and $\Delta_- = -\Delta_+$. We have $\Phi \subset \Delta$. The elements of $\Phi = \Delta_{re}$ are called *real roots* and the elements of $\Delta_{im} = \Delta \setminus \Phi$ are called *imaginary roots*.

Following [31, 4] one defines $\mathcal{U}$ as the $\mathbb{Z}$-subalgebra of $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ generated by $e_i^{(n)} := \frac{e_i^n}{n!}$, $f_i^{(n)} := \frac{f_i^n}{n!}$, $\binom{h}{n}$, for $i \in I$ and $h \in Y$ (where the e_i, f_i are the generators of $\mathfrak{g}$, see [23, 1.1]). This is a $\mathbb{Z}$-form of $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$. The algebra $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ decomposes as

$\mathcal{U}_{\mathbb{C}}(\mathfrak{g}) = \bigoplus_{\alpha\in Q} \mathcal{U}_{\mathbb{C}}(\mathfrak{g})_\alpha$ where we use the standard Q-graduation on $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ induced by the Q-graduation of $\mathfrak{g}$ (for $i \in I$, $\deg(e_i) = \alpha_i$, $\deg(f_i) = -\alpha_i$, $\deg(h) = 0$, for $h \in Y$, $\deg(xy) = \deg(x) + \deg(y)$ for all $x, y \in \mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ which can be written as a product of nonzero elements of $\mathfrak{g}$). For $\alpha \in Q$, one sets $\mathcal{U}_\alpha = \mathcal{U}_{\mathbb{C}}(\mathfrak{g})_\alpha \cap \mathcal{U}$ and $\mathcal{U}_{\alpha,\mathscr{R}} = \mathcal{U}_\alpha \otimes \mathscr{R}$.

For a ring $\mathscr{R}$, we set $\mathcal{U}_{\mathscr{R}} = \mathcal{U} \otimes_{\mathbb{Z}} \mathscr{R}$. One sets $\widehat{\mathcal{U}}^+ = \prod_{\alpha\in Q_+} \mathcal{U}_\alpha$ and $\widehat{\mathcal{U}}^+_{\mathscr{R}} = \prod_{\alpha\in Q_+} \mathcal{U}_{\alpha,\mathscr{R}}$. This is the completion of $\mathcal{U}^+$ with respect to the Q_+-gradation.

If $(u_\alpha) \in \prod_{\alpha\in Q_+} \mathcal{U}_{\alpha,\mathscr{R}}$, we write $\sum_{\alpha\in Q_+} u_\alpha$ the corresponding element of $\widehat{\mathcal{U}}^+_{\mathscr{R}}$. A sequence $(\sum_{\alpha\in Q_+} u_\alpha^{(n)})_{n\in\mathbb{N}}$ converges in $\widehat{\mathcal{U}}_{\mathscr{R}}$ if and only if for every $\alpha \in \Delta_+$, the sequence $(u_\alpha^{(n)})_{n\in\mathbb{N}}$ is stationary.

Let $(E, \leq)$ be a totally ordered set. Let $(u^{(e)}) \in (\widehat{\mathcal{U}}_{\mathscr{R}})^E$. For $e \in E$, write $u = \sum_{\alpha\in Q_+} u_\alpha^{(e)}$, with $u_\alpha^{(e)} \in \mathcal{U}_{\alpha,\mathscr{R}}$, for $\alpha \in Q_+$. We assume that for every $\alpha \in Q_+$, $\{e \in E \mid u_\alpha^{(e)} \neq 0\}$ is finite. Then one sets $\prod_{e\in E} u^{(e)} = \sum_{\alpha\in Q_+} u_\alpha$, where

$$u_\alpha = \sum_{\substack{(\beta_1,\dots,\beta_k)\in Q_+^{(\mathbb{N})},\\ \beta_1+\dots+\beta_k=\alpha}} \sum_{\substack{(e_1,\dots,e_k)\in E,\\ e_1<\dots<e_k}} u_{\beta_1}^{(e_1)} \dots u_{\beta_k}^{(e_k)} \in \mathcal{U}_{\alpha,\mathscr{R}},$$

for $\alpha \in \Delta_+$. This is well-defined since in the sum defining u_α, only finitely many nonzero terms appear.

Let $\mathcal{A} = \bigoplus_{\alpha\in Q_+} \mathcal{U}_\alpha^*$, where $\mathcal{U}_\alpha^*$ denotes the dual of $\mathcal{U}_\alpha$ (as a $\mathbb{Z}$-module). We have a natural $\mathscr{R}$-modules isomorphism between $\widehat{\mathcal{U}}^+_{\mathscr{R}}$ and $\mathrm{Hom}_{\mathbb{Z}\text{-lin}}(\mathcal{A}, \mathscr{R})$, for any ring $\mathscr{R}$ (see [24, (8.26)]) and we now identify these two spaces. The algebra $\mathcal{A}$ is equipped with a Hopf algebra structure (see [24, Definition 8.42]). This additional structure equips

$$\mathfrak{U}^{\mathrm{pma}}(\mathscr{R}) := \mathrm{Hom}_{\mathbb{Z}\text{-Alg}}(\mathcal{A}, \mathscr{R})$$

with the structure of a group (see [24, Appendix A.2.2]). Otherwise said, $\mathcal{A}$ is the representing algebra of the (infinite dimensional in general) affine group scheme $\mathfrak{U}^{\mathrm{pma}}$: $\mathbb{Z}$-Alg $\to$ Grp.

Let $\alpha \in \Delta \cup \{0\}$ and $x \in \mathfrak{g}_{\alpha,\mathbb{Z}}$. An *exponential sequence* for x is a sequence $(x^{[n]})_{n\in\mathbb{N}}$ of elements of $\mathcal{U}$ such that $x^{[0]} = 1$, $x^{[1]} = x$ and $x^{[n]} \in \mathcal{U}_{n\alpha}$ for $n \in \mathbb{Z}_{\geq 1}$ and satisfying the conditions of [24, Definition 8.45]. By [30, Proposition 2.7] or [24, Proposition 8.50], such a sequence exists. Note that it is not unique in general. However, if $\alpha \in \Phi_+$, the unique exponential sequence for x is $(x^{[n]})_{n\in\mathbb{N}} = (\frac{1}{n!}x^n)$ by [30, 2.9 2)] (this sequence is often denoted $(x^{(n)})_{n\in\mathbb{N}}$ in the literature).

For $r \in \mathscr{R}$, one then sets

$$[\exp](rx) =: \sum_{n\in\mathbb{N}} x^{[n]} \otimes r^n \in \widehat{\mathcal{U}}^+_{\mathscr{R}}.$$

This is the *twisted exponential* of rx associated with the sequence $(x^{[n]})_{n\in\mathbb{N}}$.

We fix for every $\alpha \in \Delta_+$ a $\mathbb{Z}$-basis $\mathcal{B}_\alpha$ of $\mathfrak{g}_{\alpha,\mathbb{Z}} := \mathfrak{g}_\alpha \cap \mathcal{U}$. Set $\mathcal{B} = \bigcup_{\alpha\in\Delta_+} \mathcal{B}_\alpha$. We fix an order on each $\mathcal{B}_\alpha$ and on Δ_+. Let $\alpha \in \Delta_+$. One defines $X_\alpha : \mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \mathscr{R} \to \mathfrak{U}^{\mathrm{pma}}(\mathscr{R})$ by $X_\alpha(\sum_{x\in\mathcal{B}_\alpha} \lambda_x.x) = \prod_{x\in\mathcal{B}_\alpha}[\exp]\lambda_x.x$, for $(\lambda_x) \in \mathscr{R}^{\mathcal{B}_\alpha}$. When $\alpha \in \Phi_+$, we have $\mathfrak{g}_{\alpha,\mathbb{Z}} = \mathbb{Z}e_\alpha$, where e_α is defined in [24, Remark 7.6]. One sets $x_\alpha(r) = [\exp](re_\alpha)$, for $r \in \mathscr{R}$. One has $X_\alpha(\mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \mathscr{R}) = x_\alpha(\mathscr{R}) := \mathfrak{U}_\alpha(\mathscr{R})$. By [24, Theorem 8.5.1], every $g \in \mathfrak{U}^{\mathrm{pma}}(\mathscr{R})$ can be written in a unique way as a product

$$g = \prod_{\alpha\in\Delta_+} X_\alpha(c_\alpha), \tag{2.3}$$

where $c_\alpha \in \mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \mathscr{R}$, for $\alpha \in \Delta_+$, where the product is taken in the given order on Δ_+.

Let $\Psi \subset \Delta_+$. We say that Ψ is *closed* if for all $\alpha, \beta \in \Psi$, for all $p, q \in \mathbb{N}^*$, $p\alpha + q\beta \in \Delta_+$ implies $p\alpha + q\beta \in \Psi$. Let $\Psi \subset \Delta_+$ be a closed subset. One sets

$$\mathfrak{U}^{\mathrm{pma}}_\Psi(\mathscr{R}) = \prod_{\alpha\in\Psi} X_\alpha(\mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \mathscr{R}) \subset \mathfrak{U}^{\mathrm{pma}}(\mathscr{R}).$$

This is a subgroup of $\mathfrak{U}^{\mathrm{pma}}$, which does not depend on the chosen order on Δ_+ (for the product). This is not the definition given in [30] or [24, p. 210], but it is equivalent by [24, Theorem 8.51].

2.2.3. Mathieu's group $\mathfrak{G}^{\mathrm{pma}}$

The *Borel subgroup* (it will be a subgroup of $\mathfrak{G}^{\mathrm{pma}}$) is $\mathfrak{B} = \mathfrak{B}_{\mathcal{S}} = \mathfrak{T}_{\mathcal{S}} \ltimes \mathfrak{U}^{\mathrm{pma}}$, where $\mathfrak{T}$ acts on $\mathfrak{U}^{\mathrm{pma}}$ as follows. Let $\mathscr{R}$ be a ring, $\alpha \in \Delta_+$, $t \in \mathfrak{T}(\mathscr{R})$, $r \in \mathscr{R}$ and $x \in \mathfrak{g}_{\alpha,\mathscr{R}}$,

$$t[\exp](rx)t^{-1} = [\exp](\alpha(t)rx). \tag{2.4}$$

In particular, if $\alpha \in \Phi$, we have

$$tx_\alpha(r)t^{-1} = x_\alpha(\alpha(t)r).$$

For $i \in I$, let $\mathfrak{U}^Y_{\alpha_i}$ be the reductive group associated with the root generating system $\big((2), X, Y, \alpha_i, \alpha_i^\vee\big)$. For each $i \in I$, Mathieu defines an (infinite dimensional) affine group scheme $\mathfrak{P}_i = \mathfrak{U}^Y_{\alpha_i} \ltimes \mathfrak{U}^{ma}_{\Delta_+\setminus\{\alpha_i\}}$ (see [24, Definition 8.65] for the definition of the action of $\mathfrak{U}_{-\alpha_i}$ on $\mathfrak{U}^{ma}_{\Delta_+\setminus\{\alpha_i\}}$), where $\mathfrak{U}_{-\alpha_i}$ is an affine group scheme over $\mathbb{Z}$ isomorphic to $\mathbb{G}_a$ (see [24, 7.4.3] for more details).

We do not detail the definition of $\mathfrak{G}^{\mathrm{pma}}$ and we refer to [25], [24, 8.7] or [30, 3.6]. This is an ind-group scheme containing the $\mathfrak{P}_i$ for every $i \in I$. Let $w \in W^v$ and write $w = r_{i_1} \dots r_{i_k}$, with $k = \ell(w)$ and $i_1, \dots, i_k \in I$. Then the multiplication map $\mathfrak{P}_{i_1} \times \dots \times \mathfrak{P}_{i_k} \to \mathfrak{G}^{\mathrm{pma}}$ is a scheme morphism, and we have $\mathfrak{G}^{\mathrm{pma}}(\mathscr{R}) = \bigcup_{(i_1,\dots,i_n)\in\mathrm{Red}(W^v)} \mathfrak{P}_{i_1}(\mathscr{R}) \times \dots \times \mathfrak{P}_{i_k}(\mathscr{R})$, where $\mathrm{Red}(W^v)$ is the set of reduced words of W^v (i.e. $\mathrm{Red}(W^v) = \{(i_1, \dots, i_k) \in I^{(\mathbb{N})} \mid \ell(r_{i_1} \dots r_{i_k}) = k\}$).

Let $w \in W^v$, $i \in I$ and $\alpha = w.\alpha_i$. One sets $\mathfrak{U}_\alpha = \widetilde{w}.\mathfrak{U}_{\alpha_i}.\widetilde{w}^{-1}$, where

$$\widetilde{w} = \widetilde{r}_{i_1} \dots \widetilde{r}_{i_k}, \tag{2.5}$$

if $w = r_{i_1} \dots r_{i_k}$ is a reduced decomposition of w. There is an isomorphism of group schemes $x_\alpha : \mathbb{G}_a \to \mathfrak{U}_\alpha$ (see [24, p. 262]). The group $\mathfrak{G}^{\mathrm{pma}}$ is generated by the $\mathfrak{P}_i$, $i \in I$. Moreover, if $i \in I$, then $\mathfrak{P}_i$ is generated by $\mathfrak{T}$, $\mathfrak{U}_{\pm\alpha_i}$ and $\widetilde{r}_i = x_{\alpha_i}(1)x_{-\alpha_i}(1)x_{\alpha_i}(1)$. Thus $\mathfrak{G}^{\mathrm{pma}}$ is generated by $\mathfrak{U}^{\mathrm{pma}}$, $\mathfrak{T}$, $\mathfrak{U}_{-\alpha_i}$ and the $\widetilde{r}_i$, for $i \in I$ and thus we have:

$$\mathfrak{G}^{\mathrm{pma}} = \langle \mathfrak{U}^{\mathrm{pma}}, \mathfrak{T}, \mathfrak{U}_\alpha, \alpha \in \Phi_- \rangle. \tag{2.6}$$

There is a group functor morphism $\iota : \mathfrak{G} \to \mathfrak{G}^{\mathrm{pma}}$ such that for any ring $\mathscr{R}$, $\iota_{\mathscr{R}}$ maps $x_\alpha(r)$ to $x_\alpha(r)$ and t to t, for each $\alpha \in \Phi$, $r \in \mathscr{R}$, $t \in \mathfrak{T}(\mathscr{R})$. When $\mathscr{R}$ is a field, this morphism is injective (see [30, 3.12] or [24, Proposition 8.117]).

Proposition 2.2. *Let $\mathscr{R}$ and $\mathscr{R}'$ be two rings and $\varphi : \mathscr{R} \to \mathscr{R}'$ be a ring morphism.*

Let $f_\varphi^{\widehat{\mathcal{U}}^+} : \widehat{\mathcal{U}}^+_{\mathscr{R}} \to \widehat{\mathcal{U}}^+_{\mathscr{R}'}$ and $f_\varphi : \mathfrak{G}^{\mathrm{pma}}(\mathscr{R}) \to \mathfrak{G}^{\mathrm{pma}}(\mathscr{R}')$ be the induced morphisms. Then $f_\varphi^{\widehat{\mathcal{U}}^+}(\mathfrak{U}^{\mathrm{pma}}(\mathscr{R})) \subset \mathfrak{U}^{\mathrm{pma}}(\mathscr{R}')$ and we have:

(1) *For every $(r_x) \in \mathscr{R}^{\mathcal{B}}$,*

$$f_\varphi^{\widehat{\mathcal{U}}^+}\left(\prod_{x \in \mathcal{B}} [\exp](r_x x)\right) = \prod_{x \in \mathcal{B}} [\exp](\varphi(r_x)x).$$

(2) *For $\alpha \in \Delta_+$ and $(\lambda_x) \in \mathscr{R}^{\mathcal{B}_\alpha}$, we have*

$$f_\varphi\left(X_\alpha\left(\sum_{x \in \mathcal{B}_\alpha} \lambda_x x\right)\right) = X_\alpha\left(\sum_{x \in \mathcal{B}_\alpha} \varphi(\lambda_x)x\right).$$

(3) *We have $f_\varphi(u) = f_\varphi^{\widehat{\mathcal{U}}^+}(u)$ for $u \in \mathfrak{U}^{\mathrm{pma}}(\mathscr{R})$, $f_\varphi(x_\alpha(r)) = x_\alpha(\varphi(r))$, for $\alpha \in \Phi$ and $r \in \mathscr{R}$, and $f_\varphi(\chi(r)) = \chi(\varphi(r))$, for $\chi \in Y$ and $r \in \mathscr{R}^\times$.*

(4) *If φ is surjective, then f_φ is surjective.*

Proof.

(1), (2). By definition, we have

$$f_\varphi^{\widehat{\mathcal{U}}^+}\left(\sum_{\alpha \in Q^+} \sum_{j \in J_\alpha} u_{\alpha,j} \otimes r_j\right) = \sum_{\alpha \in Q^+} \sum_{j \in J_\alpha} u_{\alpha,j} \otimes \varphi(r_j)$$

if J_α is a finite set and $(r_j) \in \mathscr{R}^{J_\alpha}$ and $u_{\alpha,j} \in \mathcal{U}_{\alpha,\mathscr{R}}$, for every $\alpha \in Q_+$. Thus φ commutes with infinite sums and product, which proves (1) and (2).

(3). Let $i \in I$. Then the morphism $\mathfrak{P}_i(\mathscr{R}) \to \mathfrak{P}_i(\mathscr{R}')$ induced by φ satisfies the formula above. Using the fact that $x_\alpha = \widetilde{w} x_{-\alpha_i} \widetilde{w}^{-1}$, for $\alpha = -w.\alpha_i$, with $w \in W^v$, $i \in I$ and $\widetilde{w}$ defined as in (2.5), we have (3).

(4). Assume φ is surjective. By (2.3) and (1), the restriction of f_φ to $\mathfrak{U}^{\mathrm{pma}}(\mathscr{R})$ is surjective. By (3), the restriction of $f_\varphi(\mathfrak{U}^-(\mathscr{R})) = \mathfrak{U}^-(\mathscr{R}')$ and $f_\varphi(\mathfrak{T}(\mathscr{R})) = \mathfrak{T}(\mathscr{R}')$. We conclude by using the fact that $\mathfrak{G}^{\mathrm{pma}}$ is generated by $\mathfrak{U}^{\mathrm{pma}}$, $\mathfrak{U}^-$ and $\mathfrak{T}$ (see (2.6)). □

2.2.4. **Minimal Kac–Moody group over rings**

For $i \in I$, there is a natural group morphism $\varphi_i : \mathrm{SL}_2 \to \mathfrak{U}^Y_{\alpha_i}$.

For a ring $\mathscr{R}$, one sets

$$\mathfrak{G}^{\min}(\mathscr{R}) = \langle \varphi_i\left(\mathrm{SL}_2(\mathscr{R})\right), \mathfrak{T}(\mathscr{R}) \rangle \subset \mathfrak{G}^{\mathrm{pma}}(\mathscr{R}).$$

This group is introduced by Marquis in [24, Definition 8.126]. By [24, Proposition 8.129], it is a nondegenerate Tits functor in the sense of [24, Definition 7.83] and we have $\mathfrak{G}^{\min}(\mathscr{R}) \simeq \mathfrak{G}(\mathscr{R})$, for any field $\mathscr{R}$.

Note that if φ is a ring morphism between two rings $\mathscr{R}$ and $\mathscr{R}'$, the induced morphism $\mathfrak{G}^{\mathrm{pma}}(\mathscr{R}) \to \mathfrak{G}^{\mathrm{pma}}(\mathscr{R}')$ restricts to a morphism $\mathfrak{G}^{\min}(\mathscr{R}) \to \mathfrak{G}^{\min}(\mathscr{R}')$.

Let $\mathscr{R}$ be a semilocal ring, i.e. a ring with finitely many maximal ideals, then by [13, 4.3.9 Theorem], $\mathrm{SL}_2(\mathscr{R})$ is generated by $\left(\begin{smallmatrix} 1 & \mathscr{R} \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ \mathscr{R} & 1 \end{smallmatrix}\right)$. Therefore,

$$\mathfrak{G}^{\min}(\mathscr{R}) = \langle \mathfrak{U}_{\pm\alpha_i}(\mathscr{R}), \mathfrak{T}(\mathscr{R}) \mid i \in I \rangle \subset \mathfrak{G}^{\mathrm{pma}}(\mathscr{R}). \tag{2.7}$$

2.3. **Split Kac–Moody groups over valued fields and masures**

We now fix a field $\mathcal{K}$ equipped with a valuation $\omega : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ such that $\Lambda := \omega(\mathcal{K}^*)$ contains $\mathbb{Z}$. Let $\mathcal{O} = \{x \in \mathcal{K} \mid \omega(x) \geq 0\}$ be its ring of valuation. We defined Mathieu's positive completion $\mathfrak{G}^{\mathrm{pma}}$. Replacing Δ_+ by $w.\Delta_+$, for $w \in W^v$, one can also define a group $\mathfrak{G}^{pma,w}$. Replacing Δ_+ by Δ_- or by $w.\Delta_-$, for $w \in W^v$, one can also define $\mathfrak{G}^{\mathrm{nma}}$ or $\mathfrak{G}^{nma,w}$.

We set $G = \mathfrak{G}(\mathcal{K})$, $G^{\mathrm{pma}} = \mathfrak{G}^{\mathrm{pma}}(\mathcal{K})$ and $G^{\mathrm{nma}} = \mathfrak{G}^{\mathrm{nma}}(\mathcal{K})$.

2.3.1. **Action of N on $\mathbb{A}$**

Let $N = \mathfrak{N}(\mathcal{K})$ and $\mathrm{Aut}(\mathbb{A})$ be the group of affine automorphism of $\mathbb{A}$. Then by [30, 4.2], there exists a group morphism $\nu : N \to \mathrm{Aut}(\mathbb{A})$ such that:

(1) for $i \in I$, $\nu(\widetilde{r}_i)$ is the simple reflection $r_i \in W^v$, it fixes 0,

(2) for $t \in \mathfrak{T}(\mathcal{K})$, $\nu(t)$ is the translation on $\mathbb{A}$ by the vector $\nu(t)$ defined by $\chi(\nu(t)) = -\omega(\chi(t))$, for all $\chi \in X$.

(3) we have $\nu(N) = W^v \ltimes (Y \otimes \Lambda) := W_\Lambda$.

2.3.2. **Affine apartment**

A *local face* in $\mathbb{A}$ is the germ $F(x, F^v) = \mathrm{germ}_x(x + F^v)$ where $x \in \mathbb{A}$ and F^v is a vectorial face (i.e., $F(x, F^v)$ is the filter of all neighbourhoods of x in $x + F^v$). It is a *local panel*, positive, or negative if F^v is. If F^v is a chamber, we call $F(x, F^v)$ an *alcove* (or a local chamber). We denote by C_0^+ the *fundamental alcove*, i.e., $C_0^+ = \mathrm{germ}_0(C_f^v)$.

A *sector* in $\mathbb{A}$ is a subset $\mathfrak{q} = x + C^v$, for x a point in $\mathbb{A}$ and C^v a vectorial chamber. Its *sector germ* (at infinity) is the filter $\mathfrak{Q} = \mathrm{germ}_\infty(\mathfrak{q})$ of subsets of $\mathbb{A}$ containing another sector $x + y + C^v$, with $y \in C^v$. It is entirely determined by its direction C^v. This sector or sector germ is said positive (resp. negative) if C^v has this property. We denote by $\pm\infty$ the germ at infinity of $\pm C_f^v$.

For $\alpha \in \Delta$ and $k \in \Lambda \cup \{+\infty\}$, we set $D(\alpha, k) = \{x \in \mathbb{A} \mid \alpha(x) + k \geq 0\}$. A set of the form $D(\alpha, k)$, for $\alpha \in \Delta$ and $k \in \Lambda$ is called a *half-apartment*.

2.3.3. **Parahoric subgroups**

In [10] and [30], the masure $\mathcal{I}$ of G is constructed as follows. To each $x \in \mathbb{A}$ is associated a group $\widehat{P}_x = G_x$. Then $\mathcal{I}$ is defined in such a way that G_x is the fixator of x in G for the action on $\mathcal{I}$. We actually associate to each filter Ω on $\mathbb{A}$ a subgroup $G_\Omega \subset G$ (with $G_{\{x\}} = G_x$ for $x \in \mathbb{A}$). Even though the masure is not yet defined, we use the terminology "fixator" to speak of G_Ω, as this will be the fixator of Ω in G. The definition of G_Ω involves the completed groups G^{pma} and G^{nma}.

If Ω is a non empty subset of $\mathbb{A}$ we sometimes regard it as a filter on $\mathbb{A}$ by identifying it with the filter consisting of the subsets of $\mathbb{A}$ containing Ω. Let $\Omega \subset \mathbb{A}$ be a non empty set or filter. One defines a function $f_\Omega : \Delta \to \mathbb{R}$ by

$$f_\Omega(\alpha) = \inf\{r \in \mathbb{R} \mid \Omega \subset D(\alpha + r)\} = \inf\{r \in \mathbb{R} \mid \alpha(\Omega) + r \subset [0, +\infty[\},$$

for $\alpha \in \Delta$. For $r \in \mathbb{R}$, one sets $\mathcal{K}_{\omega \geq r} = \{x \in \mathcal{K} \mid \omega(x) \geq r\}$, $\mathcal{K}_{\omega = r} = \{x \in \mathcal{K} \mid \omega(x) = r\}$.

If Ω is a set, we define the subgroup $U_\Omega^{\mathrm{pma}} = \prod_{\alpha \in \Delta_+} X_\alpha(\mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \mathcal{K}_{\omega \geq f_\Omega(\alpha)}) \subset G^{\mathrm{pma}}$. Actually, for $\alpha \in \Phi^+ = \Delta_{re}^+$, $X_\alpha(\mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \mathcal{K}_{\omega \geq f_\Omega(\alpha)}) = x_\alpha(\mathcal{K}_{\omega \geq f_\Omega(\alpha)}) =: U_{\alpha,\Omega}$. We then define

$$U_\Omega^{pm+} = U_\Omega^{\mathrm{pma}} \cap G = U_\Omega^{\mathrm{pma}} \cap U^+,$$

see [30, 4.5.2, 4.5.3 and 4.5.7]. When Ω is a filter, we set $U_\Omega^{\mathrm{pma}} := \bigcup_{S \in \Omega} U_S^{\mathrm{pma}}$ and $U_\Omega^{pm+} := U_\Omega^{\mathrm{pma}} \cap G$

We may also consider the negative completion $G^{\text{nma}} = \mathfrak{G}^{\text{nma}}(\mathcal{K})$ of G, and define the subgroup $U_\Omega^{ma-} = \prod_{\alpha\in\Delta_-} X_\alpha(\mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \mathcal{K}_{\omega\geq f_\Omega(\alpha)})$. For $\alpha \in \Phi^- = \Delta_{re}^-$, $X_\alpha(\mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \mathcal{K}_{\omega\geq f_\Omega(\alpha)}) = x_\alpha(\mathcal{K}_{\omega\geq f_\Omega(\alpha)}) =: U_{\alpha,\Omega}$. We then define $U_\Omega^{nm-} = U_\Omega^{ma-} \cap G = U_\Omega^{ma-} \cap U^-$.

Let Ψ be a closed subset of Δ_+. One sets $U_\Omega^{pm}(\Psi) = \mathfrak{U}_\Psi^{\text{pma}}(\mathcal{K}) \cap U_\Omega^{pm+}$. By the uniqueness in the decomposition of the elements of U^{pma} as a product, every element of $U_\Omega^{pm}(\Psi)$ belongs to $\prod_{\alpha\in\Psi} X_\alpha(\mathcal{K}_{\omega\geq f_\Omega(\alpha)})$. If Ψ is a closed subset of Δ^-, one sets $U_\Omega^{nm}(\Psi) = \mathfrak{U}_\Psi^{\text{nma}}(\mathcal{K}) \cap U_\Omega^{nm-}$. Note that $U_\Omega^{pm+} = U_\Omega^{pm}(\Delta_+)$ and $U_\Omega^{nm-} = U_\Omega^{nm}(\Delta_-)$.

Let Ω be a filter on $\mathbb{A}$. We denote by N_Ω the fixator of Ω in N (for the action of N on $\mathbb{A}$). If Ω is not a set, we have $N_\Omega = \bigcup_{S\in\Omega} N_S$. Note that we drop the hats used in [30]. When Ω is open one has $N_\Omega = N_{\mathbb{A}} = \mathfrak{T}(\mathcal{O}) := \mathfrak{T}(\mathcal{K}_{\omega\geq 0}) = \mathfrak{T}(\mathcal{K}_{\omega=0})$.

If $x \in \mathbb{A}$, we set $G_x = U_x^{pm+}.U_x^{nm-}.N_x$. This is a subgroup of G. If $\Omega \subset \mathbb{A}$ is a set, we set $G_\Omega = \bigcap_{x\in\Omega} G_x$ and if Ω is a filter, we set $G_\Omega = \bigcup_{S\in\Omega} G_S$. Note that in [30], the definition of G_x is much more complicated (see [30, Définition 4.13]). However it is equivalent to this one by [30, Proposition 4.14].

A filter is said to have a "good fixator" if it satisfies [30, Définition 5.3]. There are many examples of filters with good fixators (see [30, 5.7]): points, local faces, sectors, sector germs, $\mathbb{A}$, walls, half apartments. . . . For such a filter Ω, we have:

$$G_\Omega = U_\Omega^{pm+}.U_\Omega^{nm-}.N_\Omega = U_\Omega^{nm-}.U_\Omega^{pm+}.N_\Omega. \tag{2.8}$$

We then have:

$$U_\Omega^{pm+} = G_\Omega \cap U^+ \text{ and } U_\Omega^{nm-} = G_\Omega \cap U^-, \tag{2.9}$$

as $U^- \cap U^+.N = U^+ \cap N = \{1\}$, by [30, Remarque 3.17] and [26, 1.2.1 (RT3)].

When $\Omega = C_0^+ = \text{germ}_0(C_f^v)$ is the (fundamental) positive local chamber in $\mathbb{A}$, $K_I := G_\Omega$ is called the (fundamental) *Iwahori subgroup*. When Ω is a face of C_0^+, G_Ω is called a *parahoric subgroup*.

For Ω a set or a filter, one defines:

$$U_\Omega = \langle U_{\alpha,\Omega} \mid \alpha \in \Phi\rangle, \quad U_\Omega^\pm = U_\Omega \cap U^\pm \quad \text{and} \quad U_\Omega^{\pm\pm} = \langle U_{\alpha,\Omega} \mid \alpha \in \Phi^\pm\rangle.$$

Then one has $U_\Omega^{++} \subset U_\Omega^+ \subset U_\Omega^{pm+}$, but these inclusions are not equalities in general, contrary to the reductive case (see [30, 4.12.3a and 5.7(3)]).

Lemma 2.3. *Let $(u_+, u_-, t), (u'_+, u'_-, t') \in U^+ \times U^- \times T$. Assume that $u_+tu_- = u'_+t'u'_-$ or $u_+u_-t = u'_+u'_-t'$ or $tu_+u_- = t'u'_+u'_-$. Then $u_- = u'_-$, $u_+ = u'_+$ and $t = t'$.*

Proof. Assume $u_+tu_- = u'_+t'u'_-$. We have $(u'_+)^{-1}u_+t = t'u_-(u'_-)^{-1}$. As t normalizes U^-, we deduce the existence of u''_- such that $(u'_+)^{-1}u_+tt'^{-1} = u''_-$. By [28, Proposition 1.5 (DR5)] (there is a misprint in this proposition, Z is in fact T), we deduce $(u'_+)^{-1}u_+tt'^{-1} = 1$ and hence $u'_+ = u_+$ and $t = t'$. Therefore $u_- = u'_-$. The other cases are similar. □

By [30, 4.10] and (2.4), we have the following lemma.

Lemma 2.4. *Let Ω be a filter on $\mathbb{A}$, $t \in T$ and Ψ be a closed subset of Δ_+ (resp. Δ_-). Then $tU_{\Omega}^{pm+}t^{-1} = U_{t.\Omega}^{pm+}$, $tU_{\Omega}^{pm}(\Psi)t^{-1} = U_{t.\Omega}^{pm}(\Psi)$ (resp. $tU_{\Omega}^{nm-}(\Psi)t^{-1} = U_{t.\Omega}^{nm-}(\Psi)$).*

2.3.4. **Masure**

We now define the masure $\mathcal{I} = \mathcal{I}(\mathfrak{G}, \mathcal{K}, \omega)$. As a set, $\mathcal{I} = G \times \mathbb{A}/\sim$, where $\sim$ is defined as follows:

$$\forall\ (g,x), (h,y) \in G \times \mathbb{A}, (g,x) \sim (h,y) \iff \exists\ n \in N \mid y = \nu(n).x \text{ and } g^{-1}hn \in G_x.$$

We regard $\mathbb{A}$ as a subset of $\mathcal{I}$ by identifying x and $(1,x)$, for $x \in \mathbb{A}$. The group G acts on $\mathcal{I}$ by $g.(h,x) = (gh,x)$, for $g,h \in G$ and $x \in \mathbb{A}$. An *apartment* is a set of the form $g.\mathbb{A}$, for $g \in G$. The stabilizer of $\mathbb{A}$ in G is N and if $x \in \mathbb{A}$, then the fixator of x in G is G_x. More generally, when $\Omega \subset \mathbb{A}$, then G_Ω is the fixator of Ω in G and G_Ω permutes transitively the apartments containing Ω. If A is an apartment, we transport all the notions that are preserved by W_Λ (for example segments, walls, faces, chimneys, etc.) to A. Then by [18, Corollary 3.7], if $(\alpha_i)_{i\in I}$ and $(\alpha_i^\vee)_{i\in I}$ are free, then $\mathcal{I}$ satisfies the following axioms:

MA II. Let A, A' be two apartments. Then $A \cap A'$ is a finite intersection of half-apartments and there exists $g \in G$ such that $g.A = A'$ and g fixes $A \cap A'$.

MA III. If $\mathfrak{R}$ is the germ of a splayed chimney and if F is a local face or a germ of a chimney, then there exists an apartment containing $\mathfrak{R}$ and F.

We did not recall the definition of a chimney and we refer to [29] for such a definition. We will only use the fact that a sector-germ is a particular case of a germ of a splayed chimney.

We also have:

- The stabilizer of $\mathbb{A}$ in G is N and N acts on $\mathbb{A} \subset \mathcal{I}$ via ν.
- The group $U_{\alpha,r} := \{x_\alpha(u) \mid u \in \mathcal{K}, \omega(u) \geq r\}$, for $\alpha \in \Phi, r \in \Lambda$, fixes the half-apartment $D(\alpha,r)$. It acts simply transitively on the set of apartments in $\mathcal{I}$ containing $D(\alpha,r)$.

The first point of the next proposition extends [7, Proposition 7.4.8] to masures.

Proposition 2.5.

(1) *Let $g \in G$. Then $\mathbb{A} \cap g^{-1}.\mathbb{A}$ is a finite intersection of half-apartments and there exists $n \in N$ such that $g.x = n.x$ for every $x \in \mathbb{A} \cap g^{-1}.\mathbb{A}$.*

(2) *Let $g \in G$. Then $\{x \in \mathbb{A} \mid g.x = x\}$ is convex. In particular if Ω is a subset of $\mathbb{A}$, then $G_\Omega = G_{\mathrm{conv}(\Omega)}$, where* conv$(\Omega)$ *is the convex hull of Ω.*

Proof.

(1). Let $g \in G$. We assume that $\mathbb{A} \cap g^{-1}.\mathbb{A}$ is non-empty. Then it is a finite intersection of half-apartments by (MA II) and there exists $h \in G$ such that $hg.\mathbb{A} = \mathbb{A}$ and h fixes $\mathbb{A} \cap g.\mathbb{A}$. Then hg stabilizes $\mathbb{A}$ and thus it belongs to N, by [30, 5.7 5)]. We get (1) by setting $n = hg$.

(2). Let $g \in G$, $\Omega_1 = \mathbb{A} \cap g^{-1}.\mathbb{A}$ and $\Omega = \{x \in \mathbb{A} \mid g.x = x\}$. We have $\Omega \subset \Omega_1$. Let $n \in N$ be such that $g.x = n.x$ for all $x \in \Omega_1$. Let $f = \nu(n) : \mathbb{A} \to \mathbb{A}$. Then $\Omega = \Omega_1 \cap \{x \in \mathbb{A} \mid f(x) = x\}$. As f is affine and Ω_1 is convex, we have that Ω is convex. □

Remark 2.6. In 2.1.1, we did not assume the freeness of the families $(\alpha_i)_{i\in I}$ and $(\alpha_i^\vee)_{i\in I}$, since there are interesting Kac–Moody groups, which do not satisfy this assumption. For example, $G := \mathrm{SL}_n(\mathcal{K}[u, u^{-1}]) \rtimes \mathcal{K}^*$ is naturally equipped with the structure of a Kac–Moody group associated with a root generating system $\mathcal{S}$ having nonfree coroots. This group is particularly interesting for examples, since it is one of the only Kac–Moody groups in which we can make explicit computations. In [18], we proved that if $(\alpha_i)_{i\in I}$ and $(\alpha_i^\vee)_{i\in I}$ are free families, then the masure associated with G satisfies (MA II) and (MA III). Without this assumption we do not know. In [30, Théorème 5.16], Rousseau proves that $\mathcal{I}$ satisfies the axioms (MA2) to (MA5) of [29]. We did not introduce these axioms since they are more complicated and a bit less convenient. However it is easy to adapt the proofs of this paper to use the axioms of [29] instead of those of [18], for example, retractions are already available in [29].

2.3.5. Retraction centred at a sector-germ

Let $\mathfrak{Q}$ be a sector-germ of $\mathbb{A}$. If $x \in \mathcal{I}$, then by (MA III), there exists an apartment A of $\mathcal{I}$ containing $\mathfrak{Q}$ and x. By (MA II), there exists $g \in G$ such that $g.A = \mathbb{A}$ and g fixes $A \cap \mathbb{A}$. One sets $\rho_{\mathfrak{Q}}(x) = g.x \in \mathbb{A}$. This is well-defined, independently of the choices of A and g, by (MA II). This defines the *retraction* $\rho_{\mathfrak{Q}} : \mathcal{I} \to \mathbb{A}$ *onto* $\mathbb{A}$ *and centred at* $\mathfrak{Q}$. When $\mathfrak{Q} = +\infty$, we denote it $\rho_{+\infty}$. If $x \in \mathcal{I}$, then $\rho_{+\infty}(x)$ is the unique element of $U^+.x \cap \mathbb{A}$.

2.3.6. Topology defined by a filtration

A *filtration of* G *by subgroups* is a sequence $(V_n)_{n\in\mathbb{N}^*}$ of subgroups of G such that $V_{n+1} \subset V_n$ for all $n \in \mathbb{N}^*$. Let (V_n) be a filtration of G by subgroups. The associated topology $\mathscr{T}((V_n))$ is the topology on G for which a set V is open if for all $g \in V$, there exists $n \in \mathbb{N}^*$ such that $g.V_n \subset V$.

Let $(V_n), (\widetilde{V}_n)$ be two filtrations of G by subgroups. We say that (V_n) and $(\widetilde{V}_n)$ are equivalent if for all $n \in \mathbb{N}$, there exist $m, \widetilde{m} \in \mathbb{N}$ such that $V_m \subset \widetilde{V}_n$ and $\widetilde{V}_{\tilde{m}} \subset V_n$. This defines an equivalence relation on the set of filtrations of G by subgroups. Then (V_n) and $(\widetilde{V}_n)$ are equivalent filtrations, if and only if $\mathscr{T}((V_n)) = \mathscr{T}\big((\widetilde{V}_n)\big)$.

We say that (V_n) is *conjugation-invariant* if for all $g \in G$, (gV_ng^{-1}) is equivalent to (V_n). Then $\mathscr{T}((V_n))$ equips G with the structure of a topological group if and only if (V_n) is conjugation invariant, by [24, Exercise 8.5].

3. Congruence subgroups

In this section, we define and study the congruence subgroups. They will be a key tool in order to define the topology $\mathscr{T}$ in Section 4. We prove however that the filtration $(\ker \pi_n)_{n\in\mathbb{N}^*}$ is not conjugation-invariant. We also study how they decompose.

3.1. Definition of the congruence subgroup

Proposition 3.1. *The fixator G_0 of 0 in G is the group $\mathfrak{G}^{\min}(\mathcal{O})$.*

Proof. For $i \in I$, $x_{\alpha_i}(\mathcal{O}), x_{-\alpha_i}(\mathcal{O})$ and $\mathfrak{T}(\mathcal{O})$ fix 0. Therefore by (2.7), $\mathfrak{G}^{\min}(\mathcal{O}) \subset G_0$.

By [30, Proposition 4.14]

$$G_0 = U_0^{pm+} U_0^{nm-} N_0, \tag{3.1}$$

where $N_0 = \{n \in N \mid n.0 = 0\}$. By [3, 2.4.1 2)], we have

$$U_0^{pm+} = U_0^+ := \langle x_\alpha(u) \mid \alpha \in \Phi, u \in \mathcal{O}\rangle \cap U^+ \subset \mathfrak{G}^{\min}(\mathcal{O})$$

and

$$U_0^{nm-} = U_0^- := \langle x_\alpha(u) \mid \alpha \in \Phi, u \in \mathcal{O}\rangle \cap U^- \subset \mathfrak{G}^{\min}(\mathcal{O}).$$

For $i \in I$, set $\widetilde{r}_i = x_{\alpha_i}(1)x_{-\alpha_i}(-1)x_{\alpha_i}(1) \in \mathfrak{G}^{\min}(\mathcal{O})$. We have $N = \langle \mathfrak{T}(\mathcal{K}), \widetilde{r}_i \mid i \in I\rangle$. Let $n \in N_0$. Write $\nu^v(n) = w \in W^v$, where ν^v was defined in 2.2.1. Write $w = r_{i_1} \dots r_{i_k}$, with $k = \ell(w)$ and $i_1, \dots, i_k \in I$. Let $n' = \widetilde{r}_{i_1} \dots \widetilde{r}_{i_k} \in N_0$. By [30, 1.6 4)] $\nu^v(n') = w$ and $t := n'^{-1}n \in T \cap \ker(\nu)$. By [30, 4.2 3)], $t \in \mathfrak{T}(\mathcal{O})$. Therefore

$$N_0 = \langle \widetilde{r}_i \mid i \in I\rangle.\mathfrak{T}(\mathcal{O}). \tag{3.2}$$

and in particular, $N_0 \subset \mathfrak{G}^{\min}(\mathcal{O})$. Proposition follows. □

Recall that we assumed that $\Lambda = \omega(\mathcal{K}^*) \supset \mathbb{Z}$. If $\omega(\mathcal{K}^*)$ is discrete, we can normalize ω so that $\Lambda = \mathbb{Z}$. We fix $\varpi \in \mathcal{O}$ such that $\omega(\varpi) = 1$.

For $n \in \mathbb{N}^*$, we denote by $\pi_n^{\mathrm{pma}} : \mathfrak{G}^{\mathrm{pma}}(\mathcal{O}) \to \mathfrak{G}^{\mathrm{pma}}(\mathcal{O}/\varpi^n\mathcal{O})$ and $\pi_n^{\mathrm{nma}} : \mathfrak{G}^{\mathrm{nma}}(\mathcal{O}) \to \mathfrak{G}^{\mathrm{nma}}(\mathcal{O}/\varpi^n\mathcal{O})$ the morphisms associated with the canonical projection $\mathcal{O} \twoheadrightarrow \mathcal{O}/\varpi^n\mathcal{O}$. We denote by π_n the restriction of π_n^{pma} to $\mathfrak{G}^{\min}(\mathcal{O})$. By Proposition 2.2, π_n is also the restriction of π_n^{nma} to $\mathfrak{G}^{\min}(\mathcal{O})$ (it is also the restrictions of $\pi_n^{pma,w} : \mathfrak{G}^{pma,w}(\mathcal{O}) \to \mathfrak{G}^{pma,w}(\mathcal{O}/\varpi^n\mathcal{O})$ and $\pi_n^{nma,w} : \mathfrak{G}^{nma,w}(\mathcal{O}) \to \mathfrak{G}^{nma,w}(\mathcal{O}/\varpi^n\mathcal{O})$, for $w \in W^v$). By Proposition 2.2 and (2.7), π_n, π_n^{pma} and π_n^{nma} are surjective. Their kernels are respectively called the *n-th congruence subgroups* of $\mathfrak{G}^{\min}(\mathcal{O})$, $\mathfrak{G}^{\mathrm{pma}}(\mathcal{O})$ and $\mathfrak{G}^{\mathrm{nma}}(\mathcal{O})$.

The family $(\ker \pi_n)_{n\in\mathbb{N}^*}$ is a filtration of G. We prove below that it is not conjugation-invariant when W^v is infinite, which motivates the introduction of other filtrations $(\mathcal{V}_{n\lambda})_{n\in\mathbb{N}^*}$, for $\lambda \in Y^+$ regular, in Section 4.

Lemma 3.2. *Let $x \in \mathbb{A}$ be such that $\alpha_i(x) > 0$ for all $i \in I$. Suppose that W^v is infinite. Then for all $n \in \mathbb{N}^*$, there exists $g \in \ker(\pi_n)$ such that $g.x \neq x$.*

Proof. Let $n \in \mathbb{N}^*$. As Φ^+ is infinite, there exists $\beta \in \Phi^+$ such that $\mathrm{ht}(\beta) > \frac{n}{\min_{i\in I} \alpha_i(x)}$. Then $\beta(x) > n$. Let $g = x_{-\beta}(\varpi^n) \in \ker \pi_n$. Then the subset of $\mathbb{A}$ fixed by g is $\{y \in \mathbb{A} \mid -\beta(y) + n \geq 0\}$, which does not contain x. □

Lemma 3.3. *Assume that W^v is infinite. Then $(\ker(\pi_n))_{n\in\mathbb{N}^*}$ is not conjugation-invariant.*

Proof. Suppose that $(\ker(\pi_n))$ is conjugation-invariant. Then the topology $\mathscr{T}((\ker(\pi_n))$ equips G with the structure of a topological group. We have $\ker(\pi_1) \subset \mathfrak{G}^{\min}(\mathcal{O}) = G_0$ and in particular $G_0 = \bigcup_{g\in G_0} g.\ker(\pi_1)$ is open. Let $\lambda \in Y^+$ be such that $\alpha_i(\lambda) = 1$ for all $i \in I$ and $t \in T$ be such that $t.0 = \lambda$. Then $H := tG_0t^{-1}$ is open (since G is a topological group). As $1 \in H$, we deduce the existence of $n \in \mathbb{N}^*$ such that $\ker(\pi_n) \subset H$. As H fixes λ, this implies that W^v is finite, by Lemma 3.2. □

3.2. On the decompositions of the congruence subgroups

Let $\mathfrak{m} = \{x \in \mathcal{O} \mid \omega(x) > 0\}$ be the maximal ideal of $\mathcal{O}$ and $\Bbbk = \mathcal{O}/\mathfrak{m}$. Let $\pi_\Bbbk : \mathfrak{G}^{\min}(\mathcal{O}) \to \mathfrak{G}^{\min}(\Bbbk)$ be the morphism induced by the natural projection $\mathcal{O} \twoheadrightarrow \Bbbk$. When $\omega(\mathcal{K}^*) = \mathbb{Z}$, $\pi_\Bbbk = \pi_1$.

In this subsection we study $\ker \pi_\Bbbk$: we prove that it decomposes as the product of its intersections with U^-, U^+ and T (see Proposition 3.5), using the masure $\mathcal{I}$ of G. We also describe $U^- \cap \ker \pi_\Bbbk$ and $U^+ \cap \ker \pi_\Bbbk$ through their actions on $\mathcal{I}$ and we deduce that $\ker \pi_\Bbbk$ fixes $C_0^+ \cup C_0^-$. It would be interesting to prove similar properties for $\ker \pi_n$ instead of $\ker \pi_\Bbbk$, for $n \in \mathbb{N}^*$. The difficulty is that when ω is not discrete or $n \geq 2$, $\mathcal{O}/\varpi^n\mathcal{O}$ is no longer a field and very few is known for Kac–Moody groups over rings.

Let C, C' be two alcoves of the same sign based at 0. By [16, Proposition 5.17], there exists an apartment A containing C and C'. Let $g \in G$ be such that $g.A = \mathbb{A}$ and $g.C = C_0^+$. Then $g.C'$ is an alcove of $\mathbb{A}$ based at 0 and thus there exists $w \in W^v$ such that $g.C' = w.C_0^+$. We set $d^{W^+}(C, C') = w$, which is well-defined, independently of the choices we made (note that in [2, 1.11] the "W-distance" d^{W^+} is defined for more general pairs of alcoves). Then d^{W^+} is G-invariant.

Lemma 3.4. *Let C be a positive alcove of $\mathcal{I}$ based at* 0 *and $w \in W^v$. Write $w = r_{i_1} \dots r_{i_k}$, with $k = \ell(w)$ and $i_1, \dots, i_k \in I$. Let $\beta_1 = \alpha_{i_1}$, $\beta_2 = r_{i_1}.\alpha_{i_2}$, $\dots$, $\beta_k = r_{i_1} \dots r_{i_{k-1}}.\alpha_{i_k}$. Then $\beta_1, \dots, \beta_k \in \Phi_+$ and we have $\rho_{+\infty}(C) = w.C_0^+$ if and only if there exists $a_1, \dots, a_k \in \mathcal{O}$ such that $C = x_{\beta_1}(a_1) \dots x_{\beta_k}(a_k).\widetilde{r}_{i_1} \dots \widetilde{r}_{i_k}.C_0^+$.*

Proof. As $x_{\beta_1}(\mathcal{O}). \dots .x_{\beta_k}(\mathcal{O})$ fixes 0, an element of $x_{\beta_1}(\mathcal{O}) \dots x_{\beta_k}(\mathcal{O}).\widetilde{r}_{i_1} \dots \widetilde{r}_{i_k}.C_0^+$ is a positive alcove based at 0. The fact that $\beta_1, \dots, \beta_k \in \Phi_+$ follows from [23, 1.3.14 Lemma]. Thus $x_{\beta_1}(\mathcal{O}). \dots .x_{\beta_k}(\mathcal{O}).\widetilde{r}_{i_1} \dots \widetilde{r}_{i_k}.C_0^+ \subset U^+.w.C_0^+$ and we have one implication.

We prove the reciprocal by induction on $\ell(w)$. Assume $w = 1$. Then $\rho_{+\infty}(C) = C_0^+$. Let A be an apartment containing $+\infty$ and C. Let $g \in G$ be such that $g.A = \mathbb{A}$ and g fixes $A \cap \mathbb{A}$. Then $C = g^{-1}.C_0^+$, by definition of $\rho_{+\infty}$. Moreover, A contains 0 and $+\infty$ and thus it contains $\mathrm{conv}(0, +\infty) \supset C_0^+$. Therefore $C = C_0^+$ and the lemma is clear in this case. Assume now that $\ell(w) \geq 1$.

Let $C'_0 = C_0^+$, $C'_1 = r_{i_1}.C_0^+$, $\dots$, $C'_k = r_{i_1} \dots r_{i_k}.C_0^+ = C'$. Let C be a positive alcove based at 0 and such that $\rho_{+\infty}(C) = C'_k$. Let A be an apartment containing C and $+\infty$. Let $g \in G$ be such that $g.A = \mathbb{A}$ and g fixes $A \cap \mathbb{A}$. Set $C_i = g^{-1}.C'_i$, for $i \in [\![0, k]\!]$. Then g fixes $+\infty$ and hence $\rho_{+\infty}(x) = g.x$ for every $x \in \mathbb{A}$. Therefore $\rho_{+\infty}(C_i) = C'_i$, for $i \in [\![0, k]\!]$. In particular, $\rho_{+\infty}(C_{k-1}) = C'_{k-1}$. By induction, we may assume that there exist $a_1, \dots, a_{k-1} \in \mathcal{O}$ such that $C_{k-1} = u\widetilde{v}.C_0^+$, where $u = x_{\beta_1}(a_1) \dots x_{\beta_k}(a_{k-1})$ and $\widetilde{v} = \widetilde{r}_{i_1} \dots \widetilde{r}_{i_{k-1}}$. Moreover we have

$$\begin{aligned} d^{W^+}(C_{k-1}, C_k) = d^{W^+}(C'_{k-1}, C'_k) = r_{i_k} \\ = d^{W^+}\left(u^{-1}.C_{k-1}, u^{-1}.C_k\right) = d^{W^+}\left(\widetilde{v}.C_0^+, u^{-1}.C_k\right). \end{aligned}$$

Let P be the panel common to $\widetilde{v}.C_0^+$ and $u^{-1}.C_k$. Then $P \subset \beta_k^{-1}(\{0\})$. Let D be the half-apartment delimited by $\beta_k^{-1}(\{0\})$ and containing C_{k-1}. Then as $\beta_k(C_{k-1}) = r_{i_1} \dots r_{i_{k-1}}.\alpha_{i_k}(r_{i_1} \dots r_{i_{k-1}}.C_0^+) > 0$, D contains $+\infty$. By [29, Proposition 2.9], there exists an apartment B containing D and $u^{-1}.C_k$. Let $g' \in G$ be such that $g'.B = \mathbb{A}$ and g' fixes $\mathbb{A} \cap B$. We have $g'.u^{-1}.C_k = \rho_{+\infty}(C_k) = C'_k$. By [30, 5.7 7)], $g' \in \mathfrak{T}(\mathcal{O})U_{\beta_k,0}$ and as $\mathfrak{T}(\mathcal{O})$ fixes $\mathbb{A}$, we can assume $g' \in U_{\beta_k,0} = x_{\beta_k}(\mathcal{O})$. Write $g' = x_{\beta_k}(-a_k)$, with $a_k \in \mathcal{O}$. Then $C_k = u.x_{\beta_k}(a_k).C_k = x_{\beta_1}(a_1) \dots x_{\beta_k}(a_k).\widetilde{r}_{i_1} \dots \widetilde{r}_{i_k}.C_0^+$, which proves the lemma. □

Proposition 3.5.

(1) *We have $U^{nm-}_{C_0^+} = U^- \cap \ker(\pi_\Bbbk)$ and $U^{pm+}_{-C_0^+} = U^+ \cap \ker(\pi_\Bbbk)$.*

(2) *We have* $\ker(\pi_\Bbbk) = (\ker(\pi_\Bbbk) \cap U^+) \,.\, (\ker(\pi_\Bbbk) \cap U^-) \,.\, (\ker(\pi_\Bbbk) \cap \mathfrak{T}(\mathcal{O}))$.

(3) *We have* $\ker(\pi_\Bbbk) \subset G_{C_0^+ \cup C_0^-}$.

Proof.

(2). Let $u \in U^{nm-}_{C_0^+}$. By definition, there exists $\Omega \in C_0^+$ such that $u \in U^{nm-}_{\Omega}$. Let $x \in C^v_f \cap \Omega$. Then $u \in \prod_{\alpha\in\Delta_-} X_\alpha(\mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \mathcal{K}_{\omega\geq -\alpha(x)}) \cap G_0$. As $-\alpha(x) > 0$ for every $\alpha \in \Delta_-$, Proposition 2.2 implies:

$$U^{nm-}_{C_0^+} \subset \ker(\pi_{\Bbbk}). \tag{3.3}$$

Let $g \in \ker(\pi_{\Bbbk}) \subset \mathfrak{G}^{\min}(\mathcal{O})$. Then g fixes 0 and $g.C_0^+$ is a positive alcove based at 0. Write $\rho_{+\infty}(g.C_0^+) = w.C_0^+$, with $w \in W^v$. Write $w = r_{i_1} \dots r_{i_m}$, with $m = \ell(w)$ and $i_1, \dots, i_m \in I$. Let $\widetilde{n} = \widetilde{r}_{i_1} \dots \widetilde{r}_{i_m} \in \mathfrak{N}(\mathcal{K})$. By Lemma 3.4, there exists $u \in \langle x_\beta(\mathcal{O}) \mid \beta \in \Phi_+\rangle$ such that $g.C_0^+ = u\widetilde{n}.C_0^+$. Then $g = u\widetilde{n}i$, with $i \in G_{C_0^+}$. As C_0^+ has a good fixator ([30, 5.7 2)]), we have (by (2.8):

$$G_{C_0^+} = U^{pm+}_{C_0^+} U^{nm-}_{C_0^+} N_{C_0^+}.$$

As every element of C_0^+ has non empty interior, $N_{C_0^+} = \mathfrak{T}(\mathcal{O})$. Moreover, $U^{pm+}_{C_0^+} = U^{pm+}_0$ and $\mathcal{T}(\mathcal{O})$ normalizes U^{pm+}_0 and $U^{nm-}_{C_0^+}$. Therefore,

$$G_{C_0^+} = \mathfrak{T}(\mathcal{O}).U^{pm+}_0.U^{nm-}_{C_0^+}.$$

Write $i = tu_+u_-$, with $t \in \mathfrak{T}(\mathcal{O})$, $u_+ \in U^{pm+}_0$ and $u_- \in U^{nm-}_{C_0^+}$.

Therefore by (3.3), we have

$$\pi_{\Bbbk}(g) = 1 = \pi_{\Bbbk}(u\widetilde{n}tu_+u_-) = \pi_{\Bbbk}(u)\pi_{\Bbbk}(\widetilde{n}t)\pi_{\Bbbk}(u_+)\pi_{\Bbbk}(u_-) = \pi_{\Bbbk}(u)\pi_{\Bbbk}(\widetilde{n}t)\pi_{\Bbbk}(u_+).$$

By [30, 3.16 Proposition] or [24, Theorem 8.118],

$(\mathfrak{G}^{\mathrm{pma}}(\Bbbk), \mathfrak{N}(\Bbbk), \mathfrak{U}^{\mathrm{pma}}(\Bbbk), \mathfrak{U}^-(\Bbbk), \mathfrak{T}(\Bbbk), \{r_i \mid i \in I\})$ is a refined Tits system. By [30, 3.16 Remarque], we have the Birkhoff decomposition

$$\mathfrak{G}^{\mathrm{pma}}(\Bbbk) = \bigsqcup_{n\in\mathfrak{N}(\Bbbk)} \mathfrak{U}^+(\Bbbk) n \mathfrak{U}^{\mathrm{pma}}(\Bbbk).$$

As $\pi_{\Bbbk}(u) \in \mathfrak{U}^+(\Bbbk)$, $\pi_{\Bbbk}(\widetilde{n}t) \in \mathfrak{N}(\Bbbk)$ and $\pi_{\Bbbk}(u_+) \in \mathfrak{U}^{\mathrm{pma}}(\Bbbk)$, we deduce $\pi_{\Bbbk}(\widetilde{n}t) = 1$.

By [28, 1.4 Lemme and 1.6], there exists a group morphism $\nu^v_{\Bbbk} : \mathfrak{N}(\Bbbk) \to W^v$ such that $\nu^v_{\Bbbk}(\widetilde{r}_i) = r_i$ for $i \in I$ and $\nu^v_{\Bbbk}(\mathfrak{T}(\Bbbk)) = 1$. Then $\nu^v_{\Bbbk}(\widetilde{n}t) = w = 1$. Therefore $w = 1$ and $g = utu_+u_- = u'tu_-$, for some $u' \in U^{pm+}_0$, since t normalizes U^{pm+}_0. By Lemma 2.3 and by symmetry of the roles of U^{nm-} and U^{pm+}, we have $u' \in U^{pm+}_{-C_0^+}$. By (3.3) applied to $U^{pm+}_{-C_0^+}$, we have $\pi_{\Bbbk}(g) = 1 = \pi_{\Bbbk}(u')\pi_{\Bbbk}(t)\pi_{\Bbbk}(u-) = \pi_{\Bbbk}(t)$ and thus

$$g \in U^{pm+}_{-C_0^-}.(T \cap \ker \pi_{\Bbbk}).U^{nm-}_{C_0^+} = U^{pm+}_{-C_0^-}.U^{nm-}_{C_0^+}.(T \cap \ker \pi_{\Bbbk}).$$

By (3.3), we deduce (2).

(3). We have $U^{pm+}_{-C_0^+} = U^{pm+}_{-C_0^+\cup C_f^v} \subset G_{C_0^+\cup -C_0^+}$, $U^{nm-}_{C_0^+} = U^{nm-}_{C_0^+\cup -C_f^v} \subset G_{C_0^+\cup -C_0^+}$ and $T \cap \ker \pi_{\Bbbk} \subset T \cap G_0 \subset G_{\mathbb{A}}$, which proves (3).

(1). We already proved one inclusion. Let $u \in \ker \pi_{\Bbbk} \cap U^-$. Then by what we proved above, $u \in U^{pm+}_{-C_0^-}.(T \cap \ker \pi_{\Bbbk}).U^{nm-}_{C_0^+}$. By Lemma 2.3, $u \in U^{nm-}_{C_0^+}$, and the proposition follows. □

Corollary 3.6. *Let $n \in \mathbb{N}^*$. Then* $\ker \pi_n \subset U^{pm+}_{-C_0^+}.\mathfrak{T}(\mathcal{O}).U^{nm-}_{C_0^+}$.

Remark 3.7. Let $u \in U^- \cap \ker \pi_{\Bbbk} = U^{nm-}_{C_0}$. Write $u = \prod_{\alpha\in\Delta_-} X_\alpha(v_\alpha)$, where $v_\alpha \in \mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \mathcal{O}$, for every $\alpha \in \Delta_-$. Define $\omega : \mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ by $\omega(v) = \inf\{x \in \mathbb{R} \mid v \in \mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \mathcal{K}_{\omega \geq x}\}$, for $v \in \mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \mathcal{K}$. Let $\lambda \in Y$ be such that $\alpha_i(\lambda) = 1$ for every $i \in I$. Let $\Omega \in C_0^+$ be such that $u \in U^{nm-}_\Omega$ and $\eta \in \mathbb{R}^*_+$ be such $\eta\lambda \in \Omega$. Then $u \in U^{nm-}_\Omega$ implies $\omega(v_\alpha) \geq |\alpha(\eta\lambda)| = \eta \mathrm{ht}(\alpha)$ for every $\alpha \in \Delta_-$. In particular, $\omega(v_\alpha)$ goes to $+\infty$ when $-\mathrm{ht}(\alpha)$ goes to $+\infty$.

4. Definition of topologies on G

In this section, we define two topologies $\mathscr{T}$ and $\mathscr{T}_{\mathrm{Fix}}$ on G and compare them. For the first one, we proceed as follows. We define a set $\mathcal{V}_\lambda$ for every regular $\lambda \in Y^+$. We prove that it is actually a subgroup of $\mathfrak{G}^{\min}(\mathcal{O})$ (Lemma 4.2) and we define $\mathscr{T}$ as the topology associated with $(\mathcal{V}_{n\lambda})_{n\in\mathbb{N}^*}$. We then prove that $\mathscr{T}$ does not depend on the choice of λ and that it is conjugation-invariant (Theorem 4.8) and thus that $(G, \mathscr{T})$ is a topological group. We then introduce the topology $\mathscr{T}_{\mathrm{Fix}}$ associated with the fixators of finite subsets of $\mathcal{I}$ and we end up by a comparison of $\mathscr{T}$ and $\mathscr{T}_{\mathrm{Fix}}$.

4.1. Subgroup $\mathcal{V}_\lambda$

For $n \in \mathbb{N}^*$, we set $T_n = \ker \pi_n \cap T \subset \mathfrak{T}(\mathcal{O})$. For $\lambda \in Y^+$ regular, we set

$$N(\lambda) = \min\{|\alpha(\lambda)| \mid \alpha \in \Phi\} \in \mathbb{N}^* \text{ and } \mathcal{V}_\lambda = U^{pm+}_{[-\lambda,\lambda]}.U^{nm-}_{[-\lambda,\lambda]}.T_{2N(\lambda)}.$$

By (2.9), we have

$$\mathcal{V}_\lambda = (U^+ \cap G_{[-\lambda,\lambda]}).(U^- \cap G_{[-\lambda,\lambda]}).(T \cap \ker \pi_{2N(\lambda)}) \subset G_{[-\lambda,\lambda]}. \tag{4.1}$$

The $2N(\lambda)$ appearing comes from $x_{-\alpha}(\varpi^n\mathcal{O}).x_\alpha(\varpi^n\mathcal{O}) \subset x_\alpha(\varpi^n\mathcal{O})x_{-\alpha}(\varpi^n\mathcal{O})$ $\alpha^\vee(1 + \varpi^{2n}\mathcal{O})$ for $\alpha \in \Phi$ and $n \in \mathbb{N}^*$, which follows from (2.1). To prove that $\mathcal{V}_\lambda$ is a group, the main difficulty is to prove that it is stable by left multiplication by $U^{nm-}_{[-\lambda,\lambda]}$. If G is reductive, we have $U^{nm-}_{[-\lambda,\lambda]} = U^{--}_{[-\lambda,\lambda]} := \langle x_\alpha(a) \mid \alpha \in \Phi_-, a \in \mathcal{K}, \omega(a) \geq |\alpha(\lambda)|\rangle$. By induction, it then suffices to prove that $x_{-\alpha}(\varpi^{|\alpha(\lambda)|}\mathcal{O})\mathcal{V}_\lambda \subset \mathcal{V}_\lambda$, for $\alpha \in \Phi_+$. When

G is no longer reductive, we have $U^{--}_{[-\lambda,\lambda]} \subsetneq U^{nm-}_{[-\lambda,\lambda]}$ in general (see [30, 4.12 3)]). The group $U^{nm-}_{[-\lambda,\lambda]}$ is defined as a set of infinite products and so its seems difficult to reason by induction in our case. We could try to use the group $U^{-}_{[-\lambda,\lambda]} := \langle x_\alpha(a) \mid \alpha \in \Phi, a \in \mathcal{K}, \omega \geq |\alpha(a)|\rangle \cap U^-$ since it is sometimes equal to $U^{nm-}_{[-\lambda,\lambda]}$ (for example when $\lambda \in C^v_f$, $U^{nm-}_{[-\lambda,\lambda]} = U^{nm-}_{-\lambda} = U^-_\lambda = U^-_{[-\lambda,\lambda]}$ by [3, 2.4.1 2)]). However it seems difficult since if $\alpha \in \Phi_+$, the condition $\omega(a) + \alpha(\lambda) \geq 0$ allows elements with a negative valuation. In order to overcome these difficulties, we use the morphisms π_n, for $n \in \mathbb{N}$.

By definition,

$$U^{pm+}_{[-\lambda,\lambda]} = G \cap \prod_{\alpha\in\Delta_+} X_\alpha\Big(\mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \mathcal{K}_{\omega\geq f_\Omega([-\lambda,\lambda])}\Big) \subset G \cap \prod_{\alpha\in\Delta_+} X_\alpha\Big(\mathfrak{g}_{\alpha,\mathbb{Z}} \otimes \varpi^{N(\lambda)}\mathcal{O}\Big).$$

By Proposition 2.2 we deduce $U^{pm+}_{[-\lambda,\lambda]} \subset \ker\left(\pi_{N(\lambda)}\right)$. Using a similar reasoning for $U^{nm-}_{[-\lambda,\lambda]}$ we deduce

$$\mathcal{V}_\lambda \subset \ker(\pi_{N(\lambda)}). \tag{4.2}$$

For $n \in \mathbb{N}$ and $\alpha^\vee \in \Phi^\vee$, one sets $T_{\alpha^\vee,n} = \alpha^\vee(1 + \varpi^n\mathcal{O}) \subset \mathfrak{T}(\mathcal{O})$.

Lemma 4.1. *Let $\lambda \in Y^+$ be regular and $\alpha \in \Phi$. Then $U_{\alpha,[-\lambda,\lambda]}.U_{-\alpha,[-\lambda,\lambda]}.T_{\alpha^\vee,2N(\lambda)}$ is a subgroup of G.*

Proof. Set $\Omega = [-\lambda, \lambda]$. Set $H = U_{\alpha,\Omega}.U_{-\alpha,\Omega}.T_{\alpha^\vee,2N(\lambda)}$. It suffices to prove that H is stable under left multiplication by $U_{\alpha,\Omega}, U_{-\alpha,\Omega}$ and $T_{\alpha^\vee,2N(\lambda)}$. The first stability is clear and the third follows from Lemma 2.4 and the fact that $T_{\alpha^\vee,2N(\lambda)} \subset \mathfrak{T}(\mathcal{O})$ fixes $\mathbb{A}$. Let $u_-, \widetilde{u_-} \in U_{-\alpha,\Omega}$, $u_+ \in U_{\alpha,\Omega}$ and $t \in T_{\alpha^\vee,2N(\lambda)}$. Write $u_- = x_{-\alpha}(a_-)$, $\widetilde{u}_- = x_{-\alpha}(\widetilde{a}_-)$ and $u_+ = x_\alpha(a_+)$, for $a_-, \widetilde{a}_-, a_+ \in \mathcal{K}$. We have $\omega(\widetilde{a}_-), \omega(a_-), \omega(a_+) \geq |\alpha(\lambda)| \geq N(\lambda)$. Then by (2.1), we have

$$\widetilde{u}_-u_+u_-t = x_\alpha\left(a_+(1+\widetilde{a}_-a_+)^{-1}\right)x_{-\alpha}\left(\widetilde{a}_-(1+\widetilde{a}_-a_+)^{-1} + a_-\right)\alpha^\vee(1+\widetilde{a}_-a_+)t.$$

As $\omega(1 + \widetilde{a}_-a_+) = 1$, we deduce that $\widetilde{u}_-u_+u_-t \in H$, which proves the lemma. □

Let $w \in W^v$ and Ω be a filter on $\mathbb{A}$. Recall that $\mathfrak{G}^{pma,w}$ and $\mathfrak{G}^{nma,w}$ are the completions of $\mathfrak{G}$ with respect to $w.\Delta_+$ and $w.\Delta_-$ respectively. One defines $U^{pm}_\Omega(w.\Delta_+)$ and $U^{nm}_\Omega(w.\Delta_-)$ similarly as U^{pm+}_Ω and U^{nm-}_Ω in these groups.

Lemma 4.2. *Let $\lambda \in Y^+$ be regular. Then*

(1) $\mathcal{V}_\lambda = U^{pm}_{[-\lambda,\lambda]}(w.\Delta_+).U^{nm}_{[\lambda,\lambda]}(w.\Delta_-).T_{2N(\lambda)}$ *for every $w \in W^v$,*

(2) $\mathcal{V}_\lambda$ *is a subgroup of G.*

Proof.

(1). This follows the proof of [10, Proposition 3.4]. Set $\Omega = [-\lambda, \lambda]$. Let $i \in I$ and $\alpha = \alpha_i$. By [10, 3.3.4)] and Lemma 4.1,

$$\begin{aligned}\mathcal{V}_\lambda &= U_\Omega^{pm}(\Delta_+ \setminus \{\alpha\}).U_\Omega^{nm}(\Delta_- \setminus \{\alpha\}).U_{\alpha,\Omega}.U_{-\alpha,\Omega}.T_{2N(\lambda)} \\ &= U_\Omega^{pm}(\Delta_+ \setminus \{\alpha\}).U_\Omega^{nm}(\Delta_- \setminus \{\alpha\}).U_{-\alpha,\Omega}.U_{\alpha,\Omega}.T_{2N(\lambda)} \\ &= U_\Omega^{pm}(\Delta_+ \setminus \{\alpha\}).U_{-\alpha,\Omega}.U_\Omega^{nm}(\Delta_- \setminus \{\alpha\}).U_{\alpha,\Omega}.T_{2N(\lambda)} \\ &= U_\Omega^{pm}(r_i.\Delta_+).U_\Omega^{nm}(r_i.\Delta_-).T_{2N(\lambda)}\end{aligned}$$

Therefore $\mathcal{V}_\lambda$ does not change when Δ_+ is replaced by $w.\Delta_+$, for $w \in W^v$, which proves (1).

(2). Let $w \in W^v$ be such that $\lambda \in w.C_f^v$. By (1) we have

$$\mathcal{V}_\lambda = U_\Omega^{pm}(w.\Delta_+).U_\Omega^{nm}(w.\Delta_-).T_{2N(\lambda)}.$$

Let $t \in T$ be such that $t.0 = \lambda$. By Lemma 2.4, we have

$$tU_\Omega^{pm}(w.\Delta_+)t^{-1} = U_{t.\Omega}^{pm}(w.\Delta_+) = U_{[0,2\lambda]}^{pm}(w.\Delta_+) = U_0^{pm}(w.\Delta_+).$$

Similarly, $tU_\Omega^{nm}(w.\Delta_-)t^{-1} = U_{2\lambda}^{nm}(w.\Delta_-)$. As T is commutative, we also have $tT_{2N(\lambda)}t^{-1} = T_{2N(\lambda)}$. In order to prove (2), it suffices to prove that

$$H := tU_\Omega^{pm}(w.\Delta_+)U_\Omega^{nm}(w.\Delta_-)T_{2N(\lambda)}t^{-1} = U_0^{pm}(w.\Delta_+)U_{2\lambda}^{nm}(w.\Delta_-)T_{2N(\lambda)}$$

is a subgroup of G. It suffices to prove that H is stable under left multiplication by $U_0^{pm}(w.\Delta_+)$, $U_{2\lambda}^{nm}(w.\Delta_-)$ and $T_{2N(\lambda)}$. The first stability is clear and the third follows from Lemma 2.4.

First note that by [30, 5.7 1)],

$$G_{[0,2\lambda]} = U_0^{pm}(w.\Delta_+)U_{2\lambda}^{nm}(w.\Delta_-)\mathfrak{T}(\mathcal{O}) \tag{4.3}$$

is a subgroup of G.

Let $u_-, \widetilde{u_-} \in U_{2\lambda}^{nm}(w.\Delta_-)$, $u_+ \in U_0^{pm}(w.\Delta_+)$, $t \in T_{2N(\lambda)}$. Let us prove that $\widetilde{u}_-u_+u_-t \in H$. We have $\widetilde{u}_-u_+ = u_+(u_+^{-1}\widetilde{u}_-u_+)$. By Proposition 2.2, $U_{2\lambda}^{nm}(w.\Delta_-) \subset \ker \pi_{2N(\lambda)}$. By Proposition 3.1, $u_+ \in \mathfrak{G}^{\min}(\mathcal{O})$ and we have $u_+^{-1}\widetilde{u}_-u_+ \in G_{[0,2\lambda]} \cap \ker \pi_{2N(\lambda)}$. Therefore (4.3) implies that we can write $u_+^{-1}\widetilde{u}_-u_+ = u_1^+u_1^-t_1$, where $u_1^+ \in U_0^{pm}(w.\Delta_+)$, $u_1^- \in U_{2\lambda}^{nm}(w.\Delta_-)$ and $t_1 \in \mathfrak{T}(\mathcal{O})$. We have

$$\pi_{2N(\lambda)}(u_1^+u_1^-t_1) = 1 = \pi_{2N(\lambda)}(u_1^+t_1) = \pi_{2N(\lambda)}(u_1^+)\pi_{2N(\lambda)}(t_1).$$

As $\mathfrak{B}(\mathcal{O}/\varpi^{2N(\lambda)}\mathcal{O}) = \mathfrak{T}(\mathcal{O}/\varpi^{2N(\lambda)}\mathcal{O}) \ltimes \mathfrak{U}^{pma,w}(\mathcal{O}/\varpi^{2N(\lambda)}\mathcal{O})$ and as $\pi_{2N(\lambda)}(\mathfrak{T}(\mathcal{O})) \subset \mathfrak{T}(\mathcal{O}/\varpi^{2N(\lambda)}\mathcal{O})$ and $\pi_{2N(\lambda)}(\mathfrak{U}^{pma,w}(\mathcal{O})) \subset \mathfrak{U}^{pma,w}(\mathcal{O}/\varpi^{2N(\lambda)}\mathcal{O})$, we deduce $\pi_{2N(\lambda)}(u_1^+) = \pi_{2N(\lambda)}(t_1) = 1$ and $t_1 \in T_{2N(\lambda)}$. We have

$$\widetilde{u}_-u_+u_-t = u_+u_1^+u_1^-t_1u_-t$$

and as $T_{2N(\lambda)}$ normalizes $U^{nm}_{2\lambda}(w.\Delta_-)$, $\widetilde{u}_- u_+ u_- t \in H$, which proves the lemma. □

Remark 4.3.

(1) Note that if $\lambda \in Y^+$ is regular, then $U^{pm+}_{[-\lambda,\lambda]} U^{nm-}_{[-\lambda,\lambda]} T_{2N(\lambda)+1}$ is not a subgroup of G. Indeed, take $\alpha \in \Phi^+$ such that $|\alpha(\lambda)| = N(\lambda)$. Then $x_{-\alpha}(\varpi^{N(\lambda)}) x_\alpha(\varpi^{N(\lambda)}) = x_\alpha(\varpi^{N(\lambda)}(1+\varpi^{2N(\lambda)})^{-1}) x_{-\alpha}(\varpi^{N(\lambda)}(1+\varpi^{2N(\lambda)})^{-1}) \alpha^\vee(1+\varpi^{2N(\lambda)})$ and by Lemma 2.3, this does not belong to $U^{pm+}_{[-\lambda,\lambda]} U^{nm-}_{[-\lambda,\lambda]} T_{2N(\lambda)+1}$.

(2) For every $k \in [\![0, 2N(\lambda)]\!]$, $U^{pm+}_{[-\lambda,\lambda]}.U^{nm-}_{[-\lambda,\lambda]}.T_k = \mathcal{V}_\lambda.T_k$ is a subgroup of G, since T_k normalizes $U^{pm+}_{[-\lambda,\lambda]}$ and $U^{nm-}_{[-\lambda,\lambda]}$ by Lemma 2.4. Note that for the definition of a topology, we could also have taken the filtration $(U^{pm+}_{[-n\lambda,n\lambda]}.U^{nm-}_{[-n\lambda,n\lambda]}.T_{k(n)})_{n\in\mathbb{N}^*}$, for any $k(n) \in [\![0, 2nN(\lambda)]\!]$ such that $k(n) \xrightarrow[n\to+\infty]{} +\infty$.

(3) As $\mathcal{V}_\lambda = \mathcal{V}_\lambda^{-1}$ and by Lemma 2.4, we have $\mathcal{V}_\lambda = U^{nm-}_{[-\lambda,\lambda]} U^{pm+}_{[-\lambda,\lambda]} T_{2N(\lambda)}$.

4.2. Filtration $(\mathcal{V}_{n\lambda})_{n\in\mathbb{N}^*}$

Let Ω be a filter on $\mathbb{A}$. One defines $\mathrm{cl}^\Delta(\Omega)$ as the filter on $\mathbb{A}$ consisting of the subsets Ω' of $\mathbb{A}$ for which there exists $(k_\alpha) \in \prod_{\alpha\in\Delta} \Lambda_\alpha \cup \{+\infty\}$ such that $\Omega' \supset \bigcap_{\alpha\in\Delta} D(\alpha, k_\alpha) \supset \Omega$, where $\Lambda_\alpha = \Lambda$ if $\alpha \in \Phi$ and $\Lambda_\alpha = \mathbb{R}$ otherwise. Note that cl^Δ is denoted cl in [29] and [30]. By definition of U^{pm+}_Ω and U^{nm-}_Ω, we have

$$U^{pm+}_\Omega = U^{pm+}_{\Omega'} = U^{pm+}_{\mathrm{cl}^\Delta(\Omega)} \text{ and } U^{nm-}_\Omega = U^{nm-}_{\Omega'} = U^{nm-}_{\mathrm{cl}^\Delta(\Omega)}, \tag{4.4}$$

for any filter Ω' such that $\Omega \subset \Omega' \subset \mathrm{cl}^\Delta(\Omega)$.

Lemma 4.4. *Let $\lambda \in C^v_f$ and $w \in W^v$. Then $\mathrm{cl}^\Delta([-w.\lambda, w.\lambda]) \supset (-w.\lambda + w.\overline{C^v_f}) \cap (w.\lambda - w.\overline{C^v_f})$.*

Proof. As Δ and Φ are W^v-invariant, we have $w.\mathrm{cl}^\Delta(\Omega) = \mathrm{cl}^\Delta(w.\Omega)$ for every $w \in W^v$. Thus it suffices to determine $\mathrm{cl}^\Delta([-\lambda, \lambda])$. Let $(k_\alpha) \in \prod_{\alpha\in\Delta} \Lambda_\alpha \cup \{+\infty\}$ be such that $\bigcap_{\alpha\in\Delta} D(\alpha, k_\alpha) \supset [-\lambda, \lambda]$. Let $\alpha \in \Delta_+$. Write $\alpha = \sum_{i\in I} n_i \alpha_i$, with $n_i \in \mathbb{N}$ for $i \in I$. Then $k_\alpha \geq \alpha(\lambda) = \sum_{i\in I} n_i \alpha_i(\lambda)$. Let $\alpha \in \Delta_-$. We also have $k_{-\alpha} \geq \sum_{i\in I} n_i \alpha_i(\lambda)$.

Let $x \in (-\lambda + C^v_f) \cap (\lambda - C^v_f)$. Then $-\alpha_i(\lambda) \leq \alpha_i(x) \leq \alpha_i(\lambda)$ for every $i \in I$. Then $k_{-\alpha} \leq \sum_{i\in I} n_i \alpha_i(x) \leq k_\alpha$ and thus $k_\alpha + \alpha(x) \geq 0$ and $k_{-\alpha} - \alpha(x) \geq 0$. Consequently, $x \in \bigcap_{\alpha\in\Delta} D(\alpha, k_\alpha)$ and thus

$$(-\lambda + C^v_f) \cap (\lambda - C^v_f) \subset \bigcap_{\alpha\in\Delta} D(\alpha, k_\alpha),$$

which proves the lemma. □

The following lemma will be crucial throughout the paper. This is a rewriting of [3, Lemma 3.3 and Lemma 3.6]. Although ω is assumed to be discrete in [3], the proofs of these lemma do not use this assumption.

Lemma 4.5.

(1) *Let $a \in \mathbb{A}$ and $g \in U^+$. Then there exists $b \in a - C_f^v$ such that $g^{-1} U_b^{pm+} g \subset U_a^{pm+}$.*

(2) *Let $y \in \mathcal{I}$. Then there exists $a \in \mathbb{A}$ such that U_a^{pm+} fixes y.*

(3) *Let $\lambda \in Y^+$ be regular and $y \in \mathcal{I}$. Then for $n \in \mathbb{N}$ large enough, $U_{[-n\lambda, n\lambda]}^{pm+}$ fixes y.*

Proof. By [3, Lemma 3.3 and 3.6], we have (1) and (2). Let $\lambda \in Y^+$ be regular. Write $\lambda = w.\lambda^{++}$, with $\lambda^{++} \in C_f^v$ and $w \in W^v$. Let $a \in \mathbb{A}$ be such that U_a^{pm+} fixes y. For $n \in \mathbb{N}^*$, we have $\mathrm{cl}^\Delta([-n\lambda, n\lambda]) \supset n\big((-w.\lambda^{++} + w.\overline{C_f^v}) \cap (w.\lambda^{++} - w.\overline{C_f^v})\big)$, which contains a, for $n \gg 0$. So for $n \gg 0$, we have $U_{[-n\lambda, n\lambda]}^{pm+} \subset U_a^{pm+}$, which proves (3). □

Lemma 4.6. *Let $\lambda, \mu \in Y^+$ be regular. Then $(\mathcal{V}_{n\lambda})_{n \in \mathbb{N}^*}$ and $(\mathcal{V}_{n\mu})_{n \in \mathbb{N}^*}$ are equivalent.*

Proof. Write $\mu = v.\mu^{++}$, where $v \in W^v$ and $\mu^{++} \in C_f^v$. For $m \in \mathbb{N}$, set

$$\Omega_{m\mu} = (-m\mu + v.\overline{C_f^v}) \cap (m\mu - v.\overline{C_f^v}).$$

By Lemma 4.4, $\Omega_{m\mu} \subset \mathrm{cl}^\Delta([-m\mu, m\mu])$. Let $n \in \mathbb{N}^*$. As $\Omega_{m\mu} = m\Omega_\mu$ and Ω_μ contains 0 in its interior, there exists $m \in \mathbb{Z}_{\geq n}$ such that $\Omega_{m\mu} \supset [-n\lambda, n\lambda]$. Moreover, by (4.4), $U_{[-m\mu, m\mu]}^{pm+} = U_{\Omega_{m\mu}}^{pm+} \subset U_{[-n\lambda, n\lambda]}^{pm+}$, since $\Omega' \mapsto U_{\Omega'}^{pm+}$ is decreasing for $\subset$. With the same reasoning for U^{nm-}, we deduce $\mathcal{V}_{m\mu} \subset \mathcal{V}_{n\lambda}$. By symmetry of the roles of λ and μ, we deduce the lemma. □

The end of this subsection is devoted to the proof of the fact that for every $\lambda \in Y^+$ regular, $(\mathcal{V}_{n\lambda})_{n \in \mathbb{N}^*}$ is conjugation-invariant.

Lemma 4.7. *Let $\alpha \in \Phi$ and $a \in \mathcal{K}$. Let $\lambda \in Y^+$ be regular. Then $(x_\alpha(a).\mathcal{V}_{n\lambda}.x_\alpha(-a))_{n \in \mathbb{N}^*}$ is equivalent to $(\mathcal{V}_{n\lambda})_{n \in \mathbb{N}^*}$.*

Proof. Let $\epsilon \in \{-, +\}$ be such that $\alpha \in \Phi_\epsilon$. Let $w \in W^v$ be such that $\epsilon w^{-1}.\alpha$ is simple. By symmetry, we may assume that $\epsilon = +$. By Lemma 4.6 we may assume that $\lambda \in w.C_f^v$. Let $n \in \mathbb{N}^*$. By Lemma 4.2, we have

$$\begin{aligned}\mathcal{V}_{n\lambda} &= U_{[-n\lambda, n\lambda]}^{pm}(w.\Delta_+) U_{[-n\lambda, n\lambda]}^{nm}(w.\Delta_-) T_{2nN(\lambda)}.\\ &= U_{-n\lambda}^{pm}(w.\Delta_+) U_{n\lambda}^{nm}(w.\Delta_-) T_{2nN(\lambda)}.\end{aligned}$$

Write $\alpha = w.\alpha_i$, for $i \in I$. By Lemma 4.5,

$$x_{w.\alpha_i}(a) U_{-m\lambda}^{pm}(w.\Delta_+) x_{w.\alpha_i}(-a) \subset U_{-n\lambda}^{pm}(w.\Delta_+), \tag{4.5}$$

for $m \gg 0$.

By [30, Lemma 3.3], $U^{nm}_{[-n\lambda,n\lambda]}(w.\Delta_-) = U^{nm}_{[-n\lambda,n\lambda]}(w.(\Delta_- \setminus \{-\alpha_i\})).U_{-w.\alpha_i,[-n\lambda,n\lambda]}$ and $U_{-w.\alpha_i,[-n\lambda,n\lambda]}$ normalizes $U^{nm}_{[-n\lambda,n\lambda]}(w.(\Delta_- \setminus \{-\alpha_i\}))$.

By [23, 1.3.11 Theorem (b4)], $r_i.(\Delta_- \setminus \{-\alpha_i\}) = \Delta_- \setminus \{-\alpha_i\}$ and thus for $m \in \mathbb{N}$, we have

$$\begin{aligned} U^{nm}_{[-m\lambda,m\lambda]}\left(w.(\Delta_- \setminus \{-\alpha_i\})\right) &= U^{nm}_{[-m\lambda,m\lambda]}\left(wr_i(\Delta_- \setminus \{-\alpha_i\})\right) \\ &= U^{nm}(wr_i(\Delta_- \setminus \{-\alpha_i\})) \cap U^{nm}_{[-m\lambda,m\lambda]}(wr_i.\Delta_-). \end{aligned}$$

Moreover,

$$x_{w.\alpha_i}(a)U^{nm}(wr_i(\Delta_- \setminus \{-\alpha_i\}))x_{w.\alpha_i}(-a) = U^{nm}(wr_i(\Delta_- \setminus \{-\alpha_i\})), \tag{4.6}$$

by [30, Lemma 3.3] (applied to $wr_i.(\Delta_- \setminus \{-\alpha_i\}) \subset wr_i.\Delta_-$). By Lemma 4.5, for $m \gg 0$,

$$x_{w.\alpha_i}(a)U^{nm}_{[-m\lambda,m\lambda]}(wr_i.\Delta_-)x_{w.\alpha_i}(-a) \subset U^{nm}_{[-n\lambda,n\lambda]}(wr_i.\Delta_-). \tag{4.7}$$

Combining (4.6) and (4.7), we get

$$x_{w.\alpha_i}(a)U^{nm}_{[-m\lambda,m\lambda]}(w.\Delta_- \setminus \{-\alpha_i\})x_{w.\alpha_i}(-a) \subset U^{nm}_{[-n\lambda,n\lambda]}\left(w.(\Delta_- \setminus \{-\alpha_i\})\right), \tag{4.8}$$

for $m \gg 0$.

For $b \in \mathcal{K}$ such that $1 + ab \neq 0$, we have

$$x_{w.\alpha_i}(a)x_{-w.\alpha_i}(b)x_{w.\alpha_i}(-a) = x_{-w.\alpha_i}\big(b(1+ab)^{-1}\big)\alpha^\vee(1+ab)x_{w.\alpha_i}\big(-a^2b(1+ab)^{-1}\big).$$

Therefore if $m \gg 0$, $x_{w.\alpha_i}(a)U_{-w.\alpha_i,[-m\lambda,m\lambda]}x_{w.\alpha_i}(-a) \subset \mathcal{V}_{n\lambda}$. Combined with (4.8) we get

$$x_{w.\alpha_i}(a)U^{nm}_{m\lambda}(w.\Delta_-)x_{w.\alpha_i}(-a) \subset \mathcal{V}_{n\lambda}, \tag{4.9}$$

for $m \gg 0$, since $\mathcal{V}_{n\lambda}$ is a group.

Let $m \in \mathbb{N}^*$ and $t \in T_{2m}$. Then:

$$x_{w.\alpha_i}(a)tx_{w.\alpha_i}(-a) = tx_{w.\alpha_i}\big(a\big(w.\alpha_i(t^{-1}) - 1\big)\big)$$

Therefore if $m \geq nN(\lambda)$ and $\omega\left(a\left(w.\alpha_i(t^{-1}) - 1\right)\right) \geq \omega(a) + 2mN(\lambda)$ is greater than $|w.\alpha_i(n\lambda)|$, then $x_{w.\alpha_i}\left(a\left(w.\alpha_i(t^{-1}) - 1\right)\right) \in \mathcal{V}_{n\lambda}$ and $x_{w.\alpha_i}(a)tx_{w.\alpha_i}(-a) \in \mathcal{V}_{n\lambda}$. Combined with (4.9) and (4.5), we get $x_\alpha(a).\mathcal{V}_{m\lambda}.x_\alpha(-a) \subset \mathcal{V}_{n\lambda}$, for $m \gg 0$. Applying this to $(x_\alpha(-a).\mathcal{V}_{k\lambda}.x_\alpha(a))_{k\in\mathbb{N}^*}$, we get the other inclusion needed to prove that $(\mathcal{V}_{n\lambda})$ and $(x_\alpha(a).\mathcal{V}_{n\lambda}.x_\alpha(-a))$ are equivalent. □

Theorem 4.8. *Let $\lambda \in Y^+$ be regular. For $n \in \mathbb{N}^*$, set $\mathcal{V}_{n\lambda} = U^{pm+}_{[-n\lambda,n\lambda]}.U^{nm-}_{[-n\lambda,n\lambda]}.T_{2N(n\lambda)}$. Then $(\mathcal{V}_{n\lambda})$ is conjugation-invariant. Therefore, the associated topology $\mathscr{T}((\mathcal{V}_{n\lambda}))$ equips G with the structure of a Hausdorff topological group.*

Proof. We need to prove that for every $g \in G$, $g(\mathcal{V}_{n\lambda})g^{-1}$ is equivalent to $(\mathcal{V}_{n\lambda})$. Using Lemma 4.6, we may assume $\lambda \in C_f^v$. By [28, Proposition 1.5], G is generated by T and the $x_\alpha(a)$, for $\alpha \in \Phi$ and $a \in \mathcal{K}$. By Lemma 4.7, it remains only to prove that if $t \in T$, then $(t\mathcal{V}_{n\lambda}t^{-1})_{n\in\mathbb{N}^*}$ is equivalent to $(\mathcal{V}_{n\lambda})$. Let $t \in T$ and $m \in \mathbb{N}^*$. Then by Lemma 2.4,

$$\begin{aligned} t\mathcal{V}_{m\lambda}t^{-1} &= tU^{pm+}_{[-m\lambda,m\lambda]}t^{-1}.tU^{nm-}_{[-m\lambda,m\lambda]}t^{-1}.T_{2mN(\lambda)} \\ &= U^{pm+}_{[t.(-m\lambda),t.m\lambda]}U^{nm-}_{[t.(-m\lambda),t.m\lambda]}T_{2mN(\lambda)} \end{aligned}$$

Set $\Omega = (-\lambda + \overline{C_f^v}) \cap (\lambda - \overline{C_f^v})$. Then by Lemma 4.4, $\mathrm{cl}^\Delta([-m\lambda, m\lambda]) \supset m\Omega$. Moreover, $\mathrm{cl}^\Delta([t.(-m\lambda), t.m\lambda]) = t.\mathrm{cl}^\Delta([-m\lambda, m\lambda]) \supset t.m\Omega$. Let $n \in \mathbb{N}^*$. Then as Ω contains 0 in its interior, $t.m\Omega \supset n\Omega$ for $m \gg 0$. Therefore (by (4.4)) $U^{pm+}_{[t.(-m\lambda),t.m\lambda]} \subset U^{pm+}_{[-n\lambda,n\lambda]}$ and $U^{nm-}_{[t.(-m\lambda),t.m\lambda]} \subset U^{nm-}_{[-n\lambda,n\lambda]}$ for $m \gg 0$. Consequently, $t\mathcal{V}_{m\lambda}t^{-1} \subset \mathcal{V}_{n\lambda}$ for $m \gg 0$, which proves that $(\mathcal{V}_{n\lambda})$ is conjugation-invariant.

It remains to prove that $\mathscr{T}((\mathcal{V}_{n\lambda}))$ is Hausdorff. For that it suffices to prove that $\bigcap_{n\in\mathbb{N}^*} \mathcal{V}_{n\lambda} = \{1\}$. Let $g \in \bigcap_{n\in\mathbb{N}^*} \mathcal{V}_{n\lambda}$. Let $n \in \mathbb{N}^*$. Then as $[-n\lambda, n\lambda]$ has a good fixator ([30, 5.7 1)] and (2.8)) $g \in G_{[-n\lambda,n\lambda]} = U^{pm+}_{[-n\lambda,n\lambda]}.U^{nm-}_{[-n\lambda,n\lambda]}.\mathfrak{T}(\mathcal{O})$, so we can write $g = u_n^+ u_n^- t_n$, with $(u_n^+, u_n^-, t_n) \in U^{pm+}_{[-n\lambda,n\lambda]} \times U^{nm-}_{[-n\lambda,n\lambda]} \times \mathfrak{T}(\mathcal{O})$. By Lemma 2.3, $u_+ := u_n^+$ does not depend on n and thus $u^+ \in \bigcap_{n\in\mathbb{N}^*} U^{pm+}_{[-n\lambda,n\lambda]} = U^{pm+}_{\mathbb{A}} = \{1\}$. Similarly, $u^- := u_n^- = 1$. Therefore $t \in T \cap \bigcap_{n\in\mathbb{N}^*} \ker \pi_n$. Let $(\chi_i)_{i\in[\![1,m]\!]}$ be a $\mathbb{Z}$-basis of Y. Write $t = \prod_{i=1}^m \chi_i(a_i)$, with $a_i \in \mathcal{O}^*$, for $i \in [\![1, m]\!]$. Let $n \in \mathbb{N}$. Then $\pi_n(t) = \prod_{i=1}^m \chi_i(\pi_n(a_i)) = 1$ and thus $a_i \in \bigcap_{n\in\mathbb{N}^*} \varpi^n\mathcal{O} = \{0\}$. Consequently, $t = 1$ and $g = 1$. Therefore $\bigcap_{n\in\mathbb{N}^*} \mathcal{V}_{n\lambda} = \{1\}$ and $\mathscr{T}((\mathcal{V}_{n\lambda}))$ is Hausdorff. □

We denote by $\mathscr{T}$ the topology $\mathscr{T}((\mathcal{V}_{n\lambda}))$, for any $\lambda \in Y^+$ regular.

Corollary 4.9. *Let $\lambda \in Y^+$ be regular. The filtrations $(\ker \pi_n)_n$ and $(\mathcal{V}_{n\lambda})_{n\in\mathbb{N}^*}$ are equivalent if and only if W^v is finite.*

Proof. If W^v is infinite, this follows from Lemma 3.3 and Theorem 4.8. Assume now that W^v is finite. Then by (4.2), $\mathcal{V}_{n\lambda} \subset \ker \pi_{nN(\lambda)}$ for every $n \in \mathbb{N}^*$.

Let $n \in \mathbb{N}^*$ and $g \in \ker \pi_n$. By Corollary 3.6, $g \in U^{pm+}_{-C_0^+}.\mathfrak{T}(\mathcal{O}).U^{nm-}_{C_0^+}$. As W^v is finite, $\Phi = \Delta$ (by [21, Theorem 5.6]), $U^{pm+}_{-C_0^+} = U^+_{-C_0^+} = \prod_{\alpha\in\Phi_+} x_\alpha(\mathfrak{m})$ and $U^{nm-}_{C_0^+} = U^-_{C_0^+} = \prod_{\alpha\in\Phi_-} x_\alpha(\mathfrak{m})$, for any (fixed) orders on Φ_- and Φ_+, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$. Write $g = u_+ t u_-$, with $g \in U^-_{C_0^+}$, $t \in \mathfrak{T}(\mathcal{O})$ and $u_- \in U^-_{C_0^+}$. Then $\pi_n(g) = \pi_n(u_+t)\pi_n(u_-) = 1$. Moreover, $\pi_n(u_+t) \in \mathfrak{B}(\mathcal{O}/\varpi^n\mathcal{O})$ and $\pi_n(u_-) \in \mathfrak{B}^-(\mathcal{O}/\varpi^n\mathcal{O})$, where $\mathfrak{B} = \mathcal{U}^+ \rtimes \mathfrak{T}$ and $\mathfrak{B}^- = \mathfrak{U}^- \rtimes \mathfrak{T}$ are two opposite Borel subgroups. Therefore $\pi_n(u_+t) \in \mathfrak{B}(\mathcal{O}/\varpi^n\mathcal{O}) \cap \mathfrak{B}^-(\mathcal{O}/\varpi^n\mathcal{O}) = \mathfrak{T}(\mathcal{O}/\varpi^n\mathcal{O})$, by [4, Theorem 14.1]. Therefore $\pi_n(u_+) = \pi_n(u_-) = \pi_n(t) = 1$. Write $u_+ = \prod_{\alpha\in\Phi_+} x_\alpha(u_\alpha)$, with $u_\alpha \in \mathcal{O}$ for $\alpha \in \Phi_+$. Then $\pi_n(u_+) = \prod_{\alpha\in\Phi_+} x_\alpha(\pi_n(u_\alpha)) = 1$, by Proposition 2.2, and by [24, Theorem 8.51], this implies $\pi_n(u_\alpha) = 1$, for every $\alpha \in \Phi_+$. Let $N'(\lambda) = \max_{\alpha\in\Phi} |\alpha(\lambda)|$. Then we have $u_+ \in U^+ \cap G_{[-n\lambda/N'(\lambda), n\lambda/N'(\lambda)]}$. Using a similar reasoning for U^- and (4.1), we deduce $\ker \pi_n \subset \mathcal{V}_{n\lambda/N'(\lambda)}$. Therefore $(\mathcal{V}_{n\lambda})$ and $(\ker \pi_n)$ are equivalent. □

Remark 4.10.

(1) If $n \in \mathbb{N}^*$, then $\ker \pi_n$ is open for $\mathscr{T}$, by (4.2).

(2) The Iwahori subgroup $G_{C_0^+} = K_I$ is open. Indeed, if $\lambda \in Y \cap C_f^v$, then $\mathcal{V}_\lambda \subset K_I$ by (4.1). In particular, $\mathfrak{G}^{\min}(\mathcal{O}) \supset K_I$ is open.

(3) Assume $\mathfrak{G}$ is reductive, i.e. W^v is finite. Then by Corollary 4.9, $\mathscr{T}$ is the usual topology on G.

4.3. **Topology of the fixators**

4.3.1. **Definition of the topology**

Recall that if F is a subset of $\mathcal{I}$, we denote by G_F its fixator in G. In this subsection we study the topology $\mathscr{T}_{\mathrm{Fix}}$ which is defined as follows. A subset V of G is open if for every v in V, there exists a finite subset F of $\mathcal{I}$ such that $vG_F \subset V$. Note that $G_0 = \mathfrak{G}^{\min}(\mathcal{O})$ is open for this topology.

We begin by constructing increasing sequences of finite sets of vertices $(F_n) = (F_n(\lambda))_{n\in\mathbb{N}^*}$ such that $\mathscr{T}_{\mathrm{Fix}}$ is the topology associated with $(G_{F_n})_{n\in\mathbb{N}}$.

We fix $\lambda \in Y^+$ regular. We set $F_0 = \emptyset$. For $n \in \mathbb{N}^*$, we set

$$F_n = F_n(\lambda) = \{n\lambda, -n\lambda\} \cup \{x_{\alpha_i}(\varpi^{-n}).0 \mid i \in I\} \cup F_{n-1}. \tag{4.10}$$

Let $n \in \mathbb{N}^*$. By [30, 5.7 1)], $[-n\lambda, n\lambda]$ has a good fixator and by Proposition 2.5, we have $G_{\{n\lambda,-n\lambda\}} = G_{[-n\lambda,n\lambda]}$. Therefore

$$G_{F_n} \subset G_{[-n\lambda,n\lambda]} = U^{pm+}_{[-n\lambda,n\lambda]}.U^{nm-}_{[-n\lambda,n\lambda]}.\mathfrak{T}(\mathcal{O}). \tag{4.11}$$

We chose F_n as above for the following reasons. We want that when $x \in \mathcal{I}$ and $n \gg 0$, an element of G_{F_n} fixes x. By Lemma 4.5, if $u \in U^+$ (resp. U^-), and if u fixes $-n\lambda$ (resp. $n\lambda$), for some n large, then u fixes x. However, if $t \in T$, as $\mathfrak{T}(\mathcal{O})$ fixes $\mathbb{A}$, we need to require that t fixes elements outside of $\mathbb{A}$, and the choice of $x_{\alpha_i}(\varpi^n).0$ is justified by the lemma below.

Lemma 4.11. *Let $i \in I$, $n \in \mathbb{N}$ and $t \in T$. Then t fixes $x_{\alpha_i}(\varpi^{-n}).0$ if and only if $\omega(\alpha_i(t) - 1) \geq n$.*

Proof. We have $t.x_{\alpha_i}(\varpi^{-n}).0 = x_{\alpha_i}(\varpi^{-n}).0$ if and only if $x_{\alpha_i}(-\varpi^{-n})x_{\alpha_i}(\alpha_i(t) \times \varpi^{-n}).t.0 = 0$. We have $\rho_{+\infty}(x_{\alpha_i}(-\varpi^{-n})x_{\alpha_i}(\alpha_i(t)\varpi^{-n}).t.0) = t.0$ and thus if $t.x_{\alpha_i}(\varpi^{-n}).0 = x_{\alpha_i}(\varpi^{-n}).0$, we have $t.0 = 0$. Thus $t.x_{\alpha_i}(\varpi^{-n}).0 = x_{\alpha_i}(\varpi^{-n}).0$ if and only if $x_{\alpha_i}((\alpha_i(t)-1)\varpi^{-n}).0 = 0$ if and only if $\omega(\alpha_i(t)-1) \geq n$. □

For $n \in \mathbb{N}$, we set

$$T_{n,\Phi} = \{t \in T \mid \omega(\alpha_i(t)-1) \geq n, \forall\, i \in I\} \tag{4.12}$$

Lemma 4.12. *Let $y \in \mathcal{I}$. Then there exists $M \in \mathbb{N}$ such that $T_{M,\Phi}$ fixes y.*

Proof. By the Iwasawa decomposition (MA III), $y \in U^+.z$, where $z = \rho_{+\infty}(y)$. Write $y = x_{\beta_1}(a_1)\dots x_{\beta_k}(a_k).z$, with $k \in \mathbb{N}$ and $\beta_1, \dots, \beta_k \in \Phi_+$. Let $t \in T$. We have $t.y = y$ if and only if

$$\begin{aligned} z &= x_{\beta_k}(-a_k)\dots x_{\beta_1}(-a_1)tx_{\beta_1}(a_1)\dots x_{\beta_k}(a_k).z \\ &= x_{\beta_k}(-a_k)\dots x_{\beta_2}(-a_2)tx_{\beta_1}\Big((1-\beta_1(t^{-1}))a_1\Big)x_{\beta_2}(a_2)\dots x_{\beta_k}(a_k).z \end{aligned}$$

Let $M \in \mathbb{N}^*$. Assume that $\alpha_i(t) - 1 \in \varpi^M\mathcal{O}$ for every $i \in I$.

Write $\beta_1 = \sum_{i\in I} m_i\alpha_i$, with $m_i \in \mathbb{N}$ for every $i \in I$. Then $\beta_1(t) = \prod_{i\in I}\alpha_i^{m_i}(t)$. Therefore $\beta_1(t) \in 1 + \varpi^M\mathcal{O}$. For $M \gg 0$, $x_{\beta_1}\big((1-\beta_1(t^{-1}))a_1\big)$ fixes $x_{\beta_2}(a_2)\dots x_{\beta_k}(a_k).z$, by Lemma 4.5. By induction on k we deduce that t fixes z for $M \gg 0$. □

Lemma 4.13.

(1) *Let F be a finite subset of $\mathcal{I}$. Then there exists $n \in \mathbb{N}^*$ such that G_{F_n} fixes F.*

(2) *Let $\lambda, \mu \in Y^+$ be regular. Then the filtrations $(G_{F_n(\lambda)})_{n\in\mathbb{N}^*}$ and $(G_{F_n(\mu)})_{n\in\mathbb{N}^*}$ are equivalent.*

(3) *The topology $\mathscr{T}_{\mathrm{Fix}}$ is the topology associated with $\mathscr{T}((G_{F_n})_{n\in\mathbb{N}})$.*

Proof.

(1). It suffices to prove that if $y \in \mathcal{I}$, then G_{F_n} fixes y for $n \gg 0$. Let $n \in \mathbb{N}^*$. Then $F_n \supset \{-n\lambda, n\lambda\}$ and by Proposition 2.5, we have $G_{F_n} \subset G_{[-n\lambda,n\lambda]}$.

By Lemma 4.5, there exists $n_1 \in \mathbb{N}$ such that $U^{pm+}_{[-n\lambda,n\lambda]}$ and $U^{nm-}_{[-n\lambda,n\lambda]}$ fix y, for $n \geq n_1$. Let $n \geq n_1$. Let $n_2 = n_2(n) \geq n$ be such that for every $i \in I$, $\langle U^{pm+}_{[-n_2.\lambda,n_2.\lambda]}, U^{nm-}_{[-n_2.\lambda,n_2.\lambda]}\rangle$ fixes $x_{\alpha_i}(\varpi^{-n}).0$ for every $i \in I$. Let $g \in G_{F_{n_2}}$. Using (4.11), we write $g = u_+u_-t$, with $(u_+, u_-, t) \in U^{pm+}_{[-n_2\lambda,n_2\lambda]} \times U^{nm-}_{[-n_2\lambda,n_2\lambda]} \times \mathfrak{T}(\mathcal{O})$. Then g fixes y if and only if t fixes y. Moreover, u_+u_-t fixes F_n and thus t fixes F_n. By Lemma 4.11, we deduce $\omega((\alpha_i(t)-1) \geq n$ for every $i \in I$. By Lemma 4.12 we deduce that t fixes y, for $n \gg 0$. Thus $G_{F_{n_2(n)}}$ fixes y for $n \gg 0$.

(2). It follows from (1) by applying to $F = F_m(\mu)$ and $F_n = F_n(\lambda)$, for $m, n \in \mathbb{N}^*$ and by symmetry of the roles of λ and μ.

(3). As F_n is finite for every $n \in \mathbb{N}^*$, $\mathscr{T}((G_{F_n}))$ is coarser than $\mathscr{T}_{\text{Fix}}$. But by (1), $\mathscr{T}_{\text{Fix}}$ is coarser that $(\mathscr{T}_{G_{F_n}})$. □

Proposition 4.14. *The topology $\mathscr{T}_{\text{Fix}}$ is the coarsest topology of topological group on G such that $\mathfrak{G}^{\min}(\mathcal{O})$ is open.*

Proof. For $n \in \mathbb{N}^*$, $G_{F_n} \subset \mathfrak{G}^{\min}(\mathcal{O})$ and thus $\mathfrak{G}^{\min}(\mathcal{O})$ is open for $\mathscr{T}_{\text{Fix}}$.

Let now $\mathscr{T}'$ be a topology of topological group on G such that $\mathfrak{G}^{\min}(\mathcal{O})$ is open. Let $n \in \mathbb{N}^*$. Then for every element a of F_n, there exists $g_a \in G$ such that $g_a.0 = a$. Then $G_{F_n} = \bigcap_{a\in F_n} g_a.\mathfrak{G}^{\min}(\mathcal{O}).g_a^{-1}$ is open in G. Proposition follows. □

4.3.2. Relation between $\mathscr{T}_{\text{Fix}}$ and $\mathscr{T}$

In this subsection, we compare $\mathscr{T}_{\text{Fix}}$ and $\mathscr{T}$. We prove that $\mathscr{T}$ is finer than $\mathscr{T}_{\text{Fix}}$. When $\mathcal{K}$ is Henselian, we prove that $\mathscr{T} = \mathscr{T}_{\text{Fix}}$ if and only if the fixator of $\mathcal{I}$ in G is $\{1\}$ (see Proposition 4.21).

Let $\lambda \in Y^+$ be regular. For $n \in \mathbb{N}$, we define $F_n = F_n(\lambda)$ as in (4.10).

Let $\mathcal{Z} = \bigcap_{n\in\mathbb{N}} T_{n,\Phi}$ (where the $T_{n,\Phi}$ are defined in (4.12)) and $\mathcal{Z}_{\mathcal{O}} = \mathcal{Z} \cap \mathfrak{T}(\mathcal{O})$. Then $\mathcal{Z} = \{t \in T \mid \alpha_i(t) = 1, \forall\, i \in I\}$ is the center of G by [26, 8.4.3 Lemme].

Lemma 4.15. *The fixator $G_{\mathcal{I}}$ of $\mathcal{I}$ in G is $\mathcal{Z}_{\mathcal{O}}$ and $\bigcap_{n\in\mathbb{N}^*} G_{F_n} = \mathcal{Z}_{\mathcal{O}}$.*

Proof. We have $G_{\mathcal{I}} \subset G_{\mathbb{A}} = \mathfrak{T}(\mathcal{O})$, by [30, 5.7 5)]. By Lemma 4.11, $G_{\mathcal{I}} \subset T_{n,\Phi} \cap \mathfrak{T}(\mathcal{O})$ for every $n \in \mathbb{N}$ and thus $G_{\mathcal{I}} \subset \mathcal{Z}_{\mathcal{O}}$. Let $z \in \mathcal{Z}_{\mathcal{O}}$ and $x \in \mathcal{I}$. Write $x = g.a$, with $a \in \mathbb{A}$. Then $z.x = gz.a = g.a = x$ and $z \in G_{\mathcal{I}}$.

Now let $g \in \bigcap_{n\in\mathbb{N}^*} G_{F_n}$. Then by Lemma 4.13, g fixes $\mathcal{I}$, which proves the lemma. □

Lemma 4.16. *There exists an increasing map $M : \mathbb{N} \to \mathbb{N}$ whose limit is $+\infty$ and such that for every $m \in \mathbb{N}$,*

$$G_{F_n} \subset \mathcal{V}_{m\lambda}.(T_{M(n),\Phi} \cap \mathfrak{T}(\mathcal{O})),$$

for $m, n \in \mathbb{N}^*$ such that $M(n) \geq m$ and $n \geq m$.

Proof. Let $n \in \mathbb{N}^*$. Let $M(n) \in \mathbb{N}$ be maximum such that $U^{nm-}_{[-n\lambda,n\lambda]}$, $U^{pm+}_{[-n\lambda,n\lambda]}$, fix $F_{M(n)}$. By Lemma 4.5, $M(n) \to +\infty$. Let $g \in G_{F_n}$. Using (4.11) we write $g = u_+u_-t$, with $u_+ \in U^{pm+}_{[-n\lambda,n\lambda]}$, $u_- \in U^{nm-}_{[-n\lambda,n\lambda]}$ and $t \in \mathfrak{T}(\mathcal{O})$. Let $m' = M(n)$. Then $m' \leq n$ and u_+, u_- fix $F_{m'}$. As g fixes $F_{m'}$ we deduce t fixes $F_{m'}$. By Lemma 4.12, $t \in T_{m',\Phi} \cap \mathfrak{T}(\mathcal{O})$. Therefore $g \in \mathcal{V}_{m\lambda}.(T_{M(n),\Phi} \cap \mathfrak{T}(\mathcal{O}))$, which proves the lemma. □

Lemma 4.17. *Let $\mathbb{A}^* = X \otimes \mathbb{R}$, $Q' = (\bigoplus_{i \in I} \mathbb{Q}\alpha_i) \cap X \subset \mathbb{A}^*$ and d be the dimension of Q' as a $\mathbb{Q}$-vector space. Then there exists a $\mathbb{Z}$-basis $(\chi_1, \dots, \chi_\ell)$ of X such that $(\chi_1, \dots, \chi_d)$ is a $\mathbb{Z}$-basis of Q'.*

Proof. Let $x \in X$ and $n \in \mathbb{Z} \setminus \{0\}$. Assume that $nx \in Q'$. Then $x \in Q'$. Therefore X/Q' is torsion-free. Let $(e_{d+1}, \dots, e_\ell) \in (X/Q')^{\ell-d}$ be a $\mathbb{Z}$-basis of X/Q'. For $j \in [\![d+1, \ell]\!]$, take $\chi_j \in X$ whose reduction modulo Q' is e_j. Choose a $\mathbb{Z}$-basis $(\chi_1, \dots, \chi_d)$ of X'. Then $(\chi_1, \dots, \chi_\ell)$ satisfies the condition of the lemma. □

Lemma 4.18. *Assume $\mathcal{K}$ to be Henselian. Let $a \in \mathcal{O}$ and $m \in \mathbb{N}^*$. Assume $\omega(a^m - 1) > 0$. Then we can write $a = b + c$, with $b \in \mathcal{O}$ such that $b^m = 1$ and $\omega(c) > 0$.*

Proof. Let $\mathbb{k} = \mathcal{O}/\mathfrak{m}$ be the residual field and $\pi_\mathbb{k} : \mathcal{O} \twoheadrightarrow \mathcal{O}/\mathfrak{m}$ be the natural projection. Let p be the characteristic of $\mathbb{k}$. If $p = 0$, we set $m' = m$ and $k = 0$. If $p > 0$, we write $m = p^k m'$, with $k \in \mathbb{N}$ and $m' \in \mathbb{N}$ prime to p. We have $\pi_\mathbb{k}(a^m) = \pi_\mathbb{k}(a)^m = \pi_\mathbb{k}(1)$. We have $(\pi_\mathbb{k}(a)^{m'} - 1)^{p^k} = 0$ and thus $\pi_\mathbb{k}(a)^{m'} = \pi_\mathbb{k}(1)$. Let Z be an indeterminate. We have $\overline{Z^{m'} - 1} = \overline{(Z - a)}Q_\mathbb{k}$, where the bar denotes the reduction modulo $\mathfrak{m}[Z]$ and $Q_\mathbb{k} \in \mathbb{k}[Z]$ is prime to $\overline{Z - a}$. As $\mathcal{O}$ is Henselian, we can write $Z^{m'} - 1 = (Z - b)Q$, where $b \in \mathcal{O}$ is such that $\pi_\mathbb{k}(b) = \pi_\mathbb{k}(a)$ and $Q \in \mathcal{O}[Z]$ is such that $\overline{Q} = Q_\mathbb{k}$. Then $\pi_\mathbb{k}(b - a) = 0$ and we get the lemma, with $c = b - a$. □

The following lemma was suggested to me by Guy Rousseau.

Lemma 4.19. *Assume $\mathcal{K}$ to be Henselian. Let $m \in \mathbb{N}^*$. Then there exists $K \in \mathbb{N}^*, K' \in \mathbb{N}$ such that for every $n \in \mathbb{N}^*$ and $a \in \mathcal{O}$ such that $\omega(a^m - 1) \geq n$, we can write $a = b + c$, with $b, c \in \mathcal{O}$ such that $b^m = 1$ and $\omega(c) \geq n/K - K'$.*

Proof. We first assume that $\mathcal{K}$ has characteristic $p > 0$. Let $n \in \mathbb{N}^*$ and $a \in \mathcal{O}$ be such that $\omega(a^m - 1) \geq n$. Write $m = m'p^k$, with $k \in \mathbb{N}$ and $m' \in \mathbb{N}$ prime to p. We have $a^m - 1 = (a^{m'} - 1)^{p^k}$ and thus $\omega(a^{m'} - 1) = \omega(a^m - 1)/p^k \geq n/p^k > 0$. By Lemma 4.18, we can write $a = b + c$, with $b, c \in \mathcal{O}$, $b^{m'} = 1$ and $\omega(c) > 0$. We have $a^{m'} - 1 = \overline{m'}b^{m'-1}c + \sum_{i=2}^{m'} \overline{\binom{m'}{i}} c^i b^{m-i}$, where $\overline{x}$ is the image of x in $\mathcal{K}$, if $x \in \mathbb{Z}$. As m' is prime to p, $\overline{m'}$ is a root of 1 and thus we have $\omega(\overline{m'}) = 0$. As $b^{m'} = 1$, $\omega(b) = 0$. Therefore $\omega(c) = \omega(\overline{m'}b^{m-1}c) < \omega(\overline{\binom{m'}{i}}b^{m'-i}c^i)$ for $i \in [\![2, m']\!]$. Consequently $\omega(a^{m'} - 1) = \omega(c)$ and $\omega(a^m - 1) = p^k\omega(a^{m'} - 1) = p^k\omega(c) \geq n$. This proves the lemma in this case, with $K' = 0$ and $K = p^k$.

We now assume that $\mathcal{K}$ has characteristic 0. Then by [12, Theorem 1] and [27, Annexe A4] (for the case where $\omega(\mathcal{K}^*)$ is not discrete) applied with $F = \{Z^m - 1\}$ (where Z is an indeterminate), there exist $K \in \mathbb{N}^*$, $K' \in \mathbb{N}$ such that for every $n \in \mathbb{N}^*$, for every $a \in \mathcal{O}$ such that $\omega(a^m - 1) \geq n$, we can write $a = b + c$, with $b, c \in \mathcal{O}$, $b^m = 1$ and $\omega(c) \geq n/K - K'$, which proves the lemma in this case. □

Lemma 4.20. *Assume $\mathcal{K}$ to be Henselian. There exist $K_1 \in \mathbb{R}_+^*$, $L \in \mathbb{N}$ such that for every $n \in \mathbb{Z}_{\geq L}$, $T_{n,\Phi} \cap \mathfrak{T}(\mathcal{O}) \subset \mathcal{Z}_{\mathcal{O}}.T_{n/K_1}$.*

Proof. We keep the same notation as in Lemma 4.17. Let $(\chi_1^\vee, \dots, \chi_\ell^\vee) \in Y^\ell$ be the dual basis of $(\chi_1, \dots, \chi_\ell)$. For $i \in I$, we write $\alpha_i = \sum_{j=1}^{\ell} n_{j,i}\chi_j$, with $n_{j,i} \in \mathbb{Z}$ for all i, j. We have $n_{j,i} = 0$ for $j \in [\![d+1, \ell]\!]$. Set $\widetilde{t} = \prod_{j=d+1}^{\ell} \chi_j^\vee(\chi_j(t)) \in \mathfrak{T}(\mathcal{O})$. Then $\alpha_i(\widetilde{t}) = 1$ for every $i \in I$ and thus $\widetilde{t} \in \mathcal{Z}_{\mathcal{O}}$.

For $j \in [\![1, d]\!]$, write $\chi_j = \sum_{i \in I} m_{i,j}\alpha_i$, with $m_{i,j} \in \mathbb{Q}$ for every $i \in I$. Take $m \in \mathbb{N}^*$ such that $mm_{i,j} \in \mathbb{Z}$ for every $(i, j) \in I \times [\![1, d]\!]$. Let $j \in [\![1, d]\!]$. We have

$$\chi_j(t)^m = \prod_{j=1}^{d} \alpha_j(t)^{mm_{i,j}} \in 1 + \varpi^n\mathcal{O}.$$

Using Lemma 4.19 we can write $\chi_j(t) = b_j + c_j$, with $b_j, c_j \in \mathcal{O}$ such that $b_j^m = 1$ and $\omega(c_j) \geq n/K - K'$, with the same notation as in Lemma 4.19. Set $c'_j = c_j b_j^{-1} \in \mathcal{O}$. As b_j is a root of 1, we have $\omega(b_j) = 0$ and thus $\omega(c_j) = \omega(c'_j) \geq n/K - K'$. We have $b_j + c_j = b_j(1 + c'_j)$.

Set $t' = \prod_{j=1}^{d} \chi_j^\vee(b_j)$ and $t'' = \prod_{j=1}^{d} \chi_j^\vee(1+c'_j)$. Then $\chi_j(t') = b_j$ and $\chi_j(t'') = 1+c'_j$, for $j \in [\![1, d]\!]$. For $i \in I$, we have $\alpha_i(t) = \alpha_i(t')\alpha_i(t'')\alpha_i(\widetilde{t}) = \alpha_i(t')\alpha_i(t'')$ and $\alpha_i(t'') \in 1 + \varpi^{n/K-K'}\mathcal{O}$ (when $n/K - K' \notin \mathbb{N}$, $\varpi^{n/K-K'}\mathcal{O}$ is just a notation for $\mathcal{K}_{\omega \geq n/K-K'}$). As $\alpha_i(t) \in 1 + \varpi^{n/K-K'}\mathcal{O}$ we deduce $\alpha_i(t') \in 1 + \varpi^{n/K-K'}\mathcal{O}$ (replacing K by $K + 1$ if $K \leq 1$).

Let $F = \{\xi \in \mathcal{O} \mid \xi^m = 1\}$. Then F is finite. Let $L' = \max\{\omega(\xi - 1) \mid \xi \in F \setminus \{1\}\}$. Let $L \in \mathbb{N}$ be such that $L/K - K' \geq L'$. For $n \in \mathbb{Z}_{\geq L}$, we have $n/K - K' \geq L'$, we have $\alpha_i(t') = 1$ for $i \in I$ and $t' \in \mathcal{Z}_{\mathcal{O}}$. Maybe increasing L, we can assume that $K_1 := 1/K - K'/L > 0$. Then for $n \geq L$, we have $n/K - K' \geq n(1/K - K'/L) \geq nK_1$.

Consequently, for $n \in \mathbb{Z}_{\geq L}$ and $t \in T_{n,\Phi}$, we have $t = t'\widetilde{t}t''$, with $t'\widetilde{t} \in \mathcal{Z}_{\mathcal{O}}$ and $t'' \in T_{n/K_1}$, which proves the lemma. □

Proposition 4.21. *The topology $\mathscr{T}$ is finer than $\mathscr{T}_{\mathrm{Fix}}$. If $\mathcal{K}$ is Henselian, then we have $\mathscr{T} = \mathscr{T}_{\mathrm{Fix}}$ if and only if $\mathcal{Z}_{\mathcal{O}} = \{1\}$ if and only if $\mathscr{T}_{\mathrm{Fix}}$ is Hausdorff.*

Proof. Let $x \in \mathcal{I}$ and $m \in \mathbb{N}^*$. Then by Lemma 4.5, $U^{pm+}_{[-m\lambda,m\lambda]}$ and $U^{nm-}_{[-m\lambda,m\lambda]}$ fix x, for $m \gg 0$. By Lemma 4.12, $T_{2mN(\lambda)}$ fix x and thus $\mathcal{V}_{m\lambda}$ fixes x for $m \gg 0$. Thus if $n \in \mathbb{N}^*$, $\mathcal{V}_{m\lambda} \subset G_{F_n}$ for $m \gg 0$ and $\mathscr{T}$ is finer than $\mathscr{T}_{\mathrm{Fix}}$.

If $\mathscr{T} = \mathscr{T}_{\mathrm{Fix}}$, then by Theorem 4.8, $\mathscr{T}_{\mathrm{Fix}}$ is Hausdorff. Therefore $\bigcap_{n \in \mathbb{N}^*} G_{F_n} = \mathcal{Z}_{\mathcal{O}} = \{1\}$ by Lemma 4.15.

Assume $\mathcal{K}$ is Henselian. Let $m \in \mathbb{N}^*$. Then by Lemma 4.16 and Lemma 4.20, there exist $K_1 \in \mathbb{R}_+^*$ and $L \in \mathbb{N}$ such that

$$G_{F_n} \subset \mathcal{V}_{m\lambda}.\mathcal{Z}_{\mathcal{O}}.T_{M(n)/K_1},$$

for $n \geq \min(m, L)$, with $M(n) \xrightarrow[n\to+\infty]{} +\infty$. Therefore if $\mathcal{Z}_O = 1$ we have

$$G_{F_n} \subset \mathcal{V}_{m\lambda}.T_{M(n)/K_1} \subset \mathcal{V}_{m\lambda},$$

for n such that $M(n)/K \geq 2mN(\lambda)$, and thus $(\mathcal{V}_{n\lambda})$ and (G_{F_n}) are equivalent, which proves the proposition. □

Remark 4.22.

(1) If $(\alpha_i)_{i\in I}$ is a $\mathbb{Z}$-basis of X, then $\mathscr{T} = \mathscr{T}_{\mathrm{Fix}}$. Indeed, assume that $(\alpha_i)_{i\in I}$ is a $\mathbb{Z}$-basis of X. Let $(\chi_i^\vee)_{i\in I}$ be the dual basis. Let $n \in \mathbb{N}^*$ and $t \in T_{n,\Phi} \cap \mathfrak{T}(O)$. Write $t = \prod_{i\in I} \chi_i^\vee(a_i)$, with $a_i \in O^*$ for $i \in I$. Then $\pi_n(t) = \prod_{i\in I} \chi_i^\vee(\pi_n(a_i))$ and $\pi_n(t) = 1$ if and only if $\pi_n(a_i) = 1$ for all $i \in I$. Now $\alpha_i(t) = a_i$ and thus $t \in T_{n,\Phi}$ if and only if $t \in T_n$. Therefore $\mathcal{Z}_O = \bigcap_{n\in\mathbb{N}} T_n = \{1\}$.

(2) Note that by Lemma 4.16, the set of left $\mathfrak{T}(O)$-invariant open subsets of G are the same for $\mathscr{T}_{\mathrm{Fix}}$ and $\mathscr{T}$. Indeed, let $V \subset G$ be a non empty left $\mathfrak{T}(O)$-invariant open subset of G for $\mathscr{T}_{\mathrm{Fix}}$. Then for every $v \in V$, there exists $n \in \mathbb{N}^*$ such that $vG_{F_n} \subset V$. By Lemma 4.16, $\mathfrak{T}(O).G_{F_n} \subset \mathcal{V}_{n\lambda}$ and thus V is open for $\mathscr{T}$.

(3) Assume that $\mathcal{K}$ is local. By (2), if $\tau \in \mathrm{Hom}_{\mathrm{Gr}}(Y, \mathbb{C}^*)$, then $I(\tau)_{\mathscr{T}} = I(\tau)_{\mathscr{T}_{\mathrm{Fix}}}$ (see (1.1) for the definition). Indeed, $\delta^{1/2}$ and τ are maps from $Y = T/\mathfrak{T}(O)$ to $\mathbb{C}^*$ and thus their extensions to B are left $\mathfrak{T}(O)$-invariant. Therefore any element of $\widehat{I(\tau)}$ is left $\mathfrak{T}(O)$-invariant.

5. Properties of the topologies

In this section, we study the properties of the topologies $\mathscr{T}$ and $\mathscr{T}_{\mathrm{Fix}}$. In Section 5.1, we prove that when G is not reductive, $\mathscr{T}$ is strictly coarser than the Kac–Peterson topology on G (Proposition 5.4). In Section 5.2, we prove that certain subgroups of G are closed for $\mathscr{T}$. In Section 5.3, we prove that the compact subsets of G have empty interior. In Section 5.4, we describe the topology in the case of affine SL_2, under some assumption.

5.1. Comparison with the Kac–Peterson topology on G

In [22], Kac and Peterson defined a topology of topological group on $\mathfrak{G}(\mathbb{C})$. This topology was then studied in [14] and generalized in [14, 7]: Hartnick, Köhl and Mars define a topology of topological group on $\mathfrak{G}(\mathcal{F})$ for $\mathcal{F}$ a local field (Archimedean or not), taking into account the topology of $\mathcal{F}$. The aim of this section is to prove that the topologies we defined on $G = \mathfrak{G}(\mathcal{K})$ are strictly coarser than the Kac–Peterson topology on G, unless G is reductive. As $\mathscr{T}_{\mathrm{Fix}}$ is coarser than $\mathscr{T}$ it suffices to prove that $\mathscr{T}$ is strictly coarser than

$\mathscr{T}_{KP}$. To that end, we prove that $\mathscr{T}$ is coarser than $\mathscr{T}_{KP}$ and that the topologies induced by $\mathscr{T}$ and $\mathscr{T}_{KP}$ on a subset of $B := TU^+$ differ, using the description of $\mathscr{T}_{KP}|_B$ given in [14, 7].

We assume that $\mathcal{K}$ is local, in particular, $\omega(\mathcal{K}^*) = \mathbb{Z}$. We equip $\mathrm{SL}_2(\mathcal{K})$ with the topology associated to $(\ker \pi_n^{\mathrm{SL}_2})_{n\in\mathbb{N}^*}$, where $\pi_n^{\mathrm{SL}_2} : \mathrm{SL}_2(\mathcal{O}) \to \mathrm{SL}_2(\mathcal{O}/\varpi^n\mathcal{O})$ is the natural projection. We denote by x_+ (resp. x_-) the morphism of algebraic groups $a \mapsto \left(\begin{smallmatrix} 1 & a \\ 0 & 1\end{smallmatrix}\right)$ (resp. $a \mapsto \left(\begin{smallmatrix} 1 & 0 \\ a & 1\end{smallmatrix}\right)$) for a in a ring $\mathscr{R}$. Using Corollary 3.6, it is easy to check that

$$\ker \pi_n^{\mathrm{SL}_2} = x_+(\varpi^n\mathcal{O}).x_-(\varpi^n\mathcal{O}).\left(\begin{pmatrix} 1+\varpi^n\mathcal{O} & 0 \\ 0 & 1+\varpi^n\mathcal{O}\end{pmatrix} \cap \mathrm{SL}_2(\mathcal{K})\right),$$

and thus $(\mathcal{V}_{n\lambda}^{\mathrm{SL}_2})$ is equivalent to $(\ker \pi_n^{\mathrm{SL}_2})$ for any regular $\lambda \in Y_{\mathrm{SL}_2}$.

We equip T with its usual topology $\mathscr{T}_T$, via the isomorphism $T \simeq (\mathcal{K}^*)^m$, for m the rank of X. This is the topology $\mathscr{T}((\ker \pi_n|_T)_{n\in\mathbb{N}^*})$. As we shall see (Proposition 5.4), this is the topology induced by $\mathscr{T}$ on T.

For $\alpha \in \Phi_+$, let $\varphi_\alpha : \mathrm{SL}_2(\mathcal{K}) \to G$ be defined by $\varphi_\alpha \circ x_\pm = x_{\pm\alpha}$ and let G_α be its image in G. We equip G_α with the quotient topology $\mathscr{T}_{G_\alpha}$ inherited from $\mathrm{SL}_2(\mathcal{K})$ via φ_α. Let $\Sigma = \{\alpha_i \mid i \in I\}$ and $\Sigma^{(\mathbb{N})}$ be the set of finite sequences of elements of Σ. For $\underline{\alpha} = (\alpha_0, \dots, \alpha_k) \in \Sigma^{(\mathbb{N})}$, where $k \in \mathbb{N}^*$, one sets $G_{\underline{\alpha}} = G_{\alpha_0} \dots G_{\alpha_k} \subset G$. Note that $G_{\underline{\alpha}}$ is not a subgroup of G in general. If $\underline{\alpha}, \underline{\beta} \in \Sigma^{(\mathbb{N})}$, we write $\underline{\alpha} \leq \underline{\beta}$ if $\underline{\alpha}$ appears as an ordered subtuple of $\underline{\beta}$. Then $\leq$ is a preorder on $\Sigma^{(\mathbb{N})}$ and $(\Sigma^{(\mathbb{N})}, \leq)$ is a directed poset.

We equip $TG_{\underline{\alpha}}$ with the topology $\mathscr{T}_{TG_{\underline{\alpha}}}$ obtained as the quotient topology with respect to the multiplication map

$$m_{\underline{\alpha}} : (T, \mathscr{T}_T) \times (G_{\alpha_1}, \mathscr{T}_{G_{\alpha_1}}) \times \cdots \times (G_{\alpha_k}, \mathscr{T}_{G_{\alpha_k}}) \twoheadrightarrow TG_{\underline{\alpha}}. \tag{5.1}$$

In [14, Definition 7.8], the authors define the Kac–Peterson topology $\mathscr{T}_{KP}$ on G as the direct limit of the directed system $\{(TG_{\underline{\alpha}}, \mathscr{T}_{TG_{\underline{\alpha}}}) \mid \underline{\alpha} \in \Sigma^{(\mathbb{N})}\}$. In other words, a subset V of G is open for $\mathscr{T}_{KP}$ if and only if $V \cap TG_{\underline{\alpha}}$ is open for every $\underline{\alpha} \in \Sigma^{(\mathbb{N})}$.

Lemma 5.1. *Let $w \in W^v$. Assume that $wr_i < w$ for every $i \in I$ (for the Bruhat order on W^v). Then W^v is finite.*

Proof. By [23, 1.3.13 Lemma], we have $w.\alpha_i \in \Phi_-$ for every $i \in I$. Let $\lambda \in C_f^v$. Then $\alpha_i(w^{-1}.\lambda) < 0$ for every $i \in I$ and thus $w^{-1}.\lambda \in -C_f^v$. Thus $\lambda \in \mathring{\mathcal{T}} \cap -\mathring{\mathcal{T}}$. By [23, 1.4.2 Proposition] we deduce that Φ is finite and thus W^v is finite. □

We equip W^v with the *right weak Bruhat order* $\leq$: for every $v, w \in W^v$, $v \leq w$ if $\ell(v) + \ell(v^{-1}w) = \ell(w)$. We assume that W^v is infinite. By Lemma 5.1, there exists a sequence $(w_i)_{i\in\mathbb{N}} \in (W^v)^{\mathbb{N}}$ such that $w_0 = 1$, $\ell(w_{i+1}) = \ell(w_i) + 1$ and $w_i \leq w_{i+1}$ for every $i \in \mathbb{N}$.

For $w \in W^v$, one sets $\mathrm{Inv}(w) = \{\alpha \in \Phi_+ \mid w^{-1}.\alpha \in \Phi_-\}$. Let $U_w = \langle U_\alpha \mid \alpha \in \mathrm{Inv}(w)\rangle$. By [8, Lemma 5.8], if $w = r_{i_1} \dots r_{i_k}$, with $k = \ell(w)$ and $i_1, \dots, i_k \in I$, then $U_w = U_{\alpha_{i_1}}.U_{r_{i_1}.\alpha_{i_2}} \dots U_{r_{i_1}\dots r_{i_{k-1}}.\alpha_{i_k}}$ and every element of U_w admits a unique decomposition in this product. By [14, Proposition 7.27], as a topological space, B is the colimit $\lim_{\to} TU_w$ (note that $(W^v, \leq)$ is not directed). Let $U' = \bigcup_{n\in\mathbb{N}} U_{w_n}$. Then the topology induced on TU' by $\mathscr{T}_{KP}$ is the topology of the direct limit $\lim_{\to} TU_{w_n}$: a subset V of TU' is open if and only if $V \cap TU_{w_n}$ is open for every $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, write $w_{n+1} = w_n r_i$, where $i \in I$. Set $\beta[n] = w_n.\alpha_i$. Then $\mathrm{Inv}(w_n) = \{\beta[i] \mid i \in [\![1, n]\!]\}$, by [23, 1.3.14 Lemma].

By [14, Lemma 7.26], if $n \in \mathbb{N}^*$ then the map $m = m_n : T \times (\mathcal{K})^n \to TU_{w_n}$ defined by $m(t, a_1, \dots, a_n) = t x_{\beta[1]}(a_1) \dots x_{\beta[n]}(a_n)$ is a homeomorphism, when TU_{w_n} is equipped with the restriction of $\mathscr{T}_{KP}$.

Recall that $\mathscr{T} = \mathscr{T}((\mathcal{V}_{n\lambda}))$ for any $\lambda \in Y^+ \cap C^v_f$.

Define $\mathrm{ht} : Q_+ = \bigoplus_{i\in I} \mathbb{Z}\alpha_i \to \mathbb{Z}$ by $\mathrm{ht}(\sum_{i\in I} n_i\alpha_i) = \sum_{i\in I} n_i$, for $(n_i) \in \mathbb{Z}^I$.

Lemma 5.2. *Assume that W^v is infinite. For $n \in \mathbb{N}^*$, set $V_n = T \prod_{i=1}^n x_{\beta[i]}(\varpi^{(\mathrm{ht}(\beta[i])!}\mathcal{O})$ and set $V = \bigcup_{n\in\mathbb{N}} V_n$. Then V is open in $(TU', \mathscr{T}_{KP})$ but not in $(TU', \mathscr{T})$. In particular, $\mathscr{T}$ and $\mathscr{T}_{KP}$ are different.*

Proof. Let $n \in \mathbb{N}^*$. Let $v \in V \cap TU_{w_n}$ and choose $k \in \mathbb{N}^*$ such that $v \in V_k$. If $k \leq n$, then $v \in V_k \subset V_n$. Suppose now $k \geq n$. Write $v = t \prod_{i=1}^k x_{\beta[i]}\left(\varpi^{(\mathrm{ht}(\beta_i)!}a_i\right)$, with $a_1, \dots, a_k \in \mathcal{O}$ and $t \in T$. By [14, Lemma 7.26], we have $a_i = 0$ for every $i \in [\![n+1, k]\!]$ and thus $v \in V_n$. Therefore $V \cap TU_{w_n} = V_n$. By [14, Lemma 7.26], V_n is open in TU_{w_n} and thus V is open in $(TU', \mathscr{T}_{KP})$.

Let $\lambda \in C^v_f \cap Y$ be such that $\alpha_i(\lambda) = 1$ for every $i \in I$. Let us prove that for every $n \in \mathbb{N}^*$, $U' \cap U^{pm+}_{-n\lambda}$ is not contained in V. For $k, n \in \mathbb{N}^*$, set $x_{k,n} = \prod_{i=1}^k x_{\beta[i]}(\varpi^{n\mathrm{ht}(\beta[i])}) \in U' \cap U^{pm+}_{-n\lambda}$. Let $n \in \mathbb{N}^*$. By [14, Lemma 7.26], if $x_{k,n} \in V$, then $n\mathrm{ht}(\beta[i]) \geq (\mathrm{ht}(\beta[i]))!$, for every $i \in [\![0, k]\!]$. As $\mathrm{ht}(\beta[i]) \xrightarrow[i\to+\infty]{} +\infty$, there exists $k \in \mathbb{N}$ such that $x_{k,n} \in U' \cap U^{pm+}_{-n\lambda} \setminus V$ and thus, $U' \cap U^{pm+}_{-n\lambda} \not\subset V$. Using Lemma 2.3 we deduce that there exists no $n \in \mathbb{N}^*$ such that $\mathcal{V}_{n\lambda} \cap TU' \subset V$ and thus V is not open for $\mathscr{T}$. □

Proposition 5.3. *Let $\underline{V} = (V_n)_{n\in\mathbb{N}^*}$ be a conjugation-invariant filtration of G. Let $\mathscr{T}_{\underline{V}}$ be the associated topology on G. We assume that for every $\alpha \in \Sigma$, the induced topology on G_α is $\mathscr{T}_{G_\alpha}$ and that the induced topology on T is $\mathscr{T}_T$. Then $\mathscr{T}_{\underline{V}}$ is coarser than $\mathscr{T}_{KP}$.*

Proof. Let $n \in \mathbb{N}^*$. Let us prove that V_n is open for $\mathscr{T}_{KP}$. Let $\underline{\alpha} = (\alpha_k, \dots, \alpha_1) \in \Sigma^{(\mathbb{N})}$. Let us prove that $m_{\underline{\alpha}}^{-1}(V_n)$ is open in $T \times G_{\alpha_k} \times \dots \times G_{\alpha_1}$, with the notation of (5.1). Let $v \in TG_{\underline{\alpha}} \cap V_n$ and $(t, v_k, \dots, v_1) \in m_{\underline{\alpha}}^{-1}(\{v\})$. We have $v = tv_k \dots v_1$. We set $n_1 = n$ and we choose $n_2, \dots, n_k, n_{k+1} \in \mathbb{N}^*$ such that for all $j \in [\![2, k]\!]$, we have

$V_{n_j} v_{j-1} \dots v_1 \subset v_{j-1} \dots v_1 V_n$, which is possible since $\underline{V}$ is conjugation-invariant. Then we have:

$$\begin{aligned} V_{n_{k+1}} v_k V_{n_k} v_{k-1} V_{n_{k-1}} \dots V_{n_3} v_2 V_{n_2} v_1 V_{n_1} &\subset V_{n_{k+1}} v_k V_{n_k} v_{k-1} V_{n_{k-1}} \dots V_{n_3} v_2 v_1 V_n \\ &\subset \dots \subset v_k \dots v_1 V_n. \end{aligned}$$

Consequently

$$\left(tV_{n_{k+1}} \cap T\right)\left(v_k V_{n_k} \cap G_{\alpha_k}\right) \dots \left(v_1 V_{n_1} \cap G_{\alpha_1}\right) \subset t v_k \dots v_1 V_n = v V_n = V_n$$

and hence

$$m_{\underline{\alpha}}\left(\left(tV_{n_{k+1}} \cap T\right) \times \left(v_k V_{n_k} \cap G_{\alpha_k}\right) \times \dots \times \left(v_1 V_{n_1} \cap G_{\alpha_1}\right)\right) \subset V_n \cap TG_{\underline{\alpha}}.$$

Therefore $m_{\underline{\alpha}}^{-1}(V_n \cap TG_{\underline{\alpha}})$ is open or equivalently $V_n \cap TG_{\underline{\alpha}}$ is open. As this is true for every $\underline{\alpha} \in \Sigma^{(\mathbb{N})}$, we deduce that V_n is open for $\mathscr{T}_{KP}$. As $(G, \mathscr{T}_{KP})$ is a topological group, we deduce that xV_n is open in $\mathscr{T}_{KP}$ for every $x \in G$ and $n \in \mathbb{N}^*$ and we deduce that $\mathscr{T}_{\underline{V}}$ is coarser than $\mathscr{T}_{KP}$. □

Proposition 5.4.

(1) *Let $\alpha \in \Phi_+$ and $\varphi_\alpha : \mathrm{SL}_2(\mathcal{K}) \to G$ be the group morphism defined by $\varphi_\alpha \circ x_\pm = x_{\pm\alpha}$. Fix a basis $(\chi_1^\vee, \dots, \chi_\ell^\vee)$ of Y and define $\iota : (\mathcal{K}^*)^\ell \xrightarrow{\sim} T \subset G$ by $\iota((a_1, \dots, a_\ell)) = \chi_1^\vee(a_1) \dots \chi_\ell^\vee(a_\ell)$, for $a_1, \dots, a_\ell \in \mathcal{K}^*$. Then the φ_α, $\alpha \in \Phi$ and ι are continuous when G is equipped with $\mathscr{T}$.*

(2) *The topology induced by $\mathscr{T}$ on T is $\mathscr{T}_T$ and if $\alpha \in \Phi_+$, then the topology induced by $\mathscr{T}$ on G_α is $\mathscr{T}_{G_\alpha}$*

Proof. Let $\alpha \in \Phi$ and $\lambda \in Y^+$ be regular. Let $g \in \varphi_\alpha^{-1}(\mathcal{V}_\lambda)$. By [30, 3.16] and [26, 1.2.4 Proposition], we have the Birkhoff decomposition in $\mathrm{SL}_2(\mathcal{K})$ (where N_{SL_2} is the set of monomial matrices with coefficient in $\mathcal{K}^*$) and G:

$$\mathrm{SL}_2(\mathcal{K}) = \bigsqcup_{n \in N_{\mathrm{SL}_2}} x_+(\mathcal{K}) n x_-(\mathcal{K}) \quad \text{and} \quad G = \bigsqcup_{n \in N} U^+ n U^-.$$

Let $n \in N_{\mathrm{SL}_2}$ be such that $g \in x_+(\mathcal{K}) n x_-(\mathcal{K})$. If $n \notin T_{\mathrm{SL}_2}$, then $\varphi_\alpha(g) \in U^+ \varphi_\alpha(n) U^-$ and $\nu^v(\varphi_\alpha(n))$ acts as the reflection with respect to α on $\mathbb{A}$. Then $\varphi_\alpha(g) \notin U^+ T U^-$ which contradicts Corollary 3.6. Therefore $n \in T$. Write $g = x_+(a_+) n x_-(a_-)$, with $a_+, a_- \in \mathcal{K}$. Then by Lemma 2.3, we have $x_\alpha(a_+) \in U_{[-\lambda,\lambda]}^{pm+}$, $x_{-\alpha}(a_-) \in U_{[-\lambda,\lambda]}^{nm-}$ and $\varphi_\alpha(n) \in T_{2N(\lambda)}$. Consequently, $\omega(a_+), \omega(a_-) \geq |\alpha(\lambda)|$ and

$$t \in T_{\mathrm{SL}_2, 2N(\lambda)} := \begin{pmatrix} 1 + \varpi^{2N(\lambda)}\mathcal{O} & 0 \\ 0 & 1 + \varpi^{2N(\lambda)}\mathcal{O} \end{pmatrix} \cap \mathrm{SL}_2(\mathcal{K}).$$

Therefore $\varphi_\alpha^{-1}(\mathcal{V}_\lambda) \subset x_+(\mathcal{K}_{\omega\geq|\alpha(\lambda)|})T_{\mathrm{SL}_2,2N(\lambda)}x_-(\mathcal{K}_{\omega\geq|\alpha(\lambda)|})$. Conversely, $\varphi_\alpha(x_-(\mathcal{K}_{\omega\geq|\alpha(\lambda)|}))$, $\varphi_\alpha(x_+(\mathcal{K}_{\omega\geq|\alpha(\lambda)|})$, $\varphi_\alpha(T_{\mathrm{SL}_2,2N(\lambda)}) \subset \mathcal{V}_\lambda$ and thus $\varphi_\alpha^{-1}(\mathcal{V}_\lambda) = x_+(\mathcal{K}_{\omega\geq|\alpha(\lambda)|})T_{\mathrm{SL}_2,2N(\lambda)}x_-(\mathcal{K}_{\omega\geq|\alpha(\lambda)|})$ is open in $\mathrm{SL}_2(\mathcal{K})$. Therefore φ_α is continuous and $\mathcal{V}_\lambda \cap G_\alpha$ is open.

Let $n \in \mathbb{N}^*$. Then $\iota^{-1}(T_n) = (1+\varpi^n\mathcal{O})^\ell$ and thus ι is continuous. Moreover, we have $T \cap \mathcal{V}_\lambda = T_{2N(\lambda)}$, by the Birkhoff decomposition, which proves that $\mathscr{T}$ induces $\mathscr{T}_T$ on T. □

Corollary 5.5. *If Φ is infinite, the topologies $\mathscr{T}$ and $\mathscr{T}_{\mathrm{Fix}}$ are strictly coarser than $\mathscr{T}_{KP}$.*

Proof. Let $\lambda \in Y^+$ be any regular element. By Propositions 5.4 and 5.3, applied with $\underline{V} = (\mathcal{V}_{n\lambda})$, $\mathscr{T}$ is coarser than $\mathscr{T}_{KP}$. By Lemma 5.2, $\mathscr{T}$ is different from $\mathscr{T}_{KP}$. As $\mathscr{T}_{\mathrm{Fix}}$ is coarser than $\mathscr{T}$ (by Proposition 4.21), we deduce the result. □

5.2. Properties of usual subgroups of G for $\mathscr{T}$ and $\mathscr{T}_{\mathrm{Fix}}$

In this subsection, we prove that many subgroups important in this theory (such as B, T, U_α, $\alpha \in \Phi$, etc.) are open or closed. We have $\mathcal{T}_{\mathrm{Fix}} \subset \mathcal{T}$ and thus every subset of G open or closed for $\mathcal{T}_{\mathrm{Fix}}$ is open or closed for $\mathcal{T}$. As the Kac–Peterson topology $\mathscr{T}_{KP}$ is finer than $\mathscr{T}$, this improves the corresponding results of [14]. Note that we consider $B = B^+$ and U^+, but the same results hold for B^- and U^-, by symmetry.

If $g \in G$, we say that g *stabilizes* (resp. *pointwise fixes*) $+\infty$ if $g.+\infty = +\infty$ (resp. if there exists $Q \in +\infty$ such that g pointwise fixes Q). We denote by $\mathrm{Stab}_G(+\infty)$ the stabilizer of $+\infty$ in G.

By [15, 3.4.1], $\mathrm{Stab}_G(+\infty) = B := TU^+$. We denote by $\mathrm{Ch}(\partial\mathcal{I}^+)$ the set of positive sector-germs at infinity of $\mathcal{I}$. For $c \in \mathrm{Ch}(\partial\mathcal{I}^+)$ and $x \in \mathcal{I}$, there exists an apartment A containing x and c. We denote by

$$x + c \tag{5.2}$$

the convex hull of x and c in this apartment. This does not depend on the choice of A, by (MA II). Fix $\lambda_0 \in C_f^v$. For $r \in \mathbb{R}_+$, we set $C_r = \{c \in \mathrm{Ch}(\partial\mathcal{I}^+) \mid [0, r.\lambda_0] \subset 0 + c\}$. This set is introduced in [9, Definition 3.1] where it is denoted $U_{0,r,c}$ or $U_{r,c}$.

Proposition 5.6.

(1) *The subgroup B is closed in G for $\mathscr{T}$ and $\mathscr{T}_{\mathrm{Fix}}$.*

(2) *The subgroup U^+ of G is closed for $\mathscr{T}$. It is closed for $\mathscr{T}_{\mathrm{Fix}}$ if and only if $\mathscr{T} = \mathscr{T}_{\mathrm{Fix}}$.*

Proof.

(1). Let $g \in G \setminus B$. Then $g.(+\infty) \neq +\infty$. By [9, Lemma 7.6], $\bigcap_{r \in \mathbb{R}_+} \mathcal{C}_r = \{+\infty\}$. Thus there exists $n \in \mathbb{N}^*$ such that $g^{-1}. + \infty \notin \mathcal{C}_n$. Then $\mathcal{V}_{n\lambda_0}$ fix $[0, n\lambda_0]$. Let $v \in \mathcal{V}_{n\lambda_0}$. Then

$$v.(0 + \infty) = v.0 + (v. + \infty) = 0 + (v. + \infty) \supset v.[0, n\lambda_0] = [0, n\lambda_0].$$

Therefore $\mathcal{V}_{n\lambda_0}.(+\infty) \subset \mathcal{C}_n$. Consequently, $g^{-1}.(+\infty) \notin \mathcal{V}_{n\lambda_0}.+\infty$ and thus $g\mathcal{V}_{n\lambda_0}.+\infty \not\ni +\infty$. Thus $g.\mathcal{V}_{n\lambda_0} \subset G \setminus B$, which proves that $G \setminus B$ is open for $\mathscr{T}_{\mathrm{Fix}}$.

(2). Let $g \in G \setminus U^+$. If $g \in G \setminus U^+T$, then by (1), there exists $V \in \mathcal{T}$ such that $gV \subset G \setminus U^+T \subset G \setminus U^+$. We now assume $g \in U^+T \setminus U$. Write $g = u_+t$, with $u_+ \in U^+$ and $t \in T \setminus \{1\}$. Let $\lambda \in Y \cap C_f^v$ and assume that $g\mathcal{V}_\lambda \cap U^+ \neq \emptyset$. Then there exists $(u'_+, u'_-, t') \in U^{pm+}_{[-\lambda,\lambda]} \times U^{nm-}_{[-\lambda,\lambda]} \times T_{2N(\lambda)}$ such that $u_+tu'_+u'_-t' = u''_+$, where $u''_+ \in U^+$. As t normalizes U^+ and U^-, we can write $tu'_+u'_- = u_+^{(3)}u_-^{(3)}t$, for some $u_+^{(3)} \in U^+, u_-^{(3)} \in U^-$. Then we have

$$u_+''^{-1}u_+u_+^{(3)}u_-^{(3)}tt' = 1.$$

By Lemma 2.3 we deduce $tt' = 1$. Therefore $t \in T_{2N(\lambda)}$. Thus if λ is sufficiently dominant, $g\mathcal{V}_\lambda \cap U^+ = \emptyset$ and $g\mathcal{V}_\lambda \subset G \setminus U^+$. We deduce that $G \setminus U^+$ is closed for $\mathcal{T}$.

Suppose now $\mathscr{T} \neq \mathscr{T}_{\mathrm{Fix}}$. Then by Proposition 4.21, $\mathcal{Z}_\mathcal{O} \neq \{1\}$. Then every non empty open subset of G for $\mathcal{T}_{\mathrm{Fix}}$ contains $\mathcal{Z}_\mathcal{O}$. Take $z \in \mathcal{Z}_\mathcal{O} \setminus \{1\}$. As $z \in \mathfrak{T}(\mathcal{O})$, $z \in G \setminus U^+$. Moreover, for any non empty open subset V of G, $zV \ni 1 \in U^+$. Therefore $G \setminus U^+$ is not open, which completes the proof of the proposition. □

Proposition 5.7.

(1) *Let $x \in \mathcal{I}$. Then the fixator G_x of x in G is open (for $\mathcal{T}_{\mathrm{Fix}}$ and $\mathcal{T}$). In particular, $\mathfrak{G}^{\min}(\mathcal{O})$ is open in G.*

(2) *Let $E \subset \mathcal{I}$. Then the fixator and the stabilizer of E in G are closed for $\mathcal{T}_{\mathrm{Fix}}$ and $\mathcal{T}$.*

Proof.

(1). By Lemma 4.13, $G_{F_n} \subset G_x$ for $n \gg 0$, which proves (1).

(2). Let $g \in G \setminus G_E$. Let $x \in E$ be such that $g.x \neq x$. Then $g.G_x \subset G \setminus G_E$ and hence G_E is open. Let $g \in G \setminus \mathrm{Stab}_G(E)$. Let $x \in E$ be such that $g.x \notin E$. Then $g.G_x \subset G \setminus \mathrm{Stab}_G(E)$ and hence $\mathrm{Stab}_G(E)$ is closed. □

Corollary 5.8. *The subgroups N and T are closed in G for $\mathcal{T}_{\mathrm{Fix}}$ and $\mathcal{T}$.*

Proof. By Proposition 5.7, $N = \mathrm{Stab}_G(\mathbb{A})$ is closed. We have $T = \mathrm{Stab}_G(+\infty) \cap N$. Indeed, it is clear that $T \subset \mathrm{Stab}_G(+\infty) \cap N$. Conversely, let $g \in \mathrm{Stab}_G(+\infty) \cap N$. Let $w \in W^v$ and $\lambda \in \Lambda$ be such that $g.x = \lambda + w.x$ for every $x \in \mathbb{A}$. Then $w.C_f^v = C_f^v$ and thus $w = 1$. Therefore g acts by translation on $\mathbb{A}$ and hence $g \in T$. This proves that $T = \mathrm{Stab}_G(+\infty) \cap N$ and we conclude with Proposition 5.6. □

Remark 5.9.

(1) The fixator K_I of C_0^+ is open for $\mathscr{T}_{\mathrm{Fix}}$. Indeed, let $\lambda \in C_f^v \cap Y$. Then $G_{[0,\lambda]} = G_0 \cap G_\lambda$ is open for $\mathscr{T}_{\mathrm{Fix}}$ and $G_{[0,\lambda]} \subset K_I$.

(2) For $x, y \in \mathcal{I}$, one writes $x \leq y$ if there exists $g \in G$ such that $g.x, g.y \in \mathbb{A}$ and $g.y - g.x \in \mathcal{T} = \bigcup_{w \in W^v} w.\overline{C_f^v}$. By W^v-invariance of $\mathcal{T}$, this does not depend on the choice of g and by [29, Théorème 5.9], $\leq$ is a preorder on $\mathcal{I}$. One sets

$$G^+ = \{g \in G \mid g.0 \geq 0\}.$$

This is a subsemigroup of G which is crucial for the definition of the Hecke algebras associated with G (when $\mathcal{K}$ is local), see [2], [5], [6] or [11]. Then $G^+ \supset G_0 = \mathfrak{G}^{\min}(\mathcal{O})$ and thus G^+ is open in G.

Lemma 5.10. *Let $g \in G$. Then there exists $n \in N$ such that $g.a = n.a$ for every $a \in \mathbb{A} \cap g^{-1}.\mathbb{A}$.*

Proof. Let $h \in G$ be such that $hg.\mathbb{A} = \mathbb{A}$ and h fixes $\mathbb{A} \cap g.\mathbb{A}$, which exists by (MA II). Then $n := hg$ stabilizes $\mathbb{A}$ and thus it belongs to N. Moreover, $hg.a = n.a = g.a$ for every $a \in \mathbb{A} \cap g^{-1}.\mathbb{A}$, which proves the lemma. □

Lemma 5.11. *Let $\alpha \in \Phi$. Write $\alpha = \epsilon w.\alpha_i$, for $w \in W^v$, $\epsilon \in \{-, +\}$ and $i \in I$. Let $\mathfrak{Q}$ be the sector-germ at infinity of $-\epsilon w r_i(C_f^v)$. Then $U_\alpha T = \mathrm{Stab}_G(w.\epsilon\infty) \cap \mathrm{Stab}_G(\mathfrak{Q})$.*

Proof. There is no loss of generality in assuming that $w = 1$ and $\epsilon = +$. Let $u \in U_{\alpha_i}$. Then u fixes a translate of $\alpha_i^{-1}(\mathbb{R}_+)$. Therefore TU_{α_i} stabilizes $\mathfrak{Q}$ and $+\infty$. Conversely, let $g \in \mathrm{Stab}_G(+\infty) \cap \mathrm{Stab}_G(\mathfrak{Q})$. Then there exist $x, x' \in \mathbb{A}$ such that $g.(x + C_f^v) = x' + C_f^v$. Then by Lemma 5.10, there exists $n \in N$ such that $g.x'' = n.x''$ for every $x'' \in \mathbb{A} \cap g^{-1}.\mathbb{A}$. Then n fixes $+\infty$ and thus $n \in T$ (by the proof of Corollary 5.8). Then $n^{-1}g.x = x$. Considering $n^{-1}g$ instead of g, we may assume that g pointwise fixes $+\infty$. Therefore g pointwise fixes $\mathfrak{Q}$. There exists $a, a' \in \mathbb{A}$ such that g fixes $a + C_f^v$ and g fixes $a' - r_i(C_f^v)$. Let $A = g.\mathbb{A}$. Then $A \cap \mathbb{A}$ is a finite intersection of half-apartments by (MA II) and thus either $A = \mathbb{A}$ or $A \cap \mathbb{A}$ is a translate of $\alpha_i^{-1}(\mathbb{R}_+)$. Moreover, g fixes $A \cap \mathbb{A}$ since it fixes an open subset of $A \cap \mathbb{A}$. By [30, 5.7 3)], $g \in U_{\alpha_i}\mathfrak{T}(\mathcal{O})$. Consequently $\mathrm{Stab}_G(+\infty) \cap \mathrm{Stab}_G(\mathfrak{Q}) \subset U_{\alpha_i}T$, and the lemma follows. □

Proposition 5.12. *Let $\alpha \in \Phi$.*

(1) *The group $U_\alpha T$ is closed for $\mathcal{T}$ and $\mathcal{T}_{\mathrm{Fix}}$.*

(2) *The group U_α is closed for $\mathcal{T}$.*

(3) *If $\mathscr{T} \neq \mathscr{T}_{\mathrm{Fix}}$, then U_α is not closed for $\mathscr{T}_{\mathrm{Fix}}$.*

Proof.

(1). Let $\mathfrak{Q}$ be a sector-germ of $\mathcal{I}$ (positive or negative). Then by Proposition 5.6 (or the similar proposition for B^- if $\mathfrak{Q}$ is negative), $\mathrm{Stab}_G(\mathfrak{Q})$ is closed in G for $\mathscr{T}_{\mathrm{Fix}}$. By Lemma 5.11 and Proposition 5.7, we have (1).

(2). We have $U_\alpha = U_\alpha T \cap U^+$, by Lemma 2.3 and thus (2) follows from (1) and Proposition 5.6.

(3). It is similar to the proof of the corresponding result of Proposition 5.6. □

5.3. Compact subsets have empty interior

By [1, Theorem 3.1], for any topology of topological group on G, $G_{C_0^+}$ or G_0 are not compact and open. In particular, G_0 and $G_{C_0^+}$ are not compact for $\mathscr{T}$. With a similar reasoning, we have the following.

Proposition 5.13. *Assume that W^v is infinite.*

(1) *Let $n \in \mathbb{N}^*$ and $\lambda \in Y^+$ regular. Then $\mathcal{V}_{n\lambda}/\mathcal{V}_{(n+1)\lambda}$ is infinite.*

(2) *Every compact subset of $(G, \mathscr{T})$ has empty interior.*

Proof.

(1). Set $H = G_{[-n\lambda,(n+1)\lambda]} \subset G$. Then $H \supset \mathcal{V}_{(n+1)\lambda}$, by (4.1). Thus $|\mathcal{V}_{n\lambda}/\mathcal{V}_{(n+1)\lambda}| \geq |\mathcal{V}_{n\lambda}/(H \cap \mathcal{V}_{n\lambda})|$ and it suffices to prove that $\mathcal{V}_{n\lambda}/(H \cap \mathcal{V}_{n\lambda})$ is infinite. We have $\mathcal{V}_{n\lambda} = \bigsqcup_{v \in \mathcal{V}_{n\lambda}/(H \cap \mathcal{V}_{n\lambda})} v.(H \cap \mathcal{V}_{n\lambda})$. Moreover if $v, v' \in \mathcal{V}_{n\lambda}$, then $v.((n+1)\lambda) = v'.((n+1)\lambda)$ if and only if $v'.(G_{(n+1)\lambda} \cap \mathcal{V}_{n\lambda}) = v.(G_{(n+1)\lambda} \cap \mathcal{V}_{n\lambda})$.

Let us prove that $G_{(n+1)\lambda} \cap \mathcal{V}_{n\lambda} = H \cap \mathcal{V}_{n\lambda}$. Let $g \in G_{(n+1)\lambda} \cap \mathcal{V}_{n\lambda}$. Then by (4.1), g fixes $[-n\lambda, n\lambda]$ and $(n+1)\lambda$. Then $g.\mathbb{A}$ is an apartment containing $[-n\lambda, n\lambda] \cup \{(n+1)\lambda\}$. As $g.\mathbb{A} \cap \mathbb{A}$ is convex, $g.\mathbb{A}$ contains $[-n\lambda, (n+1)\lambda]$. By (MA II), there exists $h \in G$ such that $g.\mathbb{A} = h.\mathbb{A}$ and h fixes $\mathbb{A} \cap g.\mathbb{A}$. Then $h^{-1}g.\mathbb{A} = \mathbb{A}$ and $h^{-1}g$ acts on $\mathbb{A}$ by an affine map. As $h^{-1}g$ fixes $[-n\lambda, n\lambda]$, it fixes $[-n\lambda, (n+1)\lambda]$. Therefore g fixes $[-n\lambda, (n+1)\lambda]$ and thus $g \in H$. Therefore $G_{(n+1)\lambda} \cap \mathcal{V}_{n\lambda} = H \cap \mathcal{V}_{n\lambda}$. Consequently,

$$\mathcal{V}_{n\lambda}.((n+1)\lambda) = \bigsqcup_{v \in \mathcal{V}_{n\lambda}/(H\cap\mathcal{V}_{n\lambda})} \{v.(n+1)\lambda\} \text{ and } |\mathcal{V}_{n\lambda}/(H\cap\mathcal{V}_{n\lambda})| = |\mathcal{V}_{n\lambda}.\,((n+1)\lambda)\,|.$$

Let $(\beta_\ell) \in (\Phi_+)^{\mathbb{N}}$ be an injective sequence. Write $\beta_\ell = \sum_{i\in I} m_i^{(\ell)}\alpha_i$, with $m_i^{(\ell)} \in \mathbb{N}$ for $\ell \in \mathbb{N}$. Then $\beta_\ell(\lambda) \geq (\sum_{i\in I} m_i^{(\ell)})(\min_{i\in I}\alpha_i(\lambda)) \xrightarrow[\ell\to\infty]{} +\infty$. Let $\ell \in \mathbb{N}$. For $k \in [\![\beta_\ell(n\lambda), \beta_\ell((n+1)\lambda) - 1]\!]$, $x_{-\beta_\ell}(\varpi^k) \in U^{nm-}_{[-n\lambda,n\lambda]} \subset \mathcal{V}_{n\lambda}$. Set

$$x_k = x_{-\beta_\ell}(\varpi^k).((n+1)\lambda) \in \mathcal{V}_{n\lambda}.((n+1)\lambda).$$

Let $k' \in [\![\beta_\ell(n\lambda), \beta_\ell((n+1)\lambda)-1]\!]$. Then $x_k = x_{k'}$ if and only if $x_{-\beta_\ell}(\varpi^k).((n+1)\lambda) = x_{-\beta_\ell}(\varpi^{k'}).((n+1)\lambda)$ if and only if $x_{-\beta_\ell}(\varpi^k - \varpi^{k'}).((n+1)\lambda) = (n+1)\lambda$ if and only if $\omega(\varpi^k - \varpi^{k'}) \geq (n+1)\beta_\ell(\lambda)$ if and only $k = k'$. Therefore $|\mathcal{V}_{n\lambda}.((n+1)\lambda)| \geq \beta_\ell(\lambda)$. As this is true for every $\ell \in \mathbb{N}$, $|\mathcal{V}_{n\lambda}.((n+1)\lambda)|$ if infinite, which proves (1).

(2). Let V be a compact subset of G and assume that V has non empty interior. Considering $v^{-1}.V$ instead of V, we may assume $1 \in V$. Then there exists $\lambda \in Y^+ \cap C_f^v$ such that $\mathcal{V}_\lambda \subset V$, and we have $\mathcal{V}_{2\lambda} \subset \mathcal{V}_\lambda$. As $\mathcal{V}_\lambda$ is closed, it is compact. By (1), $\mathcal{V}_\lambda/\mathcal{V}_{2\lambda}$ is infinite. Therefore $\mathcal{V}_\lambda = \bigsqcup_{v\in\mathcal{V}_\lambda/\mathcal{V}_{2\lambda}} v.\mathcal{V}_{2\lambda}$ is a cover of $\mathcal{V}_\lambda$ by open subsets from which we can not extract a finite subcover: we reach a contradiction. Thus every compact subset of G has empty interior. □

5.4. Example of affine SL_2

In this subsection, we determine an explicit filtration equivalent to $(\mathcal{V}_{n\lambda})$ in the case of affine SL_2 (quotiented by the central extension).

Let $Y = \mathbb{Z}\mathring{\alpha}^\vee \oplus \mathbb{Z}d$, where $\mathring{\alpha}^\vee, d$ are some symbols, corresponding to the positive root of $\mathrm{SL}_2(\mathcal{K})$ and to the semi-direct extension by $\mathcal{K}^*$ respectively. Let $X = \mathbb{Z}\mathring{\alpha} \oplus \mathbb{Z}\delta$, where $\mathring{\alpha}, \delta : Y \to \mathbb{Z}$ are the $\mathbb{Z}$-module morphisms defined by $\mathring{\alpha}(\mathring{\alpha}^\vee) = 2$, $\mathring{\alpha}(d) = 0$, $\delta(\mathring{\alpha}^\vee) = 0$ and $\delta(d) = 1$. Let $\alpha_0 = \delta - \mathring{\alpha}$, $\alpha_1 = \mathring{\alpha}$, $\alpha_0^\vee = -\mathring{\alpha}^\vee$ and $\alpha_1^\vee = \mathring{\alpha}^\vee$. Then $\mathcal{S} = \left(\left(\begin{smallmatrix} 2 & -2 \\ -2 & 2\end{smallmatrix}\right), X, Y, \{\alpha_0, \alpha_1\}, \{\alpha_0^\vee, \alpha_1^\vee\}\right)$ is a root generating system. Let $\mathfrak{G}$ be the Kac–Moody group associated with $\mathcal{S}$ and $G = \mathfrak{G}(\mathcal{K})$. Then by [23, 13] and [24, 7.6], $G = \mathrm{SL}_2\left(\mathcal{K}[u, u^{-1}]\right) \rtimes \mathcal{K}^*$, where u is an indeterminate and if $(M, z), (M_1, z_1) \in G$, with $M = \left(\begin{smallmatrix} a(\varpi,u) & b(\varpi,u) \\ c(\varpi,u) & d(\varpi,u)\end{smallmatrix}\right)$, $M_1 = \left(\begin{smallmatrix} a_1(\varpi,u) & b_1(\varpi,u) \\ c_1(\varpi,u) & d_1(\varpi,u)\end{smallmatrix}\right)$, we have

$$(M, z).(M_1, z_1) = \left(M \begin{pmatrix} a_1(\varpi, zu) & b_1(\varpi, zu) \\ c_1(\varpi, zu) & d_1(\varpi, zu)\end{pmatrix}, zz_1\right). \tag{5.3}$$

Note that the family $(\alpha_0^\vee, \alpha_1^\vee)$ is not free. We have $\Phi = \{\alpha + k\delta \mid \alpha \in \{\pm\mathring{\alpha}\}, k \in \mathbb{Z}\}$ and (α_0, α_1) is a basis of this root system. We denote by Φ^+ (resp. Φ^-) the set $\Phi \cap (\mathbb{N}\alpha_0 + \mathbb{N}\alpha_1)$ (resp $-\Phi_+$). For $k \in \mathbb{Z}$ and $y \in \mathcal{K}$, we set $x_{\mathring{\alpha}+k\delta}(y) = \left(\begin{pmatrix} 1 & u^k y \\ 0 & 1 \end{pmatrix}, 1\right) \in G$ and $x_{-\mathring{\alpha}+k\delta}(y) = \left(\begin{pmatrix} 1 & 0 \\ u^k y & 1 \end{pmatrix}, 1\right) \in G$.

Let $f, g \in \mathcal{K}$ be such that $\omega(f) = \omega(g) = 0$. Let $\ell, n \in \mathbb{Z}$. Then $\left(\begin{pmatrix} f\varpi^\ell & 0 \\ 0 & f^{-1}\varpi^{-\ell} \end{pmatrix}, g\varpi^n\right)$ acts on $\mathbb{A}$ by the translation of vector $-\ell\mathring{\alpha}^\vee - nd$. For $\mu = \ell\mathring{\alpha}^\vee + nd \in Y$, we set $t_\mu = \left(\begin{pmatrix} \varpi^{-\ell} & 0 \\ 0 & \varpi^\ell \end{pmatrix}, \varpi^{-n}\right)$, which acts by the translation of vector μ on $\mathbb{A}$. We set $\lambda = \mathring{\alpha}^\vee + 3d$. We have $\alpha_0(\lambda) = 1$, $\alpha_1(\lambda) = 2$ and thus $\lambda \in C_f^v$.

By [30, 4.12 3 b], $U_0^{pm+} = \left(\begin{pmatrix} 1+u\mathcal{O}[u] & \mathcal{O}[u] \\ u\mathcal{O}[u] & 1+u\mathcal{O}[u] \end{pmatrix}, 1\right) \cap G$ and similarly

$$U_0^{nm-} = \left(\begin{pmatrix} 1 + u^{-1}\mathcal{O}[u^{-1}] & u^{-1}\mathcal{O}[u^{-1}] \\ \mathcal{O}[u^{-1}] & 1 + u^{-1}\mathcal{O}[u^{-1}] \end{pmatrix}, 1\right) \cap G.$$

We make the following assumption:

$$\forall\, n \in \mathbb{N}^*, \ker \pi_n \subset \begin{pmatrix} 1 + \varpi^n\mathcal{O}[u, u^{-1}] & 1 + \varpi^n\mathcal{O}[u, u^{-1}] \\ 1 + \varpi^n\mathcal{O}[u, u^{-1}] & 1 + \varpi^n\mathcal{O}[u, u^{-1}] \end{pmatrix} \rtimes (1 + \varpi^n\mathcal{O}). \tag{5.4}$$

If for any $n \in \mathbb{N}^*$, we have $\mathfrak{G}(\mathcal{O}/\varpi^n\mathcal{O}) \simeq \begin{pmatrix} (\mathcal{O}/\varpi^n\mathcal{O})[u,u^{-1}] & (\mathcal{O}/\varpi^n\mathcal{O})[u,u^{-1}] \\ (\mathcal{O}/\varpi^n\mathcal{O})[u,u^{-1}] & (\mathcal{O}/\varpi^n\mathcal{O})[u,u^{-1}] \end{pmatrix} \rtimes (\mathcal{O}/\varpi^n\mathcal{O})^\times$ and π_n is the canonical projection, then the assumption is satisfied. However we do not know if it is true. In [24, 7.6], $\mathfrak{G}$ is described only on fields and in [23, 13], only on $\mathbb{C}$.

For $n \in \mathbb{N}^*$, we set $H_n = \ker(\pi_n) \cap \left(\begin{pmatrix} \mathcal{O}[(\varpi u)^n, (\varpi u^{-1})^n] & \mathcal{O}[(\varpi u)^n, (\varpi u^{-1})^n] \\ \mathcal{O}[(\varpi u)^n, (\varpi u^{-1})^n] & \mathcal{O}[(\varpi u)^n, (\varpi u^{-1})^n] \end{pmatrix}, \mathcal{K}^*\right)$.

Proposition 5.14. *If* (5.4) *is true, then the filtrations* $(H_n)_{n\in\mathbb{N}^*}$ *and* $(\mathcal{V}_{n\lambda})_{n\in\mathbb{N}^*}$ *are equivalent.*

Proof. Let $n \in \mathbb{N}^*$. By Lemma 2.4, we have $U_{[-n\lambda,n\lambda]}^{pm+} = U_{-n\lambda}^{pm+} = t_{-n\lambda}U_0^{pm+}t_{n\lambda}$. We have $t_{n\lambda} = t_{n\mathring{\alpha}^\vee}t_{3nd}$. We have

$$t_{-3nd}U_0^{pm+}t_{3nd} \subset \begin{pmatrix} 1 + (\varpi^{3n}u)\mathcal{O}[\varpi^{3n}u] & \mathcal{O}[\varpi^{3n}u] \\ (\varpi^{3n}u)\mathcal{O}[\varpi^{3n}u] & 1 + (\varpi^{3n}u)\mathcal{O}[\varpi^{3n}u] \end{pmatrix} \ltimes \{1\}.$$

We have

$$t_{-n\mathring{\alpha}^\vee}\left(\begin{pmatrix}1+\varpi^{3n}u\mathcal{O}[\varpi^{3n}u] & \mathcal{O}[\varpi^{3n}u]\\ (\varpi^{3n}u)\mathcal{O}[\varpi^{3n}u] & 1+\varpi^{3n}u\mathcal{O}[\varpi^{3n}u]\end{pmatrix}\ltimes\{1\}\right)t_{n\mathring{\alpha}^\vee}$$
$$\subset\left(\begin{pmatrix}1+\varpi^{3n}u\mathcal{O}[\varpi^{3n}u] & \varpi^{2n}\mathcal{O}[\varpi^{3n}u]\\ \varpi^{-2n}(\varpi^{3n}u)\mathcal{O}[\varpi^{3n}u] & 1+\varpi^{3n}u\mathcal{O}[\varpi^{3n}u]\end{pmatrix}\ltimes\{1\}\right)$$
$$\subset\left(\begin{pmatrix}1+(\varpi^{n}u)\mathcal{O}[\varpi^{n}u] & \mathcal{O}[\varpi^{n}u]\\ \varpi^{n}u\mathcal{O}[\varpi^{n}u] & 1+\varpi^{n}u\mathcal{O}[\varpi^{n}u]\end{pmatrix}\ltimes\{1\}\right)$$
$$\subset \mathrm{SL}_2(\mathcal{O}[\varpi^n u])\ltimes\{1\}\subset \mathrm{SL}_2(\mathcal{O}[\varpi^n u,\varpi^n u^{-1}])\ltimes\{1\}.$$

Similarly, $U^{nm-}_{[-n\lambda,n\lambda]}\subset \mathrm{SL}_2(\mathcal{O}[\varpi^n u,\varpi^n u^{-1}])\ltimes\{1\}$.

As $T_{2n}\subset \mathrm{SL}_2(\mathcal{O}[\varpi^n u,\varpi^n u])\ltimes\{1+\varpi^n\mathcal{O}\}$, we deduce $\mathcal{V}_{n\lambda}\subset H_n$, since $\mathcal{V}_{n\lambda}\subset\ker(\pi_n)$.

Now let $M\in \mathrm{SL}_2(\mathcal{O}[\varpi^{2n}u,\varpi^{2n}u^{-1}])\cap\ker(\pi_{2n})$ and $a\in\mathcal{K}^*$. Using (5.4), we write

$$M=\begin{pmatrix}1+a_0\varpi^{2n}+\sum_{|i|\geq 1}a_i\varpi^{2n|i|}u^i & b_0\varpi^{2n}+\sum_{|i|\geq 1}a_i\varpi^{2n|i|}u^i\\ c_0\varpi^{2n}+\sum_{|i|\geq 1}c_i\varpi^{2n|i|}u^i & 1+d_0\varpi^{2n}+\sum_{|i|\geq 1}d_i\varpi^{2n|i|}u^i\end{pmatrix},$$

with $a_i,b_i,c_i,d_i\in\mathcal{O}$, for all i. Then

$$t_{-nd}(M,a)t_{nd}$$
$$=\left(\begin{pmatrix}1+a_0\varpi^{2n}+\sum_{|i|\geq 1}a_i\varpi^{n(2|i|-i)}u^i & b_0\varpi^{2n}+\sum_{|i|\geq 1}a_i\varpi^{n(2|i|-i)}u^i\\ c_0\varpi^{2n}+\sum_{|i|\geq 1}c_i\varpi^{n(2|i|-i)}u^i & 1+d_0\varpi^{2n}+\sum_{|i|\geq 1}d_i\varpi^{n(2|i|-i)}u^i\end{pmatrix},a\right).$$

Therefore

$$t_{-n\mathring{\alpha}^\vee-nd}(M,a)t_{n(\mathring{\alpha}^\vee+d)}$$
$$=\left(\begin{pmatrix}1+a_0\varpi^{2n}+\sum_{|i|\geq 1}a_i\varpi^{n(2|i|-i)}u^i & \varpi^{2n}(b_0\varpi^{2n}+\sum_{|i|\geq 1}a_i\varpi^{n(2|i|-i)}u^i)\\ \varpi^{-2n}\left(c_0\varpi^{2n}+\sum_{|i|\geq 1}c_i\varpi^{n(2|i|-i)}u^i\right) & 1+d_0\varpi^{2n}+\sum_{|i|\geq 1}d_i\varpi^{n(2|i|-i)}u^i\end{pmatrix},a\right)$$
$$\in \mathrm{SL}_2(\mathcal{O}[u,u^{-1}])\ltimes\mathcal{O}^*.$$

By [3, Lemma 6.11], H_{2n} fixes $n\lambda'$, where $\lambda'=\mathring{\alpha}^\vee+d$. Similarly, it fixes $-n\lambda'$. Therefore $H_{2n}\subset G_{[-n\lambda',n\lambda']}\cap\ker\pi_{2n}$. We have $G_{[-n\lambda',n\lambda']}=U^{pm+}_{[-n\lambda',n\lambda']}.U^{nm-}_{[-n\lambda',n\lambda']}.\mathfrak{T}(\mathcal{O})$, by (2.8). Using the inclusion $\langle U^{pm+}_{[-n\lambda',n\lambda']},U^{nm-}_{[-n\lambda',n\lambda']}\rangle\subset\ker\pi_n$, we deduce that $G_{[-n\lambda',n\lambda']}\cap\ker\pi_n\subset\mathcal{V}_{n\lambda'}$. As $(\mathcal{V}_{m\lambda})$ and $(\mathcal{V}_{m\lambda'})$ are equivalent, we deduce that (H_m) and $(\mathcal{V}_{n\lambda})$ are equivalent. □

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