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Existence and multiplicity results for a fractional curvature problem

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Abstract

We consider the existence problem of conformal metrics with prescribed fractional curvature on the standard sphere S^n, $n \geq 2$. It is equivalent to solving a fractional nonlinear variational equation involving a critical nonlinearity. By studying the lack of compactness of the associated variational problem, we extend the existence results of [2] and [3] to any fractional order $\sigma \in (0, \frac{n}{2})$ and prove a general existence and multiplicity Theorem under an Euler–Hopf type criterion.

1. Introduction

In recent decades, mathematicians and physicists are interested in certain problems of conformal metrics with prescribed curvatures. We study the problem of finding conformal metrics with prescribed fractional curvature, which is of interest in geometry, physics and engineering. See [21] and [26] and references therein.

Let g_0 be the standard metric of the unite sphere S^n, $n \geq 2$, and let g be a new metric conformally equivalent to g_0. Writing $g = u^{\frac{4}{n-2\sigma}} g_0$, where $\sigma \in (0, \frac{n}{2})$ and u is a smooth positive function on S^n, then the fractional curvature R_g^σ of (S^n, g) is given by:

$$P_{g_0}^\sigma(u) = c(n,\sigma) R_g^\sigma u^{\frac{n+2\sigma}{n-2\sigma}}, \quad \text{on } S^n,$$

where $c(n,\sigma) = \Gamma(\frac{n}{2}+\sigma)/\Gamma(\frac{n}{2}-\sigma)$ and $P_{g_0}^\sigma$ is the conformal fractional operator on (S^n, g_0) defined by

$$P_{g_0}^\sigma = \frac{\Gamma(B+\frac{1}{2}+\sigma)}{\Gamma(B+\frac{1}{2}-\sigma)}, \qquad B = \sqrt{-\Delta_{g_0} + \left(\frac{n-1}{2}\right)^2}.$$

$P_{g_0}^\sigma$ can be seen as the pull back operator of the fractional Laplacian $(-\Delta)^\sigma$ on $\mathbb{R}^n$ via the stereographic projection.

Let $K : S^n \to \mathbb{R}$ be a given function. According to the formula above, the problem of finding conformal metrics g on S^n with a fractional curvature R_g^σ equals K is equivalent

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to the solving of the fractional nonlinear equations

$$\begin{cases} P^{\sigma}_{g_0}(u) = c(n,\sigma)Ku^{\frac{n+2\sigma}{n-2\sigma}}, \\ u > 0 \quad \text{ on } S^n, \end{cases} \tag{E_σ}$$

where $\sigma \in (0, \frac{n}{2})$.

For $\sigma = 1$, (E_σ) corresponds to the Nirenberg problem. For $\sigma = 2$, it corresponds to the Paneitz–Branson curvature problem. For $\sigma = k \in \mathbb{N}$, it is the higher order Nirenberg problems related to the so-called *GJMS* operators. For these topics, we refer to [7, 9, 10, 14, 18, 23, 24, 30, 31] and the references therein.

For $\sigma \notin \mathbb{N}$, the fractional curvature problems and related conformally invariant operators were introduced in the works of Case–Chang [12], Chang–Gonzalez [13] and Graham–Zworski [25] and have been the subject of various studies. We may refer to [1] for $\sigma = \frac{1}{2}$, [2, 3, 16, 22, 27, 28, 37] for $\sigma \in (0, 1)$, [29, 34] for $\sigma \in (0, \frac{n}{2})$, [11] for $\sigma = \frac{n}{2}$, and [38] for $\sigma > \frac{n}{2}$. For the problem on general manifolds, we refer to [20].

The purpose of the present paper is to study problem (E_σ) on S^n, for $n \geq 2$ and $\sigma \in (0, \frac{n}{2})$. We are interested in the lack of compactness of the associated variational problem. We describe the asymptotic behavior of non-compact gradient lines; identify the locations of blow-up, which are the so-called critical points at infinity; compute the index of the associated energy functional at each blow-up point; and derive a criterion for the existence of solutions in terms of an Euler–Hopf index.

Our main assumption is the following:

$(f)_\beta$. *Assume that K is of class C^1 on S^n such that for any critical point y, there exists a real $\beta = \beta(y) \in (1, n)$, and a neighborhood N_y of y in which the following expansion holds (in some geodesic normal coordinates system around y).*

$$K(x) = K(y) + \sum_{k=1}^{n} b_k |(x-y)_k|^\beta + o(|x-y|^\beta),$$

where, $b_k = b_k(y) \neq 0$, $\forall\ k = 1, \ldots, n$, and $\sum_{k=1}^{n} b_k \neq 0$.

Under condition $(f)_\beta$, any critical point of K is isolated in S^n, therefore, K admits a finite number of critical points. We denote

$$y_1, \ldots, y_{s_0},$$

all the critical points of K. For any $1 \leq i \neq j \leq s_0$, we denote

$$\mathcal{L}_{ij} = \beta(y_i) + \beta(y_j) - 2\frac{\beta(y_i)\beta(y_j)}{n - 2\sigma}.$$

For any real $r \geq 1$, we define

$$\begin{aligned} \mathcal{K}_r &= \{y_i, i = 1, \dots, s_0, \text{ s.t. } \beta(y_i) = r\}, \\ \mathcal{K}_{<r} &= \{y_i, i = 1, \dots, s_0, \text{ s.t. } \beta(y_i) < r\}, \\ \mathcal{K}_{>r} &= \{y_i, i = 1, \dots, s_0, \text{ s.t. } \beta(y_i) > r\}. \end{aligned}$$

For $y_i \in \mathcal{K}_{>n-2\sigma}$, we denote

$$S_{y_i} = \left\{y_j \in \mathcal{K}_{<n-2\sigma}, \text{ s.t. } \mathcal{L}_{ij} = 0\right\}.$$

(H$_1$). *Assume that for any q-tuple* $\tau_q = (z_1, \dots, z_q) \in (S_{y_i})^q, 1 \leq q \leq \# S_{y_i}$, *such that* $z_i \neq z_j, \forall\, i \neq j$, *we have*

$$\begin{aligned} &\widetilde{\rho}(\tau_q) \\ &= \sum_{j=1}^{q} \frac{c(z_j)\beta(z_j)}{n} \frac{\left|\sum_{k=1}^n b_k(z_j)\right|}{K(z_j)^{1+\frac{n-2\sigma}{n}}} \left(\frac{n\widetilde{c}\, 2^{\frac{n-2\sigma}{2}} K(z_j)^{1+\frac{n-2\sigma}{2}} G(z_j, y_i)}{c(z_j)\beta(z_j)(K(z_j)K(y_i))^{\frac{n-2\sigma}{4}} \left|\sum_{k=1}^n b_k(z_j)\right|} \right)^{\frac{2\beta(y_i)}{n-2\sigma}} \\ &\qquad + \frac{c(y_i)\beta(y_i) \sum_{k=1}^n b_k(y_i)}{nK(y_i)^{1+\frac{n-2\sigma}{2}}} \neq 0, \end{aligned}$$

where

$$(c(z_j) = \int_{\mathbb{R}^n} \frac{|x_1|^{\beta(z_j)}}{(1+|x|^2)^n} \mathrm{d}x \quad \textit{and} \quad \widetilde{c} = \int_{\mathbb{R}^n} \frac{\mathrm{d}x}{(1+|x|^2)^{\frac{n+2\sigma}{2}}}.$$

For any $\tau_q = (z_1, \dots, z_q) \in (\mathcal{K}_{n-2\sigma})^q, 1 \leq q \leq \# \mathcal{K}_{n-2\sigma}$, such that $z_i \neq z_j, \forall\, i \neq j$, we define a $q \times q$ matrix $M(\tau_q) = (m_{ij})$ by:

$$\begin{aligned} m_{ii} &= m(z_i, z_i) = -\frac{n-2\sigma}{2} c(z_j) \frac{\sum_{k=1}^n b_k(z_i)}{K(z_i)^{\frac{n}{2\sigma}}}, \quad i = 1 \dots, q, \\ m_{ij} &= m(z_i, z_j) = \widetilde{c} 2^{\frac{n-2\sigma}{2}} \frac{G(z_i, z_j)}{\left(K(z_i)K(z_j)\right)^{\frac{n-2\sigma}{4\sigma}}}, \quad 1 \leq i \neq j \leq q, \end{aligned}$$

where, $G(z_i, z_j) = \frac{1}{(1-\cos d(z_i,z_j))^{\frac{n-2\sigma}{2}}}$.

(H$_2$). *Let* $\rho(\tau_q)$ *be the least eigenvalue of* $M(\tau_q)$. *We assume that* $\rho(\tau_q) \neq 0$, *for any* $q = 1, \dots, \# \mathcal{K}_{n-2\sigma}$.

It has been first pointed out by A. Bahri [4], that when the interaction between different bubbles is of the same order as the self interaction, the functions $\widetilde{\rho}(\tau_q)$ and $\rho(\tau_q)$ play a fundamental role in the theory of the critical points at infinity. For problem (E_σ), such kind of phenomenon appears when $\mathcal{L}_{ij} = 0$. Note that conditions like (H_1) and (H_2)

were used first in [15, Theorem 10.3] for the study of the Nirenberg problem, as standard conditions to guarantee the existence of solutions. See also [17]. Let

$$B_\infty = \left\{ \begin{array}{l} (y_1, \dots, y_p),\ 1 \le p \le s_0, \text{ s.t. } -\sum_{k=1}^{n} b_k(y_i) > 0,\ \forall\ i = 1, \dots, p,\ y_i \neq y_j \\ \text{and } \mathcal{L}_{ij} \ge 0,\ \forall\ 1 \le i \neq j \le p. \text{ Moreover, if we denote, } y_1, \dots, y_q, \text{ all the} \\ \text{components of } (y_1, \dots, y_p), \text{ such that, for any } i = 1, \dots, q, \text{ there exists,} \\ j = 1, \dots, q,\ j \neq i, \text{ satisfying, } \mathcal{L}_{ij} = 0, \text{ then, } (y_1, \dots, y_q) \in \mathcal{K}^q_{n-2\sigma} \text{ and} \\ \rho(y_1 \dots, y_q) > 0 \end{array} \right\}.$$

We shall prove the following result.

Theorem 1.1. *Let K be a positive function satisfying conditions $(f)_\beta$, $\beta \in (1, n)$,* (H_1) *and* (H_2). *If*

$$\sum_{(y_1,\dots,y_p)\in B_\infty} (-1)^{p-1+\sum_{j=1}^{p}(n-\tilde{i}(y_j))} \neq 1,$$

then (E_σ) has a solution. Moreover, in a generic case, if $\beta(y_i) > \frac{n-2\sigma}{2}, \forall\ i = 1, \dots, s_0$, then

$$\#S \ge \left| 1 - \sum_{(y_1,\dots,y_p)\in B_\infty} (-1)^{p-1+\sum_{j=1}^{p}(n-\tilde{i}(y_j))} \right|.$$

Here, S is the set of solutions of (E_σ) and

$$\tilde{i}(y_j) = \#\big\{ b_k(y_j), k = 1, \dots, n, \ \text{s.t. } b_k(y_j) < 0 \big\}.$$

Note that the criteria of existence of solutions given by Theorem 1.1 extends the ones of [2] and [3] to any fractional order $\sigma \in (0, \frac{n}{2})$. In addition, Theorem 1.1 provides a lower bound of the number of solutions. It holds under the assumption that all the critical points of the variational functional J are non-degenerate. Such an assumption is valid for generic K (modulo a perturbation of the function K) by Sard–Smale Theorem [35]. For the degenerate case, (degenerate critical points of J), the result of Theorem 1.1 remains open, since the topological contribution of a degenerate critical point is unknown in general, see [33].

The proof of Theorem 1.1 is based on a refined analysis of the compactness defect of the variational structure associated to problem (E_σ). Theorem 3.6 in Section 3 provides a precise characterization of the critical points at infinity (blowup points) under condition $(f)_\beta$, $\beta \in (1, n)$.

Unlike the case of $\sigma \in (0, 1)$, many interesting curvature problems in differential geometry arise in studying the non linear fractional equations (E_σ), $\sigma \in (0, \frac{n}{2})$. Namely, the celebrate scalar curvature problem for $\sigma = 1$ and $n \ge 3$ and the Q-curvature problem for $\sigma = 2$ and $n \ge 5$. Moreover, there is a qualitative difference in results and configurations of blow-up points between the cases $\sigma \in (0, 1)$ and $\sigma \in (0, \frac{n}{2})$. Indeed, if we place for example in dimension $n = 4$ and if the prescribed function K satisfies the classical non degenerate condition near its critical points, that it is the case of $\beta(y) = 2$, for all the critical points, we find that the behavior of the concentration phenomenon of the associated gradient flow depends to the value of σ. Precisely, (see Section 3), the configuration of blow-up points differ with respect to $\sigma \in (0, 1)$, $\sigma \in (1, 2)$ and $\sigma = 1$. Namely, for $\sigma \in (0, 1)$, the gradient flow concentrates at several distinct critical points of K, for $\sigma \in (1, 2)$, the concentration phenomenon happens at only one critical point and for $\sigma = 1$, the concentration phenomenon depend to the matrix defined in (H_2). Although the results on problem (E_σ), $\sigma \in (0, \frac{n}{2})$ differ to the ones of [2] and [3], the same kind of methods allows to conclude.

In the next Section, we state the variational formulation of problem (E_σ) and we recall some known results.

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2. Preliminaries

(E_σ) is a variational problem. The solutions are the critical points (up to positive multiplicative constants) of the following functional:

$$J(u) = \frac{\int_{S^n} P^\sigma_{g_0} u u \mathrm{d}v_{g_0}}{\left(\int_{S^n} K u^{\frac{2n}{n-2\sigma}} \mathrm{d}v_{g_0}\right)^{\frac{n-2\sigma}{n}}}, \quad u \in H^\sigma(S^n),$$

subjected to the constraint $u \in \Sigma^+$. Here,

$$\Sigma^+ = \{u \in \Sigma, u \geq 0\}, \quad \Sigma = \{u \in H^\sigma(S^n), \|u\| = 1\}$$

and $H^\sigma(S^n)$ is the fractional Sobolev space defined as the closure of $C^\infty(S^n)$ with respect to the norm

$$\|u\|^2 = \int_{S^n} P^\sigma_{g_0} u u \mathrm{d}v_{g_0}.$$

J does not satisfy the Palais–Smale condition (P.S). This is a consequence of the lack of compactness of the embedding $H^\sigma(S^n) \hookrightarrow L^{\frac{2n}{n-2\sigma}}(S^n)$. The sequences violating (P.S) condition are characterized as follows. For $a \in S^n$ and $\lambda > 0$, define

$$\delta_{(a,\lambda)}(x) = \frac{\lambda^{\frac{n-2\sigma}{2}}}{(1 + \lambda^2 + (1 - \lambda^2)\cos d_{g_0}(a, x))^{\frac{n-2\sigma}{2}}}.$$

Up to a positive multiplicative constant, $\delta_{(a,\lambda)}$ satisfies

$$P^{\sigma}_{g_0}\delta_{(a,\lambda)} = \delta_{(a,\lambda)}^{\frac{n+2\sigma}{n-2\sigma}}, \text{ on } S^n,$$

see [27]. Let ω be a solution of (E_σ) or zero. Let $p \in \mathbb{N}$ and $\lambda > 0$. We set,

$$V(p,\varepsilon,\omega) = \left\{ \begin{array}{l} u \in \Sigma,\ \exists\ \alpha_0, \alpha_1, \ldots, \alpha_p > 0,\ \exists\ a_1, \ldots, a_p \in S^n,\ \exists\ \lambda_1, \ldots, \lambda_p > \varepsilon^{-1}, \\ \text{s.t., } \left\|u - \alpha_0\omega - \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)}\right\| \leq \varepsilon, \text{ with } \left|\alpha_0^{\frac{4\sigma}{n-2\sigma}} J(u)^{\frac{n}{n-2\sigma}} - 1\right| < \varepsilon, \\ \left|\alpha_i^{\frac{4\sigma}{n-2\sigma}} J(u)^{\frac{n}{n-2\sigma}} K(a_i) - 1\right| < \varepsilon, \forall\ i = 1, \ldots, p \text{ and } \varepsilon_{ij} < \varepsilon, \forall\ 1 \leq i \neq \\ j \leq p \end{array} \right\}.$$

Here, $\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \frac{\lambda_i\lambda_j}{2}\big(1 - \cos \mathrm{d}(a_i, a_j)\big)\right)^{-\frac{n-2\sigma}{2}}$.

Proposition 2.1 ([5, 32, 36])**.** *For any sequence $(u_k)_k$ in Σ^+ such that $J(u_k) \to c, c \in \mathbb{R}$ and $\partial J(u_k) \to 0$, there exists $p \in \mathbb{N}$ and $(\varepsilon_k)_k > 0 \varepsilon_k \to 0$ and an extracted subsequence denoted again $(u_k)_k$ such that $u_k \in V(p, \varepsilon_k, \omega)$, where ω is a solution of (E_σ) or zero.*

It is known that for any $u \in V(p, \varepsilon, \omega)$, there exists a unique representation as follows:

$$u = \alpha_0(\omega + h) + \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} + v,$$

up to a permutation, see [4, Proposition 5.2] and [5, p. 348–350]. Here $h \in T_\omega W_u(\omega)$ and $v \in H^\sigma(S^n) \cap T_\omega W_s(\omega)$ belonging to V_0, where

$$V_0 = \{v : \|v\| < \varepsilon,\ \langle v, \phi\rangle = 0,\ \ \forall\ \phi \in E\}.$$

Here $E = \left\{\omega, h, \delta_{(a_i,\lambda_i)}, \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i}, \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial a_i}, i = 1, \ldots, p\right\}$ and $\langle \cdot, \cdot \rangle$ denotes the inner product related to the norm $\|\cdot\|$. The following Morse Lemma gets rid of the v-contributions.

Proposition 2.2 ([5, 6])**.** *There exists a C^1-mapping which to any $(\alpha_i, a_i, \lambda_i, h)$ such that $\alpha_0(\omega + h) + \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} \in V(p, \varepsilon, \omega)$, associates $\bar{v} = \bar{v}(\alpha_i, a_i, \lambda_i, h) \in H^\sigma(S^n)$, where $\bar{v}$ is the unique solution of*

$$J\left(\alpha_0(\omega + h) + \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} + \bar{v}\right) = \min_{v \text{ satisfying } (V_0)} J\left(\alpha_0(\omega + h) + \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} + v\right).$$

In addition there is a change of variables $v - \bar{v} \equiv V$, such that

$$J\left(\alpha_0(\omega + h) + \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} + v\right) = J\left(\alpha_0(\omega + h) + \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} + \bar{v}\right) + \|V\|^2.$$

As shown in [5, p. 328], by applying the differential equation $\dot{V} = -\mu V, \mu \gg 1$, the norm of the flow $V(s)$ decreases and tends to zero. Therefore, in order to establish our deformation Lemma, we can work as if $V = 0$.

Following [2, p. 1291], the estimate of $\|\overline{v}\|$ is given as follows.

Proposition 2.3.

$$\|\overline{v}\| \leq c \sum_{i=1}^{p} \left(\frac{1}{\lambda_i^{\frac{n}{2}}} + \frac{1}{\lambda_i^{\beta}} + \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{(\log \lambda_i)^{\frac{n+2\sigma}{2}}}{\lambda_i^{\frac{n+2\sigma}{2}}} \right) + c \begin{cases} \sum_{k \neq r} \varepsilon_{kr}^{\frac{n+2\sigma}{2(n-2\sigma)}} \left(\log \varepsilon_{kr}^{-1}\right)^{\frac{n+2\sigma}{2n}}, & \text{if } n \geq 6\sigma \\ \sum_{k \neq r} \varepsilon_{kr} \left(\log \varepsilon_{kr}^{-1}\right)^{\frac{n-2\sigma}{n}}, & \text{if } n < 6\sigma. \end{cases}$$

We now introduce the definition of critical point at infinity of J.

Definition 2.4 ([4]). A critical point at infinity of J in Σ^+ is a limit of flow line $u(s)$ of the gradient vector field $(-\partial J)$ such that $u(s)$ lies in $V(p, \varepsilon, \omega)$, $p \geq 1$, for any s large. Writing

$$u(s) = \alpha_0(s)(\omega + h(s)) + \sum_{i=1}^{p} \alpha_i(s) \delta_{(a_i(s), \lambda_i(s))} + v(s),$$

$$\overline{\alpha}_i = \lim \alpha_i(s) \quad \text{and} \quad \overline{y}_i = \lim a_i(s),$$

we then denote

$$\overline{\alpha}_0 \omega + \sum_{i=1}^{p} \overline{\alpha}_i \delta_{(\overline{y}_i, \infty)}$$

such a critical point at infinity.

3. Critical points at infinity

In this Section, we characterize the critical points at infinity of problem (E_σ) under conditions $(f)_\beta$, (H_1), (H_2). Such a characterization hinges on an analysis of the gradient flow of J in all the possible neighborhoods of these critical points at infinity. First, according to the above definition, these neighborhoods correspond to the sets $V(p, \varepsilon, \omega)$ such that $p \geq 1$ and ω is zero or a solution of (E_σ). Second, following the results of [2, p. 1300–1304], the possible neighborhoods of the critical points at infinity are reduced to the sets:

$$V_\delta(p, \varepsilon, \omega) = \begin{Bmatrix} u = \alpha_0(\omega + h) + \sum_{i=1}^{p} \alpha_i \delta_{(a_i, \lambda_i)} + v \in V(p, \varepsilon, \omega) \text{ s.t., } \forall\, i = 1, \dots, p, \\ \exists\, y_i \in \mathcal{K} \text{ with } |a_i - y_i| < \frac{\delta}{\lambda_i} \text{ and } y_i \neq y_j, \forall\, 1 \leq i \neq j \leq p \end{Bmatrix}.$$

Here, $\mathcal{K}$ denotes the set of all critical points of K.

3.1. **Asymptotic analysis in** $V_\delta(p,\varepsilon,\omega)$

In this Subsection, we study the variation of the energy functional J with respect to $\lambda_i, i = 1,\dots,p$ and h variables. We follow the computations of [19] and [2], (see also Propositions 3.1 and 3.2 of [19]).

Proposition 3.1. *For any* $u = \alpha_0(\omega + h) + \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} \in V_\delta(p,\varepsilon,\omega)$*, we have*

$$\langle \partial J(u), h\rangle \le -c\left[\|h\|^2 + \sum_{i=1}^{p} o\left(\frac{1}{\lambda_i^{\frac{n-2\sigma}{2}}}\right)\right],$$

where c is a fixed positive constant independent of u.

Proof. Following [19, Proposition 3.3], we have

$$\langle \partial J(u), h\rangle = 2J(u)\left(\alpha_0\|h\|^2 - \frac{n+2\sigma}{n-2\sigma}\alpha_0^{\frac{n+2\sigma}{n-2\sigma}} J(u)^{\frac{n}{n-2\sigma}} \int_{S^n} K\omega^{\frac{4\sigma}{n-2\sigma}} h^2 \mathrm{d}v_{g_0}\right) + o(\|h\|^2) + \sum_{i=1}^{p} o\left(\frac{1}{\lambda_i^{\frac{n-2\sigma}{2}}}\right).$$

Since $\alpha_0^{\frac{4\sigma}{n-2\sigma}} J(u)^{\frac{n}{n-2\sigma}} = 1 + o(1)$, we obtain,

$$\langle \partial J(u), h\rangle = \alpha_0 Q(h,h) + o(\|h\|^2) + \sum_{i=1}^{p} o\left(\frac{1}{\lambda_i^{\frac{n-2\sigma}{2}}}\right),$$

where $Q(h,h)$ is a quadratic form defined by

$$Q(h,h) = \|h\|^2 - \frac{n+2\sigma}{n-2\sigma}\int_{S^n} K\omega^{\frac{4\sigma}{n-2\sigma}} h^2 \mathrm{d}v_{g_0},$$

which is definite and negative, see [5, p. 354]. This finishes the proof. $\square$

Proposition 3.2. *For any* $u = \alpha_0(\omega + h) + \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} \in V_\delta(p,\varepsilon,\omega)$ *and for any* $i = 1,\dots,p$*, we have*

$$\left\langle \partial J(u), \alpha_i\lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i}\right\rangle$$
$$= 2J(u)\left[\alpha_0 \widehat{c}\frac{n+2\sigma}{2}\frac{\omega(a_i)}{\lambda_i^{\frac{n-2\sigma}{2}}} + \frac{n-2\sigma}{2n}\beta(y_i)c(y_i)\frac{\alpha_i^2\sum_{k=1}^{n} b_k(y_i)}{K(a_i)\lambda_i^{\beta(y_i)}} - \widetilde{c}\sum_{j\neq i}\alpha_i\alpha_j\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}\right] + o(\|h\|^2) + \sum_{j=1}^{p} o\left(\frac{1}{\lambda_j^{\beta(y_j)}}\right) + \sum_{j\neq i} o(\varepsilon_{ij}),$$

where

$$\widehat{c} = \int_{\mathbb{R}^n} \frac{|x|^2 - 1}{(1 + |x|^2)^{\frac{n+2\sigma}{2}}} \mathrm{d}x.$$

Proof. We have

$$\begin{aligned}
&\left\langle \partial J(u), \alpha_i \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i} \right\rangle \\
&\qquad = 2J(u)\left[\left\langle u, \alpha_i \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i} \right\rangle - \alpha_i J(u)^{\frac{n}{n-2\sigma}} \int_{S^n} K u^{\frac{n+2\sigma}{n-2\sigma}} \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i} \mathrm{d}v_{g_0}\right]
\end{aligned}$$

We follow the computation of [19, Proposition 3.4]. We have,

$$\left\langle u, \alpha_i \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i} \right\rangle = \alpha_0 \alpha_i \left\langle \omega + h, \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i} \right\rangle + \sum_{j=1}^{p} \alpha_i \alpha_j \left\langle \delta_{(a_j,\lambda_j)}, \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i} \right\rangle.$$

By expanding ω around a_i, we have

$$\left\langle \omega + h, \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i} \right\rangle = -\frac{n+2\sigma}{2} \widehat{c} \frac{\omega(a_i)}{\lambda_i^{\frac{n-2\sigma}{2}}} + o\left(\frac{1}{\lambda_i^{\frac{n-2\sigma}{2}}}\right).$$

By a computation similar to [4, Sections 1 and 2], we have

$$\sum_{j=1}^{p} \alpha_i \alpha_j \left\langle \delta_{(a_j,\lambda_j)}, \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i} \right\rangle = \sum_{i \neq j} \alpha_i \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \sum_{j \neq i} o(\varepsilon_{ij}).$$

Expanding $u^{\frac{n+2s}{n-2s}}$, the integral term reduces to

$$\begin{aligned}
&\int_{S^n} K u^{\frac{n+2\sigma}{n-2\sigma}} \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i} \mathrm{d}v_{g_0} \\
&= \int_{S^n} K(\alpha_0 \omega)^{\frac{n+2\sigma}{n-2\sigma}} \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i} \mathrm{d}v_{g_0} + \int_{S^n} K\left(\sum_{j=1}^{p} \alpha_j \delta_{(a_j,\lambda_j)}\right)^{\frac{n+2\sigma}{n-2\sigma}} \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i} \mathrm{d}v_{g_0} \\
&\qquad + \frac{n+2\sigma}{n-2\sigma} \int_{S^n} K\left(\sum_{j=1}^{p} \alpha_j \delta_{(a_j,\lambda_j)}\right)^{\frac{4\sigma}{n-2\sigma}} (\alpha_0 \omega) \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i} \mathrm{d}v_{g_0} \\
&\qquad + o(\|h\|^2) + o\left(\sum_{i \neq j} \varepsilon_{ij}\right). \\
&= I_1 + I_2 + I_3 + R.
\end{aligned}$$

Stereographic projection combined with expansions of K and ω around ai yield

$$I_1 = -\frac{n+2\sigma}{2}\widehat{c}\alpha_0^{\frac{n+2\sigma}{n-2\sigma}}\frac{\omega(a_i)}{\lambda_i^{\frac{n-2\sigma}{2}}} + o\left(\frac{1}{\lambda_i^{\frac{n-2\sigma}{2}}}\right),$$

$$I_3 = -\frac{n+2\sigma}{2}\widehat{c}\alpha_0\alpha_i^{\frac{4\sigma}{n-2\sigma}}K(a_i)\frac{\omega(a_i)}{\lambda_i^{\frac{n-2\sigma}{2}}} + o\left(\frac{1}{\lambda_i^{\frac{n-2\sigma}{2}}}\right) + \sum_{j\neq i} o(\varepsilon_{ij}),$$

and by the computation of [2, Proposition A-1], we have

$$\begin{aligned} I_2 &= \sum_{j=1}^{p}\alpha_j^{\frac{n+2\sigma}{n-2\sigma}}\int_{S^n} K(x)\delta_{(a_j,\lambda_j)}^{\frac{n+2\sigma}{n-2\sigma}}\lambda_i\frac{\partial\delta_{(a_i,\lambda_i)}}{\partial\lambda_i}\mathrm{d}v_{g_0} \\ &\quad + \frac{n+2\sigma}{n-2\sigma}\sum_{j\neq i}\int_{S^n} K(x)\alpha_j\delta_{(a_j,\lambda_j)}(\alpha_i\delta_{(a_i,\lambda_i)})^{\frac{4\sigma}{n-2\sigma}}\lambda_i\frac{\partial\delta_{(a_i,\lambda_i)}}{\partial\lambda_i}\mathrm{d}v_{g_0} + \sum_{j\neq i} o(\varepsilon_{ij}). \\ &= -\frac{n-2\sigma}{n}\alpha_i^{\frac{n+2\sigma}{n-2\sigma}}\beta(y_i)c(y_i)\frac{\sum_{k=1}^n b_k(y_i)}{\lambda_i^{\beta(y_i)}} + \sum_{j\neq i}\alpha_j^{\frac{n+2\sigma}{n-2\sigma}}K(a_j)\widetilde{c}\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} \\ &\quad + \sum_{j\neq i}\alpha_j\alpha_i^{\frac{4\sigma}{n-2\sigma}}K(a_i)\widetilde{c}\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + \sum_{j\neq i} o(\varepsilon_{ij}). \end{aligned}$$

Using the fact that $\alpha_i^{\frac{4\sigma}{n-2\sigma}}K(a_i)J(u)^{\frac{n}{n-2\sigma}} = 1 + o(1)$ for any $i = 1, \ldots, p$, Proposition 3.2 follows. □

3.2. Concentration phenomena in $V_\delta(p, \varepsilon, \omega)$

The objective of this Subsection is to provide a quantitative description of the concentration phenomena of problem (E_σ) and determine the locations of the critical points at infinity. As mentioned above these points take places in $V_\delta(p, \varepsilon, \omega)$, $p \geq 1$, ω is a solution of (E_σ) or zero and ε and δ are positive and small. We shall prove the following result.

Proposition 3.3. *Assume that K satisfies condition $(f)_\beta$ with $\beta \in (\frac{n-2\sigma}{2}, n)$. There exists a decreasing pseudo gradient W in $V_\delta(p, \varepsilon, \omega)$, $p \geq 1$, $\omega \neq 0$ and ε and δ positive and small such that for any $u = \alpha_0(\omega + h) + \sum_{i=1}^{p}\alpha_i\delta_{(a_i,\lambda_i)} \in V_\delta(p, \varepsilon, \omega)$, we have*

(i) $$\langle \partial J(u), W(u)\rangle \leq -c\left(\sum_{i=1}^{p}\left(\frac{1}{\lambda_i^{\frac{n-2\sigma}{2}}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + \|h\|^2 + \sum_{i\neq j}\varepsilon_{ij}\right),$$

(ii) $$\left\langle \partial J(u+\bar{v}), W(u) + \frac{\partial \bar{v}}{\partial(\alpha, a, \lambda, h)}(W(u)) \right\rangle \leq -c\left(\sum_{i=1}^{p}\left(\frac{1}{\lambda_i^{\frac{n-2\sigma}{2}}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + \|h\|^2 + \sum_{i\neq j}\varepsilon_{ij}\right),$$

(iii) $\|W(u)\| \leq \frac{1}{c}$ *and* $\max_{1\leq i\leq p}\lambda_i(s)$ *is a bounded function on* $\mathbb{R}^+$.

Here c is a positive constant independent of u.

The above result shows that a deconcentration phenomenon occurs in $V_\delta(p, \varepsilon, \omega)$, $\omega \neq 0$. This yields to the following result.

Theorem 3.4. *Assume that K satisfies condition $(f)_\beta$, $\beta \in (\frac{n-2\sigma}{2}, n)$. Then for any solution ω of (E_σ), $V_\delta(p, \varepsilon, \omega)$ contains no critical point at infinity.*

We now state the proof of Proposition 3.3.

Proof of Proposition 3.3. Let $u = \alpha_0(\omega + h) + \sum_{i=1}^{p}\alpha_i\delta_{(a_i,\lambda_i)} \in V_\delta(p, \varepsilon, \omega)$.
We order theλ_i's parameters. Assume that

$$\lambda_1 \leq \cdots \leq \lambda_p.$$

We decrease all the λ_i with different speeds. For any $i = 1, \ldots, p$, we set

$$\dot{\lambda}_i = -2^i\lambda_i.$$

Using Proposition 3.2 and the fact $\beta(y_i) > \frac{n-2\sigma}{2}$, we have

$$\left\langle \partial J(u), \dot{\lambda}_i\frac{\partial\delta_{(a_i,\lambda_i)}}{\partial\lambda_i}\right\rangle \leq -c\frac{\omega(a_i)}{\lambda_i^{\frac{n-2\sigma}{2}}} + c\sum_{j\neq i}2^i\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + \sum_{j\neq i}o(\varepsilon_{ij}) + o(\|h\|^2).$$

Define

$$W(u) = \alpha_0 h - \sum_{i=1}^{p}\alpha_i 2^i\lambda_i\frac{\partial\delta_{(a_i,\lambda_i)}}{\partial\lambda_i}.$$

By Proposition 3.1,

$$\langle\partial J(u), h\rangle \leq -c\|h\|^2 + \sum_{i=1}^{p} o\left(\frac{1}{\lambda_i^{\frac{n-2\sigma}{2}}}\right).$$

Therefor by the above two inequalities, we get

$$\langle\partial J(u), W(u)\rangle \leq -c\left(\|h\|^2 + \sum_{i=1}^{p}\frac{1}{\lambda_i^{\frac{n-2\sigma}{2}}}\right) + \sum_{j\neq i}2^i\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + \sum_{j\neq i}o(\varepsilon_{ij}).$$

Observe that,

$$2^i \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + 2^j \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \leq -c\varepsilon_{ij}, \quad \forall\ 1 \leq i < j \leq p.$$

Moreover,

$$|\nabla K(a_i)| \sim |a_i - \lambda_i|^{\beta(y_i)-1}$$

Therefore,

$$\frac{|\nabla K(a_i)|}{\lambda_i} < \frac{\delta}{\lambda_i^{\beta(y_i)}}, \quad \forall\ i = 1, \ldots, p.$$

Thus,

$$\langle \partial J(u), W(u) \rangle \leq -c\left(\|h\|^2 + \sum_{i=1}^{p} \frac{1}{\lambda_i^{\frac{n-2\sigma}{2}}} + \sum_{j \neq i} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} \right).$$

Claim (i) of the Proposition follows. Arguing as in [5, Appendix 2], Claim (ii) follows from (i) and the estimate of $\|\overline{v}\|$ given in Lemma 2.3, $\|W\|$ is bounded since $\left\|\lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i}\right\|$, $i = 1, \ldots, p$ are bounded and by construction the $\max_{1 \leq i \leq p} \lambda_i(s)$ is a bounded function, since all the parameters $\lambda_i(s), i = 1, \ldots, p$ decrease along the flow lines of W. Estimate (iii) follows and the proof of proposition 3.1 is thereby completed. □

Next, we focus on the neighborhoods of the form $V_\delta(p, \varepsilon, 0), p \geq 1$

Proposition 3.5. *Assume that K is positive and satisfies $(f)_\beta, \beta \in (1, n), (H_1)$ and (H_2) conditions. For any $p \geq 1, \varepsilon$ and δ positive and small, there exists a decreasing pseudo gradient W in $V_\delta(p, \varepsilon, 0)$ such that for any $u = \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} \in V_\delta(p, \varepsilon, \omega)$, we have*

(i) $\displaystyle \langle \partial J(u), W(u) \rangle \leq -c\left(\sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \left(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^{\beta(y_i)}} \right) \right),$

(ii)
$$\left\langle \partial J(u + \overline{v}), W(u) + \frac{\partial \overline{v}}{\partial(\alpha_i, a_i, \lambda_i)}(W(u)) \right\rangle \leq -c\left(\sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \left(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^{\beta(y_i)}} \right) \right).$$

(iii) $\|W(u)\| \leq \frac{1}{c}$ *and the only case where the* $\max_{1 \leq i \leq p} \lambda_i(s)$ *is not bounded, is when $a_i(s) \to y_i, \forall\ i = 1, \ldots, p$, with $(y_1, \ldots, y_p) \in B_\infty$.*

The above proposition shows that the flow lines of W can only be concentrated when $a_i(s)$ tends to y_i, as $s \to +\infty$, for any $i = 1, \ldots, p$, with $(y_1, \ldots, y_p) \in B_\infty$. We therefore have the following result.

Theorem 3.6. *Under the assumption of Proposition 3.5, the critical points at infinity of* (E_σ) *are*

$$\sum_{i=1}^{p} \frac{1}{K(y_i)^{\frac{n-2\sigma}{n}}} \delta(y_i, \infty), \quad (y_1, \dots, y_p) \in B_\infty.$$

We now state the proof of Proposition 3.5.

Proof of Proposition 3.5. Let $u = \sum_{i=1}^{p} \alpha_i \delta_{(a_i, \lambda_i)} \in V_\delta(p, \varepsilon, 0), p \geq 1$. The construction of $W(u)$ will depend to the following three statements.

Statement 1. We assume that $\mathcal{L}_{ij} > 0, \forall\ 1 \leq i \neq j \leq p$.

In this case, the following Claim holds.

Claim 1. $\forall\ 1 \leq i \neq j \leq p$,

$$\varepsilon_{ij} = o\left(\frac{1}{\lambda_i^{\beta(y_i)}}\right) + o\left(\frac{1}{\lambda_j^{\beta(y_j)}}\right).$$

Indeed, for $i \neq j$, we have

$$\varepsilon_{ij} \sim \frac{1}{(\lambda_i \lambda_j)^{\frac{n-2\sigma}{2}}}.$$

Let $\gamma > 0$ and small.

If $\lambda_j^{\beta(y_j) - \frac{n-2\sigma}{2}} \leq \gamma \lambda_i^{\frac{n-2\sigma}{2}}$, we get $\varepsilon_{ij} = o\big(\frac{1}{\lambda_j^{\beta(y_j)}}\big)$, for γ small enough.

If $\lambda_j^{\beta(y_j) - \frac{n-2\sigma}{2}} \geq \gamma \lambda_i^{\frac{n-2\sigma}{2}}$, then for ε small enough $\beta(y_j)$ strictly larger than $\frac{n-2\sigma}{2}$. In this case

$$\varepsilon_{ij} \leq c(\gamma) \frac{1}{\lambda_i^{\frac{n-2\sigma}{2}\left(1 + \frac{n-2\sigma}{2\beta(y_j) - (n-2\sigma)}\right)}}.$$

Using the fact that

$$\frac{n-2\sigma}{2}\left(1 + \frac{n-2\sigma}{2\beta(y_j) - (n-2\sigma)}\right) > \beta(y_i),$$

we get $\varepsilon_{ij} = o\big(\frac{1}{\lambda_i^{\beta(y_i)}}\big)$ and therefore Claim 1 is valid.

The construction of the pseudo gradient in the current statement depends to the following two cases.

Case 1: $\forall\ i = 1, \dots, p, -\sum_{k=1}^{n} b_k(y_i) > 0$. We set

$$\dot{\lambda}_i = \lambda_i,$$

for any, $i = 1, \dots, p$. The corresponding pseudo gradient is

$$W_1^1(u) = \sum_{i=1}^{p} \alpha_i \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i}.$$

By the expansion of Proposition 3.2, we get

$$\left\langle \partial J(u), W_1^1(u) \right\rangle \leq c \sum_{i=1}^{p} \frac{\sum_{k=1}^{n} b_k(y_i)}{\lambda_i^{\beta(y_i)}} + \sum_{i \neq j} O\big(\varepsilon_{ij}\big)$$

Using the estimation of Claim 1, we obtain

$$\begin{aligned}
\left\langle \partial J(u), W_1^1(u) \right\rangle &\leq -c \left(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta(y_i)}} + \sum_{i \neq j} \varepsilon_{ij} \right) \\
&\leq -c \left(\sum_{i=1}^{p} \left(\frac{1}{\lambda_i^{\beta(y_i)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{i \neq j} \varepsilon_{ij} \right).
\end{aligned}$$

Note that, under the action of W_1^1, the parameters $\lambda_i(s) \to \infty$, for any $i = 1, \dots, p$.
It is a concentration phenomena.

Case 2: $\exists\, i, 1 \leq i \leq p$, *such that* $-\sum_{k=1}^{n} b_k(y_i) < 0$. Let

$$I = \left\{ i, 1 \leq i \leq p, \text{ s.t. } -\sum_{k=1}^{n} b_k(y_i) < 0 \right\}.$$

We set $\dot{\lambda}_i = -\lambda_i$, for any $i \in I$. By the expansion of Proposition 3.2 we have

$$\begin{aligned}
\left\langle \partial J(u), -\sum_{i \in I} \alpha_i \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i} \right\rangle &\leq -\sum_{i \in I} \frac{1}{\lambda_i^{\beta(y_i)}} + \sum_{\substack{i \in I \\ j \neq i}} O(\varepsilon_{ij}) \\
&\leq -\sum_{i \in \tilde{I}} \frac{1}{\lambda_i^{\beta(y_i)}} + \sum_{\substack{i \in I \\ j \neq i}} O(\varepsilon_{ij}),
\end{aligned}$$

where,

$$\widetilde{I} := \left\{ i, 1 \leq i \leq p, \quad \lambda_i^{\beta(y_i)} \geq \frac{1}{2} \min_{j \in I} \lambda_j^{\beta(y_j)} \right\}.$$

Of course, any index i of $\widetilde{I}^c$ satisfies $-\sum_{k=1}^{n} b_k(y_i) > 0$. Let $\widehat{u} = \sum_{i \in \tilde{I}^c} \alpha_i \delta_{(a_i,\lambda_i)}$. $\widehat{u}$ satisfies the condition of Case 1. Let $W_1^1(\widehat{u})$ be the corresponding vector field. We have,

$$\left\langle \partial J(u), W_1^1(\widehat{u}) \right\rangle \leq -c \sum_{i \in \tilde{I}^c} \frac{1}{\lambda_i^{\beta(y_i)}} + \sum_{\substack{i \in \tilde{I} \\ j \neq i}} O(\varepsilon_{ij}).$$

Let
$$W_1^2(u) = W_1^1(\widehat{u}) - \sum_{i\in I} \alpha_i \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i}.$$
By the above two inequalities, W_1^2 satisfies
$$\left\langle \partial J(u), W_1^2(u) \right\rangle \leq -c\left(\sum_{i=1}^{p}\left(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{{\lambda_i}^{\beta(y_i)}}\right) + \sum_{i\neq j} \varepsilon_{ij}\right).$$
Note that the $\max_{1\leq i\leq p} \lambda_i(s)$ is a decreasing function along W_1^2. It is a deconcentration phenomenon. In this statement, we denote W_1 the pseudo gradient defined by a convex combination of W_1^1 and W_1^2

Statement 2. We assume that $\mathcal{L}_{ij} \geq 0, \forall\ 1 \leq i \neq j \leq p$ and exist $i_0 \neq j_0$ such that $\mathcal{L}_{i_0 j_0} = 0$.

Denote
$$I = \left\{i, 1 \leq i \leq p,\ \text{s.t. } \exists\ j \neq i \text{ with } \mathcal{L}_{ij} = 0\right\}.$$
The construction of the required vector field depends to the following two cases.

Case 1: $\exists\ i \in I$ *such that* $\beta(y_i) = n - 2\sigma$. Under the assumption that $\mathcal{L}_{ij} \geq 0, \forall\ i \neq j$, we obtain the following:
$$\beta(y_i) = n - 2\sigma, \qquad \forall\ i \in I.$$
Let $\widehat{u} = \sum_{i\in I} \alpha_i \delta_{(a_i,\lambda_i)}$. We introduce the following Lemma.

Lemma 3.7. *Under condition* (H$_2$)*, there exists a decreasing pseudo gradient* $\widehat{W}$ *satisfying:*

(a) $$\left\langle \partial J(u), \widehat{W}(\widehat{u}) \right\rangle \leq -c\left[\sum_{i\in I} \frac{1}{\lambda_i^{n-2\sigma}} + \sum i \neq j \in I \varepsilon_{ij}\right] + \sum_{\substack{i\in I\\ j\in \tilde{I}}} O(\varepsilon_{ij}).$$

(b) $\|\widehat{W}(\widehat{u})\| \leq \frac{1}{c}$ *and the only case where the* $\max_{i\in I} \lambda_i(s)$ *is not bounded is when* $\rho\big((y_i)_{i\in I}\big)$ *is positive.*

Here, $\rho\big((y_i)_{i\in I}\big)$ *is the least eigenvalue of the matrix* $\big(m(y_i, y_j)\big)_{i,j\in I}$ *defined in the introduction.*

Proof. See [2, Proposition 3.3]. □

According the above Lemma, three Subcases may occur.

Subcase 1: $\rho\big((y_i)_{i\in I}\big) > 0$ *and* $-\sum_{k=1}^{n} b_k(y_i) > 0, \forall\, i \in I^c$. In this case, we define

$$W_2^1(u) = \widehat{W}(\widehat{u}) + \sum_{i\in I^c} \alpha_i \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i}.$$

Using the fact that $\mathcal{L}_{ij} > 0, \forall\, i \in I$ and $j \in I^c$, we get

$$\big\langle \partial J(u), W_2^1(u)\big\rangle \leq -c\left(\sum_{i=1}^{p}\left(\frac{1}{\lambda_i^{\beta(y_i)}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + \sum_{i\neq j} \varepsilon_{ij}\right).$$

A concentration phenomenon happens in this case.

Subcase 1: $\rho\big((y_i)_{i\in I}\big) > 0$. By Lemma 3.7, the $\max_{i\in I} \lambda_i(s)$ remains bounded along the flow lines $\widehat{u}(s)$ of $\widehat{W}$. Let

$$\widetilde{I} := \left\{ i, 1 \leq i \leq p,\ \text{s.t. } \lambda_i^{\beta(y_i)} \geq \frac{1}{2} \min j \in I \lambda_j^{n-2\sigma} \right\}.$$

By the estimation of Lemma 3.7, we have

$$\Big\langle \partial J(u), \widehat{W}(\widehat{u})\Big\rangle \leq -c\left(\sum_{i\in\tilde{I}}\left(\frac{1}{\lambda_i^{\beta(y_i)}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + \sum_{i\neq j\in I} \varepsilon_{ij}\right) + \sum_{\substack{i\in I\\ j\in I^c}} O\big(\varepsilon_{ij}\big).$$

Let $U_1 = \sum_{i\in\tilde{I}^c} \alpha_i \delta(a_i,\lambda_i)$. U_1 satisfies the condition of the above Statement 1. We apply the corresponding vector field $W_1(u_1)$. It verifies

$$\langle \partial J(u), W_1(u_1)\rangle \leq -c\left(\sum_{i\in\tilde{I}^c}\left(\frac{1}{\lambda_i^{\beta(y_i)}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + \sum_{i\neq j\in\tilde{I}^c} \varepsilon_{ij}\right) + \sum_{\substack{i\in\tilde{I}^c\\ j\in\tilde{I}}} O\big(\varepsilon_{ij}\big).$$

Let $W_2^2 = \widehat{W}(\widehat{u}) + W_1(u_1)$. The above two inequalities and the estimation of Claim 1 yield

$$\big\langle \partial J(u), W_2^2(u)\big\rangle \leq -c\left(\sum_{i=1}^{p}\left(\frac{1}{\lambda_i^{\beta(y_i)}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + \sum_{i\neq j} \varepsilon_{ij}\right).$$

Moreover the $\max_{1\leq i\leq p} \lambda_i(s)$ is bounded along the flow of W_2^2.

Subcase 1: $\exists\, i \in I^c$ *such that* $-\sum_{k=1}^{n} b_k(y_i) < 0$. We denote

$$I_1 = \left\{ i \in I^c, -\sum_{k=1}^{n} b_k(y_i) < 0 \right\}.$$

We decrease all the $\lambda_i, i \in I_1$. We set

$$Z(u) = -\sum_{i\in I_1} \alpha_i \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i}.$$

By the expansion of Proposition 3.2 and Claim 1, we have

$$\langle \partial J(u), Z(u)\rangle \leq -c\left[\sum_{i\in I_1}\left(\frac{1}{\lambda_i{}^{\beta(y_i)}}+\frac{|\nabla K(a_i)|}{\lambda_i}\right)+\sum_{i\in I_1,\, j\neq i}\varepsilon_{ij}\right]+\sum_{i\in I_1^c} o\left(\frac{1}{\lambda_i{}^{\beta(y_i)}}\right).$$

Let

$$\widetilde{I}_1=\left\{i, 1\leq i\leq p, \quad \lambda_i^{\beta(y_i)}\geq \frac{1}{2}\min_{j\in I_1}\lambda_j^{\beta(y_j)}\right\}.$$

The above expansion can be improved as follows.

$$\langle \partial J(u), Z(u)\rangle \leq -c\left[\sum_{i\in \tilde{I}_1}\left(\frac{1}{\lambda_i{}^{\beta(y_i)}}+\frac{|\nabla K(a_i)|}{\lambda_i}\right)+\sum_{i\in \tilde{I}_1\, j\neq i}\varepsilon_{ij}\right]+\sum_{i\in \tilde{I}_1^c} o\left(\frac{1}{\lambda_i{}^{\beta(y_i)}}\right).$$

Let $\widetilde{u}=\sum_{i\in\tilde{I}_1^c}\alpha_i\delta_{(a_i,\lambda_i)}$. For any $i\in\widetilde{I}_1^c, \beta(y_i)=n-2\sigma$. We apply the corresponding vector field $\widehat{W}$ of Lemma 3.7. We have

$$\left\langle \partial J(u), \widehat{W}(\widetilde{u})\right\rangle \leq -c\left(\sum_{i\in\tilde{I}_1^c}\left(\frac{1}{\lambda_i{}^{\beta(y_i)}}+\frac{|\nabla K(a_i)|}{\lambda_i}\right)+\sum_{i\neq j\in\tilde{I}_1^c}\varepsilon_{ij}\right)+\sum_{\substack{i\in\tilde{I}_1^c\\ j\neq i}} O(\varepsilon_{ij}).$$

For $m>0$ small enough, let $W_2^3(u)=m\widehat{W}(\widetilde{u})+Z(u)$. From the above two inequalities, W_2^3 satisfies

$$\left\langle \partial J(u), W_2^3(u)\right\rangle \leq -c\left(\sum_{i=1}^{p}\left(\frac{1}{\lambda_i{}^{\beta(y_i)}}+\frac{|\nabla K(a_i)|}{\lambda_i}\right)+\sum_{i\neq j}\varepsilon_{ij}\right).$$

It is clear that the $\max_{1\leq i\leq p}\lambda_i(s)$ remains bounded along W_2^3.

Case 2: $\forall\, i\in I, \beta(y_i)\neq n-2\sigma$. Under the assumption $\mathcal{L}_{ij}\geq 0, \forall\, i\neq j$, there is only one index $j_0, 1\leq j_0\leq p$ such that $\beta(y_{j_0})>n-2\sigma$. The indices set I is then reduced to:

$$I=\{j_0\}\cup\left\{i=1,\ldots,p,\ \text{s.t. } \mathcal{L}_{ij_0}=0\right\}.$$

Setting,

$$N_{j_0}=\left\{i=1,\ldots,p,\ \text{s.t. } \mathcal{L}_{ij_0}=0\right\}.$$

It is easy to see that $\beta(y_i)<n-2\sigma, \forall\, i\in N_{j_0}$. We introduce the following Lemma that we will prove in the appendix of this paper.

Lemma 3.8. *Under condition* (H_1)*, there exists a pseudo gradient* $Z(u)$ *with the following properties.*

(a) $\langle \partial J(u), Z(u)\rangle \leq -c\displaystyle\sum_{i\in N_{j_0}}\left(\frac{1}{\lambda_i^{\beta(y_i)}}+\varepsilon_{ij_0}\right)+\sum_{i\in N_{j_0}^c} o\left(\frac{1}{\lambda_i^{\beta(y_i)}}\right).$

(b) $\|Z(u)\| \leq \frac{1}{c}$ *and* $\max_{1\leq i\leq p} \lambda_i(s)$ *remains bounded along the flow lines of* Z.

Define,

$$\widetilde{N}_{j_0} = \left\{ i = 1, \dots, p, \ \text{s.t. } \lambda_i^{\beta(y_i)} \geq \frac{1}{2} \min_{j \in N_{j_0}} \lambda_j^{\beta(y_j)} \right\}.$$

The first inequality of Lemma 3.8 yields

$$\langle \partial J(u), Z(u) \rangle \leq -c \left[\sum_{i \in \widetilde{N}_{j_0}} \frac{1}{\lambda_i^{\beta(y_i)}} + \sum_{i \in N_{j_0}} \varepsilon_{i j_0} \right] + \sum_{i \in \widetilde{N}_{j_0}^c} o\left(\frac{1}{\lambda_i^{\beta(y_i)}} \right)$$

Let $\widehat{u} = \sum_{i \in \widetilde{N}_{j_0}^c} \alpha_i \delta_{(a_i, \lambda_i)}$. Of course $\mathcal{L}_{ij} > 0, \forall\, i \neq j \in \widetilde{N}_{j_0}^c$. We apply $W_1(\widehat{u})$, where W_1 is the vector field defined in statement 1. We have,

$$\langle \partial J(u), W_1(\widehat{u}) \rangle \leq -c \left[\sum_{i \in \widetilde{N}_{j_0}^c} \frac{1}{\lambda_i^{\beta(y_i)}} + \sum_{i \neq j \in \widetilde{N}_{j_0}^c} \varepsilon_{ij} \right] + \sum_{i \in \widetilde{N}_{j_0}^c \ j \in \widetilde{N}_{j_0}} o(\varepsilon_{ij})$$

For $i \in \widetilde{N}_{j_0}^c$ and $j \in \widetilde{N}_{j_0}$, $\mathcal{L}_{ij} > 0$. Therefore, $\varepsilon_{ij} = o\big(\frac{1}{\lambda_i^{\beta(y_i)}}\big) + o\big(\frac{1}{\lambda_j^{\beta(y_i)}}\big)$. Thus,

$$\langle \partial J(u), W_1(\widehat{u}) \rangle \leq -c \left[\sum_{i \in \widetilde{N}_{j_0}^c} \frac{1}{\lambda_i^{\beta(y_i)}} + \sum_{i \neq j \in \widetilde{N}_{j_0}^c} \varepsilon_{ij} \right] + \sum_{i \in \widetilde{N}_{j_0}} o\left(\frac{1}{\lambda_i^{\beta(y_i)}} \right)$$

Setting $W_2^4(u) = Z(u) + W_1(\widehat{u})$. From the above inequalities, we have

$$\left\langle \partial J(u), W_2^4(u) \right\rangle \leq -c \left[\sum_{i=1}^{p} \left(\frac{1}{\lambda_i^{\beta(y_i)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{i \neq j} \varepsilon_{ij} \right].$$

Note that $\max_{1\leq i\leq p} \lambda_i(s)$ is a bounded function along the flow lines of W_2^4. In this statement, we denote W_2 the pseudo gradient defined by a convex combination of $W_2^i, i = 1, 2, \dots, 4$.

Statement 3. We assume that there exist $i \neq j$ such that $\alpha_{ij} < 0$.

We order $\lambda_i^{\beta(y_i)}, i = 1, \dots, p$. Assume that

$$\lambda_{i_1}^{\beta(y_{i_1})} \leq \dots \leq \lambda_{i_p}^{\beta(y_{i_p})}.$$

Let $M \gg 1$ and let

$$I_1 = \left\{ i = 1, \dots, p, \ \text{s.t. } \lambda_i^{\beta(y_i)} \leq M \lambda_{i_1}^{\beta(y_{i_1})} \right\}.$$

We discuss the construction of the pseudo gradient in the current statement with respect to the following three cases.

Case 1: $\# I_1 = 1$. It follows that

$$\frac{1}{\lambda_{ij}^{\beta(y_{ij})}} = o\left(\frac{1}{\lambda_{i_1}^{\beta(y_{i_1})}}\right), \forall\ j = 2, \dots, p,$$

For M large enough. We decrease all the λ_{ij}'s, $j = 2, \dots, p$. By the expansion of Proposition 3.2, we have

$$\left\langle \partial J(u), -\sum_{j=1}^{p} \alpha_{ij}\lambda_{ij}\frac{\partial \delta_{(a_{ij},\lambda_{ij})}}{\partial \lambda_{ij}}\right\rangle \le -c\sum_{i\neq j}\varepsilon_{ij} + o\left(\frac{1}{\lambda_{i_1}^{\beta(y_{i_1})}}\right).$$

Let

$$W_3^1(u) = \left(-\sum_{k=1}^{n} b_k(y_{i_1})\right)\lambda_{i_1}\frac{\partial \delta_{(a_{i_1},\lambda_{i_1})}}{\partial \lambda_{i_1}} - \sum_{j=1}^{p}\alpha_{ij}\lambda_{ij}\frac{\partial \delta_{(a_{ij},\lambda_{ij})}}{\partial \lambda_{ij}}.$$

The $\max_{1\le i\le p}\lambda_i(s)$ remains bounded along W_3^1. Moreover,

$$\left\langle \partial J(u), W_3^1(u)\right\rangle \le -c\left(\sum_{i=1}^{p}\left(\frac{1}{\lambda_i^{\beta(y_i)}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + \sum_{i\neq j}\varepsilon_{ij}\right).$$

Case 2: $\# I_1 \ge 2$ *and* $\mathcal{L}_{ij} \ge 0, \forall\ i \neq j \in I_1$. We denote $\widehat{u} = \sum_{i\in I_1}\alpha_i\delta_{(a_i,\lambda_i)}$. $\widehat{u}$ verifies the properties of statement 1 or statement 2.

Let $W_i(\widehat{u}), i = 1$ or 2 be the corresponding vector field. We have

$$\langle \partial J(u), W_i(\widehat{u})\rangle \le -c\left[\sum_{i\in I_1}\left(\frac{1}{\lambda_i^{\beta(y_i)}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + \sum_{i\neq j\in I_1}\varepsilon_{ij}\right] + \sum_{i\in I_1, j\in I_1^c} O(\varepsilon_{ij}).$$

Note that $I_1^c \neq \emptyset$, since there exists at least $i \neq j$ satisfying $\mathcal{L}_{ij} < 0$. We decrease all λ_i's, $i \in I_1^c$. We get

$$\left\langle \partial J(u), -\sum_{i\in I_1^c}\alpha_i\lambda_i\frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i}\right\rangle \le -c\sum_{\substack{i\in I_1^c\\ j\neq i}}\varepsilon_{ij} + o\left(\frac{1}{\lambda_{i_1}^{\beta(y_{i_1})}}\right).$$

Let $m > 0$ small enough and let

$$W_3^2(u) = mW_i(\widehat{u}) - \sum_{i\in I_1^c}\alpha_i\lambda_i\frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i}.$$

We have,

$$\left\langle \partial J(u), W_3^2(u)\right\rangle \le -c\left(\sum_{i=1}^{p}\left(\frac{1}{\lambda_i^{\beta(y_i)}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + \sum_{i\neq j}\varepsilon_{ij}\right).$$

Case 3: $\#I \geq 2$ and there exists $i \neq j \in I$ such that $\mathcal{L}_{ij} < 0$. In this case, $\beta(y_i) > n-2\sigma$ or $\beta(y_j) > n-2\sigma$. Assume for example that $\beta(y_j) > n-2\sigma$.

The following Claim holds.

Claim 2.

$$\frac{1}{\lambda_j^{\beta(y_j)}} = o(\varepsilon_{ij}), \text{ as } \varepsilon \text{ is small.}$$

Indeed,

$$\frac{1}{\lambda_j^{\beta(y_j)}}\varepsilon_{ij}^{-1} \sim \frac{\lambda_i^{\frac{n-2\sigma}{2}}}{\lambda_j^{\beta(y_j)-\frac{n-2\sigma}{2}}}.$$

Using the fact that $\lambda_i^{\beta(y_i)} \leq M\lambda_j^{\beta(y_j)}$, since $i, j \in I$, we obtain

$$\begin{aligned}\frac{1}{\lambda_j^{\beta(y_j)}}\varepsilon_{ij}^{-1} &\leq M\frac{\lambda_j^{\frac{\beta(y_j)}{\beta(y_i)}\frac{n-2\sigma}{2}}}{\lambda_j^{\beta(y_j)-\frac{n-2\sigma}{2}}} \\ &\leq \frac{M}{\lambda_j^{-\frac{n-2\sigma}{2\beta(y_i)}\left(\beta(y_i)+\beta(y_j)-\frac{2\beta(y_i)\beta(y_j)}{n-2\sigma}\right)}}\end{aligned}$$

Therefore,

$$\frac{1}{\lambda_j^{\beta(y_j)}}\varepsilon_{ij}^{-1} \longrightarrow 0, \text{ as } \varepsilon \longrightarrow 0,$$

since $\mathcal{L}_{ij} < 0$. Claim 22 is valid.

Let

$$W_3^3(u) = -\sum_{r=1}^{p}\alpha_r\lambda_r\frac{\partial\delta_{(a_r,\lambda_r)}}{\partial\lambda_r}.$$

By the expansion of Proposition 3.2, we have

$$\left\langle \partial J(u), W_3^3\right\rangle \leq -c\sum_{k\neq r}\varepsilon_{kr} + \sum_{r=1}^{p} O\left(\frac{1}{\lambda_r^{\beta(y_r)}}\right).$$

Observe that, for $r \in I_1$, we have $\lambda_r^{\beta(y_r)} \sim \lambda_j^{\beta(y_j)}$ and for $r \in I^c$, we have $\frac{1}{\lambda_r^{\beta(y_r)}} = o\left(\frac{1}{\lambda_{i_1}^{\beta(y_{i_1})}}\right)$.

Therefore, by Claim 2

$$\frac{1}{\lambda_r^{\beta(y_r)}} = o(\varepsilon_{ij}), \quad \forall\, r = 1, \ldots, p.$$

Thus,

$$\left\langle \partial J(u), W_3^3(u) \right\rangle \leq -c\left[\sum_{i=1}^{p}\left(\frac{1}{\lambda_i^{\beta(y_i)}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + \sum_{i\neq j}\varepsilon_{ij}\right].$$

In the current statement, we denote W_3 the vector field defined by a convex combination of $W_3^i, i = 1, 2, 3$. By construction the $\max_{1\leq i\leq p}\lambda_i(s)$ remains bounded along the flow lines of W_3.

The required pseudo gradient W of Proposition 3.3, is defined by a convex combination of W_1, W_2 and W_3. By construction it verifies (i) and (iii) for any $u \in V(p, \varepsilon, 0)$. Concerning (ii), it follows from (i) and the estimate of $\|\overline{v}\|$, following [8, Appendix 2]. This completes the proof of Proposition 3.3. $\square$

4. **Proof of Theorem 1.1**

We now prove our existence and multiplicity result.

Proof. Under the assumption of Theorem 3.6, the critical points at infinity of in $V(p, \varepsilon, 0), p \geq 1$, are:

$$(y_1, \dots, y_p)_\infty = \sum_{i=1}^{p}\frac{1}{K(y_i)^{\frac{n-2\sigma}{n}}}\delta_{(y_i,\infty)}, \text{ with } (y_1, \dots, y_p) \in B_\infty.$$

Following [8, Lemma 4.2], J can be expanded near each $(y_1, \dots, y_p)_\infty$ as follows.

$$J\left(\sum_{i=1}^{p}\alpha_i\delta_{(a_i,\lambda_i)+\overline{v}}\right)$$

$$= \left(\sum_{i=1}^{p}\frac{1}{K(y_i)^{\frac{n-2\sigma}{n}}}\right)^{\frac{2\sigma}{n}}\left[1 - \left(|h|^2 + \sum_{k=1}^{n}b_k(y_i)|(a_i - y_i)_k|^{\beta(y_i)}\right) + \sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta(y_i)}}\right] + \|v\|^2,$$

where $h \in \mathbb{R}^{p-1}$ are the related coordinates of the expansion of J according to the α_i's variables.

Using the fact that $b_k(y_i) \neq 0, \forall\ i = 1, \dots, p$ and $\forall\ k = 1, \dots, n$, the index of J at $(y_1, \dots, y_p)_\infty$ is given by

$$ind(J, y_1, \dots, y_p) = \sum_{i=1}^{p}(n - \widetilde{i}(y_i)) + p - 1,$$

where $\widetilde{i}(y_i) = \#\{b_k(y_i), k = 1, \dots, n, \text{ s.t. } b_k(y_i) < 0\}$.

For any critical point at infinity $(\omega, y_1, \dots, y_p)_\infty$ of J in $V(p, \varepsilon, \omega), p \geq 1, \omega$ is a solution of (E_σ) or zero, we denote $W_u(\omega, y_1, \dots, y_p)_\infty$ the unstable manifold of the gradient vector field at $(\omega, y_1, \dots, y_p)_\infty$. We apply the deformation Lemma to deform the variational space Σ^+. We have

$$\Sigma^+ \simeq \bigcup_{\partial J(\omega)=0} W_u(\omega) \cup \bigcup_{\substack{\partial J(\omega)=0,\\ p\geq 1}} W_u(\omega, y_1, \dots, y_p)_\infty \cup \bigcup_{(y_1,\dots,y_p)\in B_\infty} W_u(y_1, \dots, y_p)_\infty. \tag{4.1}$$

The symbol $\simeq$ designates: retracts by deformation. This implies that (E_σ) has at least one solution. Otherwise, the above retraction will be reduced to

$$\Sigma^+ \simeq \bigcup_{(y_1,\dots,y_p)\in B_\infty} W_u(y_1, \dots, y_p)_\infty \tag{4.2}$$

and thus,

$$1 = \sum_{(y_1,\dots,y_p)\in B_\infty} (-1)^{ind(J,y_1,\dots,y_p)},$$

by computing the Euler−Poincare characteristic of the both sides of(4.2). This contradicts the assumption of Theorem 1.1.

Now, if $\beta(y) > \frac{n-2\sigma}{2}$, for any critical point y of K, Theorem 3.4 rules out the existence of the critical points at infinity of J in $V(p, \varepsilon, \omega), p \geq 1$ and $\omega \neq 0$. Thus (4.1) is reduced to

$$\Sigma^+ \simeq \bigcup_{\partial J(\omega)=0} W_u(\omega) \cup \bigcup_{(y_1,\dots,y_p)\in B_\infty} W_u(y_1, \dots, y_p)_\infty. \tag{4.3}$$

By computing the EulerPoincare characteristic of the both sides of (4.3), we get

$$1 = \sum_{\partial J(\omega)=0} (-1)^{ind(J,\omega)} + \sum_{(y_1,\dots,y_p)\in B_\infty} (-1)^{ind(J,y_1,\dots,y_p)},$$

Therefore

$$\#S \geq \left| 1 - \sum_{(y_1,\dots,y_p)\in B_\infty} (-1)^{ind(J,y_1,\dots,y_p)} \right|,$$

where $S = \{\omega \in \Sigma^+, \partial J(\omega) = 0\}$. This finishes the proof of Theorem 1.1. □

5. **Appendix**

We now prove Lemma 3.8

Proof of Lemma 3.8. Let $i \in N_{j_0}$. By the expansion of Proposition 3.2, we have

$$\left\langle \partial J(u), \alpha_i \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i} \right\rangle = J(u)^{\frac{2-n}{2}} \left[\frac{n-2\sigma}{n} \frac{\beta(y_i) c(y_i)}{K(y_i)^{\frac{2(1-\sigma)+n}{2}}} \frac{\sum_{k=1}^{n} b_k(y_i)}{\lambda_i^{\beta(y_i)}} \right.$$
$$\left. + 2^{\frac{n-2\sigma}{2}} (n-2\sigma) \widetilde{c} \sum_{j \neq i} \frac{G(y_i, y_j)}{(K(y_i)K(y_j))^{\frac{n-2\sigma}{4}}} \frac{1}{(\lambda_i \lambda_j)^{\frac{n-2\sigma}{2}}} \right]$$
$$+ \sum_{i \neq j} o(\varepsilon_{ij}) + o\left(\frac{1}{\lambda_i^{\beta(y_i)}} \right),$$

since

$$\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = -\frac{n-2\sigma}{2} 2^{\frac{n-2\sigma}{2}} \frac{G(y_i, y_j)}{(\lambda_i \lambda_j)^{\frac{n-2\sigma}{2}}}, \text{ for any } i \neq j$$

and

$$K(a_i) J(u)^{\frac{n}{n-2\sigma}} \alpha_i^{\frac{4\sigma}{n-2\sigma}} = 1 + o(1), \text{ for any } i = 1, \ldots, p.$$

Let $\delta_0 > 0$ small enough and let

$$I_{\delta_0} = \left\{ \begin{array}{l} i \in N_{j_0}, \text{ s.t. } \dfrac{\beta(y_i) c(y_i)}{nK(y_i)^{\frac{2(1-\sigma)+n}{2}}} \dfrac{\left| \sum_{k=1}^{n} b_k(y_i) \right|}{\lambda_i^{\beta(y_i)}} \\ \qquad > \dfrac{1}{(1-\delta_0)} \dfrac{2^{\frac{n-2\sigma}{2}} \widetilde{c}}{(K(y_i)K(y_{j_0}))^{\frac{n-2\sigma}{4}}} \dfrac{G(y_i, y_{j_0})}{(\lambda_i \lambda_{j_0})^{\frac{n-2\sigma}{2}}} \end{array} \right\}.$$

The construction of the required pseudo gradient of Lemma 3.8 is related to the following 2 cases.

Case 1: $I_{\delta_0} \neq \emptyset$. For any $i \in I_{\delta_0}$, we set

$$V_i(u) = \left(-\sum_{k=1}^{n} b_k(y_i) \right) \alpha_i \lambda_i \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i}.$$

From the above expansion, we have

$$\begin{aligned}\langle \partial J(u), V_i(u)\rangle = J(u)^{\frac{2-n}{2}}\Bigg(&-\frac{n-2\sigma}{n}\frac{\beta(y_i)c(y_i)}{K(y_i)^{\frac{2(1-\sigma)+n}{2}}}\frac{\left(\sum_{k=1}^n b_k(y_i)\right)^2}{\lambda_i^{\beta(y_i)}}\\
&-2^{\frac{n-2\sigma}{2}}(n-2\sigma)\widetilde{c}\sum_{j\neq i}\frac{\left(\sum_{k=1}^n b_k(y_i)\right)}{(K(y_i)K(y_j))^{\frac{n-2\sigma}{4}}}\frac{G(y_i,y_j)}{(\lambda_i\lambda_j)^{\frac{n-2\sigma}{2}}}\Bigg)\\
&+o\left(\frac{1}{\lambda_i^{\beta(y_i)}}\right)+\sum_{j\neq i}o(\varepsilon_{ij})\\
=J(u)^{\frac{2-n}{2}}\Bigg(&-\frac{n-2\sigma}{n}\frac{\beta(y_i)c(y_i)}{K(y_i)^{\frac{2(1-\sigma)+n}{2}}}\frac{\left(\sum_{k=1}^n b_k(y_i)\right)^2}{\lambda_i^{\beta(y_i)}}\\
&-2^{\frac{n-2\sigma}{2}}(n-2\sigma)\widetilde{c}\frac{\sum_{k=1}^n b_k(y_i)}{(K(y_i)K(y_{j_0}))^{\frac{n-2\sigma}{4}}}\frac{G(y_i,y_{j_0})}{(\lambda_i\lambda_{j_0})^{\frac{n-2\sigma}{2}}}\Bigg)\\
&+\sum_{j=1}^p o\left(\frac{1}{\lambda_j^{\beta(y_j)}}\right),\end{aligned}$$

since $\forall\ j=1,\dots,p, j\neq j_0$ and $j\neq i$, we have $\mathcal{L}_{ij}>0$, so by Claim 1,

$$\varepsilon_{ij}=o\left(\frac{1}{\lambda_i^{\beta(y_i)}}\right)+o\left(\frac{1}{\lambda_j^{\beta(y_j)}}\right).$$

If $\left(\sum_{k=1}^n b_k(y_i)\right)>0$, we get

$$\langle \partial J(u), V_i(u)\rangle \leq -c\left(\frac{1}{\lambda_j^{\beta(y_j)}}+\varepsilon_{ij_0}\right)+\sum_{j=1}^p o\left(\frac{1}{\lambda_j^{\beta(y_j)}}\right).$$

If $\left(\sum_{k=1}^n b_k(y_i)\right)<0$, we get (using the fact that $i\in I_{\delta_0}$),

$$\begin{aligned}\langle \partial J(u), V_i(u)\rangle \leq J(u)^{\frac{2-n}{2}}\Bigg(&-\frac{n-2\sigma}{n}\frac{\beta(y_i)c(y_i)}{K(y_i)^{\frac{2(1-\sigma)+n}{2}}}\frac{\left(\sum_{k=1}^n b_k(y_i)\right)^2}{\lambda_i^{\beta(y_i)}}\\
&+(1-\delta_0)\frac{n-2\sigma}{n}\frac{\beta(y_i)c(y_i)}{nK(y_i)^{\frac{2(1-\sigma)+n}{2}}}\frac{\left(\sum_{k=1}^n b_k(y_i)\right)^2}{\lambda_i^{\beta(y_i)}}\Bigg)+\sum_{j=1}^p o\left(\frac{1}{\lambda_j^{\beta(y_j)}}\right),\\
\leq \frac{-c}{\lambda_j^{\beta(y_j)}}&+\sum_{j=1}^p o\left(\frac{1}{\lambda_j^{\beta(y_j)}}\right).\end{aligned}$$

Thus,

$$\left\langle \partial J(u), \sum_{i\in I_{\delta_0}} V_i(u) \right\rangle \le -c \sum i \in I_{\delta_0} \frac{1}{\lambda_i^{\beta(y_i)}} + \sum_{j=1}^{p} o\left(\frac{1}{\lambda_j^{\beta(y_j)}}\right).$$

Observe that, under the action of $\sum_{i\in I_{\delta_0}} V_i(u)$, the $\max_{1\le i\le p}\lambda_i(s)$ remains bounded, since for any $i \in I_{\delta_0}, \lambda_i^{\beta(y_i)-\frac{n-2\sigma}{n}}$ is upper bounded by $M\lambda_{j_0}^{\frac{n-2\sigma}{n}}$. Now, for the indices $i \in N_{j_0} \setminus I_{\delta_0}$, we have $\lambda_i \ge m\lambda_{i_0}$, where $i_0 \in I_{\delta_0}$ and $m > 0$, small enough.

Therefore,

$$\left\langle \partial J(u), \sum_{i\in I_{\delta_0}} V_i(u) \right\rangle \le -c \sum_{i\in N_{j_0}} \frac{1}{\lambda_i^{\beta(y_i)}} + \sum_{j=1}^{p} o\left(\frac{1}{\lambda_j^{\beta(y_j)}}\right).$$

Let $m_1 > 0$ small enough and let

$$X_1(u) = \sum_{i\in I_{\delta_0}} V_i(u) - m \sum_{i\in N_{j_0}} \lambda_i \frac{\partial\delta_{(a_i,\lambda_i)}}{\partial\lambda_i}.$$

We have

$$\begin{aligned}\langle \partial J(u), X_1(u)\rangle &\le -c \sum_{i\in N_{j_0}} \left(\frac{1}{\lambda_i^{\beta(y_i)}} + \sum_{j\ne i}\varepsilon_{ij}\right) + \sum_{j=1}^{p} o\left(\frac{1}{\lambda_j^{\beta(y_j)}}\right),\\ &\le -c \sum_{i\in N_{j_0}} \left(\frac{1}{\lambda_i^{\beta(y_i)}} + \varepsilon_{ij_0}\right) + \sum_{j=1}^{p} o\left(\frac{1}{\lambda_j^{\beta(y_j)}}\right).\end{aligned}$$

Lemma 3.8, follows in this case.

Case 2: $I_{\delta_0} \ne \emptyset$. We move λ_{j_0}. By the expansion of Proposition 3.2, we have

$$\begin{aligned}&\left\langle \partial J(u), \alpha_{j_0}\lambda_{j_0}\frac{\partial\delta_{(a_{j_0},\lambda_{j_0})}}{\partial\lambda_{j_0}} \right\rangle\\ &= J(u)^{\frac{2-n}{2}}\left[\frac{n-2\sigma}{n}\ \frac{\beta(y_{j_0})c(y_{j_0})}{K(y_{j_0})^{\frac{2(1-\sigma)+n}{2}}}\ \frac{\sum_{k=1}^{n} b_k(y_{j_0})}{\lambda_{j_0}^{\beta(y_{j_0})}}\right.\\ &\qquad \left. + 2^{\frac{n-2\sigma}{2}}(n-2\sigma)\widetilde{c} \sum_{i\in N_{j_0}} \frac{G(y_i,y_{j_0})}{(K(y_i)K(y_{j_0}))^{\frac{n-2\sigma}{4}}} \frac{1}{(\lambda_i\lambda_{j_0})^{\frac{n-2\sigma}{2}}}\right] + \sum_{i=1}^{p} o\left(\frac{1}{\lambda_i^{\beta(y_i)}}\right),\end{aligned}$$

since for any $i \in N_{j_0}^c, i \ne j_0$, we have $\mathcal{L}_{ij_0} > 0$, therefore by Claim 1,

$$\varepsilon_{ij_0} = o\left(\frac{1}{\lambda_{j_0}^{\beta(y_{j_0})}}\right) + o\left(\frac{1}{\lambda_i^{\beta(y_i)}}\right).$$

We decompose N_{j_0} into two parts.

$$N^1_{j_0} = \left\{ \begin{array}{l} i \in N_{j_0}, \dfrac{1-2\delta_0}{n} \dfrac{\beta(yi)c(y_i)}{K(y_i)^{\frac{2(1-\sigma)+n}{2}}} \dfrac{\left|\sum_{k=1}^n b_k(y_i)\right|}{\lambda_i^{\beta(y_i)}} < \widetilde{c} \dfrac{2^{\frac{n-2\sigma}{2}}}{\left(K(y_i)K(y_{j_0})\right)^{\frac{n-2\sigma}{4}}} \\ \qquad \dfrac{G(y_i, y_{j_0})}{\left(\lambda_i \, \lambda_{j_0}\right)^{\frac{n-2\sigma}{2}}} < \dfrac{1+2\delta_0}{n} \dfrac{\beta(yi)c(y_i)}{K(y_i)^{\frac{2(1-\sigma)+n}{2}}} \dfrac{\left|\sum_{k=1}^n b_k(y_i)\right|}{\lambda_i^{\beta(y_i)}} \end{array} \right\}$$

and

$$N^2_{j_0} = \left\{ \begin{array}{l} i \in N_{j_0}, \widetilde{c} \dfrac{2^{\frac{n-2\sigma}{2}}}{\left(K(y_i)K(y_{j_0})\right)^{\frac{n-2\sigma}{4}}} \dfrac{G(y_i, y_{j_0})}{\left(\lambda_i \, \lambda_{j_0}\right)^{\frac{n-2\sigma}{2}}} \\ \qquad \geq \dfrac{1+2\delta_0}{n} \dfrac{\beta(yi)c(y_i)}{K(y_i)^{\frac{2(1-\sigma)+n}{2}}} \dfrac{\left|\sum_{k=1}^n b_k(y_i)\right|}{\lambda_i^{\beta(y_i)}} \end{array} \right\}.$$

We have,

$$\begin{aligned} &\left\langle \partial J(u), \lambda_{j_0} \frac{\partial \delta_{(a_{j_0}, \lambda_{j_0})}}{\partial \lambda_{j_0}} \right\rangle \\ &\qquad = J(u)^{\frac{2-n}{2}} \left[\frac{n-2\sigma}{n} \frac{\beta(y_{j_0})c(y_{j_0})}{K(y_{j_0})^{\frac{2(1-\sigma)+n}{2}}} \frac{\sum_{k=1}^n b_k(y_{j_0})}{\lambda_{j_0}^{\beta(y_{j_0})}} \right. \\ &\qquad\qquad \left. + \widetilde{c}(n-2\sigma) 2^{\frac{n-2\sigma}{2}} \sum_{i \in N^1_{j_0}} \frac{G(y_i, y_{j_0})}{(K(y_i)K(y_{j_0}))^{\frac{n-2\sigma}{4}}} \frac{1}{(\lambda_i \lambda_{j_0})^{\frac{n-2\sigma}{2}}} \right] \\ &\qquad\quad + \sum_{i \in N^2_{j_0}} O(\varepsilon_{i j_0}) + \sum_{j=1}^{p} o\Big(\frac{1}{\lambda_j^{\beta(y_j)}}\Big). \end{aligned}$$

Observe that for any $i \in N^1_{j_0}$,

$$\widetilde{c} \frac{2^{\frac{n-2\sigma}{2}}}{(K(y_i)K(y_{j_0}))^{\frac{n-2\sigma}{4}}} \frac{G(y_i, y_{j_0})}{(\lambda_i \lambda_{j_0})^{\frac{n-2\sigma}{2}}} \sim \frac{\beta(y_{j_0})c(y_{j_0})}{K(y_{j_0})^{\frac{2(1-\sigma)+n}{2}}} \frac{\sum_{k=1}^n b_k(y_{j_0})}{\lambda_{j_0}^{\beta(y_{j_0})}},$$

as δ_0 is small enough. Therefore

$$\left\langle \partial J(u), \lambda_{j_0} \frac{\partial \delta_{(a_{j_0},\lambda_{j_0})}}{\partial \lambda_{j_0}} \right\rangle \sim J(u)^{\frac{2-n}{2}} \Bigg[\frac{n-2\sigma}{n} \frac{\beta(y_{j_0}) c(y_{j_0}) \sum_{k=1}^{n} b_k(y_{j_0})}{K(y_{j_0})^{\frac{2(1-\sigma)+n}{2}}} + \sum_{i \in N_{j_0}^1} \frac{n-2\sigma}{n} \frac{\beta(y_i) c(y_i)}{K(y_i)^{\frac{2(1-\sigma)+n}{2}}} \frac{\left|\sum_{k=1}^{n} b_k(y_i)\right|}{K_{ij_0}} \Bigg] \frac{1}{\lambda_{j_0}^{\beta(y_{j_0})}} + \sum_{i \in N_{j_0}^2} O(\varepsilon_{ij_0}) + \sum_{j=1}^{p} o\Big(\frac{1}{\lambda_j^{\beta(y_j)}}\Big),$$

where

$$K_{ij_1} = \left[\frac{\beta(y_i) c(y_i) \left|\sum_{k=1}^{n} b_k(y_i)\right| (K(y_i) K(y_{j_0}))^{\frac{n-2\sigma}{4}}}{n 2^{\frac{n-2\sigma}{2}} K(y_i)^{\frac{2(1-\sigma)+n}{2}} G(y_i, y_{j_0})} \right]^{\frac{2\beta(y_{j_0})}{n-2\sigma}}.$$

Let,

$$\rho(j_0, N_{j_0}) := \frac{\beta(y_{j_0}) c(y_{j_0}) \sum_{k=1}^{n} b_k(y_{j_0})}{K(y_{j_0})^{\frac{2(1-\sigma)+n}{2}}} + \sum i \in N_{j_0}^2 \frac{\beta(y_i) c(y_i)}{K(y_i)^{\frac{2(1-\sigma)+n}{2}}} \frac{\left|\sum_{k=1}^{n} b_k(y_i)\right|}{K_{ij_1}}.$$

Under condition (H_1), $\rho(j_0, N_{j_0}) \neq 0$. We set

$$Z_{j_0}(u) = -\operatorname{sign} \rho(j_0, N_{j_0}) \lambda_{j_0} \frac{\partial \delta_{(a_{j_0},\lambda_{j_0})}}{\partial \lambda_{j_0}}.$$

Along $Z_{j_0}(u)$, $\max_{1 \le i \le p}$ is bounded since $\lambda_{j_0}^{\beta(y_{j_0})}$ is upper bound by $\lambda_i^{\beta(y_i)}$, $i \in N_{j_0}$ and λ_i does not move under the action of $Z_{j_0}(u)$.

The preceding expansion yields

$$\begin{aligned} \left\langle \partial J(u), Z_{j_0}(u) \right\rangle &\le -\frac{c}{\lambda_{j_0}^{\beta(y_{j_0})}} + \sum_{i \in N_{j_0}^2} O\left(\varepsilon_{ij_0}\right) + \sum_{j=1}^{p} o\left(\frac{1}{\lambda_j^{\beta(y_j)}}\right) \\ &\le -c \sum_{i \in N_{j_0}^1} \left(\frac{1}{\lambda_i^{\beta(y_i)}} + \varepsilon_{ij_0} \right) + \sum_{i \in N_{j_0}^2} O\left(\varepsilon_{ij_0}\right) + \sum_{j=1}^{p} o\left(\frac{1}{\lambda_j^{\beta(y_j)}}\right) \end{aligned}$$

For the indices $i \in N_{j_0}^2$,

$$\left\langle \partial J(u), -\sum_{i\in N_{j_0}^2} \alpha_i\lambda_i \frac{\partial\delta_{(a_i,\lambda_i)}}{\partial\lambda_i}\right\rangle \leq -c \sum_{\substack{i\in N_{j_0}^2\\ j\neq i}} \varepsilon_{ij} + \sum_{i\in N_{j_0}^2} O\left(\frac{1}{\lambda_i^{\beta(y_i)}}\right).$$

$$\leq -c \sum_{i\in N_{j_0}^2}\left(\varepsilon_{ij_0} + \frac{1}{\lambda_i^{\beta(y_i)}}\right) + \sum_{j=1}^{p} o\left(\frac{1}{\lambda_j^{\beta(y_j)}}\right)$$

Let $m_2 > 0$ small enough and let

$$X_2(u) = m_2 Z_{j_0}(u) - \sum_{i\in N_{j_0}^2} \alpha_i\lambda_i \frac{\partial\delta_{(a_i,\lambda_i)}}{\partial\lambda_i}.$$

It satisfies

$$\langle \partial J(u), X_2(u)\rangle \leq -c \sum_{i\in N_{j_0}}\left(\frac{1}{\lambda_i^{\beta(y_i)}} + \varepsilon_{ij_0}\right) + \sum_{j=1}^{p} o\left(\frac{1}{\lambda_j^{\beta(y_j)}}\right).$$

The required pseudo gradient Z of Lemma 3.8 is defined by a convex combination of $X_1(u)$ and $X_2(u)$. □

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