ANNALES MATHÉMATIQUES

BLAISE PASCAL

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Volume 31, nº 2 (2024), p. 239-271.

https://doi.org/10.5802/ambp.430

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Publication éditée par le laboratoire de mathématiques Blaise Pascal
de l'université Clermont Auvergne, UMR 6620 du CNRS
Clermont-Ferrand — France

CENTRE MERSENNE

Publication membre du
Centre Mersenne pour l'édition scientifique ouverte
http://www.centre-mersenne.org/
e-ISSN : 2118-7436

Annales mathématiques Blaise Pascal **31**, 239-271 (2024)

Null-controllability of cascade reaction-diffusion systems with odd coupling terms

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Abstract

In this paper, we consider a nonlinear system of two parabolic equations, with a distributed control in the first equation and an odd coupling term in the second one. We prove that the nonlinear system is locally null-controllable for any arbitrary small time. The main difficulty is that the linearized system is not null-controllable. To overcome this obstacle, we extend in a nonlinear setting the strategy introduced in [18] that consists in constructing odd controls for the linear heat equation. The proof relies on three main steps. First, we obtain from the classical L^2 parabolic Carleman estimate, conjugated with maximal regularity results, a weighted L^p observability inequality for the nonhomogeneous heat equation. Secondly, we perform a duality argument, close to the well-known Hilbert Uniqueness Method in a reflexive Banach setting, to prove that the heat equation perturbed by a source term is null-controllable thanks to odd controls. Finally, the nonlinearity is handled with a Schauder fixed-point argument.

1. Introduction

Let $T > 0$ be a positive time, $d \in \mathbb{N}^*$, Ω be a bounded, connected, open subset of $\mathbb{R}^d$ of class C^2 corresponding to the spatial domain and ω be a nonempty open subset such that $\overline{\omega} \subset \Omega$. In what follows, we use the notation 1_ω for the characteristic function of ω.

The null-controllability of the heat equation described below was first obtained by Fattorini and Russell [12] for $d = 1$ and by Lebeau, Robbiano [19] and Fursikov, Imanuvilov [15] for $d \geqslant 1$. More precisely for any $y_0 \in L^2(\Omega)$, there exists $h \in L^2((0,T)\times\omega)$ such that the solution y of the system

$$\begin{cases} \partial_t y - \Delta y = h 1_\omega & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,\cdot) = y_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

satisfies $y(T,\cdot) = 0$. These results were then extended to a large number of other parabolic systems, linear or nonlinear. For instance, the null-controllability of linear coupled parabolic systems has been a challenging issue for the control community in the last two decades. In that direction, we can quote, among the large literature devoted to this

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The authors were partially supported by the French National Research Agency (ANR), Project TRECOS, ANR-20-CE40-0009.
Keywords: Null-controllability, parabolic system, nonlinear coupling, Carleman estimate.
2020 *Mathematics Subject Classification*: 35K45, 35K58, 93B05, 93C10.

problem, [2], where Ammar-Khodja, Benabdallah, Dupaix, González-Burgos exhibit sharp conditions for the null-controllability of systems of the form

$$\begin{cases} \partial_t Y - D\Delta Y = AY + Bh1_\omega & \text{in } (0,T)\times\Omega, \\ Y = 0 & \text{on } (0,T)\times\partial\Omega, \\ Y(0,\cdot) = Y_0 & \text{in } \Omega. \end{cases} \tag{1.2}$$

Here, at time $t \in (0,T]$, $Y(t,\cdot) : \Omega \to \mathbb{R}^n$ is the state, $h = h(t,\cdot) : \Omega \to \mathbb{R}^m$ is the control, $D := \mathrm{diag}(d_1, \dots, d_n)$ with $d_i \in (0,+\infty)$ is the *diffusion matrix*, $A \in \mathbb{R}^{n\times n}$ is the *coupling matrix* and $B \in \mathbb{R}^{n\times m}$ represents the *distribution of controls*. One objective is to reduce the number of controls m (and in particular to have $m < n$) by using the coupling matrices A and B. Let us also quote the survey [3] for other results and open problems in that direction. For nonlinear systems, a standard strategy consists in deducing local controllability results from the null-controllability of the linearized system: see, for instance, [1, 5, 16, 23], etc.

In this article, we consider the following controlled semi-linear reaction-diffusion system

$$\begin{cases} \partial_t y_1 - d_1\Delta y_1 = a_{11}y_1^{N_1} + h1_\omega & \text{in } (0,T)\times\Omega, \\ \partial_t y_2 - d_2\Delta y_2 = a_{21}y_1^{N_2} + a_{22}y_2^{N_3} & \text{in } (0,T)\times\Omega, \\ y_1 = y_2 = 0 & \text{on } (0,T)\times\partial\Omega, \\ y_1(0,\cdot) = y_{1,0}, \quad y_2(0,\cdot) = y_{2,0} & \text{in } \Omega, \end{cases} \tag{1.3}$$

where $d_1, d_2 \in (0,+\infty)$, $N_1, N_2, N_3 \in \mathbb{N}^*$ and $a_{ij} \in \mathbb{R}$. In (1.3), at time $t \in [0,T]$, $(y_1,y_2)(t,\cdot) : \Omega \to \mathbb{R}^2$ is the state while $h(t,\cdot) : \omega \to \mathbb{R}$ is the control. We are interested in the null-controllability of (1.3), that is find a control $h = h(t,x)$, supported in $(0,T)\times\omega$, that steers the state (y_1,y_2) to zero at time T, i.e. $(y_1,y_2)(T,\cdot) = 0$. Note that (1.3) is a so-called "cascade system" because the first equation is decoupled from the second equation. For such a system, the basic idea is to use the *nonlinear coupling* term $a_{21}y_1^{N_2}$, as an indirect control term, that acts on the second component y_2. From a modeling point of view, this type of system with polynomial nonlinearities naturally appears when considering chemical reactions. In this case, y_1, y_2 denote relative concentrations of two chemical species Y_1, Y_2, the control h represents the action of adding or extracting another chemical component at a specified location (in ω) of the chemical medium (in Ω). Then the goal is to steer the two concentrations y_1, y_2 to the chemical equilibrium $(0,0)$ at a given time T.

1.1. Main results

Our control results on (1.3) are written in the framework of weak solutions. More precisely, we define the Banach space

$$\mathcal{W} := L^2\left(0,T;H_0^1(\Omega)\right) \cap H^1\left(0,T;H^{-1}(\Omega)\right) \cap L^\infty((0,T)\times\Omega), \qquad (1.4)$$

and we consider solutions of (1.3) such that $y_1, y_2 \in \mathcal{W}$. The precise definition of the weak solutions of (1.3) is given in Definition 2.7 and a corresponding well-posedness result is stated in Theorem 2.8 for controls $h \in L^p((0,T)\times\omega)$ with

$$p \in \left(\frac{d+2}{2}, \infty\right], \quad \text{and} \quad p \geqslant 2 \text{ if } d = 1. \qquad (1.5)$$

Our first main result can be stated as follows.

Theorem 1.1. *Let p satisfies* (1.5). *Assume*

$$a_{2,1} \neq 0, \quad N_2 \text{ is odd}. \qquad (1.6)$$

Then there exists $\delta > 0$ such that for any initial data satisfying

$$\|y_{1,0}\|_{L^\infty(\Omega)} + \|y_{2,0}\|_{L^\infty(\Omega)} \leqslant \delta, \qquad (1.7)$$

there exists a control $h \in L^p((0,T)\times\omega)$ satisfying

$$\|h\|_{L^p((0,T)\times\omega)} \lesssim \delta, \qquad (1.8)$$

such that the solution $(y_1, y_2) \in \mathcal{W}\times\mathcal{W}$ of (1.3) *satisfies*

$$(y_1, y_2)(T, \cdot) = 0. \qquad (1.9)$$

Here and in all that follows, we use the notation $X \lesssim Y$ if there exists a constant $C > 0$ such that we have the inequality $X \leqslant CY$. In the whole paper, we use C as a generic positive constant that does not depend on the other terms of the inequality. The value of the constant C may change from one appearance to another. Our constants may depend on the geometry (Ω, ω), on the time T and on the dimension d. If we want to emphasize the dependence on a quantity k, we write $X \lesssim_k Y$.

As we will see, the smallness conditions on the initial data i.e. (1.7) and on the control i.e. (1.8) are sufficient conditions to guarantee the well-posedness of the system (1.3), see Theorem 2.8 below.

Before continuing, let us make some comments related to Theorem 1.1.

- Theorem 1.1 is a small-time local null-controllability result in the sense that for any time $T > 0$ (arbitrarily small), we can impose smallness conditions on the initial data $(y_{1,0}, y_{2,0})$ i.e. (1.7) so that the system (1.3) is null-controllable. The global null-controllability in small time or even in large time is an open problem. Actually, some partial answers can already be made. Indeed, if N_1 is even, then (1.3) is not null-controllable because one cannot prevent the blow-up from happening of the first equation. For a proof of this fact, one can see [14, Theorem 1] for weaker semilinearities. If N_1 is odd, the problem is largely open because we still do not know if the first equation of (1.3) is null-controllable or not, see for instance [6, Open Problems 7.14, 7.15].

- The sufficient condition (1.6) ensuring the local null-controllability of (1.3) is actually necessary. Indeed, if $a_{21} = 0$ then the second equation of (1.3) is decoupled from the first equation so y_2 cannot be driven to 0 at time T. Moreover, if N_2 is even, the strong maximum principle shows that we can not control y_2: assume for instance that $a_{21} \geqslant 0$, then
$$\partial_t y_2 - d_2 \Delta y_2 - a_{22} y_2^{N_3} = a_{21} y_1^{N_2} \geqslant 0 \text{ in } (0,T) \times \Omega \tag{1.10}$$
and thus $\widetilde{y_2}(t,x) := y_2(t,x) e^{-\lambda t}$, with $\lambda \geqslant |a_{22}| \, \|y_2\|^{N_3-1}_{L^\infty((0,T)\times\Omega)}$ satisfies
$$\partial_t \widetilde{y_2} - d_2 \Delta \widetilde{y_2} + c \widetilde{y_2} \geqslant 0 \text{ in } (0,T) \times \Omega$$
with $c \geqslant 0$ and we can apply the standard strong maximum principle (see, for instance, [11, Theorem 12, p. 397]): if $y_{2,0} \geqslant 0$ and $y_{2,0} \neq 0$ then for all $t \in (0,T]$, $y_2(t,\cdot) > 0$ in Ω.

- The linear case
$$N_1 = N_2 = N_3 = 1,$$
is already treated in [9] by de Teresa. To obtain such a result, the author shows a Carleman estimate and deduce from it an observability inequality for the adjoint system.

- For the semi-linear case, the main idea is to linearize the system in order to use the previous result. However, if $N_2 \geqslant 2$, in the linearized system around the trajectory $((\overline{y_1}, \overline{y_2}), \overline{h}) = ((0,0),0)$, we can see that the second equation is decoupled from the first one and thus can not be controlled; the linearized system is thus not null-controllable.

- To overcome this difficulty, Coron, Guerrero, Rosier [7] use the return method in the case
$$N_2 = 3, \quad N_3 = 1.$$
More precisely, they construct a reference trajectory $((\overline{y_1}, \overline{y_2}), \overline{h})$ of (1.3) starting from $(\overline{y_1}, \overline{y_2})(0,\cdot) = 0$, reaching $(\overline{y_1}, \overline{y_2})(T,\cdot) = 0$ and satisfying $|\overline{y_1}| \geqslant \varepsilon > 0$

in $(t_1, t_2) \times \omega$. Then they linearize (1.3) around the reference trajectory and obtain for the second equation

$$\partial_t y_2 - d_2 \Delta y_2 = 3 a_{21} \overline{y_1}^2 y_1 + a_{22} y_2 \text{ in } (0,T) \times \Omega. \tag{1.11}$$

They can then use [9] to obtain that the null-controllability of the linearized system and then the local null-controllability of the nonlinear system (1.3) by a fixed-point argument. For an extension of this method to a 3×3 cascade reaction-diffusion system with cubic coupling terms, see [8].

- In [18], the first author employs a new direct strategy in order to deal with the case
$$N_2 \text{ odd and } N_3 = 1,$$
that we adapt here to prove Theorem 1.1 for the more general case
$$N_2 \text{ odd and } N_3 \geqslant 1.$$

- Our method is quite general and can be applied to other systems. For instance, one can replace the Laplace operator $-\Delta$ in (1.3) by an elliptic operator $\mathcal{A} : y \mapsto -\operatorname{div}(A \nabla y)$, with $A \in C^2(\overline{\Omega}; \mathcal{M}_d(\mathbb{R}))$ a matrix-valued function such that for some constant $\beta > 0$,
$$\sum_{k,l=1}^{d} A_{k,l}(x) \xi_k \xi_l \geqslant \beta \sum_{k=1}^{d} |\xi_k|^2 \quad \left(x \in \Omega,\ \xi \in \mathbb{R}^d\right).$$
The main hypotheses needed in our method are the corresponding parabolic operator $\partial_t - \mathcal{A}$ satisfies a Carleman estimate and a maximal regularity property. Such properties are recalled for the Laplace operator in Theorem 3.2 and Theorem 2.3 but are valid for the above operator (see, for instance, [15, Lemma 1.2, p. 5] and [17, Theorem 7.1, p. 181], [17, Theorem 9.1, p. 341]).

To simplify the work and without loss of generality, we assume in what follows that

$$d_1 = d_2 = 1, \quad a_{11} = a_{21} = a_{22} = 1, \quad N_1, N_2, N_3 \geqslant 2.$$

We describe below the idea of the proof.

Strategy of the proof. We proceed in two steps: in the first step, we control the first equation of (1.3) in the time interval $(0, T/2)$. Using that the semi-linear heat equation is locally null-controllable for any positive time, there exists a control h such that $y_1(T/2, \cdot) = 0$. Using the smallness assumptions, we can ensure that the second equation of (1.3) admits a solution on $(0, T/2)$. In the second step, we control this second equation thanks to a *fictitious odd control* H. Here and in what follows, an *odd control* H means that H can be written under the form $H = \widetilde{H}^{N_2}$, where $\widetilde{H}$ is a sufficiently smooth function. More precisely, we can consider the control problem

$$\begin{cases} \partial_t y_2 - \Delta y_2 = H\chi_\omega + y_2^{N_3} & \text{in } (T/2,T)\times\Omega, \\ y_2 = 0 & \text{on } (T/2,T)\times\partial\Omega, \end{cases} \tag{1.12}$$

where $\chi_\omega = \widetilde{\chi}_\omega^{N_2}$ and where $\widetilde{\chi}_\omega \in C^\infty(\Omega)$ has a compact support in ω, $\widetilde{\chi}_\omega \not\equiv 0$. We then need a control H such that $y_2(T,\cdot) = 0$, satisfying $H(T/2,\cdot) = H(T,\cdot) = 0$ and such that H^{1/N_2} is regular. Such a control is given by our second main result (Theorem 1.2) stated below. We can then set in $(T/2,T)$

$$y_1 := (H\chi_\omega)^{1/N_2}, \quad h := \partial_t y_1 - \Delta y_1 - y_1^{N_1}.$$

Note in particular that $y_1(T/2,\cdot) = y_1(T,\cdot) = 0$. By construction, $((y_1,y_2),h)$ is thus a trajectory of (1.3) satisfying (1.9).

From the above strategy, we see that the proof of Theorem 1.1 relies on the construction of odd controls for a semi-linear heat equation that we present now. For $N \geqslant 2$, we consider the system

$$\begin{cases} \partial_t y - \Delta y = h\chi_\omega + y^N & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,\cdot) = y_0 & \text{in } \Omega. \end{cases} \tag{1.13}$$

The definition of the weak solutions for the above system and a corresponding well-posedness result are given in Definition 2.4 and Theorem 2.5. Our second main result states as follows.

Theorem 1.2. *Assume that $N \geqslant 2$, $n \in \mathbb{N}$, and $p \geqslant 1$. There exists $\delta > 0$ such that for every initial data $y_0 \in L^\infty(\Omega)$ such that*

$$\|y_0\|_{L^\infty(\Omega)} \leqslant \delta, \tag{1.14}$$

there exists a control $h \in L^\infty((0,T)\times\omega)$ satisfying

$$\|h\|_{L^\infty((0,T)\times\omega)} \lesssim \|y_0\|_{L^\infty(\Omega)}, \tag{1.15}$$

$$\begin{gathered} h^{1/(2n+1)} \in L^p\big(0,T;W^{2,p}(\Omega)\big) \cap W^{1,p}(0,T;L^p(\Omega)), \\ h^{1/(2n+1)}(0,\cdot) = h^{1/(2n+1)}(T,\cdot) = 0, \end{gathered} \tag{1.16}$$

and such that the solution $y \in \mathcal{W}$ of (1.13) satisfies

$$\|y\|_{\mathcal{W}} \lesssim \|y_0\|_{L^\infty(\Omega)}, \tag{1.17}$$

and

$$y(T,\cdot) = 0. \tag{1.18}$$

As for Theorem 1.1, the smallness conditions (1.14) and (1.15) are sufficient to guarantee the well-posedness of the semi-linear heat equation (1.13), see Theorem 2.5 below. Note that the conditions in (1.16) at $t = 0$ and $t = T$ correspond the conditions $H(T/2, \cdot) = H(T, \cdot) = 0$ in our strategy of proof for Theorem 1.1.

Before continuing, let us make some comments related to Theorem 1.2.

- The crucial property in Theorem 1.2 is the odd behavior of the control, stated in (1.16). Actually, the small-time local null-controllability of (1.13) with controls in $L^\infty((0,T) \times \omega)$ is a consequence of [4, Lemma 6].

- For $N = 1$, that is the linear case, the result of Theorem 1.2 is still true and has already been established by the first author, see [18, Proposition 3.7]. One can even obtain a (small-time) global null-controllability result with odd controls due to the linear setting. Note that here, we extend the result of [18] in the case of a linear heat equation with a source term, see Section 3.3.

Strategy of the proof. First, we use a classical Carleman estimate for the nonhomogeneous heat equation to obtain a weighted L^2 observability inequality stated in Corollary 3.3. From this result and after that, we need to take care about the weights appearing in the norm of the adjoint system they have to be "comparable". We then deduce from this result a weighted L^p observability inequality, see Proposition 3.4 below with an arbitrary large p. As a consequence, a null-controllability result is obtained for the heat equation with a source term and with odd controls. Let us remark that taking p large enough allows us to do only one bootstrap argument for getting the desired odd behavior for the control, see Proposition 3.6 below. This is different from [18, Theorem 4.4 and Proposition 4.9] where two such arguments are used for obtaining the null-controllability of the heat equation with odd controls. Another bootstrap argument is then required in order to deal with the nonlinearity in the fixed-point argument, see Proposition 3.8 below. Finally, a Schauder fixed-point argument, see Section 3.5, is performed to obtain Theorem 1.2. We can remark that here due to our method for constructing the control, in this fixed point argument, the corresponding nonlinear mapping is α-Hölder continuous with $\alpha < 1$. In particular, a Banach fixed point argument does not seem to apply.

1.2. **Outline of the paper**

The outline of the paper is as follows. In Section 2, we recall some standard facts about well-posedness, regularity results for linear and nonlinear heat equations in various functional settings. We notably prove that (1.13) and (1.3) are locally well-posed, see

Theorem 2.5 and Theorem 2.8 below. Section 3 and Section 4 are devoted to the proofs of the main results, i.e. Theorem 1.1 and Theorem 1.2.

2. Well-posedness and regularity results for the heat equation

In this section, we give the notion of solutions that we consider in what follows. Then we recall standard well-posedness results for both linear and semi-linear heat equations in various functional settings we will use in what follows.

2.1. Functional spaces

In this article, we use in a crucial way a L^p framework with $p \in (1, \infty)$. First, we introduce the standard notation for the dual exponent $p' \in (1, \infty)$ of p defined by the relation

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

We also introduce the following functional spaces

$$\mathcal{X}^p := L^p\left(0, T; W^{2,p}(\Omega)\right) \cap W^{1,p}\left(0, T; L^p(\Omega)\right). \tag{2.1}$$

We have the following classical embedding results (see, for instance, [17, Lemma 3.3, p. 80]): for $p, q \geqslant 1$,

$$\begin{cases} \mathcal{X}^p \hookrightarrow L^q(0, T; L^q(\Omega)) & \text{if} \quad \frac{1}{q} \geqslant \frac{1}{p} - \frac{2}{d+2}, \\ \mathcal{X}^p \hookrightarrow L^\infty(0, T; L^\infty(\Omega)) & \text{if} \quad p > \frac{d+2}{2}, \end{cases} \tag{2.2}$$

$$\begin{cases} \mathcal{X}^p \hookrightarrow L^q\left(0, T; W^{1,q}(\Omega)\right) & \text{if} \quad \frac{1}{q} \geqslant \frac{1}{p} - \frac{1}{d+2}, \\ \mathcal{X}^p \hookrightarrow L^\infty\left(0, T; W^{1,\infty}(\Omega)\right) & \text{if} \quad p > d+2. \end{cases} \tag{2.3}$$

We also have, see for instance [17, Lemma 3.4, p. 82],

$$\mathcal{X}^p \longleftrightarrow C^0\left([0, T]; W^{2/p',p}(\Omega)\right), \tag{2.4}$$

where $W^{\alpha,p}(\Omega)$ denotes the fractional Sobolev spaces (see, for instance, [17, p. 70]). We recall that functions in $W^{\alpha,p}(\Omega)$ admit a trace on $\partial\Omega$ if $\alpha > 1/p$. If $\alpha > 1/p$, we denote by $W_0^{\alpha,p}(\Omega)$ the subspace of functions $f \in W^{\alpha,p}(\Omega)$ such that $f = 0$ on $\partial\Omega$. We also write $W_0^{\alpha,p}(\Omega) := W^{\alpha,p}(\Omega)$ if $\alpha \leqslant 1/p$. From [10, Corollary 4.53, p. 216], we have

$$W^{2/p',p}(\Omega) \longleftrightarrow L^\infty(\Omega) \quad \text{if} \quad p > \frac{d+2}{2},$$

and thus

$$\mathcal{X}^p \longleftrightarrow C^0\left([0, T]; L^\infty(\Omega)\right) \quad \text{if} \quad p > \frac{d+2}{2}. \tag{2.5}$$

We finish with some other classical results on the spaces $\mathcal{X}^p$, for which we give a short proof for completeness.

Lemma 2.1. *The following statements hold.*

(1) *If* $p > \frac{d+2}{2}$, *then* $\mathcal{X}^p$ *is an algebra.*

(2) *For any* $N > 1$, $q > 1$, *if*

$$\frac{1}{q}\left(1 - \frac{1}{N}\right) < \frac{2}{2+d} \tag{2.6}$$

then the embedding

$$\mathcal{X}^q \hookrightarrow L^{Nq}((0,T)\times\Omega) \quad \textit{is compact}. \tag{2.7}$$

Proof. For the first point, we consider $f, g \in \mathcal{X}^p$. Then

$$\partial_t f,\ \partial_t g,\ \nabla^2 f,\ \nabla^2 g \in L^p(0,T;L^p(\Omega)),$$

and from (2.2) and (2.3)

$$f,\ g \in L^\infty(0,T;L^\infty(\Omega)), \quad \nabla f,\ \nabla g \in L^{2p}\left(0,T;L^{2p}(\Omega)\right).$$

We thus deduce that

$$\partial_t(fg),\ \nabla^2(fg) \in L^p\left(0,T;L^p(\Omega)\right).$$

For the second point, we can use (2.6) to consider $p > 1$ such that

$$\frac{1}{qN} > \frac{1}{p} > \frac{1}{q} - \frac{2}{2+d}. \tag{2.8}$$

We thus deduce from (2.2) that

$$\mathcal{X}^q \hookrightarrow L^p((0,T)\times\Omega) \hookrightarrow L^{Nq}((0,T)\times\Omega)$$

and from the Hölder inequality, there exists $\theta \in (0,1)$ such that

$$\|f\|_{L^{Nq}((0,T)\times\Omega)} \leqslant \|f\|^{\theta}_{L^q((0,T)\times\Omega)}\,\|f\|^{1-\theta}_{L^p((0,T)\times\Omega)} \quad (f \in L^p((0,T)\times\Omega)). \tag{2.9}$$

From the Aubin–Lions lemma (see, for instance, [21, § 8, Corollary 4]), the embedding

$$\mathcal{X}^q \hookrightarrow L^q((0,T)\times\Omega) \quad \text{is compact.}$$

Consequently, if (f_n) is a bounded sequence of $\mathcal{X}^q$, it has a subsequence converging in $L^q((0,T)\times\Omega)$ and bounded in $L^p((0,T)\times\Omega)$. From (2.9), this subsequence is converging in $L^{qN}((0,T)\times\Omega)$. □

2.2. Linear heat equation

Let us first consider the linear nonhomogenenous heat equation

$$\begin{cases} \partial_t y - \Delta y = g & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,\cdot) = y_0 & \text{in } \Omega. \end{cases} \tag{2.10}$$

In this article, we need several definitions of solutions for (2.10):

Definition 2.2. We introduce three concepts of solutions for (2.10).

(1) If $y_0 \in W_0^{2/p',p}(\Omega)$ and $g \in L^p((0,T)\times\Omega)$, we say that $y \in \mathcal{X}^p$ is a strong solution of (2.10) if it satisfies (2.10) a.e. and in the trace sense.

(2) If $y_0 \in L^2(\Omega)$ and $g \in L^2(0,T;H^{-1}(\Omega))$, we say that $y \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ is a weak solution if

$$\int_0^T \langle \partial_t y(t,\cdot), \zeta(t,\cdot)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,\mathrm{d}t + \int_0^T\int_\Omega \nabla y(t,x)\cdot\nabla\zeta(t,x)\mathrm{d}t\mathrm{d}x$$
$$= \int_0^T \langle g(t,\cdot), \zeta(t,\cdot)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,\mathrm{d}t \qquad \forall\, \zeta \in L^2\left(0,T;H_0^1(\Omega)\right), \tag{2.11}$$

and

$$y(0,\cdot) = y_0 \text{ in } L^2(\Omega). \tag{2.12}$$

(3) If $y_0 \in L^1(\Omega)$ and $g \in L^1((0,T)\times\Omega)$, we say that $y \in L^p((0,T)\times\Omega)$ is a very weak solution of (2.10) if

$$\iint_{(0,T)\times\Omega} y(-\partial_t\zeta - \Delta\zeta)\mathrm{d}t\mathrm{d}x$$
$$= \iint_{(0,T)\times\Omega} g\zeta\mathrm{d}t\mathrm{d}x + \int_\Omega y_0(x)\zeta(0,x)\mathrm{d}x \qquad \forall\, \zeta \in C_c^\infty([0,T)\times\Omega).$$

We recall the following implications

strong solution $\Longrightarrow$ weak solution $\Longrightarrow$ very weak solution,

and the reverse implications are also true assuming that y is regular enough. We also note that the definition of weak solution is meaningful due to the continuous embedding (see, for instance, [11, Theorem 3, p. 303])

$$L^2\left(0,T;H_0^1(\Omega)\right) \cap H^1\left(0,T;H^{-1}(\Omega)\right) \hookrightarrow C^0\left([0,T];L^2(\Omega)\right). \tag{2.13}$$

We also state standard results for the well-posedness of (2.10) (see, for instance [11, Theorems 3 and 4, pp. 378-379], [17, Theorem 7.1, p. 181] and [17, Theorem 9.1, p. 341]):

Theorem 2.3. *The following well-posedness results hold.*

(1) *For any* $y_0 \in L^2(\Omega)$ *and* $g \in L^2(0,T;H^{-1}(\Omega))$*, the equation* (2.10) *admits a unique weak solution* y *and we have the estimate*

$$\|y\|_{L^2\left(0,T;H_0^1(\Omega)\right)\cap H^1\left(0,T;H^{-1}(\Omega)\right)} \lesssim \|y_0\|_{L^2(\Omega)} + \|g\|_{L^2\left(0,T;H^{-1}(\Omega)\right)}. \tag{2.14}$$

(2) *For* $y_0 \in L^\infty(\Omega)$ *and* $g \in L^\infty((0,T)\times\Omega)$*, the unique weak solution* y *of* (2.10) *satisfies*

$$\|y\|_{L^\infty((0,T)\times\Omega)} \lesssim \|y_0\|_{L^\infty(\Omega)} + \|g\|_{L^\infty((0,T)\times\Omega)}. \tag{2.15}$$

(3) *Assume* $p \in (1,\infty)$*. For any* $y_0 \in W_0^{2/p',p}(\Omega)$ *and* $g \in L^p((0,T)\times\Omega)$*, there exists a unique strong solution* $y \in X^p$ *of* (2.10) *and we have the estimate*

$$\|y\|_{X^p} \lesssim \|y_0\|_{W^{2/p',p}(\Omega)} + \|g\|_{L^p((0,T)\times\Omega)}. \tag{2.16}$$

2.3. Semi-linear heat equation

For $N \in \mathbb{N}$, $N \geqslant 2$, let us then consider the semi-linear heat equation

$$\begin{cases} \partial_t y - \Delta y = y^N + g & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,\cdot) = y_0 & \text{in } \Omega. \end{cases} \tag{2.17}$$

The space $\mathcal{W}$ is defined in (1.4). First we recall the definition of a weak solution for the system (2.17):

Definition 2.4. We say that $y \in \mathcal{W}$ is a weak solution of (2.17) if

$$\int_0^T \langle \partial_t y(t,\cdot), \zeta(t,\cdot)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,\mathrm{d}t + \int_0^T\int_\Omega \nabla y(t,x)\cdot\nabla\zeta(t,x)\mathrm{d}t\mathrm{d}x$$
$$= \int_0^T\int_\Omega y^N(t,x)\zeta(t,x)\mathrm{d}t\mathrm{d}x + \int_0^T \langle g(t,\cdot),\zeta(t,\cdot)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,\mathrm{d}t$$
$$\forall\, \zeta \in L^2\left(0,T;H_0^1(\Omega)\right), \tag{2.18}$$

and

$$y(0,\cdot) = y_0 \text{ in } L^2(\Omega). \tag{2.19}$$

Let us state the following well-posedness result for (2.17) for small data. This result is standard, but we recall the proof for completeness.

Theorem 2.5. *Assume p satisfies* (1.5)*. There exists $\delta > 0$ small enough such that for any $y_0 \in L^\infty(\Omega)$ and $g \in L^p((0,T)\times\Omega)$, satisfying*

$$\|y_0\|_{L^\infty(\Omega)} + \|g\|_{L^p((0,T)\times\Omega)} \leqslant \delta, \tag{2.20}$$

the system (2.17) *admits a unique weak solution. Moreover, we have*

$$\|y\|_{\mathcal{W}} + \left\|y^N\right\|_{L^\infty((0,T)\times\Omega)} \lesssim \|y_0\|_{L^\infty(\Omega)} + \|g\|_{L^p((0,T)\times\Omega)}\,. \tag{2.21}$$

Proof. First, we show that for any $F \in L^\infty((0,T)\times\Omega)$, there exists a unique weak solution to the heat equation

$$\begin{cases} \partial_t y - \Delta y = F + g & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,\cdot) = y_0 & \text{in } \Omega. \end{cases} \tag{2.22}$$

In order to do this, we can write $y = y_1 + y_2$ with

$$\begin{cases} \partial_t y_1 - \Delta y_1 = F & \text{in } (0,T)\times\Omega, \\ y_1 = 0 & \text{on } (0,T)\times\partial\Omega, \\ y_1(0,\cdot) = y_0 & \text{in } \Omega, \end{cases} \qquad \begin{cases} \partial_t y_2 - \Delta y_2 = g & \text{in } (0,T)\times\Omega, \\ y_2 = 0 & \text{on } (0,T)\times\partial\Omega, \\ y_2(0,\cdot) = 0 & \text{in } \Omega. \end{cases} \tag{2.23}$$

Applying Theorem 2.3, the above systems admit respectively a unique solution $y_1 \in \mathcal{W}$ and $y_2 \in \mathcal{X}^p$ and with the hypotheses on p, we deduce from (2.2) that $\mathcal{X}^p \hookrightarrow \mathcal{W}$. We conclude the existence and the uniqueness of a weak solution $y \in \mathcal{W}$ of (2.22) and we have the estimate

$$\|y\|_{L^\infty((0,T)\times\Omega)} \lesssim \|y_0\|_{L^\infty(\Omega)} + \|g\|_{L^p((0,T)\times\Omega)} + \|F\|_{L^\infty((0,T)\times\Omega)}\,. \tag{2.24}$$

We can thus define the following mapping

$$\mathcal{N} : L^\infty((0,T)\times\Omega) \longrightarrow L^\infty((0,T)\times\Omega), \quad F \longmapsto y^N, \tag{2.25}$$

where y is the unique weak solution to (2.22) and if y_0 and g satisfy (2.20) and if we consider

$$B_\delta := \left\{F \in L^\infty((0,T)\times\Omega)\ ;\ \|F\|_{L^\infty((0,T)\times\Omega)} \leqslant \delta\right\}, \tag{2.26}$$

then we deduce from (2.24) that for $\delta > 0$ small enough, $\mathcal{N}(B_\delta) \subset B_\delta$. We can also show in a similar way that the restriction of $\mathcal{N}$ on B_δ is a strict contraction. The Banach fixed point yields the existence of a unique fixed point F and the corresponding solution y of (2.22) is a weak solution of (2.17).

For the uniqueness, we consider $y_1, y_2 \in \mathcal{W}$ two solutions of (2.17). Then, $y := y_1 - y_2$ satisfies (in a weak sense)

$$\begin{cases} \partial_t y - \Delta y = y_1^N - y_2^N & \text{in } (0,T)\times\Omega, \\ \qquad y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,\cdot) = 0 & \text{in } \Omega. \end{cases} \tag{2.27}$$

In particular, using that $y_1, y_2 \in L^\infty((0,T)\times\Omega)$, we can write the standard energy estimate: for any $t \in [0,T]$,

$$\|y(t,\cdot)\|_{L^2(\Omega)}^2 \leqslant \int_0^t \int_\Omega y\left(y_1^N - y_2^N\right) \mathrm{d}s\mathrm{d}x \lesssim \int_0^t \|y(s,\cdot)\|_{L^2(\Omega)}^2 \,\mathrm{d}s,$$

and we conclude with the Grönwall lemma. □

We now state some regularizing effects of (2.17).

Lemma 2.6. *Assume the hypotheses of Theorem 2.5 and let us consider y the corresponding weak solution of* (2.17)*.*

(1) *If $g = 0$ then for any $t \in (0,T]$ and for any $q > 1$, $y(t,\cdot) \in W_0^{2/q',q}(\Omega)$. Moreover, we have the estimate*

$$\|y(t,\cdot)\|_{W^{2/q',q}(\Omega)} \lesssim_{t,q} \|y_0\|_{L^\infty(\Omega)}. \tag{2.28}$$

(2) *In the general case, for any $t \in (0,T]$, $y(t,\cdot) \in L^\infty(\Omega)$ and we have the estimate*

$$\|y(t,\cdot)\|_{L^\infty(\Omega)} \lesssim_{t,q} \|y_0\|_{L^\infty(\Omega)} + \|g\|_{L^p((0,T)\times\Omega)}. \tag{2.29}$$

Proof. Let us denote by θ the function $\theta(t) = t$ for $t \in \mathbb{R}$. Then we deduce from (2.17) that

$$\begin{cases} \partial_t(\theta y) - \Delta(\theta y) = y + \theta y^N & \text{in } (0,T)\times\Omega, \\ \qquad \theta y = 0 & \text{on } (0,T)\times\partial\Omega, \\ (\theta y)(0,\cdot) = 0 & \text{in } \Omega, \end{cases}$$

and from Theorem 2.5,

$$\left\|y + \theta y^N\right\|_{L^\infty((0,T)\times\Omega)} \lesssim \|y_0\|_{L^\infty(\Omega)}.$$

Applying Theorem 2.3, we deduce that $\theta y \in \mathcal{X}^q$ for any $q > 1$, and we conclude with (2.4).

The second point can be done similarly by using (2.5) and that p satisfies (1.5). □

The above definition and properties can be extended to the parabolic system

$$\begin{cases} \partial_t y_1 - \Delta y_1 = y_1^{N_1} + g & \text{in } (0,T)\times\Omega, \\ \partial_t y_2 - \Delta y_2 = y_1^{N_2} + y_2^{N_3} & \text{in } (0,T)\times\Omega, \\ y_1 = y_2 = 0 & \text{on } (0,T)\times\partial\Omega, \\ y_1(0,\cdot) = y_{1,0}, \quad y_2(0,\cdot) = y_{2,0} & \text{in } \Omega. \end{cases} \tag{2.30}$$

More precisely, we have the following definition and well-posedness results:

Definition 2.7. We say that $(y_1, y_2) \in \mathcal{W}\times\mathcal{W}$ is a weak solution of (2.30) if

$$\int_0^T \langle \partial_t y_1(t,\cdot), \zeta(t,\cdot)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,\mathrm{d}t + \int_0^T\int_\Omega \nabla y_1(t,x)\cdot\nabla\zeta(t,x)\mathrm{d}t\mathrm{d}x$$
$$= \int_0^T\int_\Omega y_1^{N_1}(t,x)\zeta(t,x)\mathrm{d}t\mathrm{d}x + \int_0^T \langle g(t,\cdot), \zeta(t,\cdot)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,\mathrm{d}t$$
$$\forall\, \zeta \in L^2\left(0,T;H_0^1(\Omega)\right),$$

$$\int_0^T \langle \partial_t y_2(t,\cdot), \zeta(t,\cdot)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,\mathrm{d}t + \int_0^T\int_\Omega \nabla y_2(t,x)\cdot\nabla\zeta(t,x)\mathrm{d}t\mathrm{d}x$$
$$= \int_0^T\int_\Omega y_1^{N_2}(t,x)\zeta(t,x)\mathrm{d}t\mathrm{d}x + \int_0^T\int_\Omega y_2^{N_3}(t,x)\zeta(t,x)\mathrm{d}t\mathrm{d}x$$
$$\forall\, \zeta \in L^2\left(0,T;H_0^1(\Omega)\right),$$

and

$$(y_1,y_2)(0,\cdot) = (y_{1,0}, y_{2,0}) \text{ in } L^2(\Omega)^2. \tag{2.31}$$

Theorem 2.8. *Assume p satisfies* (1.5). *There exists $\delta > 0$ small enough such that for any $(y_{1,0}, y_{2,0}) \in L^\infty(\Omega)^2$ and $g \in L^p((0,T)\times\Omega)$, satisfying*

$$\left\|(y_{1,0}, y_{2,0})\right\|_{L^\infty(\Omega)^2} + \|g\|_{L^p((0,T)\times\Omega)} \leqslant \delta, \tag{2.32}$$

the system (2.30) *admits a unique weak solution. Moreover, we have*

$$\|y_1\|_{\mathcal{W}} + \|y_2\|_{\mathcal{W}} + \left\|y_1^{N_1}\right\|_{L^\infty((0,T)\times\Omega)} + \left\|y_1^{N_2}\right\|_{L^\infty((0,T)\times\Omega)} + \left\|y_2^{N_3}\right\|_{L^\infty((0,T)\times\Omega)}$$
$$\lesssim \|y_0\|_{L^\infty(\Omega)} + \|g\|_{L^p((0,T)\times\Omega)}\,. \tag{2.33}$$

3. Proof of Theorem 1.2

The goal of this part is to prove Theorem 1.2.

We first set

$$\rho_0(t) := \exp\left(-\frac{1}{t(T-t)}\right) \quad (t \in (0,T)), \quad \rho_0(0) = \rho_0(T) = 0, \tag{3.1}$$

and

$$\rho(t) := \begin{cases} \exp\left(-\frac{1}{(T/2)^2}\right) & (t \in [0,T/2)), \\ \exp\left(-\frac{1}{t(T-t)}\right) & (t \in [T/2,T)), \end{cases} \quad \rho(T) = 0. \tag{3.2}$$

Using Lemma 2.6, that is taking the control $h \equiv 0$ in $[0,T/2] \times \omega$ in order to benefit from the regularizing effect of the semi-linear heat equation (1.13), we see that it is sufficient to show the following result.

Theorem 3.1. *Assume $N, n \in \mathbb{N}$, $N \geqslant 2$ and $T > 0$. Let us consider p satisfying*

$$p = (2n+1)(2k+1) + 1, \tag{3.3}$$

with $k \in \mathbb{N}$ large enough so that

$$p > \frac{d+2}{2}, \tag{3.4}$$

and $q > \max(\frac{d+2}{2}, 2)$. There exist $\delta > 0$ and $m > 0$ such that for any initial data $y_0 \in W_0^{\frac{2}{q'},q}(\Omega)$ with

$$\|y_0\|_{W^{\frac{2}{q'},q}(\Omega)} \leqslant \delta,$$

there exists a control h and a strong solution y of (1.13) *such that*

$$\frac{y}{\rho^m} \in \mathcal{X}^q, \quad y^N \in L^q(0,T;L^q(\Omega)),$$

$$h \in L^\infty(0,T;L^\infty(\Omega)), \quad \left(\frac{h}{\rho_0^m}\right)^{1/(2n+1)} \in \mathcal{X}^p, \tag{3.5}$$

together with the estimate

$$\left\|\frac{y}{\rho^m}\right\|_{\mathcal{X}^q} + \left\|\frac{y^N}{\rho^{m_1}}\right\|_{L^q(0,T;L^q(\Omega))}^{1/N} + \left\|\frac{h}{\rho_0^m}\right\|_{L^\infty(0,T;L^\infty(\Omega))} + \left\|\left(\frac{h}{\rho_0^m}\right)^{1/(2n+1)}\right\|_{\mathcal{X}^p}^{2n+1} \lesssim \|y_0\|_{W^{2/q',q}(\Omega)}. \tag{3.6}$$

In particular, h satisfies (1.15) *and* (1.16) *and y satisfies* (1.18).

The differences with Theorem 1.2 are that we can apply Lemma 2.6 to replace $y_0 \in L^\infty(\Omega)$ by $y_0 \in W_0^{\frac{2}{q'},q}(\Omega)$ and that we take here p large enough for the space $\mathcal{X}^p$ (but we have $p_1 \leqslant p_2 \implies \mathcal{X}^{p_2} \subset \mathcal{X}^{p_1}$). Note that since $q > \frac{d+2}{2}$, then, from (2.2), $\mathcal{X}^q \hookrightarrow L^\infty(0,T;L^\infty(\Omega))$ so that $y \in \mathcal{W}$.

3.1. Carleman estimate and L^2 observability inequality for the heat equation

The goal of this part is to deduce a weighted L^2 observability inequality for the heat equation from a Carleman estimate. We first recall a standard Carleman estimate for the heat equation that is due to Fursikov and Imanuvilov [15]. We start by introducing a nonempty domain ω_0 such that $\chi_\omega > 0$ on $\overline{\omega_0} \subset \omega$. By using [15], see also [22, Theorem 9.4.3], there exists $\eta^0 \in C^2(\overline{\Omega})$ satisfying

$$\eta^0 > 0 \text{ in } \Omega, \quad \eta^0 = 0 \text{ on } \partial\Omega, \quad \max_{\Omega} \eta^0 = 1, \quad \nabla\eta^0 \neq 0 \text{ in } \overline{\Omega \setminus \omega_0}. \tag{3.7}$$

We then define the following functions:

$$\alpha(t,x) = \frac{\exp(4\lambda) - \exp\left\{\lambda\left(2 + \eta^0(x)\right)\right\}}{t(T-t)}, \quad \xi(t,x) = \frac{\exp\left\{\lambda\left(2 + \eta^0(x)\right)\right\}}{t(T-t)}. \tag{3.8}$$

We can now state the Carleman estimate for the heat equation, see [13, Lemma 1.3].

Theorem 3.2. *There exist $\lambda_0, s_0, C_0 \in \mathbb{R}_+^*$ such that for any $\lambda \geqslant \lambda_0$, $s \geqslant s_0(T+T^2)$, $\zeta \in \mathcal{X}^2$ with $\zeta = 0$ on $(0,T)\times\partial\Omega$,*

$$\iint_{(0,T)\times\Omega} s^3\lambda^4\xi^3 e^{-2s\alpha} |\zeta|^2 \,\mathrm{d}x\mathrm{d}t \leqslant C_0 \left(\iint_{(0,T)\times\Omega} e^{-2s\alpha} |\partial_t\zeta + \Delta\zeta|^2 \,\mathrm{d}x\mathrm{d}t \right. \\ \left. + \iint_{(0,T)\times\Omega} s^3\lambda^4\xi^3 e^{-2s\alpha} |\chi_\omega\zeta|^2 \,\mathrm{d}x\mathrm{d}t \right). \tag{3.9}$$

From the above result, one can obtain a similar estimate with weights depending only on time. We recall that ρ_0 and ρ are defined in (3.1) and (3.2). We have that $\rho_0, \rho \in W^{1,\infty}(0,T) \cap C^0([0,T])$ and

$$\rho_0 \leqslant \rho \leqslant 1, \qquad \left| \left(\frac{\rho_0}{\rho}\right)' \right| \lesssim 1. \tag{3.10}$$

Moreover, we have the following instrumental estimates

$$m_1 < m_2 \Rightarrow \left(\rho^{m_2} \leqslant \rho^{m_1},\ |(\rho^{m_2})'| \lesssim \rho^{m_1}\right). \tag{3.11}$$

With the above notation, we can state the following corollary of Theorem 3.2.

Corollary 3.3. *Assume $r > 1$. Then, there exist $m_0, M_0 \in \mathbb{R}_+^*$ with*

$$m_0 < M_0 < r m_0, \tag{3.12}$$

such that for any $\zeta \in \mathcal{X}^2$ with $\zeta = 0$ on $(0,T)\times\partial\Omega$ the following relation holds

$$\|\zeta(0,\cdot)\|_{L^2(\Omega)} + \left\|\rho^{M_0}\zeta\right\|_{L^2(0,T;L^2(\Omega))} \\ \lesssim \|\rho^{m_0}(\partial_t\zeta + \Delta\zeta)\|_{L^2(0,T;L^2(\Omega))} + \left\|\rho_0^{m_0}\zeta\chi_\omega\right\|_{L^2(0,T;L^2(\Omega))}. \tag{3.13}$$

We want to highlight the fact that the dependence in space of the Carleman weights appearing in (3.9) has been removed in (3.13). Moreover, it is worth mentioning that the vanishing property at $t = T$ of the Carleman weights for the left-hand-side of (3.9) has been dropped. This is why one can make appeared the first left-hand-side of (3.13), that is the classical left-hand-side term for proving a L^2 observability inequality. The same remark applies for the first right-hand-side term of (3.9) to get the first right-hand-side term of (3.13). Finally, the fact that m_0 and M_0 are comparable is quantified in (3.12).

Proof of Corollary 3.3. We consider s_0 and λ_0 from Theorem 3.2. Then, we deduce from (3.7) and (3.8) that for any $\lambda \geqslant \lambda_0$, $s \geqslant s_0(T + T^2)$,

$$1 \lesssim s^3\lambda^4\xi^3 \lesssim \left(s\lambda^2\xi\right)^3 \lesssim e^{s\lambda^2 \frac{e^{3\lambda}}{t(T-t)}}.$$

Therefore combining these estimates with (3.7) and (3.8) and taking $s = s_0(T + T^2)$, we deduce that

$$\rho_0^{M_0} \lesssim s^3\lambda^4\xi^3 e^{-s\alpha}, \quad e^{-s\alpha} \lesssim s^3\lambda^4\xi^3 e^{-s\alpha} \lesssim \rho_0^{m_0},$$

with

$$M_0 := s_0\left(T + T^2\right)\left(e^{4\lambda} - e^{2\lambda}\right), \quad m_0 := s_0\left(T + T^2\right)\left(e^{4\lambda} - e^{3\lambda}\left(1 + \lambda^2\right)\right).$$

We now fix $\lambda = \lambda_0$ large enough, so that (3.12) holds. Applying (3.9), we obtain

$$\left\|\rho_0^{M_0}\zeta\right\|_{L^2\left(0,T;L^2(\Omega)\right)} \lesssim \left\|\rho_0^{m_0}\left(\partial_t\zeta + \Delta\zeta\right)\right\|_{L^2\left(0,T;L^2(\Omega)\right)} + \left\|\rho_0^{m_0}\zeta\chi_\omega\right\|_{L^2\left(0,T;L^2(\Omega)\right)}. \tag{3.14}$$

Using (3.1) and (3.2), the above relation yields

$$\left\|\rho^{M_0}\zeta\right\|_{L^2\left(T/2,T;L^2(\Omega)\right)} \lesssim \left\|\rho_0^{m_0}\left(\partial_t\zeta + \Delta\zeta\right)\right\|_{L^2\left(0,T;L^2(\Omega)\right)} + \left\|\rho_0^{m_0}\zeta\chi_\omega\right\|_{L^2\left(0,T;L^2(\Omega)\right)}. \tag{3.15}$$

Let us consider $\chi_T \in C^\infty([0,T])$, $\chi_T \equiv 1$ in $[0,T/2]$, $\chi_T \equiv 0$ in $[3T/4,T]$ and $\left|\chi_T'\right| \lesssim 1/T$. Then

$$\begin{cases} -\partial_t\left(\chi_T\zeta\right) - \Delta\left(\chi_T\zeta\right) = -\left(\chi_T\right)'\zeta - \chi_T\left(\partial_t\zeta + \Delta\zeta\right) & \text{in } (0,T)\times\Omega, \\ \left(\chi_T\zeta\right) = 0 & \text{on } (0,T)\times\partial\Omega, \\ \left(\chi_T\zeta\right)(T,\cdot) = 0 & \text{in } \Omega. \end{cases} \tag{3.16}$$

By using the maximal regularity of the heat equation in L^2 i.e. Theorem 2.3 with $p = 2$ to (3.16) and the Sobolev embedding (2.4) we deduce

$$\|\zeta(0,\cdot)\|_{L^2(\Omega)} + \|\zeta\|_{L^2(0,T/2;L^2(\Omega))} \lesssim \|\partial_t\zeta + \Delta\zeta\|_{L^2(0,3T/4;L^2(\Omega))} + \|\zeta\|_{L^2(T/2,3T/4;L^2(\Omega))},$$

and thus by using that $\rho(t) = \rho(T/2)$ in $(0, T/2)$ and $\rho(t) \geqslant \rho(3T/4)$ in $(0, 3T/4)$, we obtain

$$\|\zeta(0,\cdot)\|_{L^2(\Omega)} + \left\|\rho^{M_0}\zeta\right\|_{L^2(0,T/2;L^2(\Omega))} \lesssim \|\rho^{m_0}(\partial_t\zeta + \Delta\zeta)\|_{L^2(0,T;L^2(\Omega))} + \left\|\rho^{M_0}\zeta\right\|_{L^2(T/2,T;L^2(\Omega))}.$$

Combining this last estimate with (3.15) and (3.10), we deduce the expected observability inequality (3.13). □

3.2. A weighted L^p observability inequality

The goal of this part is to deduce from the weighted L^2 observability inequality in Corollary 3.3 a weighted L^p observability inequality for $p \geqslant 2$, by applying maximal regularity results for the heat equation. More precisely, we show the following result:

Proposition 3.4. *Assume $p \geqslant 2$ and $r \in (1, p')$. Then, there exist $m_0, m_1 \in \mathbb{R}_+^*$ with*

$$m_0 < m_1 < rm_0, \tag{3.17}$$

such that for any $\zeta \in \mathcal{X}^p$ with $\zeta = 0$ on $(0,T) \times \partial\Omega$, the following relation holds

$$\|\zeta(0,\cdot)\|_{L^p(\Omega)} + \|\rho^{m_1}\zeta\|_{L^p(0,T;L^p(\Omega))} \lesssim \|\rho^{m_0}(\partial_t\zeta + \Delta\zeta)\|_{L^p(0,T;L^p(\Omega))} + \left\|\rho_0^{m_0}\zeta\chi_\omega\right\|_{L^p(0,T;L^p(\Omega))}. \tag{3.18}$$

The main difference between (3.18) and (3.13) is the L^p framework. We want to highlight that M_0 of (3.12) has been transformed into m_1 of (3.17). Basically, the proof is as follows. By a bootstrap argument, we apply recursively maximal regularity results in L^r, starting from $r = 2$ together with Sobolev embeddings to obtain (3.18). During the induction process, M_0 becomes $M_1 \in (M_0, rm_0)$ then $M_2 \in (M_1, rm_0)$, etc. to finally take the value $m_1 \in (m_0, rm_0)$.

Proof. First, we apply Corollary 3.3 to obtain $m_0, M_0 \in \mathbb{R}_+^*$ satisfying (3.12) and such that (3.13) holds for any $\zeta \in \mathcal{X}^2$ with $\zeta = 0$ on $(0,T) \times \partial\Omega$. We then set $g := -\partial_t\zeta - \Delta\zeta$

so that for any $M_1 > 0$,

$$\begin{cases} -\partial_t \left(\rho^{M_1}\zeta\right) - \Delta\left(\rho^{M_1}\zeta\right) = -\left(\rho^{M_1}\right)'\zeta + \rho^{M_1} g & \text{in } (0,T)\times\Omega, \\ \left(\rho^{M_1}\zeta\right) = 0 & \text{on } (0,T)\times\partial\Omega, \\ \left(\rho^{M_1}\zeta\right)(T,\cdot) = 0 & \text{in } \Omega. \end{cases} \tag{3.19}$$

In particular, if we consider $M_1 \in (M_0, rm_0)$ then by (3.12) and (3.11)

$$\left|\left(\rho^{M_1}\right)'\right| \lesssim \rho^{M_0}, \quad \rho^{M_1} \leqslant \rho^{m_0},$$

so that

$$\begin{aligned} &\left\|\left(\rho^{M_1}\right)'\zeta\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\rho^{M_1} g\right\|_{L^2(0,T;L^2(\Omega))} \\ &\qquad \lesssim \left\|\rho^{M_0}\zeta\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\rho^{m_0} g\right\|_{L^2(0,T;L^2(\Omega))}. \end{aligned} \tag{3.20}$$

We can apply the maximal regularity result in L^2, i.e. Theorem 2.3 with $p = 2$ to (3.19), and use (3.20) and the L^2 observability inequality (3.13) to deduce

$$\left\|\rho^{M_1}\zeta\right\|_{\mathcal{X}^2} \lesssim \left\|\rho^{m_0} g\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\rho_0^{m_0}\zeta\chi_\omega\right\|_{L^2(0,T;L^2(\Omega))}. \tag{3.21}$$

We then use the Sobolev embedding (2.2) to deduce

$$\mathcal{X}^2 \hookrightarrow L^{q_1}(0,T;L^{q_1}(\Omega)) \tag{3.22}$$

with $q_1 \geqslant 2$ defined by

$$\text{if} \quad \frac{1}{2} - \frac{2}{2+d} \leqslant \frac{1}{p} \quad \text{then } q_1 = p, \quad \text{else} \quad \frac{1}{q_1} = \frac{1}{2} - \frac{2}{2+d}.$$

Then, we consider $M_2 \in (M_1, rm_0)$ so that from (3.11),

$$\left|\left(\rho^{M_2}\right)'\right| \lesssim \rho^{M_1}, \quad \rho^{M_2} \leqslant \rho^{m_0},$$

and with (3.22) and (3.21), we deduce

$$\begin{aligned} &\left\|\left(\rho^{M_2}\right)'\zeta\right\|_{L^{q_1}(0,T;L^{q_1}(\Omega))} + \left\|\rho^{M_2} g\right\|_{L^{q_1}(0,T;L^{q_1}(\Omega))} \\ &\qquad \lesssim \left\|\rho^{M_1}\zeta\right\|_{\mathcal{X}^2} + \left\|\rho^{m_0} g\right\|_{L^{q_1}(0,T;L^{q_1}(\Omega))} \\ &\qquad \lesssim \left\|\rho^{m_0} g\right\|_{L^{q_1}(0,T;L^{q_1}(\Omega))} + \left\|\rho_0^{m_0}\zeta\chi_\omega\right\|_{L^2(0,T;L^2(\Omega))}. \end{aligned} \tag{3.23}$$

Now we apply Theorem 2.3 to

$$\begin{cases} -\partial_t \left(\rho^{M_2}\zeta\right) - \Delta \left(\rho^{M_2}\zeta\right) = -\left(\rho^{M_2}\right)' \zeta + \rho^{M_2} g & \text{in } (0,T)\times\Omega, \\ \left(\rho^{M_2}\zeta\right) = 0 & \text{on } (0,T)\times\partial\Omega, \\ \left(\rho^{M_2}\zeta\right)(T,\cdot) = 0 & \text{in } \Omega, \end{cases} \tag{3.24}$$

with $p = q_1$, and using (3.23), we obtain

$$\left\|\rho^{M_2}\zeta\right\|_{\mathcal{X}^{q_1}} \lesssim \|\rho^{m_0} g\|_{L^{q_1}(0,T;L^{q_1}(\Omega))} + \left\|\rho_0^{m_0}\zeta\chi_\omega\right\|_{L^2\left(0,T;L^2(\Omega)\right)}. \tag{3.25}$$

If $q_1 = p$, then using $H^1(0,T) \hookrightarrow C^0([0,T])$ and $L^p(0,T;L^p(\Omega)) \hookrightarrow L^2(0,T;L^2(\Omega))$, we deduce from the above relation the desired observability inequality (3.18) with $m_1 = M_2$. Else, we have $q_1 < p$ and we can repeat the argument, that is we use the Sobolev embedding (2.2) to deduce

$$\mathcal{X}^{q_1} \hookrightarrow L^{q_2}(0,T;L^{q_2}(\Omega)) \tag{3.26}$$

with $q_2 \geqslant q_1$ defined by

$$\text{if} \quad \frac{1}{q_1} - \frac{2}{2+d} \leqslant \frac{1}{p} \quad \text{then } q_2 = p, \quad \text{else} \quad \frac{1}{q_2} = \frac{1}{q_1} - \frac{2}{2+d} = \frac{1}{2} - 2\cdot\frac{2}{2+d}.$$

Taking $M_3 \in (M_2, rm_0)$, and proceeding as above, applying Theorem 2.3 with $p = q_2$ and using (3.26) and (3.25), we find

$$\left\|\rho^{M_3}\zeta\right\|_{\mathcal{X}^{q_2}} \lesssim \|\rho^{m_0} g\|_{L^{q_2}(0,T;L^{q_2}(\Omega))} + \left\|\rho_0^{m_0}\zeta\chi_\omega\right\|_{L^2\left(0,T;L^2(\Omega)\right)}.$$

We can proceed by induction and since $1/q_n$ decrease by $2/(2+d)$ at each step, after a finite number of steps, we obtain $q_n = p$ and we deduce the desired observability inequality (3.18). □

3.3. Controllability of the heat equation with a source term in $L^{p'}$

We use the above observability results to show, by a duality argument, the controllability of a linear system associated with (1.13):

$$\begin{cases} \partial_t y - \Delta y = h\chi_\omega + F & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,\cdot) = y_0 & \text{in } \Omega. \end{cases} \tag{3.27}$$

In order to control the above system, we fix $p \in 2\mathbb{N}^*$ and we consider m_0 and m_1 as in Proposition 3.4. Then, we introduce

$$\mathcal{Y}_0 := \left\{\zeta \in C^\infty([0,T]\times\overline{\Omega})\ ;\ \zeta = 0 \text{ on } (0,T)\times\partial\Omega\right\}, \tag{3.28}$$

and we define the following norm for $\zeta \in \mathcal{Y}_0$,

$$\|\zeta\|_{\mathcal{Y}} := \|\rho^{m_0}(\partial_t \zeta + \Delta\zeta)\|_{L^p(0,T;L^p(\Omega))} + \left\|\rho_0^{m_0}\zeta\chi_\omega\right\|_{L^p(0,T;L^p(\Omega))}. \tag{3.29}$$

The fact that it is a norm is a consequence of the weighted L^p observability inequality (3.18). We denote by $\mathcal{Y}$ the completion of $\mathcal{Y}_0$ with respect to the norm $\|\cdot\|_{\mathcal{Y}}$.

First, we have the following result that roughly states that a function $\zeta \in \mathcal{Y}$ belongs to some suitable weighted $\mathcal{X}^p$ spaces.

Lemma 3.5. *Assume $m > m_1$. Then, for any $\zeta \in \mathcal{Y}$,*

$$\left\|\rho_0^m \zeta\right\|_{\mathcal{X}^p} \lesssim \|\rho^m \zeta\|_{\mathcal{X}^p} \lesssim \|\zeta\|_{\mathcal{Y}}. \tag{3.30}$$

Proof. Using $m > m_1$, (3.17) and (3.11), we have

$$\left|(\rho^m)'\right| \lesssim \rho^{m_1}, \quad \rho^m \lesssim \rho^{m_0}. \tag{3.31}$$

Now, if $\zeta \in \mathcal{Y}$, then

$$\begin{cases} -\partial_t\left(\rho^m\zeta\right) - \Delta\left(\rho^m\zeta\right) = -\left(\rho^m\right)'\zeta + \rho^m\left(-\partial_t\zeta - \Delta\zeta\right) & \text{in } (0,T)\times\Omega, \\ \left(\rho^m\zeta\right) = 0 & \text{on } (0,T)\times\partial\Omega, . \\ \left(\rho^m\zeta\right)(T,\cdot) = 0 & \text{in } \Omega. \end{cases} \tag{3.32}$$

Combining the observability inequality (3.18) and (3.31), we deduce

$$\left\|-\left(\rho^m\right)'\zeta + \rho^m\left(-\partial_t\zeta - \Delta\zeta\right)\right\|_{L^p(0,T;L^p(\Omega))} \lesssim \|\zeta\|_{\mathcal{Y}}.$$

Applying the maximal regularity result Theorem 2.3 on (3.32) and using the above relation, we deduce the second estimate in (3.30). For the first estimate, we use (3.10) to obtain that $\|\rho_0/\rho\|_{W^{1,\infty}(0,T)} \lesssim 1$ and this allows us to conclude the proof. □

We now introduce some functional spaces: for $m > 0$ and $p \in [1,\infty]$, we set

$$L^p_m(0,T;L^p(\Omega)) := \left\{ f \in L^p(0,T;L^p(\Omega))\ ;\ \frac{f}{\rho^m} \in L^p(0,T;L^p(\Omega)) \right\}, \tag{3.33}$$

$$L^p_{m,0}(0,T;L^p(\Omega)) := \left\{ f \in L^p(0,T;L^p(\Omega))\ ;\ \frac{f}{\rho_0^m} \in L^p(0,T;L^p(\Omega)) \right\}, \tag{3.34}$$

endowed with the following norm

$$\begin{aligned} \|f\|_{L^p_m(0,T;L^p(\Omega))} &:= \left\|\frac{f}{\rho^m}\right\|_{L^p(0,T;L^p(\Omega))}, \\ \|f\|_{L^p_{m,0}(0,T;L^p(\Omega))} &:= \left\|\frac{f}{\rho_0^m}\right\|_{L^p(0,T;L^p(\Omega))}. \end{aligned} \tag{3.35}$$

From now on, we assume $p \in 2\mathbb{N}^*$. This assumption allows us to use, in what follows, that $|x|^p = x^p$ for $x \in \mathbb{R}$. Let us consider, for any $y_0 \in L^{p'}(\Omega)$ and $F \in L^{p'}_{m_1}(0,T;L^{p'}(\Omega))$, the functional $J = J_{y_0,F}$ defined as follows:

$$J(\zeta) := \frac{1}{p}\iint_{(0,T)\times\Omega} \rho^{m_0 p}(-\partial_t\zeta - \Delta\zeta)^p \mathrm{d}t\mathrm{d}x + \frac{1}{p}\iint_{(0,T)\times\Omega} \rho_0^{m_0 p}\zeta^p \chi_\omega^p \mathrm{d}t\mathrm{d}x - \iint_{(0,T)\times\Omega} F\zeta \mathrm{d}t\mathrm{d}x - \int_\Omega y_0(x)\zeta(0,x)\mathrm{d}x. \quad (3.36)$$

Using the L^p observability inequality (3.18), we can check that $J \in C^1(\mathcal{Y};\mathbb{R})$ is a strictly convex and coercive functional on $\mathcal{Y}$. In particular, J admits a unique minimum $\overline{\zeta}$. We can thus define, for $y_0 \in L^{p'}(\Omega)$ and $F \in L^{p'}_{m_1}(0,T;L^{p'}(\Omega))$, the following maps

$$\begin{aligned} \mathcal{M}_1(y_0,F) &:= \overline{\zeta}, \\ \mathcal{M}_2(y_0,F) &:= \rho^{m_0 p}\left(-\partial_t\overline{\zeta} - \Delta\overline{\zeta}\right)^{p-1}, \\ \mathcal{M}_3(y_0,F) &:= -\rho_0^{m_0 p}\chi_\omega^{p-1}\overline{\zeta}^{p-1}. \end{aligned} \quad (3.37)$$

Proposition 3.6. *Assume $p \in 2\mathbb{N}^*$ and $r \in (1,p')$ and let us consider m_0 and m_1 given by Proposition 3.4. For any $y_0 \in L^{p'}(\Omega)$ and $F \in L^{p'}_{m_1}(0,T;L^{p'}(\Omega))$, let us set*

$$\overline{\zeta} = \mathcal{M}_1(y_0,F), \quad y = \mathcal{M}_2(y_0,F), \quad h = \mathcal{M}_3(y_0,F). \quad (3.38)$$

(1) ***Existence of a solution.*** *We have that $y \in L^{p'}_{m_0}(0,T;L^{p'}(\Omega))$ and $h \in L^{p'}_{m_0,0}(0,T;L^{p'}(\Omega))$, together with the estimates*

$$\left\|\overline{\zeta}\right\|_{\mathcal{Y}}^p \lesssim \|F\|^{p'}_{L^{p'}_{m_1}(0,T;L^{p'}(\Omega))} + \|y_0\|^{p'}_{L^{p'}(\Omega)}, \quad (3.39)$$

$$\|y\|_{L^{p'}_{m_0}(0,T;L^{p'}(\Omega))} + \|h\|_{L^{p'}_{m_0,0}(0,T;L^{p'}(\Omega))} \lesssim \|F\|_{L^{p'}_{m_1}(0,T;L^{p'}(\Omega))} + \|y_0\|_{L^{p'}(\Omega)}. \quad (3.40)$$

Moreover, y is the very weak solution of (3.27) *associated with F, h and y_0 in the sense of Definition 2.2.*

(2) ***Odd behavior of the control.*** *The control h satisfies $h^{1/(p-1)} \in \mathcal{X}^p$ and*

$$h^{1/(p-1)}(0,\cdot) = h^{1/(p-1)}(T,\cdot) = 0 \quad \text{in } \Omega, \quad (3.41)$$

together with the estimate

$$\left\|h^{1/(p-1)}\right\|_{\mathcal{X}^p} \lesssim \|F\|^{1/(p-1)}_{L^{p'}_{m_1}(0,T;L^{p'}(\Omega))} + \|y_0\|^{1/(p-1)}_{L^{p'}(\Omega)}. \quad (3.42)$$

(3) ***Regularity of the solution.*** *Assume that* $y_0 \in W^{2/p,p'}(\Omega)$ *and that* $y_0 = 0$ *on* $\partial\Omega$ *if* $p = 2$*. Then for any* $m < m_0$*,* $y/\rho^m \in \mathcal{X}^{p'}$ *together with the estimate*

$$\left\| \frac{y}{\rho^m} \right\|_{\mathcal{X}^{p'}} \lesssim \|F\|_{L^{p'}_{m_1}(0,T;L^{p'}(\Omega))} + \|y_0\|_{W^{2/p,p'}(\Omega)} . \tag{3.43}$$

In particular, $y(T,\cdot) = 0$*.*

The first point will be obtained from Euler–Lagrange equation. The odd behavior of the control, i.e. (3.42), remarking that $p-1$ is odd, comes from the identification of h in (3.38), (3.37) and from a weighted $\mathcal{X}^p$ estimate of $\overline{\zeta}$. Finally, the regularity result on the solution comes from a maximal parabolic regularity result. Note that if $p \neq 2$, then $p \geqslant 4$ and $p' < 3/2$ so that we do not need to impose the compatibility condition $y_0 = 0$ on $\partial\Omega$.

Proof of Proposition 3.6. We start by writing the Euler–Lagrange equation for J at $\overline{\zeta}$ to obtain

$$\iint_{(0,T)\times\Omega} \rho^{m_0 p}(-\partial_t\overline{\zeta} - \Delta\overline{\zeta})^{p-1}(-\partial_t\zeta - \Delta\zeta)\mathrm{d}t\mathrm{d}x + \iint_{(0,T)\times\Omega} \rho_0^{m_0 p}\chi_\omega^p\overline{\zeta}^{p-1}\zeta\mathrm{d}t\mathrm{d}x$$
$$= \iint_{(0,T)\times\Omega} F\zeta\mathrm{d}t\mathrm{d}x + \int_\Omega y_0(x)\zeta(0,x)\mathrm{d}x \quad (\zeta \in \mathcal{Y}). \tag{3.44}$$

Taking $\zeta = \overline{\zeta}$ in the above relation and using Young's inequality and the L^p observability inequality (3.18), we deduce (3.39).

Then, (3.38) and (3.37) yield

$$\left|\frac{y}{\rho^{m_0}}\right|^{p'} = \rho^{m_0 p}\left|\partial_t\overline{\zeta} + \Delta\overline{\zeta}\right|^p, \quad \left|\frac{h}{\rho_0^{m_0}}\right|^{p'} = \left|\rho_0^{m_0}\chi_\omega\overline{\zeta}\right|^p,$$

and we deduce (3.40) from (3.39).

Moreover, (3.44) and (3.38) imply

$$\iint_{(0,T)\times\Omega} y(-\partial_t\zeta - \Delta\zeta)\mathrm{d}t\mathrm{d}x = \iint_{(0,T)\times\Omega} \chi_\omega h\zeta\mathrm{d}t\mathrm{d}x + \iint_{(0,T)\times\Omega} F\zeta\mathrm{d}t\mathrm{d}x$$
$$+ \int_\Omega y_0(x)\zeta(0,x)\mathrm{d}x \quad \left(\zeta \in C_c^\infty([0,T)\times\Omega)\right), \tag{3.45}$$

that is y is the very weak solution to (3.27) associated with the control h, F and y_0 in the sense of Definition 2.2.

For the second point, from (3.38) and (3.37), we have

$$h^{1/(p-1)} = -\rho_0^{p' m_0}\overline{\zeta}\chi_\omega, \tag{3.46}$$

and since $p'm_0 > m_1$, we can apply Lemma 3.5 with $m = p'm_0$ and we deduce (3.42) from (3.39) and (3.30). Since $p'm_0 > m_1$, there exists $r > 0$ such that $p'm_0 - r > m_1$, we then obtain (3.41) because

$$\left\|\rho_0^{-r} h^{1/(p-1)}\right\|_{C\left([0,T];W^{2/p',p}(\Omega)\right)} \lesssim \left\|\rho_0^{-r} h^{1/(p-1)}\right\|_{\mathcal{X}^p} \lesssim \left\|\rho_0^{p'm_0-r}\overline{\zeta}\right\|_{\mathcal{X}^p} \lesssim \|\zeta\|_{\mathcal{Y}}.$$

Finally, for the last point, we write the system satisfied by y/ρ^m:

$$\begin{cases} \partial_t\left(\dfrac{y}{\rho^m}\right) - \Delta\left(\dfrac{y}{\rho^m}\right) = \dfrac{h}{\rho^m}\chi_\omega + \dfrac{F}{\rho^m} - m\dfrac{\rho'}{\rho^{m+1}}y & \text{in } (0,T)\times\Omega, \\ \dfrac{y}{\rho^m} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \dfrac{y}{\rho^m}(0,\cdot) = \dfrac{y_0}{\rho^m(0)} & \text{in } \Omega. \end{cases} \tag{3.47}$$

By using $m < m_0 < m_1$, (3.10), (3.11) and (3.40), we have

$$\left\|\frac{h}{\rho^m}\chi_\omega + \frac{F}{\rho^m} - m\frac{\rho'}{\rho^{m+1}}y\right\|_{L^{p'}\left(0,T;L^{p'}(\Omega)\right)} \lesssim \|F\|_{L^{p'}_{m_1}\left(0,T;L^{p'}(\Omega)\right)} + \|y_0\|_{L^{p'}(\Omega)}$$

Applying Theorem 2.3 to (3.47) with the above estimate, we deduce the regularity estimate on y, i.e. (3.43). □

3.4. L^∞ bound on the control and L^q estimate of the nonlinearity

From now on, we assume $r \in (1, p')$ and we assume that m_0 and m_1 are given by Proposition 3.4 with this r. In particular they satisfy (3.17) which yields

$$0 < m_0 p - m_1(p-1) < m_0.$$

First we have the following result on the control h.

Lemma 3.7. *Assume p satisfies* (3.3) *and* (3.4). *Then for any*

$$0 \leqslant m < m_0 p - m_1(p-1), \tag{3.48}$$

the control h given by (3.38) *satisfies $h^{1/(2n+1)} \in \mathcal{X}^p$ and $h \in L^\infty_{m,0}(0,T;L^\infty(\Omega))$ with the estimate*

$$\|h\|_{L^\infty_{m,0}(0,T;L^\infty(\Omega))} + \left\|\left(\frac{h}{\rho_0^m}\right)^{1/(2n+1)}\right\|_{\mathcal{X}^p}^{2n+1} \lesssim \|F\|_{L^{p'}_{m_1}\left(0,T;L^{p'}(\Omega)\right)} + \|y_0\|_{L^{p'}(\Omega)}. \tag{3.49}$$

In the above result, p has to be sufficiently large to get that $\mathcal{X}^p$ is an algebra, and this enables us to get that $h^{1/(2n+1)}$ is sufficiently smooth, because $p-1 = (2n+1)(2k+1)$, as expected in (1.16).

Proof of Lemma 3.7. Since p satisfies (3.4), we can apply Lemma 2.1 and deduce that $\mathcal{X}^p$ is an algebra. On the other hand, from Proposition 3.6, $h^{1/(p-1)} \in \mathcal{X}^p$, we can thus conclude by using that

$$h^{1/(2n+1)} = \left(h^{1/(p-1)}\right)^{2k+1}.$$

Now, from (3.38) and (3.37), we can write

$$\frac{h}{\rho_0^m} = -\left(\rho_0^{(m_0 p-m)/(p-1)} \chi_\omega \bar{\zeta}\right)^{p-1}. \tag{3.50}$$

If m satisfies (3.48), then $(m_0 p - m)/(p-1) > m_1$, we can apply Lemma 3.5 and use (3.50), (3.10), (3.39) to obtain

$$\left\|\frac{h}{\rho_0^m}\right\|_{\mathcal{X}^p} \lesssim \left\|\left(\rho_0^{(m_0 p-m)/(p-1)} \chi_\omega \bar{\zeta}\right)^{p-1}\right\|_{\mathcal{X}^p} \lesssim \left\|\rho_0^{(m_0 p-m)/(p-1)} \chi_\omega \bar{\zeta}\right\|_{\mathcal{X}^p}^{p-1}$$
$$\lesssim \left\|\bar{\zeta}\right\|_{\mathcal{Y}}^{p-1} \lesssim \|F\|_{L^{p'}_{m_1}(0,T;L^{p'}(\Omega))} + \|y_0\|_{L^{p'}(\Omega)}.$$

We obtain (3.49) by using that $\mathcal{X}^p$ is an algebra and (2.2). □

Proposition 3.8. *Let $N \in \mathbb{N}^*$, $N \geqslant 2$ and assume p, q satisfying $q \geqslant p'$,* (2.6)*,* (3.3) *and* (3.4)*. Let us consider m_0 and m_1 given by Proposition 3.4 with*

$$r := \frac{p}{p-1+\frac{1}{N}} \in (1, p'). \tag{3.51}$$

For any $y_0 \in W_0^{\frac{2}{q'},q}(\Omega)$ and $F \in L^q_{m_1}(0,T;L^q(\Omega))$, and for any m satisfying (3.48)*, y defined by* (3.38) *satisfies $y/\rho^m \in \mathcal{X}^q$ and $y^N \in L^q_{m_1}(0,T;L^q(\Omega))$ with the estimates*

$$\left\|\frac{y}{\rho^m}\right\|_{\mathcal{X}^q} \lesssim \|F\|_{L^q_{m_1}(0,T;L^q(\Omega))} + \|y_0\|_{W^{2/q',q}(\Omega)}, \tag{3.52}$$

$$\left\|y^N\right\|_{L^q_{m_1}(0,T;L^q(\Omega))} \lesssim \left(\|F\|_{L^q_{m_1}(0,T;L^q(\Omega))} + \|y_0\|_{W^{2/q',q}(\Omega)}\right)^N. \tag{3.53}$$

The goal of the above result is to get an appropriate L^q bound on the nonlinearity, this would be a first step in order to prove the local null-controllability of the semi-linear heat equation.

Proof. We define q_1 as follows

$$\text{if} \quad \frac{1}{q} \leqslant \frac{1}{p'} - \frac{2}{d+2}, \text{ then } q_1 = q, \text{ else } \frac{1}{q_1} = \frac{1}{p'} - \frac{2}{d+2}. \tag{3.54}$$

In both cases, we have $q \geqslant q_1$ and $1/q' \geqslant 1/q_1'$.

We deduce from (3.43) and the Sobolev embedding (2.2) that for any $\widetilde{m} < m_0$,

$$\|y\|_{L^{q_1}_{\widetilde{m}}(0,T;L^{q_1}(\Omega))} \lesssim \|F\|_{L^{p'}_{m_1}(0,T;L^{p'}(\Omega))} + \|y_0\|_{W^{2/p,p'}(\Omega)}. \tag{3.55}$$

We then consider m satisfying (3.48). We have in particular $m < m_0 < m_1$ and we can write

$$\begin{cases} \partial_t \left(\dfrac{y}{\rho^m}\right) - \Delta \left(\dfrac{y}{\rho^m}\right) = \dfrac{h}{\rho^m}\chi_\omega + \dfrac{F}{\rho^m} - m\dfrac{\rho'}{\rho^{m+1}}y & \text{in } (0,T)\times\Omega, \\ \dfrac{y}{\rho^m} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \dfrac{y}{\rho^m}(0,\cdot) = \dfrac{y_0}{\rho^m(0)} & \text{in } \Omega. \end{cases}$$

Applying Theorem 2.3 on the above equation and using (3.49) and (3.55) with $\widetilde{m} \in (m, m_0)$ together with (3.10), (3.11), we deduce

$$\begin{aligned} \left\| \frac{y}{\rho^m} \right\|_{\mathcal{X}^{q_1}} &\lesssim \|h\|_{L^\infty_m(0,T;L^\infty(\Omega))} + \|F\|_{L^{q_1}_{m_1}(0,T;L^{q_1}(\Omega))} + \left\| \frac{\rho'}{\rho^{m+1}} y \right\|_{L^{q_1}(0,T;L^{q_1}(\Omega))} \\ &\lesssim \|F\|_{L^{q_1}_{m_1}(0,T;L^{q_1}(\Omega))} + \|y_0\|_{W^{2/q_1',q_1}(\Omega)} + \|y\|_{L^{q_1}_{\widetilde{m}}(0,T;L^{q_1}(\Omega))} \\ &\lesssim \|F\|_{L^{q_1}_{m_1}(0,T;L^{q_1}(\Omega))} + \|y_0\|_{W^{2/q',q}(\Omega)} . \end{aligned} \tag{3.56}$$

We can proceed by induction, using again (2.2), and since the corresponding sequence $1/q_n$ decreases by $2/(d+2)$ (see (3.54)) at each step, we obtain after a finite number of steps that for any m satisfying (3.48), we have

$$\left\| \frac{y}{\rho^m} \right\|_{\mathcal{X}^q} \lesssim \|F\|_{L^q_{m_1}(0,T;L^q(\Omega))} + \|y_0\|_{W^{2/q',q}(\Omega)} . \tag{3.57}$$

Using that q satisfies (2.6) so that the Sobolev embedding (2.7) holds, we deduce that

$$\left\| \frac{y^N}{\rho^{Nm}} \right\|_{L^q((0,T)\times\Omega)} \lesssim \left(\|F\|_{L^q_{m_1}(0,T;L^q(\Omega))} + \|y_0\|_{W^{2/q',q}(\Omega)} \right)^N . \tag{3.58}$$

From (3.17) and (3.51), we have

$$\frac{m_1}{N} < m_0 p - m_1(p-1)$$

so that we can take $m = m_1/N$ in (3.57), (3.58) and we deduce (3.53). □

3.5. A Schauder fixed-point argument

Let us consider the hypotheses of Proposition 3.8 and assume $y_0 \in W_0^{2/q',q}(\Omega)$. Then, using the conclusion of Proposition 3.8, we can define the mapping

$$\mathcal{N} : L^q_{m_1}(0,T;L^q(\Omega)) \longrightarrow L^q_{m_1}(0,T;L^q(\Omega)), \quad F \longmapsto y^N, \tag{3.59}$$

where $y = \mathcal{M}_2(y_0, F)$. Moreover, using (3.53), we deduce that if $R_0 := \|y_0\|_{W^{2/q',q}(\Omega)}$ is small enough, then the closed set

$$B_{R_0} := \left\{F \in L^q_{m_1}(0,T;L^q(\Omega)) \ ; \ \|F\|_{L^q_{m_1}(0,T;L^q(\Omega))} \leqslant R_0\right\} \tag{3.60}$$

is invariant by $\mathcal{N}$.

Proposition 3.9. *The mapping $\mathcal{N} : B_{R_0} \to B_{R_0}$ defined above is continuous and $\mathcal{N}(B_{R_0})$ is relatively compact into B_{R_0}.*

Proof. Let us consider a sequence $(F_n)_n$ of B_{R_0}. We write $y_n = \mathcal{M}_2(y_0, F_n)$. Then we can use (3.52) to obtain that $\left(y_n/\rho^{(m_1/N)}\right)_n$ is bounded in $\mathcal{X}^q$. Applying Lemma 2.1, we deduce that, up to a subsequence,

$$\frac{y_n}{\rho^{(m_1/N)}} \longrightarrow \frac{y}{\rho^{(m_1/N)}} \quad \text{in } L^{qN}((0,T)\times\Omega),$$

for some $y \in L^{qN}_{m_1/N}(0,T;L^{qN}(\Omega))$. We deduce that $\mathcal{N}(B_{R_0})$ is relatively compact into B_{R_0}.

To show the continuity of $\mathcal{N}$, we consider $F_1, F_2 \in B_{R_0}$ and we write (see (3.37) and (3.38)) for $i = 1, 2$,

$$\overline{\zeta}_i := \mathcal{M}_1(y_0, F_i), \quad y_i := \mathcal{M}_2(y_0, F_i), \quad h_i := \mathcal{M}_3(y_0, F_i).$$

From the Euler–Lagrange equation (3.44) for J_{y_0,F_1} and J_{y_0,F_2}, we deduce

$$\iint_{(0,T)\times\Omega} \rho^{m_0 p}\left[\left(-\partial_t\overline{\zeta}_1 - \Delta\overline{\zeta}_1\right)^{p-1} - \left(-\partial_t\overline{\zeta}_2 - \Delta\overline{\zeta}_2\right)^{p-1}\right](-\partial_t\zeta - \Delta\zeta)\mathrm{d}t\mathrm{d}x$$
$$+ \iint_{(0,T)\times\Omega} \rho_0^{m_0 p}\chi_\omega^p\left(\overline{\zeta}_1^{p-1} - \overline{\zeta}_2^{p-1}\right)\zeta\mathrm{d}t\mathrm{d}x$$
$$= \iint_{(0,T)\times\Omega} (F_1 - F_2)\,\zeta\mathrm{d}t\mathrm{d}x \quad (\zeta \in \mathcal{Y}). \tag{3.61}$$

In the above relation, we take $\zeta = \overline{\zeta}_1 - \overline{\zeta}_2$ in the above relation and we combine it with the observability inequality (3.18) and with the relation

$$(x_1 - x_2)^p \lesssim \left(x_1^{p-1} - x_2^{p-1}\right)(x_1 - x_2) \quad (x_1, x_2 \in \mathbb{R}),$$

to deduce

$$\left\|\overline{\zeta}_1 - \overline{\zeta}_2\right\|_{\mathcal{Y}}^p \lesssim \|F_1 - F_2\|^{p'}_{L^{p'}_{m_1}(0,T;L^{p'}(\Omega))}. \tag{3.62}$$

Moreover, using that

$$\left|x_1^{p-1} - x_2^{p-1}\right| \lesssim |x_1 - x_2|\left(|x_1|^{p-2} + |x_2|^{p-2}\right) \quad (x_1, x_2 \in \mathbb{R}),$$

we obtain from (3.10)

$$\left|\frac{h_1-h_2}{\rho^m}\right| = \left(\frac{\rho_0}{\rho}\right)^m \left|\left(\rho^{(m_0p-m)/(p-1)}\chi_\omega\overline{\zeta}_1\right)^{p-1} - \left(\rho^{(m_0p-m)/(p-1)}\chi_\omega\overline{\zeta}_2\right)^{p-1}\right|$$
$$\lesssim \left|\rho^{(m_0p-m)/(p-1)}\left(\overline{\zeta}_1-\overline{\zeta}_2\right)\right|\left(\left(\rho^{(m_0p-m)/(p-1)}\overline{\zeta}_1\right)^{p-2} + \left(\rho^{(m_0p-m)/(p-1)}\overline{\zeta}_2\right)^{p-2}\right).$$

Thus, if m satisfies (3.48), the above relation combined with (3.10), (3.4) that guarantees that $\mathcal{X}^p$ is an algebra and Lemma 3.5 yield

$$\|h_1-h_2\|_{L^\infty_m(0,T;L^\infty(\Omega))}$$
$$\lesssim \left\|\rho^{(m_0p-m)/(p-1)}\left(\overline{\zeta}_1-\overline{\zeta}_2\right)\right\|_{\mathcal{X}^p}\left(\left\|\rho^{(m_0p-m)/(p-1)}\overline{\zeta}_1\right\|^{p-2}_{\mathcal{X}^p} + \left\|\rho^{(m_0p-m)/(p-1)}\overline{\zeta}_2\right\|^{p-2}_{\mathcal{X}^p}\right)$$
$$\lesssim \left\|\overline{\zeta}_1-\overline{\zeta}_2\right\|_{\mathcal{Y}}\left(\left\|\overline{\zeta}_1\right\|^{p-2}_{\mathcal{Y}} + \left\|\overline{\zeta}_2\right\|^{p-2}_{\mathcal{Y}}\right).$$

Therefore, using (3.39) and (3.62), we find

$$\|h_1-h_2\|_{L^\infty_m(0,T;L^\infty(\Omega))} \lesssim \|F_1-F_2\|^{1/(p-1)}_{L^{p'}_{m_1}(0,T;L^{p'}(\Omega))}$$
$$\times\left(\|F_1\|_{L^{p'}_{m_1}(0,T;L^{p'}(\Omega))} + \|F_2\|_{L^{p'}_{m_1}(0,T;L^{p'}(\Omega))} + \|y_0\|_{L^{p'}(\Omega)}\right)^{(p-2)/(p-1)}. \quad (3.63)$$

Note that y_1-y_2 satisfies the following system

$$\begin{cases} \partial_t\left(\frac{y_1-y_2}{\rho^m}\right) - \Delta\left(\frac{y_1-y_2}{\rho^m}\right) = \frac{h_1-h_2}{\rho^m}\chi_\omega + \frac{F_1-F_2}{\rho^m} \\ \qquad\qquad - m\frac{\rho'}{\rho^{m+1}}(y_1-y_2) & \text{in } (0,T)\times\Omega, \\ \frac{y}{\rho^m} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \frac{y}{\rho^m}(0,\cdot) = \frac{y_0}{\rho^m(0)} & \text{in } \Omega. \end{cases}$$

Now, we follow the same proof as in Proposition 3.8 and we use that $m = m_1/N$ satisfies (3.48) to deduce from (3.63) that

$$\left\|\frac{y_1-y_2}{\rho^{m_1/N}}\right\|_{\mathcal{X}^q} \lesssim \|F_1-F_2\|_{L^q_{m_1}(0,T;L^q(\Omega))}$$
$$+ R_0^{(p-2)/(p-1)}\|F_1-F_2\|^{1/(p-1)}_{L^{p'}_{m_1}(0,T;L^{p'}(\Omega))}. \quad (3.64)$$

We then write

$$\left|\frac{y_1^N-y_2^N}{\rho^{m_1}}\right| \lesssim \frac{|y_1-y_2|}{\rho^{m_1/N}}\frac{|y_1|^{N-1}+|y_2|^{N-1}}{\rho^{m_1(N-1)/N}}$$

so that from Hölder's inequality, we have

$$\left\|y_1^N - y_2^N\right\|_{L^q_{m_1}(0,T;L^q(\Omega))} \lesssim \|y_1 - y_2\|_{L^{qN}_{m_1/N}\left(0,T;L^{qN}(\Omega)\right)} \times \left(\|y_1\|^{N-1}_{L^{qN}_{m_1/N}\left(0,T;L^{qN}(\Omega)\right)} + \|y_1\|^{N-1}_{L^{qN}_{m_1/N}\left(0,T;L^{qN}(\Omega)\right)}\right).$$

Combining this relation with the Sobolev embedding (2.2), (3.52), (3.64), we deduce that

$$\begin{aligned}\left\|y_1^N - y_2^N\right\|_{L^q_{m_1}(0,T;L^q(\Omega))} &\lesssim \left(\|F_1 - F_2\|_{L^q_{m_1}(0,T;L^q(\Omega))} + R_0^{(p-2)/(p-1)}\, \|F_1 - F_2\|^{1/(p-1)}_{L^{p'}_{m_1}\left(0,T;L^{p'}(\Omega)\right)}\right) \\ &\quad \times \left(\|F_1\|_{L^q_{m_1}(0,T;L^q(\Omega))} + \|F_2\|_{L^q_{m_1}(0,T;L^q(\Omega))} + \|y_0\|_{W^{2/q',q}(\Omega)}\right)^{N-1} \end{aligned} \tag{3.65}$$

which implies the continuity of $\mathcal{N}$. □

Remark 3.10. In the above proof, let us remark that we show that the mapping $\mathcal{N} : B_{R_0} \to B_{R_0}$ is α-Hölder continuous with $\alpha = 1/(p-1)$ (see (3.65)). It is not clear if this mapping is Lipschitz continuous or if we can show that for R_0 small enough it is contractive. As a consequence, in the proof of Theorem 3.1, we do not apply the Banach fixed-point theorem (as it can be done with the method proposed in [20]) and we use instead the Schauder fixed-point theorem.

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Since $q > \max(\frac{d+2}{2}, 2)$, then we can check that $q \geqslant p'$ and satisfies (2.6). We can thus apply Proposition 3.9: if $R_0 := \|y_0\|_{W^{2/q',q}(\Omega)}$ is small enough, then the mapping $\mathcal{N} : B_{R_0} \to B_{R_0}$ defined by (3.59) is continuous, where B_{R_0} is the closed convex set defined by (3.60). Moreover, $\mathcal{N}(B_{R_0})$ is relatively compact in B_{R_0} so that we can apply the Schauder fixed point theorem to deduce the existence of a fixed point $F \in B_{R_0}$. Setting $y = \mathcal{M}_2(y_0, F)$ and $h = \mathcal{M}_3(y_0, F)$, we use Proposition 3.6, Lemma 3.7 and Proposition 3.8 and obtain that h satisfies (1.16), that y is the strong solution of (1.13) associating with h and y_0 and that for any m satisfying (3.48), $y/\rho^m \in \mathcal{X}^q$, $y^N \in L^q_{m_1}(0,T;L^q(\Omega))$, $h^{1/(2n+1)} \in \mathcal{X}^p$, $h \in L^\infty_{m,0}(0,T;L^\infty(\Omega))$ together with the estimates (3.6). □

4. **Proof of Theorem 1.1**

The goal of this part is to prove the local null-controllability of (1.3).

Proof of Theorem 1.1. As explained in the introduction, the proof is divided into two steps.

Step 1. Control of the first equation in $(0, T/2)$**.** First we apply Theorem 2.5: there exists $\widetilde{\delta} > 0$ small enough such that if

$$\|y_{2,0}\|_{L^\infty(\Omega)} \leqslant \widetilde{\delta}, \quad \|g\|_{L^\infty((0,T/2)\times\Omega)} \leqslant \widetilde{\delta}, \tag{4.1}$$

the system

$$\begin{cases} \partial_t y_2 - \Delta y_2 = y_2^{N_3} + g & \text{in } (0,T/2)\times\Omega, \\ y_2 = 0 & \text{on } (0,T/2)\times\partial\Omega, \\ y_2(0,\cdot) = y_{2,0} & \text{in } \Omega, \end{cases} \tag{4.2}$$

admits a unique weak solution in the sense of Definition 2.4.

Now we apply Theorem 1.2 to

$$\begin{cases} \partial_t y_1 - \Delta y_1 = y_1^{N_1} + h\chi_\omega & \text{in } (0,T/2)\times\Omega, \\ y_1 = 0 & \text{on } (0,T/2)\times\partial\Omega, \\ y_1(0,\cdot) = y_{1,0} & \text{in } \Omega. \end{cases} \tag{4.3}$$

There exists $\delta > 0$ such that for any $y_{1,0} \in L^\infty(\Omega)$ with

$$\|y_{1,0}\|_{L^\infty(\Omega)} \leqslant \delta, \tag{4.4}$$

there exists a control $h \in L^\infty(0,T/2;L^\infty(\Omega))$ such that $y_1(T/2,\cdot) = 0$ and

$$\|y_1\|_{L^\infty(0,T/2;L^\infty(\Omega))} \lesssim \|y_{1,0}\|_{L^\infty(\Omega)}.$$

Assuming (1.7) with $\delta > 0$ possibly smaller, we have that

$$g := y_1^{N_2} \in L^\infty((0,T)\times\Omega)$$

satisfies (4.1) so that we have obtained at this step a control $h \in L^\infty(0,T/2;L^\infty(\Omega))$, such that (1.3) admits a weak solution (y_1, y_2) in $(0,T/2)$ and $y_1(T/2,\cdot) = 0$. By using Lemma 2.6, $y_{2,T/2} := y_2(T/2,\cdot)$ satisfies

$$\|y_{2,T/2}\|_{L^\infty(\Omega)} \lesssim \delta.$$

Step 2. Control of the second equation in $(T/2, T)$ ***through a fictitious odd control.*** By taking $\delta > 0$ possibly smaller, we can apply Theorem 1.2 to

$$\begin{cases} \partial_t y_2 - \Delta y_2 = H\chi_\omega + y_2^{N_3} & \text{in } (T/2,T)\times\Omega, \\ y_2 = 0 & \text{on } (T/2,T)\times\partial\Omega, \\ y_2(T/2,\cdot) = y_{2,T/2} & \text{in } \Omega. \end{cases} \tag{4.5}$$

We deduce the existence of a control H such that $y_2(T,\cdot) = 0$ and such that

$$H^{1/N_2} \in L^p\left(T/2, T; W^{2,p}(\Omega)\right) \cap W^{1,p}\left(T/2, T; L^p(\Omega)\right),$$
$$H^{1/N_2}(T/2,\cdot) = H^{1/N_2}(T,\cdot) = 0.$$

We then set, in $(T/2, T)$,

$$y_1 := (H\chi_\omega)^{1/N_2}, \quad h := \partial_t y_1 - \Delta y_1 - y_1^{N_1} \in L^p((T/2,T)\times\Omega).$$

Concatenating y_1, y_2 and h between the two steps, we can check that $h \in L^p((0,T)\times\Omega)$, that (y_1, y_2) is the weak solution of (1.3) and that (1.9) holds. This concludes the proof of Theorem 1.1. □

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