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ABSOLUTE VALUES ON ALGEBRAS $H(D)$

by Kamal Boussaf and Alain Escassut

Abstract. Let K be an algebraically closed complete ultrametric field, and let D be an infraconnected set in K such that the set $H(D)$ of the analytic elements on D is a ring. Among the continuous multiplicative semi-norms on $H(D)$, we look for the ones that are absolute values. They are characterized by the location of the T -filters on D . Besides, we characterize the sets D such that $H(D)$ admits at least one continuous absolute value $|\cdot|$.

Notations: Let K be an algebraically closed field complete for an ultrametric absolute value.

Given $a \in K$ and $r > 0$, $d(a, r)$ (resp. $d(a, r^-)$, resp. $C(a, r)$) denotes the disk $\{x \in K \mid |x - a| \leq r\}$ (resp. $\{x \in K \mid |x - a| < r\}$, resp. the circle $\{x \in K \mid |x - a| = r\}$).

Given $a \in K$, $r' > 0$ and $r'' > r'$, $\Gamma(a, r', r'')$ denotes the annulus $\{x \in K \mid r' < |x - a| < r''\}$.

Given a set A in K and a point $a \in K$, we denote by $\delta(a, A)$ the distance from a to A . Let E be an infinite set in K , and let $a \in E$. If E is bounded of diameter r , we denote by \tilde{E} the disk $d(a, r)$, and if E is not bounded, we put $\tilde{E} = K$. Then, $\tilde{E} \setminus \bar{E}$ is known to admit a partition of the form $(d(a_i, r_i^-))_{i \in J}$, with $r_i = \delta(a_i, D)$ for each $i \in J$. The disks $d(a_i, r_i^-)_{i \in J}$, are named *the holes of E* .

$R(E)$ denotes the set of rational functions $h \in K(x)$ with no poles in E . This is a K -subalgebra of the algebra K^D of all functions from E into K . Then $R(E)$ is provided with the topology \mathcal{U}_E of uniform convergence on E , and is a topological group for this topology. $H(E)$ denotes the completion of $R(E)$ for this topology and its elements are named *the analytic elements on E* [1], [2], [3], [9].

By [3], we remember that $H(E)$ is a K -subalgebra of the algebra K^D if and only if E satisfies the following conditions:

A) $\tilde{E} \setminus \bar{E}$ is bounded,

B) $\bar{E} \setminus E \subset \overset{\circ}{\bar{E}}$.

Henceforth, D will denote an infraconnected set satisfying Conditions A) and B).

In [7], [4], the continuous multiplicative semi-norms of an algebra $H(D)$ were characterized by means of the circular filters on D . So we have to recall the definitions of monotonous and circular filters.

Definitions and notations: The set of those multiplicative semi-norms that are continuous with respect to the topology of uniform convergence on D is denoted by $Mult(H(D), \mathcal{U}_D)$. Given a continuous multiplicative semi-norm $\psi \in Mult(H(D), \mathcal{U}_D)$ we denote by $Ker\psi$ the closed prime ideal of the $f \in H(D)$ such that $\psi(f) = 0$.

ψ will be said to be *punctual* if $Ker\psi$ is a maximal ideal of codimension 1 of $H(D)$. We know that there exists a bijection M from D onto the set of maximal ideals of codimension 1 of $H(D)$, defined as $M(a) = \{f \in H(D) | f(a) = 0\}$ (indeed, this was shown in [3], Proposition II.6, when D is closed and bounded, and it is easily extended to all sets D satisfying Conditions A) and B)). As a consequence, there exists a bijection S from D onto the set of punctual continuous multiplicative semi-norms of $H(D)$ defined as $S(a)(f) = |f(a)|$, whenever $f \in H(D)$.

In order to recall the characterization of the continuous multiplicative semi-norms of $H(D)$, we first have to recall the definition of monotonous and circular filters.

Given a filter \mathcal{F} on D , we will denote by $\mathcal{I}(\mathcal{F})$ the ideal of the $f \in H(D)$ such that $\lim_{\mathcal{F}} f(x) = 0$.

Let $a \in \tilde{D}$ and $S \in \mathbb{R}_+^*$ be such that $\Gamma(a, r, S) \cap D \neq \emptyset$ whenever $r \in]0, S[$ (resp. $\Gamma(a, S, r) \cap D \neq \emptyset$ whenever $r > S$). We call an *increasing* (resp. a *decreasing*) *filter of center a and diameter S , on D* the filter \mathcal{F} on D that admits for base the family of sets $\Gamma(a, r, S) \cap D$ (resp. $\Gamma(a, S, r) \cap D$). For every sequence $(r_n)_{n \in \mathbb{N}}$ such that $r_n < r_{n+1}$ (resp. $r_n > r_{n+1}$) and $\lim_{n \rightarrow \infty} r_n = S$, it is seen that the sequence $\Gamma(a, r_n, S) \cap D$ (resp. $\Gamma(a, S, r_n) \cap D$) is a base of \mathcal{F} and such a base is called a *canonical base*.

We call a *decreasing filter with no center of canonical base* $(D_n)_{n \in \mathbb{N}}$ and diameter $S > 0$, on D a filter \mathcal{F} on D that admits for base a sequence $(D_n)_{n \in \mathbb{N}}$ of the form $D_n = d(a_n, r_n) \cap D$ with $D_{n+1} \subset D_n$, $r_{n+1} < r_n$, $\lim_{n \rightarrow \infty} r_n = S$, and $\bigcap_{n \in \mathbb{N}} d(a_n, r_n) = \emptyset$.

Given an increasing (resp. a decreasing) filter \mathcal{F} on D of center a and diameter r , we will denote by $\mathcal{P}(\mathcal{F})$ the set $\{x \in D | |x - a| \geq r\}$ (resp. the set $\{x \in D | |x - a| \leq r\}$) and by $\mathcal{C}(\mathcal{F})$ the set $\{x \in D | |x - a| < r\}$ (resp. the set $\{x \in D | |x - a| > r\}$). Besides $\mathcal{C}(\mathcal{F})$ will be named *the body of \mathcal{F}* and $\mathcal{P}(\mathcal{F})$ will be named *the beach of \mathcal{F}* .

We call a *monotonous filter on D* a filter which is either an increasing filter or a decreasing filter (with or without a center).

Given a monotonous filter \mathcal{F} we will denote by $diam(\mathcal{F})$ its diameter.

The field K is said to be *spherically complete* if every decreasing filter on K has a center in K . (The field \mathbb{C}_p for example is not spherically complete). However, every algebraically closed complete ultrametric field admits a spherically complete algebraically closed extension [10], [11].

Two monotonous filters \mathcal{F} and \mathcal{G} are said to be *complementary* if $\mathcal{P}(\mathcal{F}) \cup \mathcal{P}(\mathcal{G}) = D$. Let \mathcal{F} be an increasing (resp. a decreasing) filter of center a and diameter S on D . \mathcal{F} is said to be *pierced* if for every $r \in]0, S[$, (resp. $r < S$), $\Gamma(a, r, S)$ (resp. $\Gamma(a, S, r)$) contains some hole T_m of D . A decreasing filter with no center \mathcal{F} , and canonical base $(D_n)_{n \in \mathbb{N}}$, on D is said to be *pierced* if for every $m \in \mathbb{N}$, $\tilde{D}_m \setminus \tilde{D}_{m+1}$ contains some hole T_m of D .

Let $a \in \tilde{D}$, let $\rho = \delta(a, D)$ be such that $\rho \leq S \leq \text{diam}(D)$. We call *circular filter of center a and diameter S on D* the filter \mathcal{F} which admits as a generating system the family of sets $\Gamma(\alpha, r', r'') \cap D$ with $\alpha \in d(a, S)$, $r' < S < r''$, i.e. \mathcal{F} is the filter which admits for base the family of sets of the form $D \cap \left(\bigcap_{i=1}^q \Gamma(\alpha_i, r'_i, r''_i) \right)$ with $\alpha_i \in d(a, S)$, $r'_i < S < r''_i$ ($1 \leq i \leq q$, $q \in \mathbb{N}$).

A decreasing filter with no center, of canonical base $(D_n)_{n \in \mathbb{N}}$ is also called *circular filter on D with no center, of canonical base $(D_n)_{n \in \mathbb{N}}$* .

Finally the filter of the neighbourhoods of a point $a \in D$ will be called *circular filter of the neighbourhoods of a on D* . It will be also named *circular filter of center a and diameter 0*.

A circular filter on D will be said to be *large* if it has diameter different from 0.

Given a circular filter \mathcal{F} , its diameter will be denoted by $\text{diam}(\mathcal{F})$.

The set of the circular filters on D will be denoted by $\Phi(D)$.

Now let \mathcal{F} be a circular filter on D . By [7], [4], we have the following characterization of continuous multiplicative semi-norms of $H(D)$.

Theorem 0: *Let \mathcal{F} be a circular filter on D . For every $f \in H(D)$, $|f(x)|$ admits a limit along \mathcal{F} , and this limit, denoted by $\varphi_{\mathcal{F}}(f)$, defines a continuous multiplicative semi-norm $\varphi_{\mathcal{F}}$ on $H(D)$. Further, the mapping Θ from $\Phi(D)$ into $\text{Mult}(H(D), \mathcal{U}_D)$ defined as $\Theta(\mathcal{F}) = \varphi_{\mathcal{F}}$ is a bijection.*

Notations: For convenience, when \mathcal{F} is the circular filter of center a and diameter r , we also denote by $\varphi_{a,r}$ the multiplicative semi-norm $\varphi_{\mathcal{F}}$.

Here, assuming $H(D)$ to be a K -algebra, we study what continuous multiplicative semi-norms of $H(D)$ are norms, i.e. are absolute values on $H(D)$. Of course, this requires $H(D)$ to have no divisors of zero. But then, as a transcendental extension of the field K , the field of quotients L of $H(D)$ does admit absolute values extending the

one of K . Hence so does $H(D)$. The problem, here, is whether such absolute values are continuous with respect to the topology of $H(D)$, i.e. are defined by circular filters on D . So, we will give the condition a circular filter has to satisfy in order that its continuous multiplicative semi-norm be an absolute value, and next, we will characterize the sets D such that at least one of the continuous multiplicative semi-norms is an absolute value.

All this study involves T -filters, and now we have to introduce them.

Definition: Let \mathcal{F} be an increasing (resp. a decreasing) filter on D , of center a and diameter s . An element $f \in \mathcal{I}(\mathcal{F})$ is said to be *strictly vanishing along \mathcal{F}* if there exists $t < s$ (resp. $t > s$) such that $\varphi_{a,r}(f) > 0$ for all $r \in [t, s[$, (resp. $]s, t]$).

Let \mathcal{F} be a decreasing filter on D , with no center, of diameter r , of canonical base $(D_n)_{n \in \mathbb{N}}$, with $D_n = d(a_n, r_n) \cap D$. Then an element $f \in \mathcal{I}(\mathcal{F})$ is said to be *strictly vanishing along \mathcal{F}* if there exists $t > s$ such that $\varphi_{a_n,r}(f) > 0$ for all $r \in [r_n, t]$, for every $n \in \mathbb{N}$.

Let \mathcal{F} be a filter on D , and let $A \subset D$. \mathcal{F} will be said to be *secant with A* if for every $F \in \mathcal{F}$, $A \cap F$ is not empty.

T -filters are certain pierced monotonous filters satisfying particular properties linked to the holes of D , and were defined in [1], [2], [5]. Here we will only use the following characterization:

A monotonous filter \mathcal{F} is a T -filter if and only if there exists $f \in H(D)$ strictly vanishing along \mathcal{F} .

From [2], [5], [6]. we can easily deduce the following technical propositions that will be indispensable.

Proposition P: *Let $b \in D$, $l > 0$ and let $f \in H(D)$ satisfy $f(b) \neq 0$ and $\varphi_{b,l} = 0$. There exists an increasing T -filter \mathcal{F} of center b and diameter $t \in]0, l[$ such that f is strictly vanishing along \mathcal{F} and satisfies $\varphi_{b,s}(f) > 0$ for every $s \in]0, t[$.*

Let $a \in \tilde{D}$ and let $r, s \in \mathbb{R}$ satisfy $\delta(a, D) \leq r \leq \text{diam}(D)$. If f satisfies $f(x) = 0$ whenever $x \in d(a, r) \cap D$, then f is strictly vanishing along a T -filter \mathcal{F} such that $d(a, r) \cap D \subset \mathcal{P}(\mathcal{F})$ and $b \in \mathcal{C}(\mathcal{F})$.

Now, we can characterize absolute values among continuous multiplicative semi-norms.

Theorem 1: *Let \mathcal{F} be a large circular filter on D . Then $\varphi_{\mathcal{F}}$ is not an absolute value if and only if it satisfies one of the following conditions:*

- a) *There exists a T -filter \mathcal{G} on D such that \mathcal{F} is secant with $\mathcal{P}(\mathcal{G})$,*
- b) *\mathcal{F} is a T -filter.*

Proof: First, suppose that there exists a T -filter \mathcal{G} satisfying a). By Lemma I.6 A of [5], there exists $f \in H(D)$, strictly vanishing along \mathcal{G} , equal to 0 in all of $\mathcal{P}(\mathcal{G})$. Hence we have $\varphi_{\mathcal{F}}(f) = 0$ and then $\varphi_{\mathcal{F}}$ is not a norm. Now if \mathcal{F} is a T -filter, then there exists

$f \in H(D)$ strictly vanishing along \mathcal{F} and therefore we have $\lim_{\mathcal{F}} f(x) = 0$, hence $\varphi_{\mathcal{F}}$ is not a norm.

Now we suppose that there exists no T -filter \mathcal{G} satisfying a) and that \mathcal{F} is not a T -filter, and we suppose that $\varphi_{\mathcal{F}}$ is not a norm. Let $f \in H(D) \setminus \{0\}$ satisfying $\varphi_{\mathcal{F}}(f) = 0$. Let $S = \text{diam}(\mathcal{F})$. Let $b \in D$ be such that $f(b) \neq 0$.

We first assume that \mathcal{F} has a center a .

On the first hand, we suppose that $b \in d(a, S)$. Since $\varphi_{a,r}(f) \neq 0$, when r approaches 0 there does exist $s \in]0, S[$ such that $\varphi_{b,s}(f) = 0$ and $\varphi_{b,r}(f) \neq 0$ whenever $r \in]0, s[$. Hence f is strictly vanishing along the increasing filter \mathcal{G} of center b and diameter s , and therefore \mathcal{F} is secant with $\mathcal{P}(\mathcal{G})$.

On the second hand, we suppose that $|a - b| > S$. Let $t = |a - b|$. If $\varphi_{b,t}(f) = 0$, there exists $s \in]0, t[$ such that $\varphi_{b,s}(f) = 0$ and $\varphi_{b,r}(f) \neq 0$ whenever $r \in]0, s[$, hence f is strictly vanishing along an increasing T -filter \mathcal{G} of center b and diameter s and therefore \mathcal{F} is secant with $\mathcal{P}(\mathcal{G})$. Now we may assume $\varphi_{b,t}(f) \neq 0$. But $\varphi_{b,t}(f) = \varphi_{a,t}(f)$ and therefore there exists $s \in]S, t[$ such that $\varphi_{a,s}(f) = 0$ and $\varphi_{a,r}(f) \neq 0$ whenever $r \in]s, t[$. Hence f is strictly vanishing along a decreasing T -filter \mathcal{G} of center a and diameter s , and \mathcal{F} is secant with $\mathcal{P}(\mathcal{G})$.

Now, assume that \mathcal{F} is a decreasing filter with no center. Let $(D_n)_{n \in \mathbb{N}}$ be a canonical base of \mathcal{F} , and for each $n \in \mathbb{N}$, let $D_n = d(a_n, r_n) \cap D$ and let $u_n = \varphi_{a_n, r_n}(f)$. If there exists $q \in \mathbb{N}$ such that $u_q = 0$, by Proposition P, D admits a T -filter \mathcal{G} such that D_q is included in $\mathcal{P}(\mathcal{G})$, and therefore, \mathcal{F} is obviously secant with $\mathcal{P}(\mathcal{G})$. Hence we can assume that $u_n > 0$ for every $n \in \mathbb{N}$. Now, suppose that there exist $q \in \mathbb{N}$ and $r \in [r_{q+1}, r_q]$ such that $\varphi_{a_{q+1}, r}(f) = 0$. As we just saw, there exists a T -filter \mathcal{G} on D such that the circular filter \mathcal{F}_q is secant with $\mathcal{P}(\mathcal{G})$, and therefore so is \mathcal{F} .

Thus, without loss of generality, we can assume that $\varphi_{a_{q+1}, r}(f) > 0$ for every $r \in [r_{q+1}, r_q]$, for every $q \in \mathbb{N}$. Hence f is just strictly vanishing along the decreasing filter \mathcal{F} and therefore \mathcal{F} is a T -filter. This ends the proof of Theorem 1.

Corollary a: *All the not punctual continuous multiplicative semi-norms of $H(D)$ are absolute values if and only if D has no T -filter.*

Definitions and notations: Let $\text{inc}T(D)$ (resp. $\text{dec}T(D)$) be the set of increasing (resp. decreasing) T -filters on D . We will denote by \preceq the relation defined on $\text{inc}T(D)$ (resp. $\text{dec}T(D)$) by $\mathcal{F}_1 \preceq \mathcal{F}_2$ if $\mathcal{C}(\mathcal{F}_2) \subset \mathcal{C}(\mathcal{F}_1)$. This relation is obviously seen to be an order relation on $\text{inc}T(D)$ (resp. $\text{dec}T(D)$).

An increasing (resp. a decreasing) T -filter \mathcal{F} will be said to be *maximal* if it is maximal in $\text{inc}T(D)$ (resp. in $\text{dec}T(D)$) with respect to this relation.

We will denote by \prec the strict order associated to \preceq by $\mathcal{F}_1 \prec \mathcal{F}_2$ if $\mathcal{F}_1 \preceq \mathcal{F}_2$ and $\mathcal{F}_1 \neq \mathcal{F}_2$.

We will call *an ascending chain of increasing (resp. decreasing) T -filters* a sequence of increasing (resp. decreasing) T -filters $(\mathcal{F}_n)_{n \in \mathbb{N}}$ such that $\mathcal{F}_n \prec \mathcal{F}_{n+1}$ whenever $n \in \mathbb{N}$.

Let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be an ascending chain of increasing T -filters. For each $n \in \mathbb{N}$ let $r_n = \text{diam}(\mathcal{F}_n)$. Since the sequence $(r_n)_{n \in \mathbb{N}}$ is decreasing, we put $r = \lim_{n \rightarrow \infty} r_n$, and then r will be named *the diameter of the chain*.

We put $A := \bigcap_{n \in \mathbb{N}} \mathcal{C}(\mathcal{F}_n)$ and for each $n \in \mathbb{N}$, $D_n := \mathcal{C}(\mathcal{F}_n) \setminus A$. The sequence $(D_n)_{n \in \mathbb{N}}$

is then a base of a filter \mathcal{F} on D of diameter r .

If $r = 0$, since D Condition Condition B), A is a point a of D , hence \mathcal{F} is the filter of the neighbourhoods of a in D .

If $r > 0$ \mathcal{F} is a decreasing filter on D of diameter r .

In both cases \mathcal{F} will be called *the returning filter of the ascending chain* $(\mathcal{F}_n)_{n \in \mathbb{N}}$.

Now let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be an ascending chain of decreasing T -filters and let $a \in \mathcal{P}(\mathcal{F}_n)$ for some $n \in \mathbb{N}$. The sequence $(r_n)_{n \in \mathbb{N}}$ is an increasing sequence of limit $r \in]0, +\infty[$, and r will be named *the diameter of the chain*. Since D belongs to \mathcal{A} , by Condition A) we notice that $r < +\infty$, and then we will call *the returning filter of the ascending chain* $(\mathcal{F}_n)_{n \in \mathbb{N}}$ the increasing filter \mathcal{F} of center a and diameter r (it is seen that \mathcal{F} does not depend on the point $a \in \mathcal{P}(\mathcal{F}_n)$, whenever $n \in \mathbb{N}$).

Lemma 1: *Let $H(D)$ have no divisors of zero. Then $\text{inc}T(D)$ is totally ordered with respect to the order \preceq .*

Proof: Suppose that \mathcal{F}, \mathcal{G} are increasing T -filters on D that are not comparable. We put $A = \mathcal{C}(\mathcal{F})$ and $B = \mathcal{C}(\mathcal{G})$. Then A, B are two disks of D which satisfy neither $A \subset B$, nor $B \subset A$. Hence we have $A \cap B = \emptyset$. As a consequence, $\mathcal{P}(\mathcal{F}) \cup \mathcal{P}(\mathcal{G})$ is equal to D and then by [6] $H(D)$ has divisors of zero. Hence this contradicts the hypothesis and ends the proof.

Corollary b: *Let $H(D)$ have no divisors of zero, and let \mathcal{F} (resp. \mathcal{G}) be an increasing T -filter on D , of diameter r (resp. s). If $r > s$ then $\mathcal{F} \prec \mathcal{G}$. If $r = s$, then $\mathcal{F} = \mathcal{G}$.*

We are now able to characterize the sets D such that $H(D)$ admits continuous absolute values. Let us recall the following theorem of [6]:

The algebra $H(D)$ has no divisors of zero if and only if D does not admit two complementary T -filters.

We will use comparison between filters. Here, a filter \mathcal{F} will be said *thinner than* a filter \mathcal{G} every element of \mathcal{G} belongs to \mathcal{F} .

Theorem 2: *Let $H(D)$ have no divisors of zero. Then $\text{Mult}(H(D), \mathcal{U}_D)$ contains no norm if and only if D admits an ascending chain of T -filters $(\mathcal{F}_n)_{n \in \mathbb{N}}$ whose returning filter is either a T -filter or a Cauchy filter.*

Proof : On the first hand, we suppose that D admits an ascending chain of T -filters $(\mathcal{F}_n)_{n \in \mathbb{N}}$ whose returning filter \mathcal{F} is either a T -filter or a Cauchy filter and we will prove that $\text{Mult}(H(D), \mathcal{U}_D)$ contains no norm. We denote by r the diameter

of this ascending chain $(\mathcal{F}_n)_{n \in \mathbb{N}}$. Let \mathcal{G} be a circular filter on D , of diameter $s > 0$. By Theorem 1, we only have to show that either \mathcal{G} is a T -filter, or \mathcal{G} is secant with the beach of a T -filter.

First we suppose \mathcal{F} is a Cauchy filter. Then the \mathcal{F}_n are increasing T -filters. Let $q \in \mathbb{N}$ be such that $r_q < s$. Then \mathcal{G} is clearly secant with $\mathcal{P}(\mathcal{F}_q)$.

Now we suppose that \mathcal{F} is a T -filter. Then we just have to consider the case when \mathcal{G} is secant with $\mathcal{C}(\mathcal{F})$ and is not equal to \mathcal{F} .

First we suppose \mathcal{F} increasing, of center b and diameter r . Hence we have $\mathcal{C}(\mathcal{F}) = d(b, r^-) \cap D$. Then \mathcal{G} has a diameter $s \in]0, r[$, and then it admits elements E of diameter $t \in]s, r[$, included in $d(b, r^-) \cap D$. Since $E \cap \mathcal{C}(\mathcal{F}) \neq \emptyset$, given a point $a \in E \cap \mathcal{C}(\mathcal{F})$ we have $E \subset d(a, t)$. Let $q \in \mathbb{N}$ be such that $r_q > \max(t, |a - b|)$. Then E is included in $d(b, r_q)$ and therefore is included in $\mathcal{P}(\mathcal{F}_q)$. Hence \mathcal{G} is secant with $\mathcal{P}(\mathcal{F}_q)$.

Finally we suppose \mathcal{F} decreasing, of diameter r . Since \mathcal{G} is secant with $\mathcal{C}(\mathcal{F})$, and is not less thin than \mathcal{F} , we have $r < s$, and there exists $t \subset]r, s[$ and $a \in D$ such that \mathcal{G} is secant with $(K \setminus d(a, t)) \cap D$ while \mathcal{F} is secant with $d(a, t^-) \cap D$. Let $q \in \mathbb{N}$ be such that $r_q < t$ and let a_q be a center of \mathcal{F}_q . Then a_q belongs to $d(a, t)$ and we have $\mathcal{P}(\mathcal{F}_q) = (K \setminus d(a, r_q^-)) \cap D$. Hence \mathcal{G} is secant with $\mathcal{P}(\mathcal{F}_q)$ and this finishes showing that $\text{Mult}(H(D), \mathcal{U}_D)$ contains no norm.

On the second hand, reciprocally, we suppose that $\text{Mult}(H(D), \mathcal{U}_D)$ contains no norm and we will show that D admits an ascending chain of T -filters $(\mathcal{F}_n)_{n \in \mathbb{N}}$ whose returning filter is either a T -filter or a Cauchy filter.

We denote by \mathcal{R}' the set of the diameters of the $\mathcal{F} \in \text{inc}T(D)$, by \mathcal{R}'' the set of the diameters of the $\mathcal{F} \in \text{dec}T(D)$, and we put $\mathcal{R} = \mathcal{R}' \cup \mathcal{R}''$. Since $H(D)$ has no norm, by Theorem 1 \mathcal{R} is not empty. Since D belongs to \mathcal{A} , by Condition A) \mathcal{R} is obviously bounded. We put $t = \sup(\mathcal{R})$. Let $a \in D$. We will show that $\text{inc}T(D) \neq \emptyset$. Indeed, suppose $\text{inc}T(D) = \emptyset$. First let D be bounded, of diameter S . Any decreasing filter on D has a diameter $r < S$, and therefore the circular filter \mathcal{G} of center a and diameter S is secant with D , but (of course) is not a T -filter on D , and is not secant with the beach of any decreasing T -filter on D . As a consequence, by Theorem 1 $\varphi_{\mathcal{G}}$ is a norm. Thus we see that D is not bounded. Then, any circular filter \mathcal{G} of center a and diameter $r > t$ is not a T -filter and is not secant with the beach of any T -filter. Finally this shows that $\varphi_{\mathcal{G}}$ is a norm again. Thus we see that $\text{inc}T(D)$ is not empty, and neither is \mathcal{R}' .

Now, we put $s = \inf(\mathcal{R}')$. First we suppose $s \in \mathcal{R}'$. Let $\mathcal{T} \in \text{inc}T(D)$ satisfy $\text{diam}(\mathcal{T}) = s$, and let $b \in \mathcal{C}(\mathcal{T})$. Then for every $r \in]0, s[$, the circular filter of center b and diameter r is not a decreasing T -filter, and therefore is secant with the beach of a decreasing T -filter. Hence there exists a decreasing T -filter \mathcal{F} of center b and diameter $\ell \geq s$. But since $H(D)$ has no divisors of zero, \mathcal{F} is not complementary with \mathcal{T} , hence we have $s \leq \ell$, i.e. $s \leq \ell < r$. So, we clearly deduce the existence of a sequence of decreasing T -filters $(\mathcal{F}_n)_{n \in \mathbb{N}}$, such that each one admits b as a center and has a diameter r_n satisfying $r_n < r_{n+1} < s$, $\lim_{n \rightarrow \infty} r_n = s$. Therefore the sequence $(\mathcal{F}_n)_{n \in \mathbb{N}}$

is an ascending chain of decreasing T -filters such that $\bigcap_{n \in \mathbb{N}} \mathcal{P}(\mathcal{F}_n) = \mathcal{C}(T)$, hence the returning filter of the ascending chain $(\mathcal{F}_n)_{n \in \mathbb{N}}$ is a T -filter.

Now, we suppose $s \notin \mathcal{R}'$. Let $(T_n)_{n \in \mathbb{N}}$ be a sequence in $\text{inc}T(D)$ such that $\lim_{n \rightarrow \infty} \text{diam}(T_n) = s$, with $\text{diam}(T_{n+1}) < \text{diam}(T_n)$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, we put $A_n = \widetilde{\mathcal{C}(T_n)}$, and $r_n = \text{diam}(A_n)$. By Lemma 1 the sequence $\mathcal{C}(T_n)_{n \in \mathbb{N}}$ is strictly decreasing, and so is the sequence $(A_n)_{n \in \mathbb{N}}$. Besides, the sequence $(T_n)_{n \in \mathbb{N}}$ is an ascending chain of increasing T -filters. Obviously each A_n contains a hole T_n of D .

If $s = 0$, then we have $\delta(b, T_n) \leq r_n$, and therefore b does not belong to $\overset{\circ}{D}$, but then, by Condition A) b must belong to \overline{D} . Thus, the sequence $(T_n)_{n \in \mathbb{N}}$ is an ascending chain of increasing T -filters that converges to b , and therefore the returning filter of the ascending chain $(T_n)_{n \in \mathbb{N}}$ is a Cauchy filter.

Finally, it only remains to consider the case when $s > 0$, with $s \notin \mathcal{R}'$. Let \mathcal{F} be the returning filter of the sequence $(T_n)_{n \in \mathbb{N}}$. If \mathcal{F} were a T -filter, or were secant with the beach of an increasing T -filter, this increasing T -filter would have a diameter inferior or equal to s . Hence \mathcal{F} either is a decreasing T -filter or is secant with the beach of a decreasing T -filter. Of course, if \mathcal{F} is a T -filter, it is just the returning filter of the chain $(T_n)_{n \in \mathbb{N}}$. Finally if \mathcal{F} is secant with the beach of a decreasing T -filter \mathcal{G} , then we have $\text{diam}(\mathcal{G}) \leq s$ because if $\text{diam}(\mathcal{G})$ were strictly superior to s , then \mathcal{G} would be complementary to T_n when n is big enough, and therefore $H(D)$ would have divisors of zero. Hence we have $\text{diam}(\mathcal{G}) = s$, and therefore \mathcal{G} is just the returning filter of the sequence $(T_n)_{n \in \mathbb{N}}$. This finishes proving that D admits an ascending chain of T -filters $(\mathcal{F}_n)_{n \in \mathbb{N}}$ whose returning filter is either a T -filter or a Cauchy filter, and this ends the proof of Theorem 2.

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