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Biharmonic Steklov operator on differential forms

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Abstract

We introduce the biharmonic Steklov problem on differential forms by considering suitable boundary conditions. We characterize its smallest eigenvalue and prove elementary properties of the spectrum. We obtain various estimates for the first eigenvalue, some of which involve eigenvalues of other problems such as the Dirichlet, Neumann, Robin and Steklov ones. Independently, new inequalities relating the eigenvalues of the latter problems are proved.

Opérateur de Steklov biharmonique sur les formes différentielles

Résumé

Nous introduisons le problème de Steklov biharmonique sur les formes différentielles en considérant des conditions de bord adaptées. Nous caractérisons sa plus petite valeur propre et établissons des propriétés élémentaires de son spectre. Nous obtenons des estimations diverses de la première valeur propre, dont certaines font intervenir des valeurs propres d'autres problèmes tels que ceux de Dirichlet, de Neumann, de Robin et de Steklov. Indépendamment, nous montrons de nouvelles inégalités reliant les valeurs propres de ces derniers problèmes.

1. Introduction

Let (M^n, g) be an n-dimensional compact Riemannian manifold with nonempty smooth boundary ∂M. Denote by ν the inward unit vector field normal to the boundary and by $\Delta f := -\operatorname{tr}(\nabla^2 f)$ the Laplace operator applied to a smooth function f on M. The following fourth order eigenvalue boundary problem

$$\begin{cases} \Delta^2 f = 0 & \text{on } M \\ f = 0 & \text{on } \partial M \\ \Delta f - q \frac{\partial f}{\partial \nu} = 0 & \text{on } \partial M, \end{cases} \tag{1.1}$$

also called biharmonic Steklov problem I, or biharmonic Steklov for simplicity, was first introduced by Kuttler and Sigillito [16] and Payne [19]. Its physical interpretation in terms of the deformation of an elastic plate under the action of transversal forces can be found in e.g. [6, p. 316] and [25, p. 2637]. When M is a bounded domain in $\mathbb{R}^n$, the spectrum of this problem has been studied in [6] and proved to be discrete consisting of positive eigenvalues of finite multiplicities (see also [2] for the case when the boundary is not smooth). Variational characterizations of the first eigenvalue q_1 have been also given in [6] (see also [15]). Moreover, in the case of a ball, the spectrum has been calculated explicitly.

In the case of a compact n-dimensional Riemannian manifold M with smooth boundary ∂M, sharp estimates for the first eigenvalue q_1 of the biharmonic Steklov operator are given in [25] and [22]. It is shown that

$$q_1 \leq \frac{\text{Vol}(\partial M)}{\text{Vol}(M)}. \tag{1.2}$$

In [22, Thm. 2], Raulot and Savo proved that, if M is a geodesic ball in a space form, then (1.2) is an equality (see also [25, Thm. 1.3]). Under some assumptions on the Ricci curvature of M and if the mean curvature of the boundary is bounded below by a positive constant H_0, then the first eigenvalue q_1 also satisfies

$$q_1 \geq nH_0.$$

Moreover, equality holds if and only if M is isometric to a ball of radius $\frac{1}{H_0}$ in $\mathbb{R}^n$ (see [25, Thm. 1.2] and [22, Thm. 2]). For other recent results on the biharmonic Steklov eigenvalue problem, we refer to [8, 10] and the references listed therein.

On the other hand, recall that a compact Riemannian manifold M with smooth nonempty boundary ∂M is called a *harmonic domain* [21, p. 893] if and only if it supports a (necessarily unique) solution f to the Serrin boundary value problem

$$\begin{cases} \Delta f = 1 & \text{on } M \\ f = 0 & \text{on } \partial M \\ \partial_\nu f = c & \text{on } \partial M \end{cases} \tag{1.3}$$

for some constant $c \in \mathbb{R}$. From [24, 26], we know that the only harmonic domains in $\mathbb{R}^n$ are the Euclidean balls of radius $nc = \frac{n\,\text{Vol}(M)}{\text{Vol}(\partial M)}$. Independently, it is not difficult to check that a solution to the Serrin problem (1.3) is an eigenfunction of (1.1).

The aim of this paper is first to extend the biharmonic Steklov problem (1.1) to the context of differential forms. As we mentioned above, there is a relationship between problems (1.1) and (1.3), hence the idea is to also define the Serrin problem on differential forms. For this purpose, we assume that the manifold M carries a non-trivial parallel form and introduce the generalization of (1.3) (see (2.1)). In this case, we show that

M is a harmonic domain if and only if there exists on M a solution to (2.1). Also in Section 2, we provide a natural extension of problem (1.1) to the case of differential forms (problem (2.2)). Applying the techniques used in [6] by Ferrero, Gazzola and Weth, we show in Theorem 2.3 that problem (2.2) has a discrete spectrum consisting of a countable number of positive eigenvalues of finite multiplicities. This involves proving the ellipticity of problem (A.1) (see Lemma A.1 in the appendix). Moreover, we give two variational characterizations of the first eigenvalue of this problem (Theorem 2.6) which will be useful to establish inequalities in the sequel.

In Section 3, we obtain different estimates regarding the eigenvalues of problem (2.2). As a preliminary step, we prove an interesting property of that problem, namely its invariance by the Hodge star operator. On the other hand, under curvature assumptions, using a previous result by Raulot and Savo [22, Thm. 10], we derive a lower bound (3.1) for the first eigenvalue $q_{1,p}$, which generalizes the estimates [25, Thm. 1.2] and [22, Thm. 2]. On the other hand, when the manifold M supports a non-trivial parallel p-form, we show that

$$q_{1,p} \leq \frac{\mathrm{Vol}(\partial M)}{\mathrm{Vol}(M)},$$

with equality if and only if M is a harmonic domain. Note that there is no reason for a harmonic domain to admit a nontrivial parallel p-form for some $p \geq 1$. Surprisingly, when M is a domain of $\mathbb{R}^n$, the eigenvalues of the biharmonic Steklov problem on differential forms are the same as those of the scalar problem, without taking into consideration their multiplicities. It should be noted that the same type of result is true for the eigenvalues of the Dirichlet problem. We end Section 3 with an inequality relating the eigenvalues corresponding to degrees $p - 1$, p and $p + 1$ on the sphere. That inequality is established by first proving a more general result when M is isometrically immersed in a Euclidean space and by using the variational characterization (2.9) of the first eigenvalue. The computations involved in these results being rather technical, we place the details in the appendix (Lemma A.2 and Proposition A.3) to lighten the text.

In Sections 4 and 5, we establish several bounds concerning the eigenvalues of various differential operators, again using variational characterizations. For example, we give estimates involving the eigenvalues of the Robin eigenvalue problem (4.1) introduced in [4] and those of Neumann (4.3), Dirichlet (4.4), biharmonic Steklov (2.2) eigenvalue problems on differential forms (Theorems 4.1 and 5.1). Note that similar results are known in the case of scalar problems (see [13, Thm. 1.17]). In Theorem 4.2, we also give an estimate, under some curvature condition along the boundary, for the difference between the first eigenvalues of the Robin problem on q and $(q - p)$-forms, for some p and q such that $p \leq q$.

Other inequalities involving eigenvalues of the Steklov problem on forms (see Section 5) are obtained using properties already established for the eigenvalues and eigenforms of problem (2.2).

Acknowledgments

2. Biharmonic Steklov operator on differential forms

2.1. Serrin problem

In this subsection, we extend the Serrin problem to differential forms. This extension will motivate us to define the biharmonic Steklov problem on differential forms.

Recall that the Serrin problem is given by the following [24]: Let $M \subset \mathbb{R}^n$ be a bounded domain and let f be a solution to the problem

$$\begin{cases} \Delta f = 1 & \text{on } M \\ f = 0 & \text{on } \partial M. \end{cases}$$

If the inner normal derivative of the function f is a constant c, then the domain M must be a ball of radius nc and the function f has the form $(n^2 c^2 - r^2)/2n$. Here c is equal to $\frac{\text{Vol}(M)}{\text{Vol}(\partial M)}$. The proof that the Euclidean ball is the unique domain in $\mathbb{R}^n$ supporting a solution to the Serrin problem was given in [24, Thm. 1] by using the method of moving planes, which is based on Hopf's maximum principle. In [26], H.F. Weinberger suggested an elementary proof introducing so-called P-functions for the Laplacian. Since then, the Serrin problem has been generalized to several contexts and when the ambient space is a simply connected space form [3, 14, 18].

A natural question to ask in this set-up is whether the Serrin problem can be extended to differential forms on a domain in $\mathbb{R}^n$. For this purpose, we fix $p \in \{0, \dots, n\}$ and

consider, on the set of differential p-forms $\Omega^p(M)$, a solution to the system

$$\begin{cases} \Delta\omega = \omega_0 & \text{on } M \\ \omega = 0 & \text{on } \partial M, \end{cases}$$

where ω_0 is a given parallel form on $\mathbb{R}^n$ assumed to be of norm 1. We now set the following question: If the conditions $\nu \lrcorner \, d\omega = c\iota^*\omega_0$ and $\iota^*(\delta\omega) = -c\nu \lrcorner \, \omega_0$ are satisfied on ∂M for some constant c and where ν is the inner unit normal vector field to ∂M, can one deduce that the manifold M is a ball of radius nc? Here, $\iota : \partial M \to M$ denotes the inclusion map. Notice that for $p = 0$, the problem that we propose reduces to the usual one on functions. The answer of this question is given in Proposition 2.1 below. Note that, in the whole article, we identify vectors and covectors via the musical isomorphism.

Proposition 2.1. *Let M be a compact manifold with smooth nonempty boundary ∂M and carrying a nontrivial parallel p-form ω_0.*

(1) *If M is a harmonic domain, then for the solution f to the Serrin problem on M the p-form $\omega := f \cdot \omega_0$ is the unique solution to the boundary value problem*

$$\begin{cases} \Delta\omega = \omega_0 & \text{on } M \\ \omega_{|\partial M} = 0 & \text{on } \partial M \\ \nu \lrcorner \, d\omega = c\iota^*\omega_0 & \text{on } \partial M \\ \iota^*(\delta\omega) = -c\nu \lrcorner \, \omega_0 & \text{on } \partial M \end{cases} \tag{2.1}$$

for some constant $c \in \mathbb{R}$.

(2) *Conversely, if (2.1) has a solution $\omega \in \Omega^p(M)$, then assuming w.l.o.g. that $|\omega_0| = 1$ on M, we have that $\omega = f \cdot \omega_0$ where f solves (1.3). As a consequence, M must be a harmonic domain.*

Proof. Before proving the proposition, we begin with the following fact. Given any parallel form α and a smooth function h on M, we have that $\Delta(h\alpha) = (\Delta h)\alpha$. To see this, we first have $d(h\alpha) = dh \wedge \alpha$ and $\delta(h\alpha) = -dh \lrcorner \, \alpha$, as α is parallel. Therefore, if we take $\{e_1, \ldots, e_n\}$ a local orthonormal frame of TM, we compute

$$\Delta(h\alpha) = \delta(dh \wedge \alpha) - d(dh \lrcorner \, \alpha)$$

$$= -\sum_{i=1}^n e_i \lrcorner \, (\nabla_{e_i} dh \wedge \alpha) - \sum_{i=1}^n e_i \wedge (\nabla_{e_i} dh \lrcorner \, \alpha)$$

$$= (\Delta h)\alpha + \sum_{i=1}^n \nabla_{e_i} dh \wedge (e_i \lrcorner \, \alpha) - \sum_{i=1}^n e_i \wedge (\nabla_{e_i} dh \lrcorner \, \alpha)$$

$$= (\Delta h)\alpha,$$

since $\nabla \mathrm{d}h$ is a symmetric 2-tensor field.

As a first consequence, if f solves (1.3), then for any parallel p-form ω_0 on M the p-form $\omega := f \cdot \omega_0$ solves $\Delta \omega = (\Delta f) \cdot \omega_0 = \omega_0$ on M together with $\omega_{|\partial M} = f_{|\partial M} \cdot \omega_0 = 0$. As for the other two boundary conditions, note that, if $\omega_{|\partial M} = 0$, then by $\nabla_X \omega = 0$ for all $X \in T\partial M$ we have that $\mathrm{d}\omega = \nu^\flat \wedge \nabla_\nu \omega$ and $\delta \omega = -\nu \lrcorner \nabla_\nu \omega$. Therefore,

$$\nu \lrcorner \mathrm{d}\omega = \nu \lrcorner \left(\nu^\flat \wedge \nabla_\nu \omega\right) = \nabla_\nu \omega - \nu^\flat \wedge (\nu \lrcorner \nabla_\nu \omega) = \iota^* \nabla_\nu \omega = \partial_\nu f \cdot \iota^* \omega_0 = c \iota^* \omega_0$$

and

$$\iota^*(\delta \omega) = -\iota^*(\nu \lrcorner \nabla_\nu \omega) = -\nu \lrcorner \nabla_\nu \omega = -\partial_\nu f \cdot \nu \lrcorner \omega_0 = -c \nu \lrcorner \omega_0,$$

so that ω solves (2.1). Note that, since the Dirichlet boundary condition $\omega_{|\partial M} = 0$ forces $\ker(\Delta) = \{0\}$ [1, Thm. p. 445], the p-form ω is necessarily the only solution to (2.1). This proves (1).

Conversely, let ω solve (2.1) for some nontrivial parallel p-form ω_0. Up to rescaling ω_0 we may assume that $|\omega_0| = 1$ on M. We consider the function $f := \langle \omega, \omega_0 \rangle$ on M. By the Bochner formula and $\nabla \omega_0 = 0$, we have that

$$\Delta f = \langle \nabla^* \nabla \omega, \omega_0 \rangle = \langle \Delta \omega, \omega_0 \rangle - \underbrace{\left\langle W_M^{[p]} \omega, \omega_0 \right\rangle}_{0} = |\omega_0|^2 = 1.$$

Here, we use the fact that the Bochner operator $W_M^{[p]}$ is a symmetric tensor. Recall that $W_M^{[p]}$ is defined by $W_M^{[p]} := -\sum_{i,j=1}^n e_i \wedge (e_j \lrcorner R^M(e_i, e_j))$, where $\{e_i\}_{i=1,\dots,n}$ denotes a local orthonormal frame of TM and R^M is the Riemann curvature tensor of (M, g) that is, $R_{X,Y}^M = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ for all X, Y tangent to M. Also, it is immediate to see that $f_{|\partial M} = 0$ since $\omega_{|\partial M} = 0$. Therefore, we deduce that

$$\Delta(\omega - f\omega_0) = \omega_0 - (\Delta f)\omega_0 = 0,$$

on M and $(\omega - f\omega_0)_{|\partial M} = 0$. Hence by triviality of the Dirichlet kernel, we deduce that $\omega = f \cdot \omega_0$ on M. In order to finish the proof, we still have to compute the normal derivative of f:

$$\begin{aligned}
\partial_\nu f &= \langle \nabla_\nu \omega, \omega_0 \rangle \\
&= \langle \iota^* \nabla_\nu \omega, \iota^* \omega_0 \rangle + \langle \nu \lrcorner \nabla_\nu \omega, \nu \lrcorner \omega_0 \rangle \\
&= \langle \nu \lrcorner \mathrm{d}\omega, \iota^* \omega_0 \rangle - \langle \iota^*(\delta \omega), \nu \lrcorner \omega_0 \rangle \\
&= c \langle \iota^* \omega_0, \iota^* \omega_0 \rangle + c \langle \nu \lrcorner \omega_0, \nu \lrcorner \omega_0 \rangle \\
&= c |\omega_0|^2 = c.
\end{aligned}$$

Here we used the identities

$$\begin{cases} \nu \lrcorner \nabla_\nu \omega = \delta^{\partial M}(\iota^* \omega) - \iota^*(\delta\omega) - S^{[p-1]}(\nu \lrcorner \omega) + (n-1)H\nu \lrcorner \omega, \\ \iota^* \nabla_\nu \omega = d^{\partial M}(\nu \lrcorner \omega) + \nu \lrcorner d\omega + S^{[p]}(\iota^* \omega). \end{cases}$$

stated in [20, Lem. 18]. Therefore the function f is a solution to the Serrin problem on M. This concludes the proof of (2) and of Proposition 2.1. □

Remark 2.2. We notice that if we impose that $\iota^* \omega_0$ is nowhere vanishing along the boundary, the last boundary condition in (2.1) can be dropped. Indeed, with the boundary condition $\nu \lrcorner d\omega = c\iota^* \omega_0$ and the explicit form $\omega = f \cdot \omega_0$, we compute

$$c\iota^* \omega_0 = \nu \lrcorner d\omega = \nu \lrcorner d(f\omega_0) = (\partial_\nu f)\omega_0 - df \wedge (\nu \lrcorner \omega_0) = (\partial_\nu f)\iota^* \omega_0,$$

from which $\partial_\nu f = c$ along ∂M follows. However, the condition $\iota^* \omega_0 \neq 0$ is not always assured.

The Serrin problem on functions is closely related to the biharmonic Steklov operator, that is the boundary problem (1.1). Indeed, as mentioned in the introduction, on a given compact Riemannian manifold (M^n, g) (not necessarily a domain in $\mathbb{R}^n$), any solution to the Serrin problem is a solution to (1.1) with $q = \frac{1}{c}$. Conversely, it was shown in [21, Thm. 10] that the first positive eigenvalue q_1 of problem (1.1) is bounded from below by the first eigenvalue of the Dirichlet-to-Neumann operator on n-forms (see Section 5 for the definition) and, when equality occurs, the corresponding eigenfunction f of (1.1) is a solution to the Serrin problem. Notice here that, by [6], problem (1.1) admits a discrete spectrum that consists of a nondecreasing sequence of positive eigenvalues $(q_i)_i$ of finite multiplicities.

In order to have a similar situation on differential forms, we come back to the Serrin problem defined in (2.1) for domains in $\mathbb{R}^n$. In fact, one can easily see that any solution to the Serrin problem (2.1) gives rise to a solution to the following boundary problem

$$\begin{cases} \Delta^2 \omega = 0 & \text{on } M \\ \omega = 0 & \text{on } \partial M \\ \nu \lrcorner \Delta\omega + q\iota^* \delta\omega = 0 & \text{on } \partial M \\ \iota^* \Delta\omega - q\nu \lrcorner d\omega = 0 & \text{on } \partial M, \end{cases} \tag{2.2}$$

with $q = \frac{1}{c}$. The equation $\Delta^2 \omega = 0$ comes from taking the Laplacian of $\Delta\omega = \omega_0$ and using the fact that ω_0 is a parallel form. Note here that, because of $\omega_{|\partial M} = 0$, the last two boundary conditions in (2.2) are actually equivalent to

$$\Delta\omega = q\nabla_\nu \omega$$

along ∂M since $\nu \lrcorner \, d\omega = \iota^* \nabla_\nu \omega$ and $\iota^* \delta \omega = -\nu \lrcorner \, \nabla_\nu \omega$ as we have seen in the proof of Proposition 2.1.

It is then natural to study problem (2.2) for compact Riemannian manifolds with smooth boundary that are not necessarily domains in $\mathbb{R}^n$.

2.2. **Biharmonic Steklov operator**

In this section, we will show that the spectrum of problem (2.2) is discrete and it consists entirely of eigenvalues of finite multiplicities. We mainly follow [6].

First, note that on a compact Riemannian manifold (M^n, g) with smooth boundary ∂M, we have the following integration by parts, which is valid for any $\omega, \omega' \in \Omega^p(M)$:

$$\int_M \Big(\langle \Delta \omega, \omega' \rangle - \langle \omega, \Delta \omega' \rangle \Big) \, d\mu_g = \int_{\partial M} \Big(\langle \nu \lrcorner \, d\omega, \iota^* \omega' \rangle - \langle \iota^* \omega, \nu \lrcorner \, d\omega' \rangle$$
$$+ \langle \nu \lrcorner \, \omega, \iota^* \delta \omega' \rangle - \langle \iota^* \delta \omega, \nu \lrcorner \, \omega' \rangle \Big) \, d\mu_g. \quad (2.3)$$

Thus, replacing ω by $\Delta \omega$ in (2.3), we obtain:

$$\int_M \langle \Delta^2 \omega, \omega' \rangle \, d\mu_g = \int_M \langle \Delta \omega, \Delta \omega' \rangle \, d\mu_g + \int_{\partial M} \Big(\langle \nu \lrcorner \, d\Delta \omega, \iota^* \omega' \rangle - \langle \iota^* \Delta \omega, \nu \lrcorner \, d\omega' \rangle$$
$$+ \langle \nu \lrcorner \, \Delta \omega, \iota^* \delta \omega' \rangle - \langle \iota^* \delta \Delta \omega, \nu \lrcorner \, \omega' \rangle \Big) \, d\mu_g. \quad (2.4)$$

The main result of this section is the following:

Theorem 2.3. *Let (M^n, g) be a compact Riemannian manifold with smooth boundary ∂M and let ν be the inward unit vector field normal to the boundary. Then the boundary problem*

$$\begin{cases} \Delta^2 \omega = 0 & on \ M \\ \omega = 0 & on \ \partial M \\ \nu \lrcorner \, \Delta \omega + q \iota^* \delta \omega = 0 & on \ \partial M \\ \iota^* \Delta \omega - q \nu \lrcorner \, d\omega = 0 & on \ \partial M, \end{cases}$$

on p-forms, has a discrete spectrum consisting of an unbounded monotonously nondecreasing sequence of positive eigenvalues of finite multiplicities $(q_{j,p})_{j \geq 1}$.

Proof. As in [6, Eq. (1.7)], we let

$$Z := \big\{ \omega \in \Omega^p(M) \, \big| \, \Delta^2 \omega = 0 \ \text{on} \ M \ \text{and} \ \omega_{|\partial M} = 0 \big\}.$$

We define the following Hermitian sesquilinear forms on $\Omega^p(M)$: for all $\omega, \omega' \in \Omega^p(M)$,

$$(\omega, \omega')_V := \int_M \langle \Delta \omega, \Delta \omega' \rangle \, d\mu_g$$

and

$$(\omega, \omega')_W := \int_{\partial M} \langle \nu \lrcorner \, d\omega, \nu \lrcorner \, d\omega' \rangle \, d\mu_g + \int_{\partial M} \langle \iota^* \delta \omega, \iota^* \delta \omega' \rangle \, d\mu_g.$$

We split the proof of Theorem 2.3 into the following lemmas.

Lemma 2.4. *For $\omega \in Z$, we have*

$$\int_M |\Delta \omega|^2 \, d\mu_g + \int_{\partial M} \Big(\langle \nu \lrcorner \, \Delta \omega, \iota^* \delta \omega \rangle - \langle \iota^* \Delta \omega, \nu \lrcorner \, d\omega \rangle \Big) \, d\mu_g = 0. \qquad (2.5)$$

The forms $(\,\cdot\,,\cdot\,)_V$ and $(\,\cdot\,,\cdot\,)_W$ are positive definite on Z. Moreover, there exists a positive constant C such that $\|\cdot\|_W \leq C \cdot \|\cdot\|_V$ on Z. As a consequence, if we denote by V (resp. W) the completion of Z w.r.t. $\|\cdot\|_V$ (resp. $\|\cdot\|_W$) as Hilbert spaces, then there is a natural bounded linear map $I_1 \colon V \to W$ extending the identity map Id_Z.

Proof. To prove (2.5) for $\omega \in Z$, we replace ω' in (2.4) by ω and use the fact that $\Delta^2 \omega = 0$ and $\iota^* \omega = \nu \lrcorner \, \omega = 0$. Since, if $\Delta \omega = 0$ on M and $\omega_{|\partial M} = 0$, then $\omega = 0$ (see e.g. [1, Thm. p. 445]), the sesquilinear form $(\,\cdot\,,\cdot\,)_V$ is positive definite. For $(\,\cdot\,,\cdot\,)_W$, positive definiteness is a consequence of (2.5). In fact, if $(\omega, \omega)_W = 0$, then $\nu \lrcorner \, d\omega = 0$ and $\iota^* \delta \omega = 0$ on ∂M and therefore, from equation (2.5), $\Delta \omega = 0$ on M, from which $\omega = 0$ on M follows again by [1, Thm. p. 445] since $\omega_{|\partial M} = 0$. We now show, as in [6, Sec. 2], the existence of a positive constant C such that $\|\cdot\|_W \leq C \cdot \|\cdot\|_V$ on Z. First, both $\|\cdot\|_V$ and $\|\cdot\|_{H^2(M)}$ are equivalent on Z. To see this, we have for any $\omega \in Z$

$$\|\omega\|_V = \|\Delta \omega\|_{L^2(M)} \leq C \cdot \|\omega\|_{H^2(M)}$$

for some constant C depending only on M and n. On the other hand, by the elliptic estimates and using the fact that, given any $f \in L^2(M)$, there exists a *unique* weak solution ω to the boundary value problem $\Delta \omega = f$ on M with $\omega_{|\partial M} = 0$, we have, for any $\omega \in Z$ that $\|\omega\|_{L^2(M)} \leq C \cdot \|\Delta \omega\|_{L^2(M)}$ for some constant C, so that

$$\|\omega\|_{H^2(M)} \leq C \cdot \Big(\|\Delta \omega\|_{L^2(M)} + \|\omega\|_{L^2(M)} \Big) \leq C \cdot \|\Delta \omega\|_{L^2(M)} = C \cdot \|\omega\|_V$$

for some positive constant that we also denote by C and which again depends only on M and n, see e.g. [5, Thm. 4 in Sec. 6.3]. Therefore, both $\|\cdot\|_V$ and $\|\cdot\|_{H^2(M)}$ are equivalent on Z. Finally, using the fact that both d and δ are first-order linear differential operators, we estimate, for any $\omega \in Z$,

$$\begin{aligned}
\|\omega\|_W^2 &= \|\nu \lrcorner \, d\omega\|_{L^2(\partial M)}^2 + \|\iota^* \delta \omega\|_{L^2(\partial M)}^2 \\
&\leq C \cdot \|\omega\|_{H^1(\partial M)}^2 \\
&\leq C \cdot \|\omega\|_{H^2(M)}^2 \\
&\leq C \cdot \|\omega\|_V^2
\end{aligned}$$

for some positive constant that we also denote by C, which again depends only on M and n. Here we have also used the boundedness of the trace map $T: H^2(M) \to H^1(\partial M)$. $\qquad\square$

Next we consider the linear operator $K: V \to V$ defined by

$$K := D_V^{-1} \circ {}^t I_1 \circ D_W \circ I_1,$$

where $D_V: V \to V'$ and $D_W: W \to W'$ are the natural duality isomorphisms, i.e. $D_V(\omega) := (\cdot, \omega)_V$ and $D_W(\omega) := (\cdot, \omega)_W$ for every ω in V resp. W. As usual, ${}^t I_1(\theta) := \theta \circ I_1 \in V'$ for every $\theta \in W'$. Actually K can be defined via the identity

$$(K\omega, \omega')_V = (I_1\omega, I_1\omega')_W \ (= (\omega, \omega')_W)$$

for all $\omega, \omega' \in V$. By definition, the operator K is self-adjoint and positive semi-definite. We need now to prove the following.

Lemma 2.5. *The map I_1, defined in the previous lemma, is compact and injective. Therefore K is also compact and injective.*

Proof. Let $I_3: V \to L^2(\partial M) \oplus L^2(\partial M)$ be the composition of the following linear maps:

$$
\begin{array}{ccccc}
V & \longrightarrow & H^{\frac{1}{2}}(\partial M) \oplus H^{\frac{1}{2}}(\partial M) & \longrightarrow & L^2(\partial M) \oplus L^2(\partial M) \\
\omega & \longmapsto & (\nu \lrcorner\, d\omega, \iota^*\delta\omega) & \longmapsto & (\nu \lrcorner\, d\omega, \iota^*\delta\omega).
\end{array}
$$

Note that I_3 is well-defined since the trace operator maps $H^1(M)$ into (and onto) $H^{\frac{1}{2}}(\partial M)$. Moreover, since the inclusion map $H^{\frac{1}{2}}(\partial M) \to L^2(\partial M)$ is compact by the Rellich–Kondrachov theorem, so is I_3. Now $(Z, (\cdot, \cdot)_W) \to L^2(\partial M) \oplus L^2(\partial M)$, $\omega \mapsto (\nu \lrcorner\, d\omega, \iota^*\delta\omega)$, is a linear isometry, therefore it extends to a linear isometry $I_2: W \to L^2(\partial M) \oplus L^2(\partial M)$ with $I_3 = I_2 \circ I_1$. Since I_3 is compact and I_2 is a linear isometry, I_1 must be compact.

We now prove that I_3 is injective, so that I_1 must be injective as well. First, we show the inclusion

$$V \subset \left\{ \omega \in H^2(M) \cap H_0^1(M) \,\middle|\, \Delta^2\omega = 0 \text{ weakly on } M \right\},$$

where the concept of a weak solution is defined by the following: given $f \in L^2(M)$, a weak solution ω to $\Delta^2\omega = f$ on M is a form $\omega \in H^2 \cap H_0^1(M) := H^2(M) \cap H_0^1(M)$ with

$$(\Delta\omega, \Delta\omega')_{L^2(M)} = (f, \omega')_{L^2(M)} \ \ \forall\, \omega' \in H_0^2(M), \tag{2.6}$$

where $H_0^2(M) := \{\omega \in H^2(M) \,|\, \omega_{|\partial M} = 0 \text{ and } (\nabla\omega)_{|\partial M} = 0\}$. Note that the condition $(\nabla\omega)_{|\partial M} = 0$ can be replaced by $\nabla_\nu\omega = 0$ along ∂M because of $\omega_{|\partial M} = 0$. Namely $V \subset H^2(M)$ already holds because of the equivalence of $\|\cdot\|_V$ and $\|\cdot\|_{H^2(M)}$ on Z. Moreover $V \subset H_0^1(M)$ holds as well because of the continuous inclusion map $H^2(M) \subset H^1(M)$ and the continuity of the trace operator $H^1(M) \to L^2(\partial M)$. Thus $V \subset H^2 \cap H_0^1(M)$. Furthermore, if $\omega \in V$ is given, then there exists a sequence $(\omega_m)_m$

in Z with $\|\omega_m - \omega\|_V \xrightarrow[m\to\infty]{} 0$. Because of $V \subset H^2(M)$ and the equivalence of $\|\cdot\|_V$ and $\|\cdot\|_{H^2(M)}$, the sequence $(\omega_m)_m$ goes to ω in $H^2(M)$ and hence $(\Delta\omega_m)_m$ goes to $\Delta\omega$ in $L^2(M)$. But since, as a consequence of (2.4), we have, for all $m \in \mathbb{N}$ and $\omega' \in \{\omega \in \Omega^p(M) \mid \omega_{|\partial M} = 0 \text{ and } (\nabla\omega)_{|\partial M} = 0\}, 0 = (\Delta\omega_m, \Delta\omega')_{L^2(M)}$, we can deduce that $(\Delta\omega, \Delta\omega')_{L^2(M)} = 0$ for all ω' as above and therefore for all $\omega' \in H_0^2(M)$. This shows that $\omega \in H^2 \cap H_0^1(M)$ satisfies $\Delta^2\omega = 0$ weakly on M and therefore the inclusion is proved. Now, we come back to the injectivity of I_3. Consider $\omega \in V$ such that $I_3(\omega) = 0$, that is $\nu \lrcorner\, d\omega = 0$ and $\iota^*\delta\omega = 0$ in $L^2(\partial M)$. Then both $d\omega$ and $\delta\omega$ vanish along ∂M because of $\iota^*\omega = 0$, $\nu \lrcorner\, \omega = 0$ and the identities $[d, \iota^*] = 0$ and $\{\delta, \nu\lrcorner\} = 0$. Again, because of $\omega_{|\partial M} = 0$, we have $(d\omega)_{|\partial M} = \nu^\flat \wedge \nabla_\nu\omega$ and $(\delta\omega)_{|\partial M} = -\nu \lrcorner\, \nabla_\nu\omega$, so that $\nu^\flat \wedge \nabla_\nu\omega = 0$ and $\nu \lrcorner\, \nabla_\nu\omega = 0$ along ∂M, from which $\nabla_\nu\omega = 0$ on ∂M follows. This shows that $\omega \in H_0^2(M)$. Taking $\omega' = \omega$ in (2.6), we deduce that $\Delta\omega = 0$ and therefore $\omega = 0$ on M. This shows I_3 and hence I_1 to be injective. $\qquad\square$

We end the proof of Theorem 2.3. Since K is compact, self-adjoint and positive definite in the Hilbert space V, there is a countable Hilbert o.n.b. $(\omega_i)_{i \geq 1}$ of V for which a monotonously nonincreasing positive real sequence $(\mu_i)_{i \geq 1}$ going to 0 exists such that $K\omega_i = \mu_i\omega_i$ for all $i \geq 1$. We want to show that, for each $i \geq 1$, the eigenform ω_i lies in Z and satisfies $\mu_i\iota^*\Delta\omega_i = \nu \lrcorner\, d\omega_i$ as well as $\mu_i\nu \lrcorner\, \Delta\omega_i = -\iota^*\delta\omega_i$ along ∂M. Hence, for each i, the form ω_i becomes a smooth eigenform for problem (2.2) associated with the eigenvalue $q_{i,p} = \frac{1}{\mu_i}$ which is of finite multiplicity, since μ_i is.

For this purpose, fix $i \geq 1$. Since $\omega_i \in V$, we already know that $\Delta^2\omega_i = 0$ holds weakly on M with $\omega_{i|\partial M} = 0$. It remains to show that ω_i is smooth and satisfies $\iota^*\Delta\omega_i = \frac{1}{\mu_i}\nu \lrcorner\, d\omega_i$ as well as $\nu \lrcorner\, \Delta\omega_i = -\frac{1}{\mu_i}\iota^*\delta\omega_i$ along ∂M. By definition, for every $\omega \in Z$,

$$\mu_i\,(\Delta\omega_i, \Delta\omega)_{L^2(M)} = \mu_i\,(\omega_i, \omega)_V = (K\omega_i, \omega)_V = (\omega_i, \omega)_W$$
$$= (\nu \lrcorner\, d\omega_i, \nu \lrcorner\, d\omega)_{L^2(\partial M)} + (\iota^*\delta\omega_i, \iota^*\delta\omega)_{L^2(\partial M)}. \qquad (2.7)$$

But by (2.4), we have, still for every $\omega \in Z$,

$$(\Delta\omega_i, \Delta\omega)_{L^2(M)} = \Big(\underbrace{\Delta^2\omega_i}_{0}, \omega\Big)_{L^2(M)} - \Big(\nu \lrcorner\, d\Delta\omega_i, \underbrace{\iota^*\omega}_{0}\Big)_{L^2(\partial M)}$$
$$+ (\iota^*\Delta\omega_i, \nu \lrcorner\, d\omega)_{L^2(\partial M)} - (\nu \lrcorner\, \Delta\omega_i, \iota^*\delta\omega)_{L^2(\partial M)}$$
$$+ \Big(\iota^*\delta(\Delta\omega_i), \underbrace{\nu \lrcorner\, \omega}_{0}\Big)_{L^2(\partial M)}$$
$$= (\iota^*\Delta\omega_i, \nu \lrcorner\, d\omega)_{L^2(\partial M)} - (\nu \lrcorner\, \Delta\omega_i, \iota^*\delta\omega)_{L^2(\partial M)}. \qquad (2.8)$$

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Here we see both $\iota^*\Delta\omega_i$ and $\nu \lrcorner \Delta\omega_i$ as elements in $H^{-\frac{1}{2}}(\partial M)$. Comparing (2.7) and (2.8), we deduce that

$$\left(\iota^*\Delta\omega_i - \frac{1}{\mu_i}\nu \lrcorner \,\mathrm{d}\omega_i, \nu \lrcorner \,\mathrm{d}\omega\right)_{L^2(\partial M)} - \left(\nu \lrcorner \Delta\omega_i + \frac{1}{\mu_i}\iota^*\delta\omega_i, \iota^*\delta\omega\right)_{L^2(\partial M)} = 0$$

for all $\omega \in Z$. Note that the map $Z \to \Omega^p(\partial M) \oplus \Omega^{p-1}(\partial M), \omega \mapsto (\nu \lrcorner \,\mathrm{d}\omega, \iota^*\delta\omega)$ is continuous w.r.t. $\|\cdot\|_V$ and $\|\cdot\|_{L^2(\partial M)}$ and is injective since it is the restriction to Z of the map I_3 from above. Now, Lemma A.1 in the appendix shows that this map is onto. Hence, it follows that $\iota^*\Delta\omega_i - \frac{1}{\mu_i}\nu \lrcorner \,\mathrm{d}\omega_i = 0$ and $\nu \lrcorner \Delta\omega_i + \frac{1}{\mu_i}\iota^*\delta\omega_i = 0$. Therefore, ω_i is an eigenform for (2.2) associated to the eigenvalue $q_{i,p} = \frac{1}{\mu_i}$. The smoothness of ω_i follows from the fact that both boundary conditions $\iota^*\Delta\omega_i - \frac{1}{\mu_i}\nu \lrcorner \,\mathrm{d}\omega_i = 0$ and $\nu \lrcorner \Delta\omega_i + \frac{1}{\mu_i}\iota^*\delta\omega_i = 0$ together with $\omega_{i|\partial M} = 0$ define elliptic boundary conditions for Δ^2.

In order to finish the proof of Theorem 2.3, it remains to show that there is a one-to-one correspondence between solutions of (2.2) and eigenforms of K. We have already shown that every eigenform ω of K, associated to some eigenvalue $\mu > 0$, satisfies (2.2) with $q = \frac{1}{\mu}$. Conversely, if $q \in \mathbb{R}$ is given for which a nontrivial solution ω to (2.2) exists, then by (2.4), we have, for every $\omega' \in Z$,

$$(\Delta\omega, \Delta\omega')_{L^2(M)} = q \cdot \left((\nu \lrcorner \,\mathrm{d}\omega, \nu \lrcorner \,\mathrm{d}\omega')_{L^2(\partial M)} + (\iota^*\delta\omega, \iota^*\delta\omega')_{L^2(\partial M)}\right),$$

that is, seeing both ω, ω' as elements of V,

$$(\omega, \omega')_V = q \cdot (\omega, \omega')_W .$$

Note that necessarily $q > 0$ holds, otherwise $\omega = 0$ would follow. By definition of K, we then have $(\omega, \omega')_V = q \cdot (K\omega, \omega')_V$ for all $\omega' \in Z$ and hence in V, therefore $K\omega = \frac{1}{q}\omega$. This shows ω to be an eigenform of K associated to the eigenvalue $\mu = \frac{1}{q}$. This shows the correspondence to be one-to-one. This concludes the proof of Theorem 2.3. $\square$

In the following, we give a characterization for the first eigenvalue $q_{1,p}$ on p-forms. This will be used later in order to get estimates for the eigenvalues.

Theorem 2.6. *The first eigenvalue $q_{1,p}$ of the boundary problem* (2.2) *is characterized by*

$$q_{1,p} = \inf_{\omega \in \Omega^p(M)} \left\{ \frac{\|\Delta\omega\|^2_{L^2(M)}}{\|\nu \lrcorner \,\mathrm{d}\omega\|^2_{L^2(\partial M)} + \|\iota^*\delta\omega\|^2_{L^2(\partial M)}} \,\middle|\, \omega_{|\partial M} = 0, \nabla_\nu\omega \neq 0 \right\} \qquad (2.9)$$

$$= \inf_{\omega \in \Omega^p(M)\setminus\{0\}} \left\{ \frac{\|\omega\|^2_{L^2(\partial M)}}{\|\omega\|^2_{L^2(M)}} \,\middle|\, \Delta\omega = 0 \text{ on } M \right\}. \qquad (2.10)$$

Both infima are indeed minima, (2.9) is attained by an eigenform of (2.2), associated to $q_{1,p}$ and (2.10) is attained by $\Delta\omega$, where ω is an eigenform of (2.2), associated to $q_{1,p}$. Moreover, for every $\omega \in \Omega^p(M)$ with $\omega_{|\partial M} = 0$, the inequality

$$q_{1,p} \cdot \left(\|\nu \lrcorner d\omega\|^2_{L^2(\partial M)} + \|\iota^*\delta\omega\|^2_{L^2(\partial M)} \right) \leq \|\Delta\omega\|^2_{L^2(M)}$$

holds.

Proof. As mentioned above, it follows from (2.5) that, given any nonzero eigenform ω associated to a positive eigenvalue q of (2.2), we have

$$\int_M |\Delta\omega|^2 \, d\mu_g = q \int_{\partial M} |\nu \lrcorner d\omega|^2 \, d\mu_g + q \int_{\partial M} |\iota^*\delta\omega|^2 \, d\mu_g,$$

so that

$$q_{1,p} \leq \frac{\|\Delta\omega\|^2_{L^2(M)}}{\|\nu \lrcorner d\omega\|^2_{L^2(\partial M)} + \|\iota^*\delta\omega\|^2_{L^2(\partial M)}}$$

for every such eigenform, with equality for ω associated to $q_{1,p}$ of course. More generally, for every $\omega \in V$, we can write $\omega = \sum_i (\omega, \omega_i)_V \cdot \omega_i$ because $(\omega_i)_i$ is a Hilbert orthonormal basis of V. From this, we can express

$$\|\Delta\omega\|^2_{L^2(M)} = \|\omega\|^2_V = \sum_i |(\omega, \omega_i)_V|^2$$

on the one hand, and

$$\begin{aligned}
\|\nu \lrcorner d\omega\|^2_{L^2(\partial M)} + \|\iota^*\delta\omega\|^2_{L^2(\partial M)} &= \|\omega\|^2_W \\
&= (K\omega, \omega)_V \\
&= \sum_i \frac{1}{q_{i,p}} |(\omega, \omega_i)_V|^2 \\
&\leq \frac{1}{q_{1,p}} \sum_i |(\omega, \omega_i)_V|^2 \\
&\leq \frac{1}{q_{1,p}} \|\Delta\omega\|^2_{L^2(M)}
\end{aligned}$$

on the other hand, therefore

$$q_{1,p} = \inf \left\{ \frac{\|\Delta\omega\|^2_{L^2(M)}}{\|\nu \lrcorner d\omega\|^2_{L^2(\partial M)} + \|\iota^*\delta\omega\|^2_{L^2(\partial M)}} \,\middle|\, \omega \in V \setminus \{0\} \right\}.$$

Note that, for some nonzero $\omega \in V$, the denominator $\|\nu \lrcorner d\omega\|^2_{L^2(\partial M)} + \|\iota^*\delta\omega\|^2_{L^2(\partial M)}$ in the above quotient cannot vanish since if it did, then $\|\omega\|_W = 0$ and therefore $\omega = 0$ by Lemma 2.5. Now, we will show that this infimum can be taken over all

$\omega \in (H^2 \cap H_0^1)(M) \setminus \{0\}$, as well as over all smooth forms vanishing and whose normal derivative does not vanish identically along ∂M. Recall that,

$$H_0^2(M) = \left\{ \omega \in \left(H^2 \cap H_0^1 \right)(M) \,\middle|\, \iota^* \delta \omega = 0 \text{ and } \nu \lrcorner\, d\omega = 0 \right\},$$

since, as we noticed above, if $\omega \in (H^2 \cap H_0^1)(M)$ is such that both $\iota^* \delta \omega$ and $\nu \lrcorner\, d\omega$ vanish along ∂M, then so does $\nabla_\nu \omega$. As in [6, Thm. 1.2], we have the following lemma:

Lemma 2.7. *The inner product* $(\,\cdot\,,\cdot\,)_V$ *is well defined on* $(H^2 \cap H_0^1)(M)$ *and we have the following* $(\,\cdot\,,\cdot\,)_V$*-orthogonal splitting:*

$$H^2 \cap H_0^1(M) = V \oplus H_0^2(M).$$

Proof. By its definition, $(\,\cdot\,,\cdot\,)_V$ is well defined on $(H^2 \cap H_0^1)(M)$. Furthermore, (2.4) already implies that, for all $(\omega, \omega') \in Z \times H^2 \cap H_0^1(M)$,

$$(\omega, \omega')_V = (\iota^* \Delta \omega, \nu \lrcorner\, d\omega')_{L^2(\partial M)} - (\nu \lrcorner\, \Delta \omega, \iota^* \delta \omega')_{L^2(\partial M)}, \qquad (2.11)$$

so that $(\omega, \omega')_V = 0$ as soon as $\omega' \in H_0^2(M)$. This shows that $H_0^2(M) \subset Z^\perp = V^\perp$. Conversely, let $\omega' \in V^\perp \subset H^2 \cap H_0^1(M)$. Then $(\omega_i, \omega')_V = 0$ for all i. By (2.11), this is equivalent to $(\nu \lrcorner\, d\omega_i, \nu \lrcorner\, d\omega')_{L^2(\partial M)} + (\iota^* \delta \omega_i, \iota^* \delta \omega')_{L^2(\partial M)} = 0$ for all i by $\iota^* \Delta \omega_i = \frac{1}{\mu_i} \nu \lrcorner d\omega_i$ and $\nu \lrcorner \Delta \omega_i = -\frac{1}{\mu_i} \iota^*(\delta \omega_i)$. Since the map $Z \to \Omega^p(\partial M) \oplus \Omega^{p-1}(\partial M)$, $\omega \mapsto (\nu \lrcorner\, d\omega, \iota^* \delta \omega)$ is bounded (w.r.t. $\|\cdot\|_V$ and $\|\cdot\|_{L^2(\partial M)}$), onto and the ω_i's span a dense subspace of V, we obtain that $(\omega_1, \nu \lrcorner\, d\omega')_{L^2(\partial M)} = 0$ as well as $(\omega_2, \nu \lrcorner\, d\omega')_{L^2(\partial M)} = 0$ for all $(\omega_1, \omega_2) \in \Omega^p(\partial M) \oplus \Omega^{p-1}(\partial M)$ and therefore $\nu \lrcorner\, d\omega' = 0$ and $\iota^* \delta \omega' = 0$ hold along ∂M. This shows that $\omega' \in H_0^2(M)$. On the whole, $H_0^2(M) = V^\perp$ and the orthogonal splitting is proved. $\qquad \square$

It remains to notice that, for any $\omega \in \Omega^p(M)$ with $\omega_{|\partial M} = 0$, we can split $(\,\cdot\,,\cdot\,)_V$-orthogonally $\omega = \omega_V + \widehat{\omega}$, where $\omega_V \in V$ and $\widehat{\omega} \in H_0^2(M)$. Then

$$\|\Delta \omega\|_{L^2(M)}^2 = \|\omega\|_V^2 = \|\omega_V\|_V^2 + \|\widehat{\omega}\|_V^2$$

on the one hand, and

$$\|\nu \lrcorner\, d\omega\|_{L^2(\partial M)}^2 + \|\iota^* \delta \omega\|_{L^2(\partial M)}^2 = \|\nu \lrcorner\, d\omega_V\|_{L^2(\partial M)}^2 + \|\iota^* \delta \omega_V\|_{L^2(\partial M)}^2$$

on the other hand, so that

$$\begin{aligned}
\|\Delta \omega\|_{L^2(M)}^2 &= \|\Delta \omega_V\|_{L^2(M)}^2 + \|\Delta \widehat{\omega}\|_{L^2(M)}^2 \\
&\geq \|\Delta \omega_V\|_{L^2(M)}^2 \\
&\geq q_{1,p} \cdot \left(\|\nu \lrcorner\, d\omega_V\|_{L^2(\partial M)}^2 + \|\iota^* \delta \omega_V\|_{L^2(\partial M)}^2 \right) \\
&= q_{1,p} \cdot \left(\|\nu \lrcorner\, d\omega\|_{L^2(\partial M)}^2 + \|\iota^* \delta \omega\|_{L^2(\partial M)}^2 \right),
\end{aligned}$$

which proves (2.9). Furthermore, the r.h.s. of (2.9) is actually a minimum attained exactly by those eigenforms of the biharmonic Steklov problem that are associated to the smallest positive eigenvalue $q_{1,p}$. And that same r.h.s. vanishes if and only if $\nabla_\nu \omega = 0$ along ∂M as we noticed above.

We now prove the following lemma:

Lemma 2.8. *The infimum in* (2.10)*, that we denote by* $q'_{1,p}$*, is a positive minimum.*

Proof. To prove this result, we apply the same argument as in [6, p. 318]. Namely the standard Rellich–Kondrachov compactness theorem ensures the natural inclusion map $H^{\frac{1}{2}}(\partial M) \to L^2(\partial M)$ to be compact. Hence its transpose map $L^2(\partial M) \to H^{-\frac{1}{2}}(\partial M)$ is compact, as a straightforward consequence. Moreover, the harmonic extension from ∂M to M defines a bounded linear map $H^{-\frac{1}{2}}(\partial M) \to L^2(M)$, see e.g. [17, Thm. 6.6, Chap. 2]. Therefore the composition $L^2(\partial M) \to H^{-\frac{1}{2}}(\partial M) \to L^2(M)$ of both maps defines a compact linear map $E: L^2(\partial M) \to L^2(M)$, which already shows that $q'_{1,p}$ to be positive using only the boundedness of the map. Furthermore, because the image by E of the unit sphere $\mathbb{S} := \{\omega \in L^2(\partial M), \|\omega\|_{L^2(\partial M)} = 1\}$ of $L^2(\partial M)$ is relatively compact in $L^2(M)$, there exists $\widetilde{\omega} \in \overline{E(\mathbb{S})}$ such that $\|\widetilde{\omega}\|_{L^2(M)} = \sup\{\|E\omega\|_{L^2(M)} \mid \omega \in \mathbb{S}\}$. By definition of the closure, there exists a sequence $(\omega_m)_m$ in $\mathbb{S}$ such that $E\omega_m \xrightarrow[m\to\infty]{} \widetilde{\omega}$ in $L^2(M)$. But then $\Delta(E\omega_m) \xrightarrow[m\to\infty]{} \Delta\widetilde{\omega}$ in $H^{-2}(M) = (H_0^2(M))'$, so that necessarily $\Delta\widetilde{\omega} = 0$ holds in $H^{-2}(M)$. Note that, because $\Delta\widetilde{\omega} = 0$ and $\Delta(E\omega_m) = 0$ for all m, we can also claim that $\Delta(E\omega_m) \xrightarrow[m\to\infty]{} \Delta\widetilde{\omega}$ in $L^2(M)$. By Gårding's inequality and since both $E\omega_m \xrightarrow[m\to\infty]{} \widetilde{\omega}$ and $\Delta(E\omega_m) \xrightarrow[m\to\infty]{} \Delta\widetilde{\omega}$ in $L^2(M)$, we have that $E\omega_m \xrightarrow[m\to\infty]{} \widetilde{\omega}$ in $H^2(M)$. As a consequence, because of $\nu \lrcorner E\omega_m = 0$ along ∂M and $\nu \lrcorner E\omega_m \xrightarrow[m\to\infty]{} \nu \lrcorner \widetilde{\omega}$ in $L^2(\partial M)$, necessarily $\nu \lrcorner \widetilde{\omega} = 0$ holds along ∂M. Now again $E\omega_m \xrightarrow[m\to\infty]{} \widetilde{\omega}$ in $H^2(M)$ also implies $\omega_m = (E\omega_m)_{|\partial M} \xrightarrow[m\to\infty]{} \widetilde{\omega}_{|\partial M}$ in $L^2(\partial M)$ (actually also in $H^1(\partial M)$), so that $\|\widetilde{\omega}_{|\partial M}\|_{L^2(\partial M)} = 1$. This shows that $\widetilde{\omega} \in E(\mathbb{S})$ and hence $\|\widetilde{\omega}\|_{L^2(M)} = \max\{\|E\omega\|_{L^2(M)} \mid \omega \in \mathbb{S}\}$ satisfies $\|\widetilde{\omega}\|_{L^2(M)} = \frac{1}{q'_{1,p}}$. This shows the positive r.h.s. $q'_{1,p}$ of (2.10) to be a minimum. $\square$

In order to finish the proof of Theorem 2.6, we want to show that $q_{1,p} = q'_{1,p}$. Pick any eigenform ω associated to $q_{1,p}$. Up to rescaling ω, we may assume that $\|\nu \lrcorner d\omega\|_{L^2(\partial M)}^2 + \|\iota^*\delta\omega\|_{L^2(\partial M)}^2 = 1$. Let $\widehat{\omega} \in \Omega^p(M)$ be the unique solution to $\Delta\widehat{\omega} = 0$ on M with $\iota^*\widehat{\omega}_{|\partial M} = -q_{1,p}\nu \lrcorner d\omega$ a well as $\nu \lrcorner \widehat{\omega} = q_{1,p}\iota^*\delta\omega$ along ∂M. By (2.3),

$$0 = \int_M \langle\Delta\widehat{\omega}, \omega\rangle \, d\mu_g = \int_M \langle\widehat{\omega}, \Delta\omega\rangle \, d\mu_g + q_{1,p} \int_{\partial M} \left(|\nu \lrcorner d\omega|^2 + |\iota^*\delta\omega|^2\right) d\mu_g$$

$$= \int_M \langle\widehat{\omega}, \Delta\omega\rangle \, d\mu_g + q_{1,p},$$

so that Cauchy–Schwarz inequality leads to $q_{1,p} \leq \|\Delta\omega\|_{L^2(M)} \cdot \|\widehat{\omega}\|_{L^2(M)}$. Using $q_{1,p} = \|\Delta\omega\|_{L^2(M)}^2$, we obtain $q_{1,p} \leq \|\widehat{\omega}\|_{L^2(M)}^2$. Therefore,

$$\frac{\|\widehat{\omega}\|_{L^2(\partial M)}^2}{\|\widehat{\omega}\|_{L^2(M)}^2} = \frac{q_{1,p}^2}{\|\widehat{\omega}\|_{L^2(M)}^2} \leq q_{1,p},$$

from which $q_{1,p}' \leq q_{1,p}$ follows. Conversely, if $\omega \in \Omega^p(M) \setminus \{0\}$ satisfies $\Delta\omega = 0$ on M and $\|\omega\|_{L^2(\partial M)}^2 = q_{1,p}' \|\omega\|_{L^2(M)}^2$, then let $\widehat{\omega}$ be the solution to $\Delta\widehat{\omega} = \omega$ on M with $\widehat{\omega}_{|\partial M} = 0$. Then, again by (2.3), $\omega_{|\partial M} = \iota^*\omega + \nu^\flat \wedge (\nu \lrcorner \omega)$ and Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\|\omega\|_{L^2(M)}^2 &= (\Delta\widehat{\omega}, \omega)_{L^2(M)} \\
&= \Big(\widehat{\omega}, \underbrace{\Delta\omega}_{0}\Big)_{L^2(M)} + (\nu \lrcorner d\widehat{\omega}, \iota^*\omega)_{L^2(\partial M)} - (\iota^*\delta\widehat{\omega}, \nu \lrcorner \omega)_{L^2(\partial M)} \\
&= (\nu \lrcorner d\widehat{\omega}, \omega)_{L^2(\partial M)} - \Big(\nu^\flat \wedge \iota^*\delta\widehat{\omega}, \omega\Big)_{L^2(\partial M)} \\
&= \Big(\nu \lrcorner d\widehat{\omega} - \nu^\flat \wedge \iota^*\delta\widehat{\omega}, \omega\Big)_{L^2(\partial M)} \\
&\leq \Big\|\nu \lrcorner d\widehat{\omega} - \nu^\flat \wedge \iota^*\delta\widehat{\omega}\Big\|_{L^2(\partial M)} \cdot \|\omega\|_{L^2(\partial M)}.
\end{aligned}$$

But since $\Big(\nu \lrcorner d\widehat{\omega}, \nu^\flat \wedge \iota^*\delta\widehat{\omega}\Big)_{L^2(\partial M)} = 0$, we have

$$\begin{aligned}
\Big\|\nu \lrcorner d\widehat{\omega} - \nu^\flat \wedge \iota^*\delta\widehat{\omega}\Big\|_{L^2(\partial M)}^2 &= \|\nu \lrcorner d\widehat{\omega}\|_{L^2(\partial M)}^2 + \Big\|\nu^\flat \wedge \iota^*\delta\widehat{\omega}\Big\|_{L^2(\partial M)}^2 \\
&= \|\nu \lrcorner d\widehat{\omega}\|_{L^2(\partial M)}^2 + \|\iota^*\delta\widehat{\omega}\|_{L^2(\partial M)}^2,
\end{aligned}$$

so that

$$\|\omega\|_{L^2(M)}^2 \leq \Big(\|\nu \lrcorner d\widehat{\omega}\|_{L^2(\partial M)}^2 + \|\iota^*\delta\widehat{\omega}\|_{L^2(\partial M)}^2\Big)^{\frac{1}{2}} \cdot \|\omega\|_{L^2(\partial M)}.$$

Therefore,

$$
\frac{\|\Delta\widehat{\omega}\|^2_{L^2(M)}}{\|v \lrcorner d\widehat{\omega}\|^2_{L^2(\partial M)} + \|\iota^*\delta\widehat{\omega}\|^2_{L^2(\partial M)}} = \frac{\|\omega\|^2_{L^2(M)}}{\|v \lrcorner d\widehat{\omega}\|^2_{L^2(\partial M)} + \|\iota^*\delta\widehat{\omega}\|^2_{L^2(\partial M)}}
$$

$$
\leq \frac{\|\omega\|^2_{L^2(M)} \cdot \|\omega\|^2_{L^2(\partial M)}}{\|\omega\|^4_{L^2(M)}}
$$

$$
= \frac{\|\omega\|^2_{L^2(\partial M)}}{\|\omega\|^2_{L^2(M)}}
$$

$$
= q'_{1,p},
$$

from which $q'_{1,p} \geq q_{1,p}$ follows. On the whole, we deduce that $q_{1,p} = q'_{1,p}$, as we claimed. Moreover, the p-form $\widehat{\omega}$ defined above, because it now minimizes $q_{1,p}$, must be an eigenform of the biharmonic Steklov operator associated to the eigenvalue $q_{1,p}$. As a consequence, $\omega = \Delta\widehat{\omega}$ where $\widehat{\omega}$ is an eigenform of the biharmonic Steklov operator associated to the eigenvalue $q_{1,p}$. $\qquad\square$

3. Eigenvalues of the biharmonic Steklov operator

In this section, we will establish some eigenvalue estimates for the first eigenvalue of the biharmonic Steklov operator defined in the previous section.

As before, we will consider a compact Riemannian manifold (M^n, g) with smooth boundary ∂M. Notice first the following fact:

Lemma 3.1. *The biharmonic Steklov operator is preserved by the Hodge star operator* $*_M$ *on M.*

Proof. We only need to check that the last two boundary conditions in (2.2) are preserved. For this purpose, using the equality $\iota^*(*_M\alpha) = *_{\partial M}(v \lrcorner \alpha)$ for any form α, we compute, for any solution ω of degree p to problem (2.2),

$$
v \lrcorner \Delta(*_M\omega) = v \lrcorner (*_M\Delta\omega)
$$

$$
= (-1)^p *_{\partial M} \iota^*(\Delta\omega)
$$

$$
= (-1)^p q *_{\partial M} (v \lrcorner d\omega)
$$

$$
= (-1)^p q\iota^*(*_M d\omega)
$$

$$
= -q\iota^*(\delta *_M \omega).
$$

In the last equality, we use the fact that $*_M d = (-1)^{p-1}\delta*_M$ on p-forms. For the other boundary condition, we have $\iota^*(\Delta *_M \omega) = \iota^*(*_M\Delta\omega) = *_{\partial M}(v \lrcorner \Delta\omega) = -q *_{\partial M} (\iota^*\delta\omega) =$

$(-1)^p qv \lrcorner *_M \delta\omega = qv \lrcorner \mathrm{d}(*_M\omega)$. Also here we use the fact that $d*_M = (-1)^p *_M \delta$ on p-forms. This finishes the proof. $\qquad\square$

Remark 3.2. As a direct consequence of the invariance of the biharmonic Steklov operator by the Hodge star operator is that $q_{i,p} = q_{i,n-p}$ for any $i \geq 1$ and $p \leq n$.

In the following, we recall the estimate stated by S. Raulot and A. Savo in [22] for subharmonic functions that we will use in order to get a lower bound of the first eigenvalue $q_{1,p}$. Let (M^n, g) be a compact Riemannian manifold with smooth boundary such that the Ricci curvature of M satisfies $\mathrm{Ric}_M \geq (n-1)K$ and the mean curvature of the boundary satisfies $H \geq H_0$ for some real numbers K and H_0. Let R be the inner radius of the manifold M, that is

$$R = \max \{\mathrm{dist}(x, \partial M) \mid x \in M\},$$

and $\Theta(r) = (s'_K(r) - H_0 s_K(r))^{n-1}$ for all r, where the function s_K is being given by

$$s_K(r) := \begin{cases} \frac{1}{\sqrt{K}} \sin\left(r\sqrt{K}\right) & \text{if } K > 0, \\ r & \text{if } K = 0, \\ \frac{1}{\sqrt{|K|}} \sinh\left(r\sqrt{|K|}\right) & \text{if } K < 0. \end{cases}$$

It was shown in [22, Prop. 14] (see also [12, Thm. A]) that the function Θ is smooth and positive on $[0, R[$ and $\Theta(R) = 0$ when M is a geodesic ball in M_K, the space form of sectional curvature K. The following result was proved in [22]:

Theorem 3.3 ([22, Thm. 10]). *Let (M^n, g) be a compact Riemannian manifold with smooth boundary. Assume that the Ricci curvature of M satisfies $\mathrm{Ric}_M \geq (n-1)K$ and the mean curvature of ∂M fulfils $H \geq H_0$ for some real numbers K and H_0. If h is a non-trivial, nonnegative subharmonic function on M (i.e. $\Delta h \leq 0$ on M), then*

$$\frac{\int_{\partial M} h \mathrm{d}\mu_g}{\int_M h \mathrm{d}\mu_g} \geq \frac{1}{\int_0^R \Theta(r) \, \mathrm{d}r}.$$

Using this result and the Bochner formula $\Delta = \nabla^*\nabla + W_M^{[p]}$ on p-forms, we prove the following:

Theorem 3.4. *Let (M^n, g) be a compact Riemannian manifold with smooth boundary. Assume that the Ricci curvature of M satisfies $\mathrm{Ric}_M \geq (n-1)K$ and the mean curvature $H \geq H_0$ for some real numbers K and H_0. Assume also that the Bochner operator $W_M^{[p]}$ is nonnegative for some p. Then, the inequality*

$$q_{1,p} \geq \frac{1}{\int_0^R \Theta(r) \, \mathrm{d}r} \tag{3.1}$$

holds.

Proof. Applying the Bochner formula to $\Delta\omega$ where ω is a p-eigenform of the biharmonic Steklov operator associated with $q_{1,p}$, we get after taking the pointwise scalar product with $\Delta\omega$ that

$$0 = \left\langle \Delta^2\omega, \Delta\omega \right\rangle = |\nabla\Delta\omega|^2 + \frac{1}{2}\Delta\left(|\Delta\omega|^2\right) + \left\langle W_M^{[p]}\Delta\omega, \Delta\omega \right\rangle.$$

Since $W_M^{[p]}$ is nonnegative, we deduce that $\Delta(|\Delta\omega|^2)$ is nonpositive or equivalently the function $h := |\Delta\omega|^2$ is subharmonic. Therefore, by the previous theorem, we can say that

$$\frac{\int_{\partial M}|\Delta\omega|^2\mathrm{d}\mu_g}{\int_M|\Delta\omega|^2\mathrm{d}\mu_g} \geq \frac{1}{\int_0^R \Theta(r)\,\mathrm{d}r}.$$

Now, Characterization (2.10) gives the result and finishes the proof of the theorem. □

Remark 3.5. Depending on the sign of K and H_0, we can estimate explicitly $\int_0^R \Theta(r)\,\mathrm{d}r$ in terms of R and H_0, as in [22, Thms. 12 & 13]. Therefore, one can deduce several estimates for $q_{1,p}$ in terms of R and H_0.

We will now provide an estimate for the first eigenvalue of problem (2.2) on manifolds carrying parallel forms and study the limiting case of the estimate. Recall that a harmonic domain is a compact Riemannian manifold (M^n, g) with smooth boundary ∂M supporting a solution to the Serrin problem (1.3). We have the following result:

Theorem 3.6. *Let (M^n, g) be a compact Riemannian manifold with smooth boundary. Assume that M supports a non-trivial parallel p-form ω_0 for some $p = 0, \ldots, n$. Then*

$$q_{1,p} \leq \frac{\mathrm{Vol}(\partial M)}{\mathrm{Vol}(M)}. \tag{3.2}$$

Moreover, if equality holds in (3.2), then $f \cdot \omega_0$ is an eigenform associated to $q_{1,p}$, where f is the solution of (1.3) and therefore M must be a harmonic domain (and hence a Euclidean ball if $M \subset \mathbb{R}^n$).

Conversely, if M is a harmonic domain, then $\frac{\mathrm{Vol}(\partial M)}{\mathrm{Vol}(M)}$ is an eigenvalue – not necessarily the first one – of the biharmonic Steklov problem (2.2).

Proof. As ω_0 is a parallel form, we can assume that $|\omega_0| = 1$. By using the variational characterization (2.10), we obtain that

$$q_{1,p} \leq \frac{\|\omega_0\|^2_{L^2(\partial M)}}{\|\omega_0\|^2_{L^2(M)}} = \frac{\mathrm{Vol}(\partial M)}{\mathrm{Vol}(M)}.$$

If equality occurs in (3.2), then $\omega_0 = \Delta\omega$ for some eigenform ω associated with $q_{1,p}$ by Theorem 2.6. Now Proposition 2.1 implies that M carries a solution f to the Serrin

problem (1.3) and that $\omega = f \cdot \omega_0$. To check the converse, we take a function f solution to the Serrin problem (1.3), then using again Proposition 2.1 the p-form $\omega := f \cdot \omega_0$ is an eigenform of (2.2) associated with the eigenvalue $q = \frac{1}{c}$, where $c = \frac{\mathrm{Vol}(M)}{\mathrm{Vol}(\partial M)}$. □

Remark 3.7. If a compact Riemannian manifold M with smooth boundary carries a nontrivial *harmonic form of constant length* ω_0, then (3.2) remains valid. Moreover, if (3.2) is an equality, then there still exists an eigenform ω to the biharmonic Steklov operator on p-forms such that $\Delta\omega = \omega_0$, nevertheless it is no more true in general that M must be a harmonic domain and that $\omega = f \cdot \omega_0$ for some solution f to the scalar Serrin problem.

Next we compare the first eigenvalues of the biharmonic Steklov operator for successive degrees, when the manifold M is a domain in $\mathbb{R}^n$ or $\mathbb{S}^n$. We first notice that, if f is any eigenfunction to the scalar biharmonic Steklov problem (1.1), then for any parallel p-form ω_0 on M the form $f \cdot \omega_0$ is an eigenform to the biharmonic Steklov problems on p-forms and associated to the *same eigenvalue*. Therefore, for every eigenvalue q of the scalar biharmonic Steklov operator, we have an embedding

$$\ker\left(\mathrm{BS}_0 - q\right) \otimes \mathrm{P}_p \longhookrightarrow \ker\left(\mathrm{BS}_p - q\right),$$

where $\ker(\mathrm{BS}_j - q)$ denotes the eigenspace for the biharmonic Steklov operator on j-forms and associated to the eigenvalue q, and P_p denotes the space of parallel p-forms on M. When $M \subset \mathbb{R}^n$, then conversely for any $\omega \in \ker(\mathrm{BS}_p - q)$, there exists a parallel p-form ω_0 on M with $|\omega_0| = 1$ and $\langle\omega, \omega_0\rangle \neq 0$ (non identically vanishing) on M. But then f can be easily shown to lie in $\ker(\mathrm{BS}_0 - q)$. This shows that, when $M \subset \mathbb{R}^n$, both 0- and p-biharmonic Steklov eigenvalues coincide, their multiplicities being ignored.

In what follows, we assume that (M^n, g) is isometrically immersed into the Euclidean space $\mathbb{R}^{n+m}$. For any given smooth normal vector field N to M, we denote by $I\!I_N$ the associated Weingarten map, that is, the endomorphism field of TM defined by

$$\langle I\!I_N(X), Y \rangle = \langle N, I\!I(X, Y) \rangle$$

for all X, Y tangent to M, where $I\!I$ is the second fundamental form of the immersion. Recall that any endomorphism A of TM can be extended to the set of differential p-forms on M as follows: For any p-form ω on M, we define

$$A^{[p]}\omega(X_1, \ldots, X_p) = \sum_{i=1}^{p} \omega(X_1, \ldots, A(X_i), \ldots, X_p), \tag{3.3}$$

for all $X_1, \ldots, X_p$ vector fields in TM. In particular, this applies to $I\!I_N$ for all $N \in T^\perp M$. The following lemma is technical but will be useful for the comparison.

Lemma 3.8. *Let (M^n, g) be a compact Riemannian manifold with smooth boundary ∂M. Assume that M is isometrically immersed into the Euclidean space $\mathbb{R}^{n+m}$. Let ω be any p-eigenform of the biharmonic Steklov operator. Then we have*

$$pq_{1,p-1} \int_{\partial M} \left(|\nu \lrcorner \, \mathrm{d}\omega|^2 + |\iota^*(\delta\omega)|^2 \right) \mathrm{d}\mu_g$$

$$\leq \sum_{i=1}^{n+m} \int_M \left(\left| -2II_{n\widetilde{H}} \left(\partial_{x_i}^T \right) \lrcorner \, \omega + 2 \sum_{a=1}^{m} II_{f_a} \left(\partial_{x_i}^T \right) \lrcorner \, II_{f_a}^{[p]} \omega \right. \right.$$

$$\left. \left. -2 \sum_{s=1}^{n} \left(II_{\partial_{x_i}^\perp} e_s \right) \lrcorner \, (\nabla_{e_s} \omega) - n \left(\mathrm{d} \left(\left\langle \widetilde{H}, \partial_{x_i}^\perp \right\rangle \right) \right) \lrcorner \, \omega + \partial_{x_i}^T \lrcorner \, \Delta\omega \right|^2 \right) \mathrm{d}\mu_g. \quad (3.4)$$

Here $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_m\}$ are respectively orthonormal bases of TM and $T^\perp M$ and $\widetilde{H}$ is the mean curvature field of the immersion.

Proof. Let ω be any eigenform of the biharmonic Steklov problem associated with $q_{1,p}$. For each $i = 1, \dots, n + m$, the unit parallel vector field ∂_{x_i} on $\mathbb{R}^{n+m}$ splits into $\partial_{x_i} = (\partial_{x_i})^T + (\partial_{x_i})^\perp$ where $(\partial_{x_i})^T$ is the tangent part in TM and $(\partial_{x_i})^\perp$ is the orthogonal one in $T^\perp M$. We consider the $(p-1)$-form $(\partial_{x_i})^T \lrcorner \, \omega$ on M which clearly vanishes on ∂M. By applying to it the variational characterization (2.9), we get, for each i,

$$q_{1,p-1} \cdot \int_{\partial M} \left(\left| \nu \lrcorner \, \mathrm{d} \left((\partial_{x_i})^T \lrcorner \, \omega \right) \right|^2 + \left| \iota^* \delta \left((\partial_{x_i})^T \lrcorner \, \omega \right) \right|^2 \right) \mathrm{d}\mu_g$$

$$\leq \int_M \left| \Delta \left((\partial_{x_i})^T \lrcorner \, \omega \right) \right|^2 \mathrm{d}\mu_g. \quad (3.5)$$

Now we want to sum over $i = 1, \dots, n + m$. We first begin with the l.h.s. Recall the Cartan formula: $\mathcal{L}_X \omega = \mathrm{d}(X \lrcorner \, \omega) + X \lrcorner \, \mathrm{d}\omega$, for any vector field X on M. Using this formula, we have for each i,

$$\mathrm{d}(\partial_{x_i}^T \lrcorner \, \omega) = \mathcal{L}_{\partial_{x_i}^T} \omega - \partial_{x_i}^T \lrcorner \, \mathrm{d}\omega$$

$$= \nabla_{\partial_{x_i}^T} \omega + II_{\partial_{x_i}^\perp}^{[p]} \omega - \partial_{x_i}^T \lrcorner \, \mathrm{d}\omega. \quad (3.6)$$

In the last equality, we used the splitting of the Lie derivative in terms of the connection as follows: $\mathcal{L}_{X^T} \omega = \nabla_{X^T} \omega + II_{X^\perp}^{[p]} \omega$, for a parallel vector field $X \in \mathbb{R}^{n+m}$, proved in [9, Eq. (4.3) p. 337]. Since $\omega = 0$ on ∂M, we have that for any $X \in T\partial M$ [20, Eq. (23)]

$$\nu \lrcorner \, \nabla_X \omega = \nabla_X^{\partial M} (\nu \lrcorner \, \omega) + S(X) \lrcorner \, (\iota^* \omega) = 0.$$

Here S denotes the second fundamental form of the boundary. Therefore, we deduce that

$$\nu \lrcorner \, d\left((\partial_{x_i})^T \lrcorner \, \omega\right) = \nu \lrcorner \, \nabla_{\partial_{x_i}^T} \omega + \partial_{x_i}^T \lrcorner \, (\nu \lrcorner \, d\omega)$$

$$= g((\partial_{x_i})^T, \nu)\nu \lrcorner \, \nabla_\nu \omega + \partial_{x_i}^T \lrcorner \, (\nu \lrcorner \, d\omega)$$

$$= -g(\partial_{x_i}, \nu)\iota^*(\delta\omega) + \partial_{x_i}^T \lrcorner \, (\nu \lrcorner \, d\omega). \qquad (3.7)$$

In the last equality, we use the identity [20, Lem. 18]

$$\nu \lrcorner \, \nabla_\nu \omega = \delta^{\partial M}(\iota^*\omega) - \iota^*(\delta\omega) - S^{[p-1]}(\nu \lrcorner \, \omega) + (n-1)H\nu \lrcorner \, \omega = -\iota^*(\delta\omega).$$

Independently, by a straightforward computation, we check that, for any p-form α,

$$\sum_{i=1}^{n+m} \partial_{x_i}^T \wedge \left(\partial_{x_i}^T \lrcorner \, \alpha\right) = \sum_{i=1}^{n} e_i \wedge (e_i \lrcorner \, \alpha) = p\alpha.$$

As a consequence, we obtain for any p-forms α and β on M

$$\sum_{i=1}^{n+m} \left\langle \partial_{x_i}^T \lrcorner \, \alpha, \partial_{x_i}^T \lrcorner \, \beta \right\rangle = p\langle \alpha, \beta \rangle. \qquad (3.8)$$

We take the norm of (3.7) and sum over i to get

$$\sum_{i=1}^{n+m} \left| \nu \lrcorner \, d\left((\partial_{x_i})^T \lrcorner \, \omega\right) \right|^2$$

$$\overset{(3.8)}{=} |\iota^*(\delta\omega)|^2 + p|\nu \lrcorner \, d\omega|^2 - 2\sum_{i=1}^{n+m} g(\partial_{x_i}, \nu) \left\langle \iota^*(\delta\omega), \partial_{x_i}^T \lrcorner \, (\nu \lrcorner \, d\omega) \right\rangle$$

$$= |\iota^*\delta\omega|^2 + p|\nu \lrcorner \, d\omega|^2 - 2\left\langle \iota^*\delta\omega, \nu^T \lrcorner \, (\nu \lrcorner \, d\omega) \right\rangle$$

$$= |\iota^*\delta\omega|^2 + p|\nu \lrcorner \, d\omega|^2. \qquad (3.9)$$

Here we notice that by the Gauß formula and the fact that ∂_{x_i} is parallel in $\mathbb{R}^{n+m}$, we have $\nabla_X \partial_{x_i}^T = II_{\partial_{x_i}^\perp}(X)$, so that

$$\delta\left(\partial_{x_i}^T \lrcorner \, \omega\right) = -\sum_{j=1}^{n} e_j \lrcorner \, \nabla_{e_j}\left(\partial_{x_i}^T \lrcorner \, \omega\right)$$

$$= -\sum_{j=1}^{n} e_j \lrcorner \, \left(\nabla_{e_j}\partial_{x_i}^T \lrcorner \, \omega + \partial_{x_i}^T \lrcorner \, \nabla_{e_j}\omega\right)$$

$$= -\sum_{j=1}^{n} e_j \lrcorner \, II_{\partial_{x_i}^\perp}(e_j) \lrcorner \, \omega + \sum_{j=1}^{n} \partial_{x_i}^T \lrcorner \, \left(e_j \lrcorner \, \nabla_{e_j}\omega\right)$$

$$= -\sum_{j,k=1}^{n} \left\langle II_{\partial_{x_i}^{\perp}}(e_j), e_k \right\rangle e_j \lrcorner e_k \lrcorner \omega - \partial_{x_i}^{T} \lrcorner \delta\omega$$

$$= -\partial_{x_i}^{T} \lrcorner \delta\omega$$

because of the expression $\langle II_{\partial_{x_i}^{\perp}}(e_j), e_k \rangle = \langle II(e_j, e_k), \partial_{x_i}^{\perp} \rangle$ being symmetric in j, k. Thus $\delta(\partial_{x_i}^{T} \lrcorner \omega) = -\partial_{x_i}^{T} \lrcorner \delta\omega$ for any i. Using also the fact that $\nu \lrcorner \delta(\partial_{x_i}^{T} \lrcorner \omega) = -\delta^{\partial M}(\nu \lrcorner \partial_{x_i}^{T} \lrcorner \omega) = 0$, we get that

$$\sum_{i=1}^{n+m} \left| \iota^* \delta\left(\partial_{x_i}^{T} \lrcorner \omega \right) \right|^2 = \sum_{i=1}^{n+m} \left| \delta\left(\partial_{x_i}^{T} \lrcorner \omega \right) \right|^2 \overset{(3.8)}{=} (p-1) \, |\iota^* \delta\omega|^2 . \tag{3.10}$$

Hence adding Equations (3.9) and (3.10) allows to find the l.h.s. of Inequality (3.4). We are now going to estimate the term $\sum_{i=1}^{n+m} |\Delta(\partial_{x_i}^{T} \lrcorner \omega)|^2$ in (3.5). Taking the divergence of (3.6) and the differential of the identity $\delta(\partial_{x_i}^{T} \lrcorner \omega) = -\partial_{x_i}^{T} \lrcorner \delta\omega$ along with the Cartan formula and the decomposition of the Lie derivative as in (3.6), we get that

$$\Delta\left(\partial_{x_i}^{T} \lrcorner \omega \right) = \left[\delta, \nabla_{\partial_{x_i}^{T}} \right](\omega) + \delta\left(II_{\partial_{x_i}^{\perp}}^{[p]} \omega \right) - II_{\partial_{x_i}^{\perp}}^{[p-1]}(\delta\omega) + \partial_{x_i}^{T} \lrcorner \Delta\omega. \tag{3.11}$$

In the following, we will compute each term of (3.11) separately. First, take an orthonormal frame $\{e_1, \ldots, e_n\}$ of TM such that $\nabla e_i = 0$ at some point. Then we have, for any vector field X on M,

$$\begin{aligned}
[\delta, \nabla_X](\omega) &= \delta(\nabla_X \omega) - \nabla_X \delta\omega \\
&= -\sum_{s=1}^{n} e_s \lrcorner \nabla_{e_s} \nabla_X \omega - \nabla_X \delta\omega \\
&= -\sum_{s=1}^{n} e_s \lrcorner \left(R(e_s, X)\omega + \nabla_X \nabla_{e_s} \omega + \nabla_{[e_s, X]}\omega \right) - \nabla_X \delta\omega \\
&= -\sum_{s=1}^{n} e_s \lrcorner R(e_s, X)\omega + \nabla_X \delta\omega - \sum_{s=1}^{n} e_s \lrcorner \nabla_{\nabla_{e_s} X}\omega - \nabla_X \delta\omega \\
&= -\sum_{s=1}^{n} e_s \lrcorner R(e_s, X)\omega - \sum_{s=1}^{n} e_s \lrcorner \nabla_{\nabla_{e_s} X}\omega.
\end{aligned} \tag{3.12}$$

Now we use the fact that for any tensor field A, any vector field X and any p-form α on M, $\nabla_X A^{[p]} = (\nabla_X A)^{[p]}$ and $A^{[p-1]}(X \lrcorner \alpha) = X \lrcorner A^{[p]}\alpha - A(X) \lrcorner \alpha$, which both can

be proved by a straightforward computation. Thus for any $N \in T^{\perp}M$, we can write

$$\delta\left(II_N^{[p]}\omega\right) - II_N^{[p-1]}(\delta\omega)$$

$$= -\sum_{s=1}^{n} e_s \lrcorner \nabla_{e_s}\left(II_N^{[p]}\omega\right) - II_N^{[p-1]}(\delta\omega)$$

$$= -\sum_{s=1}^{n} e_s \lrcorner (\nabla_{e_s}II_N)^{[p]}(\omega) - \sum_{s=1}^{n} e_s \lrcorner II_N^{[p]}(\nabla_{e_s}\omega) - II_N^{[p-1]}(\delta\omega)$$

$$= -\sum_{s=1}^{n} (\nabla_{e_s}II_N)^{[p-1]}(e_s \lrcorner \omega) - \sum_{s=1}^{n} \left((\nabla_{e_s}II_N)(e_s)\right) \lrcorner \omega \qquad (3.13)$$

$$\quad - \sum_{s=1}^{n} II_N^{[p-1]}(e_s \lrcorner \nabla_{e_s}\omega) - \sum_{s=1}^{n} (II_N e_s) \lrcorner (\nabla_{e_s}\omega) - II_N^{[p-1]}(\delta\omega)$$

$$= -\sum_{s=1}^{n} (\nabla_{e_s}II_N)^{[p-1]}(e_s \lrcorner \omega) + \delta(II_N) \lrcorner \omega - \sum_{s=1}^{n} (II_N e_s) \lrcorner (\nabla_{e_s}\omega).$$

By taking $X = \partial_{x_i}^T$ in (3.12) and $N = \partial_{x_i}^{\perp}$ in (3.13), Equation (3.11) reduces after using the fact that $\nabla \partial_{x_i}^T = II_{\partial_{x_i}^{\perp}}$ as a consequence of the parallelism of the vector field ∂_{x_i}, to

$$\Delta\left(\partial_{x_i}^T \lrcorner \omega\right) = -\sum_{s=1}^{n} e_s \lrcorner R\left(e_s, \partial_{x_i}^T\right)\omega - 2\sum_{s=1}^{n} \left(II_{\partial_{x_i}^{\perp}} e_s\right) \lrcorner (\nabla_{e_s}\omega)$$

$$\quad - \sum_{s=1}^{n} \left(\nabla_{e_s}II_{\partial_{x_i}^{\perp}}\right)^{[p-1]}(e_s \lrcorner \omega) + \delta\left(II_{\partial_{x_i}^{\perp}}\right) \lrcorner \omega + \partial_{x_i}^T \lrcorner \Delta\omega. \quad (3.14)$$

We proceed in the computation of (3.14) and compute $\sum_{s=1}^{n}(\nabla_{e_s}II_{\partial_{x_i}^{\perp}})^{[p-1]}(e_s \lrcorner \omega)$. Using the fact that $A^{[p]} = \sum_{l=1}^{n} e_l \wedge (A(e_l)\lrcorner)$ for any symmetric tensor A, we compute with the help of Equation (A.2) in the appendix

$$\sum_{s=1}^{n} \left(\nabla_{e_s}II_{\partial_{x_i}^{\perp}}\right)^{[p-1]}(e_s \lrcorner \omega) = \sum_{l,s=1}^{n} e_l \wedge \left(\left(\nabla_{e_s}II_{\partial_{x_i}^{\perp}}\right)(e_l) \lrcorner e_s \lrcorner \omega\right)$$

$$= \sum_{l,s=1}^{n} e_l \wedge \left(\left(\nabla_{e_l}II_{\partial_{x_i}^{\perp}}\right)(e_s) \lrcorner e_s \lrcorner \omega\right)$$

$$\quad - \sum_{l,s=1}^{n} e_l \wedge \left(\left(II_{\left(\nabla_{e_l}^{\mathbb{R}^{n+m}}\partial_{x_i}^{\perp}\right)^{\perp}}\right)(e_s) \lrcorner e_s \lrcorner \omega\right) \qquad (3.15)$$

$$\quad + \sum_{l,s=1}^{n} e_l \wedge \left(\left(II_{\left(\nabla_{e_s}^{\mathbb{R}^{n+m}}\partial_{x_i}^{\perp}\right)^{\perp}}\right)(e_l) \lrcorner e_s \lrcorner \omega\right).$$

The first two sums vanish identically, since $\sum_s A(e_s) \lrcorner e_s \lrcorner \omega = 0$ for any symmetric endomorphism A. Hence with the help of Equation (A.4), Equality (3.15) reduces to

$$
\begin{aligned}
\sum_{s=1}^{n} \left(\nabla_{e_s} II_{\partial_{x_i}^{\perp}}\right)^{[p-1]} (e_s \lrcorner \omega) &= -\sum_{l,s=1}^{n} e_l \wedge \left(II_{II(e_s, \partial_{x_i}^T)}(e_l) \lrcorner e_s \lrcorner \omega\right) \\
&= -\sum_{l,s=1}^{n} \sum_{a=1}^{m} \left\langle II\left(e_s, \partial_{x_i}^T\right), f_a\right\rangle e_l \wedge \left(II_{f_a}(e_l) \lrcorner e_s \lrcorner \omega\right) \\
&= -\sum_{l=1}^{n} \sum_{a=1}^{m} e_l \wedge \left(II_{f_a}(e_l) \lrcorner II_{f_a}\left(\partial_{x_i}^T\right) \lrcorner \omega\right) \\
&= -\sum_{a=1}^{m} II_{f_a}^{[p-1]} \left(II_{f_a}\left(\partial_{x_i}^T\right) \lrcorner \omega\right),
\end{aligned}
\tag{3.16}
$$

where $\{f_1, \ldots, f_m\}$ is an orthonormal frame of $T^{\perp}M$. In the last equality, we used again the expression $A^{[p]} = \sum_l e_l \wedge (A(e_l) \lrcorner)$ for any symmetric endomorphism A. Hence after replacing Equations (A.3), proved in the appendix, and (3.16) into Equation (3.14), we finally get

$$
\begin{aligned}
&\Delta\left(\partial_{x_i}^T \lrcorner \omega\right) \\
&= -\sum_{s=1}^{n} e_s \lrcorner R\left(e_s, \partial_{x_i}^T\right) \omega - 2\sum_{s=1}^{n} \left(II_{\partial_{x_i}^{\perp}} e_s\right) \lrcorner (\nabla_{e_s}\omega) \\
&\quad + \sum_{a=1}^{m} II_{f_a}^{[p-1]} \left(II_{f_a}\left(\partial_{x_i}^T\right) \lrcorner \omega\right) \\
&\quad + \left(\sum_{a=1}^{m} II_{f_a}^2\left(\partial_{x_i}^T\right) - nd\left(\left\langle \widetilde{H}, \partial_{x_i}^{\perp}\right\rangle\right) - nII_{\widetilde{H}}\left(\partial_{x_i}^T\right)\right) \lrcorner \omega + \partial_{x_i}^T \lrcorner \Delta\omega \\
&= -\sum_{s=1}^{n} e_s \lrcorner R\left(e_s, \partial_{x_i}^T\right) \omega - 2\sum_{s=1}^{n} \left(II_{\partial_{x_i}^{\perp}} e_s\right) \lrcorner (\nabla_{e_s}\omega) \\
&\quad + \sum_{a=1}^{m} II_{f_a}\left(\partial_{x_i}^T\right) \lrcorner II_{f_a}^{[p]}\omega + \left(-nd\left(\left\langle \widetilde{H}, \partial_{x_i}^{\perp}\right\rangle\right) - nII_{\widetilde{H}}\left(\partial_{x_i}^T\right)\right) \lrcorner \omega + \partial_{x_i}^T \lrcorner \Delta\omega \quad (3.17) \\
&= -II_{n\widetilde{H}}\left(\partial_{x_i}^T\right) \lrcorner \omega + \sum_{a=1}^{m} II_{f_a}\left(\partial_{x_i}^T\right) \lrcorner II_{f_a}^{[p]}\omega - 2\sum_{s=1}^{n} \left(II_{\partial_{x_i}^{\perp}} e_s\right) \lrcorner (\nabla_{e_s}\omega) \\
&\quad + \sum_{a=1}^{m} II_{f_a}\left(\partial_{x_i}^T\right) \lrcorner II_{f_a}^{[p]}\omega + \left(-nd\left(\left\langle \widetilde{H}, \partial_{x_i}^{\perp}\right\rangle\right) - nII_{\widetilde{H}}\left(\partial_{x_i}^T\right)\right) \lrcorner \omega + \partial_{x_i}^T \lrcorner \Delta\omega
\end{aligned}
$$

$$= -2II_{n\widetilde{H}}\left(\partial_{x_i}^T\right) \lrcorner\, \omega + 2\sum_{a=1}^m II_{f_a}\left(\partial_{x_i}^T\right) \lrcorner\, II_{f_a}^{[p]}\omega - 2\sum_{s=1}^n \left(II_{\partial_{x_i}^\perp} e_s\right) \lrcorner\, (\nabla_{e_s}\omega)$$

$$- n\left(\mathrm{d}\left(\left\langle \widetilde{H}, \partial_{x_i}^\perp\right\rangle\right)\right) \lrcorner\, \omega + \partial_{x_i}^T \lrcorner\, \Delta\omega.$$

In the second equality, we used again the relation $A^{[p-1]}(X \lrcorner\, \alpha) = X \lrcorner\, A^{[p]}\alpha - A(X) \lrcorner\, \alpha$ for any p-form α and $X \in TM$, and in the third equality, we used Proposition A.3 in the appendix. Equation (3.17), along with (3.9) and (3.10), gives the result, by using Inequality (3.5). $\qquad\square$

In general, it is difficult to control all the terms in Inequality (3.4) in order to compare $q_{1,p-1}$ and $q_{1,p}$. Therefore, we shall restrict ourselves to the case when M is a domain in $\mathbb{S}^n$. We have the following result:

Theorem 3.9. *Let (M^n, g) be a compact Riemannian manifold with smooth boundary. If M is a domain in $\mathbb{S}^n$, then we have that*

$$pq_{1,p-1} + (n-p)q_{1,p+1} < C_{p,n}q_{1,p}, \tag{3.18}$$

where $C_{p,n}$ is some constant that depends on p and n and whose explicit expression is given in (3.23).

Proof. We consider the isometric immersion $M \subset \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$. In this case, we have that $m = 1$, the orthonormal basis of $T^\perp M$ reduces to the inward unit vector field $\widetilde{\nu} = -\sum_{i=1}^{n+1} x_i \partial_{x_i}$, the second fundamental form is given by $II_{\widetilde{\nu}} = \mathrm{Id}$ and $\widetilde{H} = \widetilde{\nu}$. Therefore, Inequality (3.4) becomes

$$pq_{1,p-1}\int_{\partial M}\left(|\nu \lrcorner\, \mathrm{d}\omega|^2 + |\iota^*(\delta\omega)|^2\right)\mathrm{d}\mu_g$$

$$\leq \sum_{i=1}^{n+1}\int_M \left|(2p-2n)\partial_{x_i}^T \lrcorner\, \omega + 2\langle\partial_{x_i}^\perp, \widetilde{\nu}\rangle\delta\omega - n\mathrm{d}\left(\langle\widetilde{\nu}, \partial_{x_i}^\perp\rangle\right) \lrcorner\, \omega + \partial_{x_i}^T \lrcorner\, \Delta\omega\right|^2 \mathrm{d}\mu_g.$$

Now, an elementary computation shows that $d(\langle \widetilde{\nu}, \partial_{x_i}^\perp \rangle) = -\partial_{x_i}^T$. Therefore, the above inequality reduces to

$$
pq_{1,p-1} \int_{\partial M} \left(|\nu \lrcorner \, d\omega|^2 + |\iota^*(\delta\omega)|^2 \right) d\mu_g
$$

$$
\leq \sum_{i=1}^{n+1} \int_M \left| (2p-n)\partial_{x_i}^T \lrcorner \, \omega + 2 \left\langle \partial_{x_i}^\perp, \widetilde{\nu} \right\rangle \delta\omega + \partial_{x_i}^T \lrcorner \, \Delta\omega \right|^2 d\mu_g
$$

$$
= \int_M \left((2p-n)^2 p|\omega|^2 + 4|\delta\omega|^2 + p|\Delta\omega|^2 + 2p(2p-n)\langle\omega, \Delta\omega\rangle \right) d\mu_g \quad (3.19)
$$

$$
= \int_M \left(4|\delta\omega|^2 + p|\Delta\omega + (2p-n)\omega|^2 \right) d\mu_g.
$$

Here, we use Identity (3.8) and that

$$
\sum_{i=1}^{n+1} \left\langle \partial_{x_i}^\perp, \widetilde{\nu} \right\rangle \left\langle \partial_{x_i}^T \lrcorner \, \omega, \delta\omega \right\rangle = \sum_{i=1}^{n+1} \langle \partial_{x_i}, \widetilde{\nu} \rangle \left\langle \partial_{x_i}^T \lrcorner \, \omega, \delta\omega \right\rangle = \left\langle \widetilde{\nu}^T \lrcorner \, \omega, \delta\omega \right\rangle = 0.
$$

The same argument applies to $\sum_{i=1}^{n+1} \langle \partial_{x_i}^\perp, \widetilde{\nu} \rangle \langle \delta\omega, \partial_{x_i}^T \lrcorner \, \Delta\omega \rangle = 0$. Since Inequality (3.19) is true for any p-eigenform ω, we apply it to the $(n-p)$-eigenform $*_M\omega$ to get

$$
(n-p)q_{1,n-p-1} \int_{\partial M} \left(|\iota^*(\delta\omega)|^2 + |\nu \lrcorner \, d\omega|^2 \right) d\mu_g
$$

$$
\leq \int_M \left(4|d\omega|^2 + (n-p)|\Delta\omega + (2n-2p-n)\omega|^2 \right) d\mu_g. \quad (3.20)
$$

Summing inequalities (3.19) and (3.20) and using the fact that $q_{1,n-p-1} = q_{1,p+1}$ yield the following:

$$
(pq_{1,p-1} + (n-p)q_{1,p+1}) \int_{\partial M} \left(|\nu \lrcorner \, d\omega|^2 + |\iota^*(\delta\omega)|^2 \right) d\mu_g
$$

$$
\leq \int_M \left(4|d\omega|^2 + 4|\delta\omega|^2 + p|\Delta\omega + (2p-n)\omega|^2 + (n-p)|\Delta\omega + (n-2p)\omega|^2 \right) d\mu_g
$$

$$
= \int_M \left(4\langle\Delta\omega, \omega\rangle + p|\Delta\omega + (2p-n)\omega|^2 + (n-p)|\Delta\omega + (n-2p)\omega|^2 \right) d\mu_g. \quad (3.21)
$$

Here, we used the fact that $\int_M \langle \Delta\omega, \omega \rangle d\mu_g = \int_M (|d\omega|^2 + |\delta\omega|^2) d\mu_g$ as a consequence of the boundary condition $\omega_{|\partial M} = 0$. Now, the Bochner formula applied to the eigenform ω, with the help of the pointwise inequality $|\nabla\alpha|^2 \geq \frac{1}{p+1}|d\alpha|^2 + \frac{1}{n-p+1}|\delta\alpha|^2$ which is true

for any p-form α [7, Lem. 6.8], gives that

$$\int_M \langle \Delta\omega, \omega \rangle \mathrm{d}\mu_g$$

$$= \int_M |\nabla\omega|^2 \mathrm{d}\mu_g + \frac{1}{2} \int_M \Delta\left(|\omega|^2\right) \mathrm{d}\mu_g + \int_M \left\langle W_M^{[p]}\omega, \omega \right\rangle \mathrm{d}\mu_g$$

$$\geq \int_M \frac{1}{a(p,n)} \left(|\mathrm{d}\omega|^2 + |\delta\omega|^2\right) \mathrm{d}\mu_g + \int_{\partial M} \langle \nabla_\nu\omega, \omega \rangle \mathrm{d}\mu_g + \int_M p(n-p)|\omega|^2 \mathrm{d}\mu_g$$

$$= \frac{1}{a(p,n)} \int_M \langle \Delta\omega, \omega \rangle \mathrm{d}\mu_g + \int_M p(n-p)|\omega|^2 \mathrm{d}\mu_g.$$

Here, we have set $a(p,n) = \max(p+1, n-p+1)$ and used the fact that the Bochner operator $W_M^{[p]}$ is equal to $p(n-p)\,\mathrm{Id}$ on the round sphere $\mathbb{S}^n$, see e.g. [7, Cor. 2.6] and [7, Rem. 6.15]. Thus, we deduce that

$$p(n-p)\|\omega\|_{L^2(M)}^2 \leq \left(1 - \frac{1}{a(p,n)}\right) \int_M \langle \Delta\omega, \omega \rangle \mathrm{d}\mu_g$$

$$\leq \left(1 - \frac{1}{a(p,n)}\right) \|\Delta\omega\|_{L^2(M)} \|\omega\|_{L^2(M)},$$

so that

$$\|\omega\|_{L^2(M)} \leq \frac{a(p,n)-1}{p(n-p)a(p,n)} \|\Delta\omega\|_{L^2(M)}. \tag{3.22}$$

Coming back to Inequality (3.21) and using again the Cauchy–Schwarz inequality as well as the estimate $\|\alpha+\beta\|_{L^2(M)}^2 \leq 2(\|\alpha\|_{L^2(M)}^2 + \|\beta\|_{L^2(M)}^2)$, we obtain

$$\left(pq_{1,p-1} + (n-p)q_{1,p+1}\right) \int_{\partial M} \left(|\nu \lrcorner \mathrm{d}\omega|^2 + |\iota^*(\delta\omega)|^2\right) \mathrm{d}\mu_g$$

$$\leq 4\|\Delta\omega\|_{L^2(M)}\|\omega\|_{L^2(M)} + 2p\|\Delta\omega\|_{L^2(M)}^2 + 2p(2p-n)^2\|\omega\|_{L^2(M)}^2$$

$$+ 2(n-p)\|\Delta\omega\|_{L^2(M)}^2 + 2(n-p)(n-2p)^2\|\omega\|_{L^2(M)}^2$$

$$\overset{(3.22)}{\leq} C(p,n)\|\Delta\omega\|_{L^2(M)}^2,$$

where $C(p,n)$ is the constant given by

$$C(p,n) = 4\frac{a(p,n)-1}{p(n-p)a(p,n)} + 2n$$

$$+ \left(2p(2p-n)^2 + 2(n-p)(n-2p)^2\right)\left(\frac{a(p,n)-1}{p(n-p)a(p,n)}\right)^2 \tag{3.23}$$

$$= 4\frac{a(p,n)-1}{p(n-p)a(p,n)} + 2n + 2n(2p-n)^2\left(\frac{a(p,n)-1}{p(n-p)a(p,n)}\right)^2.$$

Finally Characterization (2.9) allows to get the result. Notice that, if (3.18) were an equality, then by the limiting case in the Cauchy–Schwarz inequality, the form $\Delta\omega$ would be parallel to ω. But by $\Delta^2\omega = 0$, this would imply that $\Delta\omega = 0$. Because of $\omega_{|\partial M} = 0$, we would deduce from [1] that $\omega = 0$ on M, which is a contradiction. Therefore, (3.18) always remains strict. □

4. Robin vs. Dirichlet and Neumann eigenvalue problems

In this section, we will establish estimates for the Robin eigenvalue problem on differential forms defined in [4]. We mainly generalize some results in [13] to differential forms. For this purpose, we recall the Robin problem on forms. Let (M^n, g) be a compact Riemannian manifold with smooth boundary ∂M. Fix a real number τ. Then the boundary value problem [4]

$$\begin{cases} \Delta\omega = \lambda\omega & \text{on } M \\ \iota^*(\nu \lrcorner \, \mathrm{d}\omega - \tau\omega) = 0 & \text{on } \partial M \\ \nu \lrcorner \, \omega = 0 & \text{on } \partial M \end{cases} \tag{4.1}$$

is elliptic and self-adjoint. Note that, when $\tau < 0$ and $\lambda = 0$, a nonzero p-form ω on M satisfying (4.1) is nothing but a Steklov-eigenform associated with the eigenvalue $-\tau$. Problem (4.1) admits an increasing unbounded sequence of real eigenvalues with finite multiplicities

$$\lambda_{1,p}(\tau) \leq \lambda_{2,p}(\tau) \leq \dots$$

where $\lambda_{1,p}(\tau) > 0$ as soon as $\tau > 0$, which we assume from now on. The first eigenvalue $\lambda_{1,p}(\tau)$ of the Robin boundary problem (4.1) can be characterized as follows:

$$\lambda_{1,p}(\tau) = \inf \left\{ \frac{\int_M \left(|\mathrm{d}\omega|^2 + |\delta\omega|^2 \right) \mathrm{d}\mu_g + \tau \int_{\partial M} |\iota^*\omega|^2 \mathrm{d}\mu_g}{\int_M |\omega|^2 \mathrm{d}\mu_g} \right\}, \tag{4.2}$$

where ω runs over all non-identically vanishing p-forms on M such that $\nu \lrcorner \, \omega = 0$. When the parameter τ tends to 0, the Robin problem (4.1) reduces to the Neumann boundary problem, that is

$$\begin{cases} \Delta\omega = \lambda\omega & \text{on } M \\ \nu \lrcorner \, \mathrm{d}\omega = 0 & \text{on } \partial M \\ \nu \lrcorner \, \omega = 0 & \text{on } \partial M. \end{cases} \tag{4.3}$$

Notice that the first eigenvalue $\lambda_{1,p}^N$ of (4.3) is nonnegative and the kernel of the operator (4.3) is isomorphic to the so-called absolute de Rham cohomology, which is

defined by

$$H_A^p(M) = \{\omega \in \Omega^p(M) \,|\, d\omega = \delta\omega = 0 \text{ on } M \text{ and } \nu \lrcorner \omega = 0 \text{ on } \partial M\}.$$

When $\tau \to \infty$, the Robin problem (4.1) reduces to the Dirichlet boundary problem

$$\begin{cases} \Delta\omega = \lambda\omega & \text{on } M \\ \quad \omega = 0 & \text{on } \partial M. \end{cases} \tag{4.4}$$

By [1], the first eigenvalue $\lambda_{1,p}^D$ of problem (4.4) is positive. We also have the estimate [4, Prop. 5.4]

$$\lambda_{1,p}^N \leq \lambda_{1,p}(\tau) \leq \lambda_{1,p}^D.$$

In the following, we will establish another estimates for $\lambda_{1,p}(\tau)$ in terms of the Neumann and Dirichlet ones. We have:

Theorem 4.1. *Let (M^n, g) be a compact Riemannian manifold with smooth boundary. We have the following estimates for the first eigenvalue of the Robin boundary problem:*

(1) *Assume that the absolute de Rham cohomology $H_A^p(M)$ does not vanish. We denote by ω_D an eigenform of the Dirichlet boundary problem associated to $\lambda_{1,p}^D$ and let ω_0 be the orthogonal projection of ω_D on the space $H_A^p(M)$, assumed to be nonzero. Then*

$$\frac{1}{\lambda_{1,p}(\tau)} \geq \frac{1}{\lambda_{1,p}^D} + \frac{\|\omega_0\|_{L^2(M)}^4}{\tau\|\omega_0\|_{L^2(\partial M)}^2}.$$

(2) *Assume that the first eigenvalue $\lambda_{1,p}^N$ of the Neumann boundary problem is positive, then*

$$\frac{1}{\lambda_{1,p}(\tau)} \geq \frac{1}{\lambda_{1,p}^N} - \frac{\tau\alpha_N\left(\lambda_{1,p}^D - \lambda_{1,p}^N\right)}{\lambda_{1,p}^N\left(\tau\alpha_N\lambda_{1,p}^D + \lambda_{1,p}^N\left(\lambda_{1,p}^D - \lambda_{1,p}^N\right)\right)},$$

where

$$\alpha_N = \frac{\|\omega_N\|_{L^2(\partial M)}^2}{\|\omega_N\|_{L^2(M)}^2}$$

and ω_N is being an eigenform of the Neumann boundary problem associated to $\lambda_{1,p}^N$.

Proof. We begin with the proof of the first point. Let ω_D be a p-eigenform associated to the first eigenvalue $\lambda_{1,p}^D$ that is assumed to be of L^2-norm equal to 1. Let ω_0 be the orthogonal projection of ω_D to $H_A^p(M)$. For any real number t, we consider the p-form

$$\omega_t = \omega_D + t\,\omega_0.$$

Clearly, we have that $\nu \lrcorner \omega_t = 0$. Therefore, by the characterization of the first eigenvalue $\lambda_{1,p}(\tau)$, we have that

$$\lambda_{1,p}(\tau) \leq \frac{\int_M \left(|\mathrm{d}\omega_t|^2 + |\delta\omega_t|^2\right)\mathrm{d}\mu_g + \tau \int_{\partial M} |\omega_t|^2 \mathrm{d}\mu_g}{\int_M |\omega_t|^2 \mathrm{d}\mu_g}.$$

By the definition of the form ω_0, we have that

$$\|\mathrm{d}\omega_t\|_{L^2(M)}^2 + \|\delta\omega_t\|_{L^2(M)}^2 = \lambda_{1,p}^D, \quad \|\omega_t\|_{L^2(\partial M)}^2 = t^2 \|\omega_0\|_{L^2(\partial M)}^2.$$

Also, we have that

$$\|\omega_t\|_{L^2(M)}^2 = 1 + t^2 \|\omega_0\|_{L^2(M)}^2 + 2t\|\omega_0\|_{L^2(M)}^2.$$

The last term comes from the fact that ω_0 is the orthogonal projection of ω_D. Thus by plugging in the above inequality, we get that

$$\lambda_{1,p}(\tau) \leq \frac{\lambda_{1,p}^D + t^2\tau\|\omega_0\|_{L^2(\partial M)}^2}{1 + t^2\|\omega_0\|_{L^2(M)}^2 + 2t\|\omega_0\|_{L^2(M)}^2}.$$

Now, we take the inverse of this last inequality, then add and subtract the term $\dfrac{t^2\tau\|\omega_0\|_{L^2(\partial M)}^2}{\lambda_{1,p}^D}$ in the numerator to find that

$$\frac{1}{\lambda_{1,p}(\tau)} \geq \frac{1}{\lambda_{1,p}^D} + \frac{t^2\left(\|\omega_0\|_{L^2(M)}^2 - \frac{\tau\|\omega_0\|_{L^2(\partial M)}^2}{\lambda_{1,p}^D}\right) + 2t\|\omega_0\|_{L^2(M)}^2}{\lambda_{1,p}^D + t^2\tau\|\omega_0\|_{L^2(\partial M)}^2}.$$

Since this is true for any real number t, we deduce that $\frac{1}{\lambda_{1,p}(\tau)} \geq \frac{1}{\lambda_{1,p}^D} + \sup_{\mathbb{R}}(f)$, where f is the function given by

$$f(t) = \frac{t^2\left(\|\omega_0\|_{L^2(M)}^2 - \frac{\tau\|\omega_0\|_{L^2(\partial M)}^2}{\lambda_{1,p}^D}\right) + 2t\|\omega_0\|_{L^2(M)}^2}{\lambda_{1,p}^D + t^2\tau\|\omega_0\|_{L^2(\partial M)}^2} = \frac{At^2 + Bt}{Ct^2 + D}$$

with

$$A = \|\omega_0\|^2_{L^2(M)} - \frac{\tau\|\omega_0\|^2_{L^2(\partial M)}}{\lambda^D_{1,p}}, \qquad\qquad B = 2\|\omega_0\|^2_{L^2(M)},$$

$$C = \tau\|\omega_0\|^2_{L^2(\partial M)}, \qquad\qquad D = \lambda^D_{1,p}.$$

The supremum of f can be checked to be attained at $t_0 = \frac{AD+\sqrt{A^2D^2+B^2CD}}{BC}$ and is equal to

$$\sup_{\mathbb{R}} f = f(t_0) = \frac{t_0(At_0 + B)}{Ct_0^2 + D} = \frac{t_0(At_0 + B)}{\frac{2ADt_0}{B} + D + D} = \frac{Bt_0}{2D} = \frac{A + \sqrt{A^2 + \frac{B^2C}{D}}}{2C}.$$

Now, by replacing A, B, C and D by their values, we estimate

$$A^2 + \frac{B^2C}{D}$$

$$= \left(\|\omega_0\|^2_{L^2(M)} - \frac{\tau\|\omega_0\|^2_{L^2(\partial M)}}{\lambda^D_{1,p}}\right)^2 + \frac{4\|\omega_0\|^4_{L^2(M)}\tau\|\omega_0\|^2_{L^2(\partial M)}}{\lambda^D_{1,p}}$$

$$= \left(\|\omega_0\|^2_{L^2(M)} - \frac{\tau\|\omega_0\|^2_{L^2(\partial M)}}{\lambda^D_{1,p}} - 2\|\omega_0\|^4_{L^2(M)}\right)^2 - 4\|\omega_0\|^8_{L^2(M)} + 4\|\omega_0\|^6_{L^2(M)}$$

$$\geq \left(\|\omega_0\|^2_{L^2(M)} - \frac{\tau\|\omega_0\|^2_{L^2(\partial M)}}{\lambda^D_{1,p}} - 2\|\omega_0\|^4_{L^2(M)}\right)^2,$$

since $\|\omega_0\|^2_{L^2(M)} \leq \|\omega_D\|^2_{L^2(M)} = 1$. Then

$$\sup_{\mathbb{R}} f \geq \frac{\|\omega_0\|^2_{L^2(M)} - \frac{\tau\|\omega_0\|^2_{L^2(\partial M)}}{\lambda^D_{1,p}} + \left|\|\omega_0\|^2_{L^2(M)} - \frac{\tau\|\omega_0\|^2_{L^2(\partial M)}}{\lambda^D_{1,p}} - 2\|\omega_0\|^4_{L^2(M)}\right|}{2\tau\|\omega_0\|^2_{L^2(\partial M)}}$$

$$\geq \frac{\|\omega_0\|^4_{L^2(M)}}{\tau\|\omega_0\|^2_{L^2(\partial M)}}.$$

This shows the required estimate. To prove the second inequality, let ω_D (resp. ω_N) an eigenform of the Dirichlet (resp. Neumann) boundary problem associated to $\lambda^D_{1,p}$ (resp. $\lambda^N_{1,p}$) such that $\|\omega_D\|_{L^2(M)} = \|\omega_N\|_{L^2(M)} = 1$. For any nonnegative number s, we consider the p-form $\omega_s = s\omega_D + \omega_N$, which clearly satisfies $\nu \lrcorner \omega_s = 0$. In order to use

the Rayleigh inequality for the eigenvalue $\lambda_{1,p}(\tau)$, we compute

$$
\int_M \left(|\mathrm{d}\omega_s|^2 + |\delta\omega_s|^2 \right) \mathrm{d}\mu_g
$$

$$
= s^2 \int_M \left(|\mathrm{d}\omega_D|^2 + |\delta\omega_D|^2 \right) \mathrm{d}\mu_g + \int_M \left(|\mathrm{d}\omega_N|^2 + |\delta\omega_N|^2 \right) \mathrm{d}\mu_g
$$

$$
+ 2s \int_M \langle \mathrm{d}\omega_D, \mathrm{d}\omega_N \rangle \mathrm{d}\mu_g + 2s \int_M \langle \delta\omega_D, \delta\omega_N \rangle \mathrm{d}\mu_g
$$

$$
= s^2 \lambda_{1,p}^D + \lambda_{1,p}^N + 2s\lambda_{1,p}^N (\omega_D, \omega_N)_{L^2(M)}.
$$

In the last equality, we use the Stokes formula. Also, we have

$$
\|\omega_s\|_{L^2(\partial M)}^2 = \|\omega_N\|_{L^2(\partial M)}^2 \quad \text{and} \quad \|\omega_s\|_{L^2(M)}^2 = s^2 + 1 + 2s(\omega_D, \omega_N)_{L^2(M)}.
$$

Therefore, after replacing we get that

$$
\lambda_{1,p}(\tau) \leq \frac{s^2 \lambda_{1,p}^D + \lambda_{1,p}^N + 2s\lambda_{1,p}^N (\omega_D, \omega_N)_{L^2(M)} + \tau \|\omega_N\|_{L^2(\partial M)}^2}{s^2 + 1 + 2s(\omega_D, \omega_N)_{L^2(M)}}.
$$

As we did before, we take the inverse of this last inequality, then add and subtract the term

$$
\frac{s^2 \lambda_{1,p}^D + \tau \|\omega\|_{L^2(\partial M)}^2}{\lambda_{1,p}^N}
$$

in the numerator to get that

$$
\frac{1}{\lambda_{1,p}(\tau)} \geq \frac{1}{\lambda_{1,p}^N} + \frac{s^2 \left(1 - \frac{\lambda_{1,p}^D}{\lambda_{1,p}^N} \right) - \frac{\tau \alpha_N}{\lambda_{1,p}^N}}{s^2 \lambda_{1,p}^D + \lambda_{1,p}^N + 2s\lambda_{1,p}^N + \tau \alpha_N}.
$$

Here, we also use the fact that $|(\omega_D, \omega_N)_{L^2(M)}| \leq 1$ by the Cauchy–Schwarz inequality and the fact that $s \geq 0$. In order to get the lower bound, we need to compute the supremum of the function g which is given by

$$
g(s) = \frac{As^2 + B}{Cs^2 + Ds + E},
$$

with

$$
A = 1 - \frac{\lambda_{1,p}^D}{\lambda_{1,p}^N}, \qquad B = -\frac{\tau \alpha_N}{\lambda_{1,p}^N},
$$

$$
C = \lambda_{1,p}^D, \qquad D = 2\lambda_{1,p}^N, \qquad E = \lambda_{1,p}^N + \tau \alpha_N.
$$

A direct computation shows that the supremum of the function g is attained at $s_2 = \frac{\tau \alpha_N}{\lambda^D_{1,p} - \lambda^N_{1,p}}$ and thus

$$\sup_{\mathbb{R}}(g) = g(s_2) = -\frac{\tau \alpha_N \left(\lambda^D_{1,p} - \lambda^N_{1,p}\right)}{\lambda^N_{1,p} \left(\tau \alpha_N \lambda^D_{1,p} + \lambda^N_{1,p} \left(\lambda^D_{1,p} - \lambda^N_{1,p}\right)\right)}.$$

This finishes the proof of the Theorem 4.1. $\qquad\square$

In the following result, we will give a gap inequality between the eigenvalues of the Robin Laplacian under some curvature conditions. Let (M^n, g) be a Riemannian manifold with smooth boundary and let $\eta_1(x), \eta_2(x), \ldots, \eta_{n-1}(x)$ be the principal curvatures at a point x of the boundary (i.e. eigenvalues of the second fundamental form of the Weingarten tensor S). We assume that $\eta_1(x) \leq \eta_2(x) \leq \cdots \leq \eta_{n-1}(x)$. For any integer $p \in \{1, \ldots, n-1\}$, the p-curvatures $\sigma_p(x)$ are defined as $\sigma_p(x) = \eta_1(x) + \cdots + \eta_p(x)$. One can easily check that for any integer p and q with $p \leq q$, we have that $\frac{\sigma_p(x)}{p} \leq \frac{\sigma_q(x)}{q}$ with equality if and only if $\eta_1(x) = \eta_2(x) = \cdots = \eta_q(x)$. Hence, we deduce that $H \geq \frac{\sigma_p(x)}{p}$ for any integer $p \in \{1, \ldots, n-1\}$. In the next theorem, we set

$$\sigma_p = \inf_{x \in \partial M} \sigma_p(x)$$

We state the result which generalizes [4, Thm. 5.8].

Theorem 4.2. *Let M be a compact domain in $\mathbb{R}^n$. Fix an integer number $q \in \{1, \ldots, n-1\}$ and Let ω be a q-eigenform of the Robin Laplacian. If $\sigma_p > 0$ for some $p \leq q$, then we have*

$$\frac{\|\omega\|^2_{L^2(\partial M)}}{\|\omega\|^2_{L^2(M)}} \leq \frac{1}{\sigma_p} \left(\lambda_{1,q} - \lambda_{1,q-p}\right).$$

Proof. We mainly follow the same computations as in [4, Thm. 5.8]. Let ω be a q-eigenform of the Robin Laplacian and, for any $p \leq q$, consider the $(q-p)$-form $\phi_{i_1, \ldots, i_p} := \partial_{x_{i_1}} \lrcorner \cdots \lrcorner \partial_{x_{i_p}} \lrcorner \omega$, for $i_k = 1, \ldots, n$ with $k = 1, \ldots, p$. Clearly, we have that $\nu \lrcorner \phi_{i_1, \ldots, i_p} = 0$. Hence by the characterization (4.2) of the first eigenvalue, we get that

$$\lambda_{1,q-p}(\tau) \int_M \left|\phi_{i_1, \ldots, i_p}\right|^2 d\mu_g$$

$$\leq \int_M \left(\left|d\phi_{i_1, \ldots, i_p}\right|^2 + \left|\delta\phi_{i_1, \ldots, i_p}\right|^2\right) d\mu_g + \tau \int_{\partial M} \left|\phi_{i_1, \ldots, i_p}\right|^2 d\mu_g. \quad (4.5)$$

Next we sum over $i_1, \ldots, i_p$. We begin with the l.h.s. Applying successively (3.8), we have

$$\sum_{i_1,\ldots,i_p} |\phi_{i_1,\ldots,i_p}|^2 = \sum_{i_1,\ldots,i_p} \left| \partial_{x_{i_1}} \lrcorner \cdots \lrcorner \partial_{x_{i_p}} \lrcorner \omega \right|^2$$

$$= (q-(p-1))\ldots q \cdot |\omega|^2 = \frac{q!}{(q-p)!} |\omega|^2.$$

(4.6)

For the r.h.s., we first compute

$$\sum_{i_1,\ldots,i_p} |\delta\phi_{i_1,\ldots,i_p}|^2 = \sum_{i_1,\ldots,i_p} \left| \partial_{x_{i_1}} \lrcorner \cdots \lrcorner \partial_{x_{i_p}} \lrcorner \delta\omega \right|^2 \overset{(3.8)}{=} \frac{(q-1)!}{(q-1-p)!} |\delta\omega|^2. \qquad (4.7)$$

In order to compute the term $\sum_{i_1,\ldots,i_p} |d\phi_{i_1,\ldots,i_p}|^2$, we proceed as in Equation (3.6). Using repeatedly the identity $d(X \lrcorner \alpha) = \nabla_X \alpha - X \lrcorner d\alpha$, true for any parallel vector field X, we get

$$d\phi_{i_1,\ldots,i_p} = (-1)^p \partial_{x_{i_1}} \lrcorner \cdots \lrcorner \partial_{x_{i_p}} \lrcorner d\omega + \sum_{l=1}^{p} (-1)^{l+1} \partial_{x_{i_1}} \lrcorner \cdots \widehat{\partial_{x_{i_l}}} \lrcorner \cdots \lrcorner \partial_{x_{i_p}} \lrcorner \nabla_{\partial_{x_{i_l}}} \omega.$$

Thus, we find that the sum $\sum_{i_1,\ldots,i_p} |d\phi_{i_1,,cdots,i_p}|^2$ is equal to

$$\sum_{i_1,\ldots,i_p} \left| \partial_{x_{i_1}} \lrcorner \ldots \lrcorner \partial_{x_{i_p}} \lrcorner d\omega \right|^2 + \sum_{i_1,\ldots,i_p} \left| \sum_{l=1}^{p} (-1)^{l+1} \partial_{x_{i_1}} \lrcorner \ldots \widehat{\partial_{x_{i_l}}} \lrcorner \ldots \lrcorner \partial_{x_{i_p}} \lrcorner \nabla_{\partial_{x_{i_l}}} \omega \right|^2$$

$$+ 2(-1)^p \sum_{i_1,\ldots,i_p} \sum_{l=1}^{p} (-1)^{l+1} \left\langle \partial_{x_{i_1}} \lrcorner \ldots \lrcorner \partial_{x_{i_p}} \lrcorner d\omega, \partial_{x_{i_1}} \lrcorner \ldots \widehat{\partial_{x_{i_l}}} \lrcorner \ldots \lrcorner \partial_{x_{i_p}} \lrcorner \nabla_{\partial_{x_{i_l}}} \omega \right\rangle$$

$$\overset{(3.8)}{=} \frac{(q+1)!}{(q+1-p)!} |d\omega|^2$$

$$+ \sum_{i_1,\ldots,i_p} \sum_{l} \left\langle \partial_{x_{i_1}} \lrcorner \ldots \widehat{\partial_{x_{i_l}}} \lrcorner \ldots \lrcorner \partial_{x_{i_p}} \lrcorner \nabla_{\partial_{x_{i_l}}} \omega, \partial_{x_{i_1}} \lrcorner \ldots \widehat{\partial_{x_{i_l}}} \lrcorner \ldots \lrcorner \partial_{x_{i_p}} \lrcorner \nabla_{\partial_{x_{i_l}}} \omega \right\rangle$$

$$- 2 \sum_{\substack{i_1,\ldots,i_p \\ l<s}} (-1)^{l+s} \left\langle \partial_{x_{i_1}} \lrcorner \ldots \widehat{\partial_{x_{i_l}}} \lrcorner \ldots \lrcorner \partial_{x_{i_p}} \lrcorner \nabla_{\partial_{x_{i_l}}} \omega, \partial_{x_{i_1}} \lrcorner \ldots \widehat{\partial_{x_{i_s}}} \lrcorner \ldots \lrcorner \partial_{x_{i_p}} \lrcorner \nabla_{\partial_{x_{i_s}}} \omega \right\rangle$$

$$- 2 \sum_{i_1,\ldots,i_p} \sum_{l=1}^{p} \left\langle \partial_{x_{i_1}} \lrcorner \ldots \widehat{\partial_{x_{i_l}}} \lrcorner \ldots \lrcorner \partial_{x_{i_p}} \lrcorner (\partial_{x_{i_l}} \lrcorner d\omega), \partial_{x_{i_1}} \lrcorner \ldots \widehat{\partial_{x_{i_l}}} \lrcorner \ldots \lrcorner \partial_{x_{i_p}} \lrcorner \nabla_{\partial_{x_{i_l}}} \omega \right\rangle$$

$$\times \frac{(q+1)!}{(q+1-p)!} |d\omega|^2 + \frac{pq!}{(q-p+1)!} |\nabla\omega|^2$$

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$$- 2 \sum_{\substack{i_1,\dots,i_p \\ l < s}} (q-1) - (p-3)) \dots (q-1) \left\langle \partial_{x_{i_s}} \lrcorner \nabla_{\partial_{x_{i_l}}} \omega, \partial_{x_{i_l}} \lrcorner \nabla_{\partial_{x_{i_s}}} \omega \right\rangle$$

$$- 2 \frac{pq!}{(q-p+1)!} |d\omega|^2.$$

In this lenghty computation, we used the fact that $\sum_i \langle \partial_{x_i} \lrcorner \alpha, \partial_{x_i} \lrcorner \beta \rangle = p \langle \alpha, \beta \rangle$ for any p-forms α and β. We also make use of the formula $d = \sum_i \partial_{x_i} \wedge \nabla_{\partial_{x_i}}$. Now, one can easily check by using the expression of d and ∇ that the sum term in the above equality is equal to

$$\sum_{i_1,\dots,i_p} \sum_{l<s} \left\langle \partial_{x_{i_s}} \lrcorner \nabla_{\partial_{x_{i_l}}} \omega, \partial_{x_{i_l}} \lrcorner \nabla_{\partial_{x_{i_s}}} \omega \right\rangle = \binom{p}{2} \left(|\nabla \omega|^2 - |d\omega|^2 \right).$$

Hence, after simplifying, we find that

$$\sum_{i_1,\dots,i_p} \left| d\phi_{i_1,\dots,i_p} \right|^2 = \frac{p(q-1)!}{(q-p)!} |\nabla \omega|^2 + \frac{(q-1)!}{(q-p-1)!} |d\omega|^2. \tag{4.8}$$

Plugging Equalities (4.6), (4.7) and (4.8) into Inequality (4.5) yields after simplifying by $\frac{(q-1)!}{(q-p)!}$

$$\lambda_{1,q-p}(\tau) q \int_M |\omega|^2 d\mu_g$$

$$\le p \int_M |\nabla \omega|^2 d\mu_g + (q-p) \int_M \left(|d\omega|^2 + |\delta\omega|^2 \right) d\mu_g + \tau q \int_{\partial M} |\omega|^2 d\mu_g. \tag{4.9}$$

Now the Bochner formula $\Delta = \nabla^* \nabla + W_M^{[q]}$ applied to the form ω gives after integrating that

$$\lambda_{1,q}(\tau) \int_M |\omega|^2 d\mu_g = \int_M \langle \Delta \omega, \omega \rangle d\mu_g = \int_M |\nabla \omega|^2 d\mu_g + \int_{\partial M} \langle \nabla_\nu \omega, \omega \rangle d\mu_g.$$

By using $\langle \nabla_\nu \omega, \omega \rangle = \langle S^{[q]}(\iota^*\omega), \iota^*\omega \rangle + \tau |\omega|^2$ as proved in [9], we get that

$$\int_M |\nabla \omega|^2 d\mu_g = \lambda_{1,q}(\tau) \int_M |\omega|^2 d\mu_g - \int_{\partial M} \left(\left\langle S^{[q]}(\iota^*\omega), \iota^*\omega \right\rangle + \tau |\omega|^2 \right) d\mu_g.$$

Plugging this last equality into Inequality (4.9), we get after using the pointwise inequality $\langle S^{[q]}(\iota^*\omega), \iota^*\omega \rangle \geq \sigma_q|\omega|^2$ that

$$\lambda_{1,q-p}(\tau)q \int_M |\omega|^2 \mathrm{d}\mu_g$$

$$\leq p\lambda_{1,q}(\tau) \int_M |\omega|^2 \mathrm{d}\mu_g - p \int_{\partial M} (\sigma_q + \tau)|\omega|^2 \mathrm{d}\mu_g$$

$$+ (q-p)\left(\lambda_{1,q}(\tau) \int_M |\omega|^2 \mathrm{d}\mu_g - \tau \int_{\partial M} |\omega|^2 \mathrm{d}\mu_g\right) + \tau q \int_{\partial M} |\omega|^2 \mathrm{d}\mu_g$$

$$= q\lambda_{1,q}(\tau) \int_M |\omega|^2 \mathrm{d}\mu_g - p\sigma_q \int_{\partial M} |\omega|^2 \mathrm{d}\mu_g$$

$$\leq q\lambda_{1,q}(\tau) \int_M |\omega|^2 \mathrm{d}\mu_g - q\sigma_p \int_{\partial M} |\omega|^2 \mathrm{d}\mu_g.$$

In the last inequality, we used the fact that $\frac{\sigma_p}{p} \leq \frac{\sigma_q}{q}$. This finishes the proof of the Theorem 4.2. $\square$

5. Robin and Steklov operators vs. biharmonic Steklov

In this section, we relate the eigenvalues of the Robin problem to those of the biharmonic Steklov operator. This extends the result stated in [13, Thm. 1.17] for functions.

Theorem 5.1. *Let (M^n, g) be a compact Riemannian manifold with smooth boundary. We have for any $\tau > 0$ the estimate*

$$\frac{1}{\lambda_{1,p}(\tau)} < \frac{1}{\lambda_{1,p}^D} + \frac{1}{\tau q_{1,p}},$$

where $\lambda_{1,p}^D$ is the first eigenvalue of the Dirichlet boundary problem (4.4).

Proof. We mainly follow the computations done in [13, Thm. 1.17]. Let ω be an eigenform of the Robin boundary problem (4.1) associated to $\lambda_{1,p}(\tau)$. We denote by ω_1 a solution to the problem

$$\begin{cases} \Delta\omega_1 = 0 & \text{on } M \\ \iota^*\omega_1 = \iota^*\omega & \text{on } \partial M \\ \nu \lrcorner \omega_1 = 0 & \text{on } \partial M. \end{cases} \tag{5.1}$$

Notice that such a problem admits a unique solution by [23]. Now, let us consider the p-form $\omega_2 := \omega - \omega_1$. It clearly satisfies

$$\begin{cases} \Delta\omega_2 = \Delta\omega & \text{on } M \\ \omega_2 = 0 & \text{on } \partial M. \end{cases} \tag{5.2}$$

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By using the triangle inequality, the characterization (2.10) and the one of the first eigenvalue $\lambda_{1,p}^D$ of the Dirichlet problem (corresponding to $\tau \to \infty$ in (4.2)), we have

$$
\begin{aligned}
\|\omega\|_{L^2(M)} &\leq \|\omega_1\|_{L^2(M)} + \|\omega_2\|_{L^2(M)} \\
&\leq \sqrt{q_{1,p}^{-1}} \|\omega_1\|_{L^2(\partial M)} + \sqrt{\left(\lambda_{1,p}^D\right)^{-1}} \left(\|d\omega_2\|_{L^2(M)}^2 + \|\delta\omega_2\|_{L^2(M)}^2\right)^{\frac{1}{2}} \quad (5.3)\\
&\leq \sqrt{q_{1,p}^{-1}} \|\omega\|_{L^2(\partial M)} + \sqrt{\left(\lambda_{1,p}^D\right)^{-1}} \left(\|d\omega\|_{L^2(M)}^2 + \|\delta\omega\|_{L^2(M)}^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

Indeed, $\|\omega_1\|_{L^2(\partial M)} = \|\omega\|_{L^2(\partial M)}$ since $\omega_2 = 0$ on ∂M and

$$
\begin{aligned}
\|d\omega\|_{L^2(M)}^2 &+ \|\delta\omega\|_{L^2(M)}^2 \\
&= \|d\omega_1\|_{L^2(M)}^2 + \|d\omega_2\|_{L^2(M)}^2 + 2(d\omega_1, d\omega_2)_{L^2(M)} + \|\delta\omega_1\|_{L^2(M)}^2 \\
&\quad + \|\delta\omega_2\|_{L^2(M)}^2 + 2(\delta\omega_1, \delta\omega_2)_{L^2(M)} \\
&= \|d\omega_1\|_{L^2(M)}^2 + \|\delta\omega_1\|_{L^2(M)}^2 + \|d\omega_2\|_{L^2(M)}^2 + \|\delta\omega_2\|_{L^2(M)}^2 \\
&\quad + 2\int_M \langle \delta d\omega_1, \omega_2\rangle d\mu_g - 2\int_{\partial M} \langle \nu \lrcorner d\omega_1, \iota^*\omega_2\rangle d\mu_g \\
&\quad + 2\int_M \langle d\delta\omega_1, \omega_2\rangle d\mu_g + 2\int_{\partial M} \langle \iota^*\delta\omega_1, \nu \lrcorner \omega_2\rangle d\mu_g \\
&= \|d\omega_1\|_{L^2(M)}^2 + \|\delta\omega_1\|_{L^2(M)}^2 + \|d\omega_2\|_{L^2(M)}^2 + \|\delta\omega_2\|_{L^2(M)}^2 \\
&\geq \|d\omega_2\|_{L^2(M)}^2 + \|\delta\omega_2\|_{L^2(M)}^2.
\end{aligned}
$$

Now, we square both sides of Inequality (5.3) to write

$$
\begin{aligned}
\|\omega\|_{L^2(M)}^2 &\leq q_{1,p}^{-1} \|\omega\|_{L^2(\partial M)}^2 + \left(\lambda_{1,p}^D\right)^{-1} \left(\|d\omega\|_{L^2(M)}^2 + \|\delta\omega\|_{L^2(M)}^2\right) \\
&\quad + 2\left(q_{1,p}^{-1} \left(\lambda_{1,p}^D\right)^{-1} \|\omega\|_{L^2(\partial M)}^2 \left(\|d\omega\|_{L^2(M)}^2 + \|\delta\omega\|_{L^2(M)}^2\right)\right)^{\frac{1}{2}} \\
&\leq q_{1,p}^{-1} \|\omega\|_{L^2(\partial M)}^2 + \left(\lambda_{1,p}^D\right)^{-1} \left(\|d\omega\|_{L^2(M)}^2 + \|\delta\omega\|_{L^2(M)}^2\right) \\
&\quad \times \tau^{-1} q_{1,p}^{-1} \left(\|d\omega\|_{L^2(M)}^2 + \|\delta\omega\|_{L^2(M)}^2\right) + \tau \left(\lambda_{1,p}^D\right)^{-1} \|\omega\|_{L^2(\partial M)}^2 \\
&= \left(\tau^{-1} q_{1,p}^{-1} + \left(\lambda_{1,p}^D\right)^{-1}\right) \left(\|d\omega\|_{L^2(M)}^2 + \|\delta\omega\|_{L^2(M)}^2 + \tau \|\omega\|_{L^2(\partial M)}^2\right).
\end{aligned}
$$

In the second above inequality, we use the fact that $2\sqrt{ab} \leq \frac{a}{\tau} + \tau b$ for any real positive τ. The characterization (4.2) allows to deduce the inequality in the broad sense. Moreover,

if that inequality were an equality, then ω_2 would be a $\lambda_{1,p}^D$-Dirichlet-eigenform. But then

$$\omega_1 = \frac{\lambda_{1,p}^D - \lambda_{1,p}(\tau)}{\lambda_{1,p}^D}\omega$$

would hold on M, which with $\Delta\omega_1 = 0$ would imply that $\Delta\omega = 0$ and hence $\omega = 0$, contradiction. Therefore, the inequality must be strict. $\qquad\square$

Now, we come back to the Serrin problem on forms. We will use the existence of solution to this problem in harmonic domains carrying parallel forms to get an estimate for the eigenvalues of the absolute Dirichlet-to-Neumann operator introduced in [21] (see also [11]). Recall the definition of this operator. Let (M^n, g) be a compact Riemannian manifold with smooth boundary ∂M. Let $p \in \{0, \ldots, n-1\}$. Given any p-form ω on ∂M, there exists a unique p-form $\widehat{\omega}$ on M such that [23]

$$\begin{cases} \Delta\widehat{\omega} = 0 & \text{on } M \\ \iota^*\widehat{\omega} = \omega & \text{on } \partial M \\ \nu \lrcorner \omega = 0 & \text{on } \partial M. \end{cases} \qquad (5.4)$$

The form $\widehat{\omega}$ is usually called the harmonic tangential extension of ω. The Dirichlet-to-Neumann operator is then defined as $T^{[p]} : \Lambda^p(\partial M) \to \Lambda^p(\partial M)$, $\omega \mapsto -\nu \lrcorner d\widehat{\omega}$. When $p = 0$, this operator reduces to the classical Dirichlet-to-Neumann operator on functions, known as Steklov operator. It is shown in [21] that $T^{[p]}$ is an elliptic self-adjoint pseudo-differential operator with discrete spectrum consisting of eigenvalues

$$0 \le \nu_{1,p}(M) \le \nu_{2,p}(M) \le \ldots .$$

The kernel of this operator is isomorphic to the absolute de Rham cohomology $H_A^p(M)$ introduced in Section 4. The dual problem to (5.4) (w.r.t. the Hodge star operator) is called the relative Dirichlet-to-Neumann operator and is defined by $T_D^{[p]} = (-1)^{p(n-1-p)} *_{\partial M} T^{[n-1-p]} *_{\partial M}$. If $\nu_{1,p}^D(M)$ is the first eigenvalue of $T_D^{[p]}$, we have

$$\nu_{1,p}^D(M) = \nu_{1,n-1-p}(M).$$

Also, we have the following characterization [22] for the first eigenvalue $\nu_{1,p}^D(M)$:

$$\nu_{1,p}^D(M) = \inf\left\{ \frac{\int_M \left(|d\phi|^2 + |\delta\phi|^2\right) d\mu_g}{\int_{\partial M} |\phi|^2 d\mu_g} \,\middle|\, \phi \in \Omega^{p+1}(M), \iota^*\phi = 0 \right\}. \qquad (5.5)$$

Theorem 5.2. *Let (M^n, g) be a compact Riemannian manifold with smooth boundary. Assume that M is a harmonic domain and carries a parallel p-form for some $p =$*

$1, \ldots, n-1$. *If moreover $\sigma_p > 0$ or $\sigma_{n-p} > 0$, then*

$$\min\left(\nu_{1,p-1}(M), \nu_{1,n-1-p}(M)\right) \le \frac{\mathrm{Vol}(\partial M)}{\mathrm{Vol}(M)}.$$

Proof. Let ω be any p-eigenform of the biharmonic Steklov operator associated to some eigenvalue, say q. We let the $(p+1)$-form $\phi := \mathrm{d}\omega$. Clearly, we have that $\iota^*\phi = 0$. Hence, by Characterization (5.5), we get that

$$\nu_{1,p}^D(M) \int_{\partial M} |\nu \lrcorner \mathrm{d}\omega|^2 \mathrm{d}\mu_g \le \int_M |\delta \mathrm{d}\omega|^2 \mathrm{d}\mu_g. \tag{5.6}$$

Now, applying the same characterization (5.5) to $\mathrm{d}(*_M\omega)$, since $*_M\omega$ is also an eigenform of the biharmonic Steklov operator, yields the inequality

$$\nu_{1,n-p}^D(M) \int_{\partial M} |\iota^*\delta\omega|^2 \mathrm{d}\mu_g \le \int_M |\mathrm{d}\delta\omega|^2 \mathrm{d}\mu_g. \tag{5.7}$$

Summing Inequalities (5.6) and (5.7) yields

$$\min(\nu_{1,p}^D(M), \nu_{1,n-p}^D(M)) \int_{\partial M} \left(|\nu \lrcorner \mathrm{d}\omega|^2 + |\iota^*\delta\omega|^2\right) \mathrm{d}\mu_g \le \int_M \left(|\delta \mathrm{d}\omega|^2 + |\mathrm{d}\delta\omega|^2\right) \mathrm{d}\mu_g.$$

Now a direct computation using the Stokes formula and the boundary conditions on ω gives that

$$\int_M |\Delta\omega|^2 \mathrm{d}\mu_g = \int_M \left(|\mathrm{d}\delta\omega|^2 + |\delta \mathrm{d}\omega|^2\right) \mathrm{d}\mu_g - \frac{2}{q^2} \int_{\partial M} \left\langle \nu \lrcorner \Delta\omega, \delta^{\partial M}(\iota^*\Delta\omega) \right\rangle \mathrm{d}\mu_g.$$

Hence after plugging this equality into the above inequality, we deduce that

$$\min\left(\nu_{1,p}^D(M), \nu_{1,n-p}^D(M)\right) \le q + \frac{2}{q^2} \frac{\int_{\partial M} \left\langle \nu \lrcorner \Delta\omega, \delta^{\partial M}(\iota^*\Delta\omega) \right\rangle \mathrm{d}\mu_g}{\int_{\partial M} \left(|\nu \lrcorner \mathrm{d}\omega|^2 + |\iota^*\delta\omega|^2\right) \mathrm{d}\mu_g}. \tag{5.8}$$

Notice that this inequality is true for any eigenform ω of the biharmonic Steklov operator. From Theorem (3.6), we know that when M is a harmonic domain carrying a parallel p-form ω_0, the form $\omega = f \cdot \omega_0$ is an eigenform associated to the eigenvalue $q = \frac{\mathrm{Vol}(\partial M)}{\mathrm{Vol}(M)}$. Hence, we will apply Inequality (5.8) to the particular form $\omega = f \cdot \omega_0$. For this purpose, we will check the sign of the integral. Assume first that $\sigma_{n-p} > 0$. We estimate

$$\int_{\partial M} \left\langle \nu \lrcorner \Delta\omega, \delta^{\partial M}(\iota^*\Delta\omega) \right\rangle \mathrm{d}\mu_g = \int_{\partial M} \left\langle \nu \lrcorner \omega_0, \delta^{\partial M}(\iota^*\omega_0) \right\rangle \mathrm{d}\mu_g$$

$$= \int_{\partial M} \left\langle \nu \lrcorner \omega_0, \left(S^{[p-1]} - (n-1)H\right)\nu \lrcorner \omega_0\right) \right\rangle \mathrm{d}\mu_g$$

$$\le \int_{\partial M} \left((\sigma_{n-1} - \sigma_{n-p}) |\nu \lrcorner \omega_0|^2 - \sigma_{n-1}|\nu \lrcorner \omega_0|^2\right) \mathrm{d}\mu_g$$

$$\le 0.$$

In the second equality, we use the identity [20, Lem. 18]

$$\delta^{\partial M}(\iota^* \omega_0) = \nu \lrcorner \nabla_\nu \omega_0 + \iota^*(\delta \omega_0) + S^{[p-1]}(\nu \lrcorner \omega_0) - (n-1)H\nu \lrcorner \omega_0.$$

Also, we use the pointwise estimate $\langle S^{[p]}\alpha, \alpha \rangle \leq (\sigma_{n-1} - \sigma_{n-p-1})|\alpha|^2$ for any p-form α. Hence, we deduce that

$$\min\left(\nu_{1,p}^D(M), \nu_{1,n-p}^D(M)\right) \leq \frac{\text{Vol}(\partial M)}{\text{Vol}(M)}.$$

Finally, the fact that $\nu_{1,p}^D(M) = \nu_{1,n-1-p}(M)$ and $\nu_{1,n-p}^D(M) = \nu_{1,p-1}(M)$ finishes the proof of the statement when $\sigma_{n-p} > 0$. If $\sigma_p > 0$, then replacing p by $n - p$ and by invariance of parallel forms under the Hodge star operator on M, we obtain the same inequality. This concludes the proof. $\qquad\square$

Remark 5.3. Notice that a similar estimate has been established in [21, Cor. 15] for compact manifolds carrying parallel forms with the assumption $H_A^p(M) = H^p(M) = 0$. The inequality is

$$\nu_{1,n-1-p}(M) + \nu_{1,p-1}(M) \leq \frac{\text{Vol}(\partial M)}{\text{Vol}(M)}.$$

Appendix A.

Lemma A.1. *Let (M^n, g) be a compact Riemannian manifold with smooth boundary ∂M and let ν be the inward unit normal vector field to the boundary. Consider the following boundary value problem*

$$\begin{cases} \Delta^2 \omega = f & \text{on } M \\ B_1 \omega = \omega_1 & \text{on } \partial M \\ B_2 \omega = \omega_2 & \text{on } \partial M \\ B_3 \omega = \omega_3 & \text{on } \partial M \end{cases} \tag{A.1}$$

for given $f \in \Omega^p(M)$, $\omega_1 \in \Gamma(\Lambda^p T^* M_{|\partial M})$, $\omega_2 \in \Omega^{p-1}(\partial M)$ and $\omega_3 \in \Omega^p(\partial M)$ and where if $E_1 := \Lambda^p T^* M_{|\partial M}$, $E_2 := \Lambda^{p-1} T^* \partial M$ and $E_3 := \Lambda^p T^* \partial M$, $B_1 : \Omega^p(M) \to \Gamma(E_1)$ such that $B_1 \omega := \omega_{|\partial M}$, $B_2 : \Omega^p(M) \to \Gamma(E_2)$ and $B_3 : \Omega^p(M) \to \Gamma(E_3)$ are either:

(1) *$B_2 = \iota^* \delta \omega$ and $B_3 = \nu \lrcorner d\omega$. In this case,* (A.1) *is elliptic in the sense of Lopatinskiĭ–Shapiro (see [23, Def. 1.6.1]), self-adjoint and its kernel is reduced to $\{0\}$. Or*

(2) *$B_2 \omega := \nu \lrcorner \Delta \omega + q\iota^*(\delta \omega)$ and $B_3 \omega := \iota^* \Delta \omega - q\nu \lrcorner d\omega$ for some real constant q. In this case, problem* (A.1) *is elliptic in the sense of Lopatinskiĭ–Shapiro.*

Proof. We denote by σ_{Δ^2} and σ_{B_j} the principal symbols of the operators Δ^2 and B_j respectively, for $j = 1, 2, 3$. Given any $v \in T_x^* \partial M \setminus \{0\}$ for a fixed $x \in \partial M$, we consider the space

$$\mathcal{M}_v^+ := \{\text{bounded solutions } y = y(t) \text{ on } \mathbb{R}_+ \text{ to the ODE } \sigma_{\Delta^2}((-iv, \partial_t))y = 0\}.$$

A direct computation shows that

$$\mathcal{M}_v^+ = \text{Span}\left(e^{-|v|t}(at + b) \cdot \omega_0 \,\Big|\, a, b \in \mathbb{R}, \omega_0 \in \Lambda^P T_x^* M_{|\partial M}\right),$$

which is hence a space of dimension $N := 2\binom{n}{P}$. We look at the pointwise map

$$\mathcal{M}_v^+ \longrightarrow \bigoplus_{j=1}^3 E_j$$
$$y \longmapsto \left(\sigma_{B_1}((-iv, \partial_t))y, \sigma_{B_2}((-iv, \partial_t))y, \sigma_{B_3}((-iv, \partial_t))y\right)(0)$$

which we want to show to be an isomorphism. Note already that space dimensions are equal on both sides. Since $\sigma_{B_1}((-iv, \partial_t)) = \text{Id}$, $\sigma_{B_2}((-iv, \partial_t)) = -\partial_t \cdot v \lrcorner \cdot + iv \lrcorner \iota^* \cdot$ and $\sigma_{B_3}((-iv, \partial t)) = \partial_t \cdot \iota^* + iv \wedge (v \lrcorner \cdot)$, we obtain that, for any fixed $\omega_0 \in \Lambda^P T_x^* M$, the element $e^{-|v|t} \cdot \omega_0$ of $\mathcal{M}_v^+$ (corresponding to $a = 0$ and $b = 1$) is sent to $(\omega_0, |v|v \lrcorner \omega_0 + iv \lrcorner \iota^* \omega_0, -|v|\iota^* \omega_0 + iv \wedge (v \lrcorner \omega_0))$; and that the element $te^{-|v|t} \cdot \omega_0$ of $\mathcal{M}_v^+$ (corresponding to $a = 1$ and $b = 0$) is sent to $(0, -v \lrcorner \omega_0, \iota^* \omega_0)$. Choosing a basis $(\omega_0^{(1)}, \ldots, \omega_0^{(N)})$ of $\Lambda^P T_x^* M$, the basis $(e^{-|v|t} \cdot \omega_0^{(1)}, \ldots, e^{-|v|t} \cdot \omega_0^{(N)}, te^{-|v|t} \cdot \omega_0^{(1)}, \ldots, te^{-|v|t} \cdot \omega_0^{(N)})$ of $\mathcal{M}_v^+$ will therefore be sent to a basis of $\bigoplus_{j=1}^3 E_j$. This shows the map $\mathcal{M}_v^+ \to \bigoplus_{j=1}^3 E_j$ to be an isomorphism. Therefore the boundary value problem (A.1) is elliptic (see [23, Def. 1.6.1]).

Using (2.3), it is easy to see that (A.1) is also self-adjoint. Moreover, the kernel of (A.1) is reduced to $\{0\}$: namely, if $\omega \in \Omega^p(M)$ solves (A.1) with $f = 0$ as well as $\omega_1 = \omega_2 = \omega_3 = 0$, then (2.4) implies that $\|\Delta\omega\|_{L^2(M)} = 0$, from which $\omega = 0$ on M follows using $\omega_{|\partial M} = 0$.

As a consequence, fixing $f = 0$ as well as $\omega_1 = 0$, for any given $(\omega_2, \omega_3) \in \Omega^{p-1}(\partial M) \oplus \Omega^p(\partial M)$, there exists a unique $\omega \in \Omega^p(M)$ solving (A.1). In particular, $\omega \in Z$, where, as in the proof of Theorem 2.3,

$$Z := \left\{\omega \in \Omega^p(M) \,\big|\, \Delta^2 \omega = 0 \text{ on } M \text{ and } \omega_{|\partial M} = 0\right\}.$$

This shows the map $Z \to \Omega^p(\partial M) \oplus \Omega^{p-1}(\partial M)$, $\omega \mapsto (v \lrcorner d\omega, \iota^* \delta\omega)$, to be onto. This proves 1.

Changing the boundary operators B_2 and B_3 via $B_2\omega := v \lrcorner \Delta\omega + q\iota^*(\delta\omega)$ and $B_3\omega := \iota^* \Delta\omega - qv \lrcorner d\omega$ for some real contant q (which actually plays no role since it is only involved in the first-order-terms of the b.c. and not in their principal symbols),

we still get elliptic boundary conditions for Δ^2: for any $v \in T_x^* \partial M \setminus \{0\}$, the pointwise map $\mathcal{M}_v^+ \to \bigoplus_{j=1}^3 E_j$ from above sends $e^{-|v|t} \cdot \omega_0$ to $(\omega_0, 0, 0)$ and sends $te^{-|v|t} \cdot \omega_0$ to $(0, 2|v|v \lrcorner \omega_0, 2|v|\iota^*\omega_0)$, which shows that map to be an isomorphism. Therefore B_1, B_2, B_3 define elliptic boundary conditions for Δ^2, this proves 2. $\qquad\square$

Lemma A.2. *Let $M^n \hookrightarrow \mathbb{R}^{n+m}$ be an isometric immersion and let II be the second fundamental form of the immersion. For all $X, Y \in TM$ and $N \in T^\perp M$, we have*

$$(\nabla_X II_N)(Y) = (\nabla_Y II_N)(X) - II_{\nabla_Y^\perp N}(X) + II_{\nabla_X^\perp N}(Y), \tag{A.2}$$

where $\nabla_X^\perp N := (\nabla_X^{\mathbb{R}^{n+m}} N)^\perp$ defines the normal connection on $T^\perp M$. As a consequence, by writing $\partial_{x_i} = \partial_{x_i}^T + \partial_{x_i}^\perp$ for all $i = 1, \ldots, n+m$, the divergence of the endomorphism $II_{\partial_{x_i}^\perp}$ is equal to

$$\delta(II_{\partial_{x_i}^\perp}) = -n\mathrm{d}\left(\left\langle \widetilde{H}, \partial_{x_i}^\perp \right\rangle\right) - nII_{\widetilde{H}}(\partial_{x_i}^T) + \sum_{a=1}^m II_{f_a}^2(\partial_{x_i}^T). \tag{A.3}$$

Here $\{f_1, \ldots, f_m\}$ is a local orthonormal frame of $T^\perp M$ and $\widetilde{H}$ is the mean curvature field of the immersion.

Proof. Let X, Y, Z be vector fields in TM that we assume to be parallel at some point in M and let $N \in T^\perp M$. We compute

$$
\begin{aligned}
(\nabla_X II_N)(Y, Z) &= X(II_N(Y, Z)) \\
&= X\langle II(Y, Z), N \rangle \\
&= \left\langle \nabla_X^{\mathbb{R}^{n+m}} II(Y, Z), N \right\rangle + \left\langle II(Y, Z), \nabla_X^{\mathbb{R}^{n+m}} N \right\rangle \\
&= \left\langle \nabla_Y^{\mathbb{R}^{n+m}} II(X, Z), N \right\rangle + \left\langle II(Y, Z), \nabla_X^{\mathbb{R}^{n+m}} N \right\rangle \\
&= Y(II_N(X, Z)) - \left\langle II(X, Z), \nabla_Y^{\mathbb{R}^{n+m}} N \right\rangle + \left\langle II(Y, Z), \nabla_X^{\mathbb{R}^{n+m}} N \right\rangle \\
&= (\nabla_Y II_N)(X, Z) - II_{\nabla_Y^\perp N}(X, Z) + II_{\nabla_X^\perp N}(Y, Z).
\end{aligned}
$$

In the fourth equality, we use the Codazzi equation for submanifolds in $\mathbb{R}^{n+m}$. Hence we get Equality (A.2). To find Equation (A.3), we decompose $\partial_{x_i} = \partial_{x_i}^T + \partial_{x_i}^\perp$ for all $i = 1, \ldots, n+m$. Then from the parallelism of the vector ∂_{x_i} and the Gauss formula, we get that

$$\nabla_X^{\mathbb{R}^{n+m}} \partial_{x_i}^\perp = -\nabla_X^{\mathbb{R}^{n+m}} \partial_{x_i}^T = -\nabla_X \partial_{x_i}^T - II\left(X, \partial_{x_i}^T\right),$$

where ∇ is the Levi-Civita connection on TM. Thus, we deduce that

$$\nabla_X^\perp \partial_{x_i}^\perp = \left(\nabla_X^{\mathbb{R}^{n+m}} \partial_{x_i}^\perp\right)^\perp = -II\left(X, \partial_{x_i}^T\right). \tag{A.4}$$

The divergence of the endomorphism $II_{\partial_{x_i}^{\perp}}$ can be computed using Equation (A.2) with $N = \partial_{x_i}^{\perp}$. For $X \in TM$, we have

$$\delta\left(II_{\partial_{x_i}^{\perp}}\right)(X)$$

$$= -\sum_{s=1}^{n}\left(\nabla_{e_s}II_{\partial_{x_i}^{\perp}}\right)(e_s, X)$$

$$= -\sum_{s=1}^{n}\left(\nabla_{e_s}II_{\partial_{x_i}^{\perp}}\right)(X, e_s)$$

$$\overset{(A.2)}{=} -\sum_{s}\left(\nabla_X II_{\partial_{x_i}^{\perp}}\right)(e_s, e_s) - \sum_{s=1}^{n}II_{II(X, \partial_{x_i}^T)}(e_s, e_s) + \sum_{s=1}^{n}II_{II(e_s, \partial_{x_i}^T)}(X, e_s)$$

$$= -nX\left(\left\langle\widetilde{H}, \partial_{x_i}^{\perp}\right\rangle\right) - nII_{\widetilde{H}}\left(X, \partial_{x_i}^T\right) + \sum_{a=1}^{m}\sum_{s=1}^{n}\left\langle II\left(e_s, \partial_{x_i}^T\right), f_a\right\rangle\langle II(X, e_s), f_a\rangle$$

$$= -nX\left(\left\langle\widetilde{H}, \partial_{x_i}^{\perp}\right\rangle\right) - nII_{\widetilde{H}}\left(\partial_{x_i}^T, X\right) + \sum_{a=1}^{m}g\left(II_{f_a}\left(\partial_{x_i}^T\right), II_{f_a}(X)\right).$$

This ends the proof of the Lemma A.2. □

Proposition A.3. *For any p-form ω, the curvature term $\sum_{s=1}^{n}e_s \lrcorner R(e_s, X)\omega$ is equal to*

$$\sum_{s=1}^{n}e_s \lrcorner R(e_s, X)\omega = -\sum_{a=1}^{m}II_{f_a}(X) \lrcorner II_{f_a}^{[p]}\omega + II_{n\widetilde{H}}(X) \lrcorner \omega,$$

where $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_m\}$ are respectively orthonormal basis of TM and $T^{\perp}M$.

Proof. In order to compute the curvature term, we use the Gauß equation. Indeed, for any $X, Y, Z, T \in TM$ we have

$$R(X, Y, Z, T) = -\langle II(X, Z), II(Y, T)\rangle + \langle II(X, T), II(Y, Z)\rangle,$$

which can be equivalently written as

$$R(X, Y)Z = -\sum_{a=1}^{m}g(II_{f_a}(X), Z)II_{f_a}(Y) + \sum_{a=1}^{m}g(II_{f_a}(Y), Z)II_{f_a}(X).$$

Now, due to linearity, we can consider that ω of the form $e_{i_1} \wedge \cdots \wedge e_{i_p}$. Then, we compute

$$\sum_{s=1}^{n} e_s \lrcorner R(e_s, X)\omega$$

$$= \sum_{s=1}^{n} \sum_{j=1}^{p} e_s \lrcorner \left(e_{i_1} \wedge \cdots \wedge R(e_s, X)e_{i_j} \wedge \cdots \wedge e_{i_p} \right)$$

$$= -\sum_{a=1}^{m} \sum_{s=1}^{n} \sum_{j=1}^{p} g\left(II_{f_a}(e_s), e_{i_j} \right) e_s \lrcorner \left(e_{i_1} \wedge \cdots \wedge \underbrace{II_{f_a}(X)}_{j^{th}} \wedge \cdots \wedge e_{i_p} \right)$$

$$+ \sum_{a=1}^{m} \sum_{s=1}^{n} \sum_{j=1}^{p} g\left(II_{f_a}(X), e_{i_j} \right) e_s \lrcorner \left(e_{i_1} \wedge \cdots \wedge \underbrace{II_{f_a}(e_s)}_{j^{th}} \wedge \cdots \wedge e_{i_p} \right)$$

$$= -\sum_{a=1}^{m} \sum_{s=1}^{n} \sum_{\substack{k,j=1 \\ k \neq j}}^{p} (-1)^{k+1} g\left(II_{f_a}(e_s), e_{i_j} \right) \delta_{s i_k} e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_k} \wedge \cdots \wedge II_{f_a}(X) \wedge \cdots \wedge e_{i_p}$$

$$- \sum_{a=1}^{m} \sum_{s=1}^{n} \sum_{j=1}^{p} (-1)^{j+1} g\left(II_{f_a}(e_s), e_{i_j} \right) g\left(II_{f_a}(X), e_s \right) e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_j} \wedge \cdots \wedge e_{i_p}$$

$$+ \sum_{a=1}^{m} \sum_{s=1}^{n} \sum_{\substack{k,j=1 \\ k \neq j}}^{p} (-1)^{k+1} g(II_{f_a}(X), e_{i_j}) \delta_{s i_k} e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_k} \wedge \cdots \wedge II_{f_a}(e_s) \wedge \cdots \wedge e_{i_p}$$

$$+ \sum_{a=1}^{m} \sum_{s=1}^{n} \sum_{j=1}^{p} (-1)^{j+1} g\left(II_{f_a}(X), e_{i_j} \right) g\left(II_{f_a}(e_s), e_s \right) e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_j} \wedge \cdots \wedge e_{i_p}.$$

By using the symmetry of the second fundamental form, the above computation reduces to

$$\sum_{s=1}^{n} e_s \lrcorner R(e_s, X)\omega$$

$$= -\sum_{a=1}^{m} \sum_{\substack{k,j=1 \\ k \neq j}}^{p} (-1)^{k+1} g\left(II_{f_a}(e_{i_k}), e_{i_j} \right) e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_k} \wedge \cdots \wedge II_{f_a}(X) \wedge \cdots \wedge e_{i_p}$$

$$- \sum_{a=1}^{m} \sum_{j=1}^{p} (-1)^{j+1} g\left(II_{f_a}^2(X), e_{i_j} \right) e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_j} \wedge \cdots \wedge e_{i_p}$$

$$+ \sum_{a=1}^{m} \sum_{\substack{k,j=1 \\ k \neq j}}^{p} (-1)^{k+1} g\left(II_{f_a}(X), e_{i_j} \right) e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge II_{f_a}(e_{i_k}) \wedge \cdots \wedge e_{i_p}$$

$$+ \sum_{a=1}^{m} \sum_{j=1}^{p} (-1)^{j+1} g\left(II_{n\widetilde{H}}(X), e_{i_j} \right) e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_p}.$$

Now, we prove that the first sum vanishes. Indeed, by decomposing with respect to $k < j$ and $k > j$, it is equal to

$$\sum_{k<j} (-1)^{k+1} g\left(II_{f_a}(e_{i_k}), e_{i_j} \right) e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge II_{f_a}(X) \wedge \cdots \wedge e_{i_p}$$

$$+ \sum_{k>j} (-1)^{k+1} g\left(II_{f_a}(e_{i_k}), e_{i_j} \right) e_{i_1} \wedge \cdots \wedge II_{f_a}(X) \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p}$$

$$= \sum_{k<j} (-1)^{k+j-1} g\left(II_{f_a}(e_{i_k}), e_{i_j} \right) II_{f_a}(X) \wedge e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_p}$$

$$+ \sum_{k>j} (-1)^{k+j} g\left(II_{f_a}(e_{i_k}), e_{i_j} \right) II_{f_a}(X) \wedge e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p}$$

$$= 0.$$

In the same way, we prove that the third sum is to $-\sum_{a=1}^{m} II_{f_a}^{[p-1]}(II_{f_a}(X) \lrcorner \, \omega)$. Indeed, it is equal to

$$\sum_{k<j} (-1)^{k+1} g\left(II_{f_a}(X), e_{i_j} \right) \left(e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge II_{f_a}(e_{i_k}) \wedge \cdots \wedge e_{i_p} \right)$$

$$+ \sum_{k>j} (-1)^{k+1} g\left(II_{f_a}(X), e_{i_j} \right) \left(e_{i_1} \wedge \cdots \wedge II_{f_a}(e_{i_k}) \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p} \right)$$

$$= \sum_{k<j} (-1)^{k+1+j-k-1} g\left(II_{f_a}(X), e_{i_j} \right) \left(e_{i_1} \wedge \cdots \wedge II_{f_a}(e_{i_k}) \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_p} \right)$$

$$+ \sum_{k>j} (-1)^{k+1+k-j-1} g\left(II_{f_a}(X), e_{i_j} \right) \left(e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge II_{f_a}(e_{i_k}) \wedge \cdots \wedge e_{i_p} \right)$$

$$= \sum_{k<j} (-1)^{j} g\left(II_{f_a}(X), e_{i_j} \right) \left(e_{i_1} \wedge \cdots \wedge II_{f_a}(e_{i_k}) \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_p} \right)$$

$$+ \sum_{k>j} (-1)^{j} g\left(II_{f_a}(X), e_{i_j} \right) \left(e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge II_{f_a}(e_{i_k}) \wedge \cdots \wedge e_{i_p} \right)$$

$$= -II_{f_a}^{[p-1]} \left(II_{f_a}(X) \lrcorner \, \omega \right).$$

In the last equality, $A^{[p]}(X_1 \wedge \cdots \wedge X_p) = \sum_{i=1}^{p} X_1 \wedge \cdots \wedge A(X_i) \wedge \cdots \wedge X_p$ is used again for any tangent vector fields $X_1, \ldots X_p$ on M. Therefore, we deduce that

$$\sum_{s=1}^{n} e_s \lrcorner R(e_s, X)\omega = -\sum_{a=1}^{m} II_{f_a}^2(X) \lrcorner \omega - \sum_{a=1}^{m} II_{f_a}^{[p-1]}\left(II_{f_a}(X) \lrcorner \omega\right) + II_{n\widetilde{H}}(X) \lrcorner \omega$$

$$= -\sum_{a=1}^{m} II_{f_a}(X) \lrcorner II_{f_a}^{[p]}\omega + II_{n\widetilde{H}}(X) \lrcorner \omega.$$

In the last equality, we used the identity $A^{[p-1]}(X \lrcorner \alpha) = X \lrcorner A^{[p]}\alpha - A(X) \lrcorner \alpha$. □

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