ANNALES MATHÉMATIQUES

BLAISE PASCAL

Benjamin Lledos

Regularity of the stress field for degenerate and/or singular elliptic problems

Volume 31, nº 1 (2024), p. 83-135.

https://doi.org/10.5802/ambp.427

Publication éditée par le laboratoire de mathématiques Blaise Pascal
de l'université Clermont Auvergne, UMR 6620 du CNRS
Clermont-Ferrand — France

Publication membre du
Centre Mersenne pour l'édition scientifique ouverte
http://www.centre-mersenne.org/
e-ISSN : 2118-7436

Annales mathématiques Blaise Pascal **31**, 83-135 (2024)

Regularity of the stress field for degenerate and/or singular elliptic problems

Benjamin Lledos

Abstract

We investigate the regularity of the solutions to degenerate and/or singular elliptic equations. We prove the continuity of $G(\nabla u)$ where u is a locally Lipschitz solution of $\operatorname{div} G(\nabla u) = \lambda \in \mathbb{R}$ in dimension two under some growth assumptions on G. Additionally, we establish a result that holds in any dimension, indicating that the separation between ∇u and the degeneracy set of G is continuous.

Régularité du stress field pour des équations elliptiques dégénérées et/ou singulières

Résumé

Nous étudions la régularité des solutions d'équations elliptiques dégénérées et/ou singulières. Nous prouvons la continuité de $G(\nabla u)$ où u est une solution localement Lipschitz de $\operatorname{div} G(\nabla u) = \lambda \in \mathbb{R}$ en dimension deux sous certaines hypothèses de croissance sur G. De plus, nous établissons un résultat valable en toute dimension, indiquant que la séparation entre ∇u et l'ensemble de dégénérescence de G est continue.

1. Introduction

1.1. A first example

In this article, we establish the continuity of certain functions of the gradients of solutions for elliptic partial differential equations. For instance, let us consider a locally Lipschitz continuous function u, which is defined on an open subset Ω of $\mathbb{R}^2$, and minimizes the following functional:

$$v \longrightarrow \int_\Omega \varphi(\nabla v) - \lambda v \tag{1.1}$$

among the functions in $W^{1,2}_u(\Omega)$. The set $W^{1,2}_u(\Omega)$ is the set of functions $v \in L^2(\Omega)$, with a distributional gradient that is also in $L^2(\Omega)$, such that u and v have the same trace on the boundary $\partial\Omega$ of Ω. Here, we assume that $\lambda \in \mathbb{R}_+$ and φ has the following form:

$$\varphi(z) := \begin{cases} \frac{1}{2}|z|^2 & \text{if } |z| \leq 1, \\ |z| - \frac{1}{2} & \text{if } 1 < |z| < 2, \\ \frac{1}{4}|z|^2 + \frac{1}{2} & \text{if } 2 \leq |z|. \end{cases} \tag{1.2}$$

Keywords: Regularity, elliptic PDE, dimension two.
2020 *Mathematics Subject Classification*: 35B65, 35J62, 35J70, 35J75, 49N99.

This convex function is not strictly convex and is not C^2. Hence, we cannot apply the classical regularity theory for smooth strictly convex functions. The case where we do not have ellipticity at only one point has also been well studied. For instance, in the case of the p-Laplacian when $\varphi(z) = |z|^p$ with $p > 1$ we know that the solutions are $C^{1,\alpha}$. However, in our example, $D^2\varphi(z)$ has an eigenvalue equal to 0 on the entire annulus $\{1 < |z| < 2\}$. Thus, we cannot use the results already known when the set of degeneracy is just a point.

For this kind of problems, we know that we can not expect u to be C^1 on Ω. In fact, by [10, Theorem 1] the function

$$u(x) := \begin{cases} C - \frac{\lambda}{4}|x|^2 & \text{if } |x| \leq \frac{2}{\lambda}, \\ C + \frac{1}{\lambda} - \frac{\lambda}{2}|x|^2 & \text{if } \frac{2}{\lambda} < |x| \leq 1 \end{cases} \tag{1.3}$$

is a minimizer of (1.1) with φ as in (1.2) on the set $W^{1,2}_u(\Omega)$ with its own boundary condition.

The problem (1.1) with φ as in (1.2) was introduced by Kawohl, Stara and Wittum in [23] where the authors want to prove the uniqueness of the solutions. They assume that Ω has several symmetries to establish the Lipschitz continuity of the level sets of the minimizers. However, we prove in this article that Ω need not have any symmetry to achieve this result. This shows that a good understanding of the regularity of the solutions can be useful to prove the uniqueness of the minimizers. Nevertheless, for (1.1) with φ as stated in (1.2) and $\lambda \in \mathbb{R}_+$, [27, Theorem 1.1] presents a direct proof of uniqueness.

Since in general, u is not C^1, one of our main objectives is to prove the continuity of $\nabla\varphi(\nabla u)$. This new result has important applications, such as the local C^∞ regularity of the solution around points where the gradient has a norm either smaller than one or larger than two. In addition, this demonstrates that, in general, the level sets of a solution are C^1 curves.

More generally, the aim of the article is to prove this kind of continuity estimates for various non-strictly convex functions defined on $\mathbb{R}^2$. These estimates are provided in the broader context of elliptic equations, which also encompass the Euler–Lagrange equations linked to minimization problems. In Theorem 1.10, we partially generalize these results to any dimension $N \in \mathbb{N}$.

1.2. **General problem**

Let $G : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function with $N \in \mathbb{N}$. In this article, we study the regularity of locally Lipschitz continuous weak solutions of the following equation:

$$\operatorname{div} G(\nabla u_0) = f \text{ in } \Omega \tag{1.4}$$

with Ω an open bounded set of $\mathbb{R}^N$ and $f : \mathbb{R}^N \to \mathbb{R}$ in $W^{1,q}(\Omega)$ with $q > N$. We assume that G is monotone in the following sense:

$$\langle G(z_1) - G(z_2), z_1 - z_2 \rangle \geq 0 \tag{1.5}$$

for every $z_1, z_2 \in \mathbb{R}^N$. By solution, we mean every locally Lipschitz function u_0 such that

$$\int_\Omega \langle G(\nabla u_0), \nabla\theta \rangle = -\int_\Omega f\theta$$

for every function $\theta \in C_0^\infty(\Omega)$. When G is the gradient of a convex function φ, we obtain a nonlinear elliptic equation that can be seen as the Euler–Lagrange equation associated to the minimization of

$$\int_\Omega \varphi(\nabla v) + fv. \tag{1.6}$$

Many results state that the solutions are locally Lipschitz under suitable growth on G at infinity, see e.g. [7], [9], [16] and [18]. The main goal of the paper is to prove the continuity of the stress field $G(\nabla u_0)$ depending on the assumptions of G and f.

If G is smooth and if there exists $C > 0$ such that

$$\frac{1}{C}|A - B|^2 \leq \langle G(A) - G(B), A - B \rangle \leq C|A - B|^2$$

for every $A, B \in \mathbb{R}^N$ then the solutions of (1.4) are C^1 when $f \in L^p(\Omega)$ with $p > N$, (see [17, Theorem 6.33]). This is the case for instance for the Poisson equation when $G = Id$. When there exist A and B two distinct vectors of $\mathbb{R}^N$ such that $\langle G(A)-G(B), A-B\rangle = 0$ we say that the equation is degenerate. If we cannot bound from above $\langle G(A) - G(B), A - B\rangle$ by a constant times the quantity $|A - B|^2$, then we say that the equation is singular. In these two critical frameworks, the C^1 regularity is not guaranteed.

However, the study of the regularity of the solutions for degenerate and/or singular equations with a large set of degeneracy and/or singularity is a recent and dynamic subject. In the seminal paper [1], the authors study the partial $C^{1,\alpha}$ regularity of the solutions. Namely, let u_0 be a minimizer of (1.6), if $x \in \Omega$ is a Lebesgue point of ∇u_0 such that φ is C^2 and $D^2\varphi$ is positive-definite on a neighborhood of $\nabla u_0(x)$, then there exists an open neighborhood U of x such that $u_0 \in C^{1,\alpha}(U)$. If we apply this result to φ as in (1.2) then there exist two open sets U_1 and U_2 such that $u_0 \in C^{1,\alpha}$ on these two sets. Moreover, $|\nabla u_0| < 1$ on U_1, $|\nabla u_0| > 2$ on U_2 and for a.e. $x \in \Omega\backslash(U_1 \cup U_2)$ we have $1 \leq |\nabla u_0(x)| \leq 2$. The drawback of this result is that we do not know the behavior of ∇u_0 at the boundary of the set where $u \in C^{1,\alpha}$. Specifically, as x approaches $\partial U_2 \cap \Omega$ from the interior of U_2, will $|\nabla u_0(x)|$ converge to 2?

Some results of our paper use ideas from [15]. In this article, De Silva and Savin prove, in particular, two theorems stating that the minimizers of (1.6) with $f \equiv 0$ and φ strictly convex, are C^1. The first one is [15, Theorem 1.1] where φ is not singular and degenerate on the same set. The second one is [15, Theorem 1.2] where φ is not singular except at a finite number of points.

There is a recent family of results when the set of degeneracy or singularity is convex. In this case, $G = \nabla\varphi$ with φ a convex function that is strongly convex outside a convex set C containing the origin. Let us quote two results of continuity everywhere on Ω. The first one is an article of Santambrogio and Vespri [30, Theorem 11] in dimension two and the second one an article of Colombo and Figalli [14, Theorem 1.1]. In the latter case, the authors prove that $F(\nabla u)$ is continuous on Ω when F is a continuous function that vanishes on C. When φ is as in (1.2), thanks to [14, Theorem 1.1] we get that $(|\nabla u_0| - 2)_+$ is continuous on Ω. In our paper we obtain a similar result for $(1 - |\nabla u_0|)_+$ even if the set of degeneracy is not convex.

In the vectorial case, the article [4] extends [14] for a particular φ that is equal to $\frac{1}{p}(|\cdot| - 1)_+^p$ with $p > 1$. It would be interesting to see if we can extend the results of our paper to the vectorial case.

1.3. Main results

We state the new results of this paper. Theorem 1.1, Theorem 1.5 and Theorem 1.7 are only valid in dimension two. Theorem 1.10 which is valid in any dimension extends [28, Theorem 2.1] with a larger class of degeneracy sets.

Theorem 1.1. *Let us assume that $G \in C^{0,1}_{\mathrm{loc}}(\mathbb{R}^2)$ and $f = 0$. We also assume that:*

for every $L > 0$ there exists $C_L > 0$ such that for every $z_1, z_2 \in B_L(0)$:

$$\langle G(z_1) - G(z_2), z_1 - z_2\rangle \geq C_L |G(z_1) - G(z_2)|^2. \quad (A_1)$$

Then for every solution u_0 of (1.4) the function $G(\nabla u_0)$ is continuous.

Remark 1.2. We point out that (A_1) does not imply that G is strictly monotone or a gradient of a convex function. However, if G is the gradient of a $C^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$ convex function, then it satisfies (A_1) with $C_L = \frac{1}{\|DG\|_{L^\infty(B_L(0))}}$.

Moreover, we have an explicit modulus of continuity:

Remark 1.3. Let u_0 be a solution of (1.4). Then, for every $x \in \Omega$, for every $y \in B_{\frac{\mathrm{dist}(x,\partial\Omega)}{2}}(x)$, we have

$$|G(\nabla u_0(x)) - G(\nabla u_0(y))| \leq C\omega(|x - y|).$$

Here, $C > 0$ is a constant depending on the Sobolev norm of $G(\nabla u_0)$:

$$C = \sqrt{2\pi}\|G(\nabla u_0)\|^2_{W^{1,2}\left(B_{\frac{\mathrm{dist}(x,\partial\Omega)}{2}}(x)\right)} \leq \frac{9\pi L}{4K}$$

where $L = \|\nabla u_0\|_{L^\infty\left(B_{\frac{3\,\mathrm{dist}(x,\partial\Omega)}{4}}(x)\right)}$ and $K > 0$ depending only on $\|DG\|_{L^\infty(B_L(0))}$ and C_L the constant introduced in (A_1).

Moreover, ω is a modulus of continuity independent of u_0, for every $0 \le r < \frac{\mathrm{dist}(x,\partial\Omega)}{2}$:

$$\omega(r) = \frac{1}{\sqrt{\ln\left(\frac{\mathrm{dist}(x,\partial\Omega)}{2}\right) - \ln(r)}}.$$

When $f \in \mathbb{R}$ is not equal to zero, we can extend the previous result under structural assumptions on G. In order to state the next result, we need the following definition:

Definition 1.4. We say that a convex function $\mathcal{N} : \mathbb{R}^N \to \mathbb{R}_+$ is a *pseudo-norm* if $\mathcal{N}(0) = 0$, $\mathcal{N}$ is positively homogeneous and $\{z \in \mathbb{R}^N$ such that $\mathcal{N}(z) < 1\}$ is an open strictly convex bounded set with a $C^{1,1}$ continuous boundary.

It is important to notice that $\mathcal{N}$ is not necessarily symmetric. Hence, the definition of a *pseudo-norm* is more general than the definition of a norm with $C^{1,1}$ level sets.

The next theorem is stated when $f \equiv \lambda \in \mathbb{R}$ and G is the sum of gradients of convex functions:

Theorem 1.5. *Let us assume that $f \equiv \lambda \in \mathbb{R}$ and $G = \sum_{i=1}^n G_i$ with $n \in \mathbb{N}$. Here, the functions $(G_i)_{1\le i\le n}$ are gradients of convex functions $(\varphi_i)_{1\le i\le n}$ that have one of the two following forms:*

$$\varphi_i(z) := f_i(\mathcal{N}_i(z - \xi_i)) \text{ and } f_i'(z) = 0 \Leftrightarrow z = 0 \tag{A_2}$$

with $f_i \in C^{1,1}_{\mathrm{loc}}(\mathbb{R})$ a convex function, $\mathcal{N}_i$ a pseudo-norm *and $\xi_i \in \mathbb{R}^2$.*

$$\varphi_i(z) := f_i(\langle z, \xi_i\rangle) \tag{A_3}$$

where $f_i \in C^{1,1}_{\mathrm{loc}}(\mathbb{R})$ is a convex function and $\xi_i \in \mathbb{R}^2\backslash\{0\}$.

Then for every solution u_0 of (1.4) *the function $G(\nabla u_0)$ is continuous.*

Remark 1.6. In this article, we use the convention that $0 \notin \mathbb{N}$.

We can apply Theorem 1.5 to $G = \nabla\varphi$ with φ as in (1.2). Hence, $\nabla\varphi(\nabla u_0)$ is continuous. In this particular case, $[1 - |\nabla u_0|]_+$ is continuous which is a new feature that can not be obtained with [14], [28] or [30] since the set of degeneracy is an annulus. Thus, we know that $|\nabla u_0(x)|$ has to go to 1 when x converges to the boundary of the open set $\{x' \in \Omega$ such that $|\nabla u_0(x')| < 1\}$ from the inside. This new result is useful to study global regularity of the level lines.

Theorem 1.5 can also be used for orthotropic type functionals. By orthotropic we mean that $G = \nabla\varphi$ and φ is the sum of convex functions $z \mapsto \varphi_i(z)$ that depends only on one coordinate of z. Hence, if $\varphi(z) = |z_1|^{p_1} + |z_2|^{p_2}$ with $2 \le p_1 \le p_2$ and with $z_i := \langle z, e_i\rangle$, then $\nabla\varphi(\nabla u_0)$ is continuous. In the singular case where $1 < p_1 \leq 2 \leq p_2$ we have the following result:

Theorem 1.7. *Let us assume that $f \equiv \lambda \in \mathbb{R}$ and $G = G_1 + G_2$ with*

$$G_i(z) := f_i'(\langle z, \xi_i \rangle)\xi_i$$

where $f_1 \in C^{1,1}_{\mathrm{loc}}(\mathbb{R})$ and $f_2 \in C^1(\mathbb{R}) \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}\backslash\{0\})$ are two convex functions and $\xi_1, \xi_2 \in \mathbb{R}^2\backslash\{0\}$ are non colinear. Moreover, we assume that there exist $r > 0$ and a modulus of convexity $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ for f_2. Namely, ω is a continuous function satisfying $\omega(t) = 0 \Leftrightarrow t = 0$ such that for every $x, y \in (-r, r)$ we have

$$(f_2'(x) - f_2'(y))(x - y) \geq \omega(|x - y|).$$

Then for every solution u_0 of (1.4) the function $G(\nabla u_0)$ is continuous.

Theorem 1.7 can be used to prove some regularity results of the solutions in the case of orthotropic functionals with more general growth than power-type growth. In fact, if $\varphi(z) = |z_1|^{p_1} + |z_2|^{p_2}$ with $1 < p_1 < 2 \leq p_2$ as in a case of [5] then the functional is singular on $\{z_1 = 0\}$ and degenerate on $\{z_2 = 0\}$. Hence, this is only when $z = 0$ that we have both problems. In our case, the set where φ is singular and degenerate at the same time can be a line. For instance, we can consider $f_1(t) := (|t| - r)_+^2$ and $f_2(t) = |t|^{\frac{3}{2}}$ for every $t \in \mathbb{R}$. But we do not have the C^1 regularity in that case.

However, the two functions $(|\partial_1 u_0| - r)_+$ and $\frac{\partial_2 u_0}{|\partial_2 u_0|^{1/2}}$ are continuous. We can see that the continuity of this last function implies the continuity of $\partial_2 u_0$. Hence, the regularity of $G(\nabla u_0)$ can be useful when we exploit the local properties of G. For instance, in the setting of Theorem 1.1, Theorem 1.5 or Theorem 1.7 we have a local C^1 regularity result:

Proposition 1.8. *Under the assumptions of Theorem 1.1, Theorem 1.5 or Theorem 1.7, $G(\nabla u_0)$ has a continuous representative σ. If G is a homeomorphism between two open sets U and V then $u_0 \in C^1(\sigma^{-1}(V))$.*

In the case where $\sigma^{-1}(V) = \Omega$ then u_0 is C^1. The above proposition can be seen as an extension of what is known in dimension one. For instance, [13, Theorem 15.5] states that a Lipschitz minimizer u_0 of $\int_a^b F(x, u(x), u'(x))\,\mathrm{d}x$ is C^1 when $y \to F(x, u_0(x), y)$ is strictly convex for a.e. $x \in (a, b)$.

This proposition is useful in the case of orthotropic functionals with $\varphi(z) = |z_1|^{p_1} + |z_2|^{p_2}$. The first result on this subject [15, Theorem 1.1] provides the C^1 regularity of the locally Lipschitz minimizers when the problem is fully singular: $1 < p_1 \leq p_2 \leq 2$ or fully degenerate: $2 \leq p_1 \leq p_2$. We point out that we can use Theorem 1.5 and Proposition 1.8 to obtain a new proof of the C^1 regularity of u_0 in the degenerate case $2 \leq p_1 \leq p_2$.

The singular and degenerate case where $1 < p_1 < 2 < p_2$ is studied in [5] using ideas of [15] but Proposition 1.8 combined with Theorem 1.7 gives a new proof of the C^1

regularity in this case. The fully singular case $1 < p_1 \leq p_2 \leq 2$ is out of the scope of Theorem 1.5 and Theorem 1.7 unless $p_2 = 2$. Some other cases with different exponents can be found in the following papers: [6], [26] and [29].

Under the assumptions of Theorem 1.5, when G is the gradient of a convex function φ that depends only on the Euclidean norm, we have that $\frac{\nabla\varphi(\nabla u_0)}{|\nabla\varphi(\nabla u_0)|} = \frac{\nabla u_0}{|\nabla u_0|}$ is continuous when $\nabla u_0 \neq 0$. This allows to define the normal of the level sets as a continuous function. Consequently, we have the following result on the regularity of the level sets of a solution:

Proposition 1.9. *Let φ be a radial $C^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$ convex function and u_0 a solution of* (1.4) *with $G = \nabla\varphi$. We denote by σ the continuous representative of $\nabla\varphi(\nabla u_0)$ obtained in Theorem 1.5. Then for a.e. $t \in \mathbb{R}$, the connected components of $[u_0 = t] \cap [\sigma \neq 0]$ are C^1 curves.*

Our last result is an improvement of [28, Theorem 2.1]. In this article, Mooney considers a C^1 convex function φ and proves that the Lipschitz minimizers of

$$u \longrightarrow \int_\Omega \varphi(\nabla u)$$

are C^1 under some assumptions on φ. He introduces the sets

$$O_k := \left\{ z \in \mathbb{R}^N,\ \frac{1}{k}|v|^2 < \langle D^2\varphi(z)v, v\rangle < k|v|^2, \forall\, v \in \mathbb{R}^N \right\} \text{ and } D_\varphi = \mathbb{R}^N \setminus \bigcup_{k\in\mathbb{N}} O_k.$$

Then [28, Theorem 2.1] establishes that if φ is C^2 outside D_φ and if D_φ is a finite set of coplanar points, then the solutions are C^1.

In order to state the last theorem, we assume that there exists a compact set D_G such that $G \in C^1(\mathbb{R}^N \setminus D_G)$ and $D_G = \mathbb{R}^N \setminus \bigcup_{k\in\mathbb{N}} O_k$ with this time

$$O_k := \left\{ z \in \mathbb{R}^N,\ \frac{1}{k}|v|^2 < \langle DG(z)v, v\rangle < k|v|^2 \text{ for every } v \in \mathbb{R}^N \right\}.$$

For every $t \geq 0$, we introduce the closed $t-$neighborhood of a set U as

$$\overline{N}_t(U) := \left\{ z \in \mathbb{R}^N \text{ such that } \operatorname{dist}(z, U) \leq t \right\}.$$

Theorem 1.10. *Let us assume that $f \in W^{1,q}(\Omega)$ with $q > N$, D_G is contained in a plane and has finitely many connected components. We assume that there exists $t_0 > 0$ such that for every $0 \leq t \leq t_0$ the connected components of $\overline{N}_t(D_G)$ are simply connected. Then, for every solution u_0 of* (1.4)*,* $\operatorname{dist}(\nabla u_0, D_G)$ *and* $\operatorname{dist}(\nabla u_0, D_G) \times \nabla u_0$ *are continuous. Moreover, if G is constant on each connected components of D_G, then $G(\nabla u_0)$ is continuous.*

This assumption on the simply connected neighborhoods is satisfied when the connected components of D_G are simply connected with a Lipschitz boundary. The main difference between this result and [28, Theorem 2.1] is that the degeneracy set D_G is not just points. However, even if the conclusion is weakened, the solutions are still C^1 around points $x \in \Omega$ such that $\nabla u(x)$ is outside this set of degeneracy. Furthermore, we prove that the distance between ∇u and the degeneracy set is a continuous function.

This extension is natural in the sense that this is an improvement of [28] comparable to the improvement of [14] and [30] to the p-Laplacian case. In fact, we view [14] or [30] as an extension of what is known for the p-Laplacian case, where the set of degeneracy is one point, to the case where the set of degeneracy is larger. This is exactly what we are doing in Theorem 1.10 with respect to [28, Theorem 2.1]. Here, it is proven that the distance between ∇u_0 and the degeneracy set is continuous.

The study of the regularity of the solutions of (1.4) with a right-hand side $f \in L^q(\Omega)$ with $q > N$ is a widely studied subject in the classical framework of *uniform* elliptic equations and degenerate problems. It is the case in [4] and [14] for instance. In our case, the right-hand side belongs to the smaller set of Sobolev functions: $f \in W^{1,q}(\Omega)$ with $q > N$.

It is important to notice that the case where G is constant on each connected components of D_G does not cover the framework of Theorem 1.1, Theorem 1.5 and Theorem 1.7 since the connected components of D_G must be simply connected. That is not the case when $G = \nabla\varphi$ with φ as in (1.2), for example.

1.4. **Ideas of the proofs**

The proof of the continuity of $G(\nabla u_0)$ in Theorem 1.1, Theorem 1.5 and Theorem 1.7 uses ideas from [15]. In this article, the authors prove the C^1 regularity for Lipschitz minimizers of the following functional:

$$v \longrightarrow \int_\Omega F(\nabla v).$$

A major difference between our article and [15] is that we do not require G to be strictly monotone, which, as expected, weakens the conclusion. The solutions are not necessarily C^1 as shown in (1.3) but Proposition 1.8 provides a partial answer to that.

We can divide the proofs of Theorem 1.1, Theorem 1.5 and Theorem 1.7 in four parts:

Part 1. We regularize G in order to work with smooth elliptic equations of the form

$$\operatorname{div} G^m(\nabla u_m) = f_m.$$

We have to be careful when we approximate our problem since the functions $(G^m)_{m\in\mathbb{N}}$ have to share some properties of G such as the *pseudo-norm* structure or the orthotropic form.

Part 2. As in [15], we want to prove that $\|G^m(\nabla u_m)\|_{W^{1,2}(\Omega)}$ can be bounded uniformly in $m \in \mathbb{N}$. Since Theorem 1.1 is stated for a function G that is not necessarily the gradient of a convex function, we have to adapt some ideas of [15] to the setting of partial differential equations. In the case of Theorem 1.1 and Theorem 1.5 we prove the following result:

Proposition 1.11. *We assume that $G \in C^1(\mathbb{R}^N)$ satisfies the assumptions of Theorem 1.1 or Theorem 1.5. Then $G(\nabla u) \in W^{1,2}_{\mathrm{loc}}(\Omega)$.*

We have an analogous result in the framework of Theorem 1.7. In our case, we have to combine some results of [15] with an adaptation of [8, Theorem 2.1] to obtain Sobolev estimates in this particular framework. Hence, we can avoid the singularity at the origin with the following result:

Proposition 1.12. *We assume that $G \in C^1(\mathbb{R}^N)$ satisfies the assumption of Theorem 1.7. Then for every $\Omega' \Subset \Omega$ and every $r > 0$:*

$$\int_{\Omega' \cap U_r} |\nabla[G(\nabla u)]|^2 \leq C(G, r, \Omega') \tag{1.7}$$

where $U_r := \{x \in \Omega$ such that $|\langle \nabla u(x), \xi_2 \rangle| \geq r\}$. Moreover, $G_1(\nabla u) \in W^{1,2}_{\mathrm{loc}}(\Omega)$.

There exists several other results about the Sobolev regularity of $G(\nabla u)$ with more general right-hand side f. We can cite the recent papers: [2], [11], [12] and [21] for instance.

Part 3. We use this uniform estimate to obtain a uniform modulus of continuity. The original idea, specific to the dimension two, is due to Lebesgue and is used e.g. in [15, Lemma 2.1] and [26, Lemma 3.1]:

Proposition 1.13. *Let $H \in W^{1,2}_{\mathrm{loc}}(\Omega)$. If for every $\epsilon > 0$ and every $x_0 \in \Omega$ there exists $C(\epsilon, x_0) > 0$ such that for every $0 < \delta < \mathrm{dist}(x_0, \partial\Omega)$:*

$$\mathrm{osc}_{B_\delta(x_0)} H \geq \epsilon \Longrightarrow \mathrm{osc}_{\partial B_\delta(x_0)} H \geq C(\epsilon, x_0),$$

then H is continuous at x_0. Here, $\mathrm{osc}_{B_\delta(x_0)} H := \sup_{x,y \in B_\delta(x_0)} |H(x) - H(y)|$.

The second tool is a classical maximum principle, see e.g. [19, Theorem 3.1]:

Proposition 1.14. *Let u be a C^3 solution of (1.4) with $G \in C^2$ and $f \equiv \lambda \in \mathbb{R}$. Then for any $e \in \mathbb{S}^{N-1}$ and any open set $\Omega' \Subset \Omega$, we have that*

$$\sup_{x \in \Omega'} \partial_e u(x) = \sup_{x \in \partial\Omega'} \partial_e u(x).$$

This maximum principle is used as in [15] to prove that $G^m(\nabla u_m)$ satisfies the assumptions of the result from Lebesgue uniformly in $m \in \mathbb{N}$. Hence, the functions $G^m(\nabla u_m)$ are uniformly continuous in $m \in \mathbb{N}$.

Part 4. We pass to the limit when m goes to $+\infty$ and we prove that the sequence $G^m(\nabla u_m)$ converges uniformly to $G(\nabla u_0)$.

The strategy of the proof of Theorem 1.10 is different. Since the result is stated in any dimension, we can not use the result from Lebesgue. The proof is an adaptation of the one from [28, Theorem 2.1]. In our case, the result is stated with partial differential equations and with a non-zero right-hand side $f \in W^{1,q}(\Omega)$ with $q > N$, which create some technical difficulties.

The proof shows that one of the two following cases occurs:

- either $\nabla u(B_r(x_0))$ is outside the degeneracy set D_G for r small enough.
- or $\nabla u(B_r(x_0))$ is inside the convex hull of D_G when r is small enough.

In the first case, we are reduced to the framework of *uniform* elliptic partial differential equations and the conclusion follows from classical results. In the second case, we use the fact that the set of degeneracy D_G is in a plane to show that either $\nabla u(B_r(x_0))$ converges to a point outside D_G when $r \to 0$ or $\nabla u(B_r(x_0))$ is contained in a neighborhood of D_G when $r \to 0$.

1.5. Plan of the paper

In the following Section 2, we approximate our equation (1.4) by smooth equations in order to work with smooth functions. We also prove that if we pass to the limit, we obtain a solution of (1.4). In Section 3, we prove a uniform continuity estimate for Theorem 1.1 and Theorem 1.5 thanks to a uniform Sobolev estimate. Section 4 is devoted to the proof of Theorem 1.7 for approximated solutions. In the subsequent Section 5, we prove an intermediate result for Theorem 1.10. Finally, we pass to the limit in Section 6 to obtain the final conclusions. Section A is an appendix about classical results for the Minkowski functional used for the *pseudo-norms*.

2. Approximations of the solutions by smooth functions

In this article, we assume a priori that the solution u_0 of (1.4) is locally Lipschitz continuous. This regularity can be obtained under a uniform convexity condition at infinity.

For instance, we can apply [16, Theorem 4.1] or [7, Theorem 2.1] when there exist $C > 0$ and $R > 0$ such that $G \in C^1(\mathbb{R}^N \backslash \overline{B_R(0)})$ and

$$\frac{1}{C}|\xi|^2 \leq \langle DG(z)\xi, \xi\rangle \leq C|\xi|^2 \tag{2.1}$$

for every $z \in \mathbb{R}^N \backslash \overline{B_R(0)}$ and every $\xi \in \mathbb{R}^N$. Under these assumptions for every $\Omega' \Subset \Omega$ there exists a constant $L := L(\Omega', R, C)$ such that $\|\nabla u_0\|_{L^\infty(\Omega')} \leq L$.

Since we want to prove some local regularity results, we can assume that u_0 is globally Lipschitz continuous on Ω. Hence, we can change G outside a sufficiently large ball in order to assume that there exist $C > 0$ and $R > 0$ such that G satisfies (2.1).

In this section, we describe an approximation argument for the proofs of the main theorems, which must be adapted for each theorem in order to have smooth approximations $(G^m)_{m\in\mathbb{N}}$ that have the same properties as G and (2.1) uniformly in $m \in \mathbb{N}$.

We begin with an infinitesimal version of the assumption (A_1):

Lemma 2.1. *Let $L > 0$ and H be a C^1 function that satisfies (A_1). Then there exists $C_L > 0$ such that for every $z \in B_L(0)$ and every $v \in \mathbb{R}^N$ we have:*

$$\langle DH(z)v, v\rangle \geq C_L|DH(z)v|^2.$$

Proof. This result is true when $v = 0$. By assumption (A_1), we have that for every $z \in B_L(0)$, every $v \in \mathbb{R}^N \backslash \{0\}$ and every $0 < h < \frac{L-|z|}{|v|}$:

$$\langle H(z + hv) - H(z), hv\rangle \geq C_L|H(z + hv) - H(z)|^2.$$

By dividing this last equation by h^2 and letting h go to 0, we get:

$$\langle DH(z)v, v\rangle \geq C_L|DH(z)v|^2. \qquad \square$$

We use this new version of (A_1) in order to approximate G in the framework of Theorem 1.1. In this section, the constant $L > 0$ is such that $\|\nabla u_0\|_{L^\infty(\Omega)} \leq L$.

Proposition 2.2. *If G satisfies the assumptions of Theorem 1.1 then there exists a sequence of smooth functions $(G^m)_{m\in\mathbb{N}}$ converging uniformly to G on $B_L(0)$ that satisfy the same assumptions as G and (2.1) uniformly in $m \in \mathbb{N}$. Namely, there exists $C_1 > 0$ independent of $m \in \mathbb{N}$ such that for every $z, \xi \in \mathbb{R}^N$ we have*

$$\langle DG^m(z)\xi, \xi\rangle \geq C_1|DG^m(z)\xi|^2. \tag{A_1'}$$

Moreover, DG^m is invertible everywhere for every $m \in \mathbb{N}$.

Proof. Let $(\rho_m)_{m\in\mathbb{N}}$ be a standard radial mollifying sequence with support in $B_{\frac{1}{m}}(0)$. We introduce the convex function $\Phi(z) := (|z| - 2L)_+^2$ for every $z \in \mathbb{R}^2$ with L the Lipschitz constant of u_0 and $\theta \in C_0^\infty(B_{4L}(0))$ such that $0 \leq \theta \leq 1$ on $\mathbb{R}^N$, $\theta \equiv 1$ on $B_{3L}(0)$ and $\|\nabla\theta\|_{L^\infty(\mathbb{R}^N)} \leq \frac{2}{L}$.

We set

$$G^m := \theta(G * \rho_m) + K\nabla\Phi^m + \frac{1}{m}\,\mathrm{Id}$$

where $\Phi^m := \Phi * \rho_m$ and $K > 0$ to be fixed later. Thanks to Φ and the regularity of G, G^m satisfies (2.1) uniformly in $m \in \mathbb{N}$. Since we add the identity in G^m, we have that DG^m is invertible everywhere. It remains to check that G^m satisfies the assumption (A_1') uniformly in $m \in \mathbb{N}$.

For every $z_1, z_2 \in \mathbb{R}^2$ we have

$$\langle G*\rho_m(z_1)-G*\rho_m(z_2), z_1-z_2\rangle \geq \int_{\mathbb{R}^2} \langle G(z_1-y)-G(z_2-y), (z_1-y)-(z_2-y)\rangle \rho_m(y)\,\mathrm{d}y.$$

Thus, by assumption (A_1) we obtain that

$$\langle G * \rho_m(z_1) - G * \rho_m(z_2), z_1 - z_2\rangle \geq C_{3L+1} \int_{\mathbb{R}^2} |G(z_1 - y) - G(z_2 - y)|^2 \rho_m(y)\,\mathrm{d}y$$

for every $z_1, z_2 \in B_{3L}(0)$. By Jensen's inequality we get that

$$\langle G * \rho_m(z_1) - G * \rho_m(z_2), z_1 - z_2\rangle \geq C_{3L+1} |G * \rho_m(z_1) - G * \rho_m(z_2)|^2.$$

Hence, by Lemma 2.1 we obtain that

$$\langle DG * \rho_m(z)\xi, \xi\rangle \geq C_{3L+1} |DG * \rho_m(z)\xi|^2$$

for every $z \in B_{3L}(0)$, every $\xi \in \mathbb{R}^N$ and every $m \in \mathbb{N}$. In (A_1') we can assume that $|\xi| = 1$. For every $z, \xi \in \mathbb{R}^N$ with $|\xi| = 1$ we have that:

$$\langle DG^m(z)\xi, \xi\rangle \geq \theta\langle D(G * \rho_m)(z)\xi, \xi\rangle + K\langle D^2\Phi^m(z)\xi, \xi\rangle - |\nabla\theta||D(G * \rho_m)(z)|$$

and

$$|DG^m(z)\xi|^2 \leq 4\big(|\nabla\theta|^2 |G * \rho_m(z)|^2 + \theta^2 |D(G * \rho_m)(z)|^2 + K^2 |D^2\Phi^m(z)\xi|^2\big).$$

If $z \in B_{3L}(0)$, then $\theta(z) = 1$ and $\nabla\theta(z) = 0$. Hence,

$$\langle DG^m(z)\xi, \xi\rangle \geq \frac{1}{4} \min\left\{C_{3L+1}, \frac{1}{2K}\right\} |DG^m(z)\xi|^2.$$

If $z \notin B_{4L}(0)$ then $DG^m = KD^2\Phi^m$. Thus,

$$\langle DG^m(z)\xi, \xi\rangle \geq \frac{1}{2K} |DG^m(z)\xi|^2.$$

Finally, if $z \in B_{4L}(0)\backslash B_{3L}(0)$ then we can bound $|\nabla\theta|$ from above by $\frac{2}{L}$. Hence, we want to find $C > 0$ such that:

$$\begin{aligned} &K\langle D^2\Phi^m(z)\xi, \xi\rangle \\ &\geq 4C\left(\frac{4}{L^2} |G * \rho_m(z)|^2 + |D(G * \rho_m)(z)|^2 + K^2 |D^2\Phi^m(z)|^2\right) + \frac{2}{L} |D(G * \rho_m)(z)|. \end{aligned}$$

By definition of Φ, when m is large enough this is equivalent to

$$\frac{4K}{3} \geq \frac{2}{L}|D(G*\rho_m)(z)| + 4C\left(\frac{4}{L^2}|G*\rho_m(z)|^2 + |D(G*\rho_m)(z)|^2 + 4K^2\right).$$

By global Lipschitz regularity of G on $B := B_{4L}(0)$ we can choose the two constants $K := K(L, \|DG\|_{L^\infty(B)}) > 0$ and $C := C(L, \|DG\|_{L^\infty(B)}) > 0$ such that this last inequality is true. Hence, by taking C_1 as $\min\{C, \frac{1}{8K}, \frac{C_{3L+1}}{4}\}$ the assumption (A_1') is satisfied uniformly in $m \in \mathbb{N}$. □

In the case of Theorem 1.5 we proceed as follows:

Proposition 2.3. *If G satisfies the assumptions of Theorem 1.5 then there exists a sequence of C^4 functions $(G^m)_{m\in\mathbb{N}}$ converging to G uniformly on $B_L(0)$ that satisfy the same assumptions as G and* (2.1) *uniformly in $m \in \mathbb{N}$. Moreover, DG^m is invertible everywhere for every $m \in \mathbb{N}$.*

Proof. In the framework of Theorem 1.5, we have that $G = \sum_{i=1}^n \nabla\varphi_i$ with $\varphi_i(\,\cdot\,) = f_i(\mathcal{N}_i(\,\cdot\, - \xi_i))$ or $\varphi_i(\,\cdot\,) = f_i(\langle\,\cdot\,, \xi_i\rangle)$.

For $L \geq \|\nabla u_0\|_{L^\infty(\Omega)}$ and every $1 \leq i \leq n$, we introduce:

$$\widetilde{f}_i(t) := \begin{cases} f_i(-2L) + f_i'(-2L)(t+2L) + (t+2L)^2 & \text{if } t < -2L \\ f_i(t) & \text{if } -2L \leq t \leq 2L, \\ f_i(2L) + f_i'(2L)(t-2L) + (t-2L)^2 & \text{if } t > 2L, \end{cases}$$

and $\Phi(z) = (|\cdot| - 2L)_+^2$. We divide the rest of the proof in four steps.

Step 1. If $\varphi_i(\,\cdot\,) = f_i(\langle\,\cdot\,, \xi_i\rangle)$ then we set $G_i^m(\,\cdot\,) := \nabla[f_i^m(\langle\,\cdot\,, \xi_i\rangle)]$ for every $m \in \mathbb{N}$ with $f_i^m(\,\cdot\,) := \widetilde{f}_i * \rho_m(\,\cdot\,) + \frac{1}{m}|\cdot|^2$.

Step 2. If $\varphi_i(\,\cdot\,) = f_i(\mathcal{N}_i(\,\cdot\, - \xi_i))$, we proceed as follows. We introduce $C := (\mathcal{N}_i)^{-1}(\{[0,1)\})$, then $\mathcal{N}_i$ is the convex gauge γ_C of the convex set C. We regularize γ_C by convolution: $\gamma_C^m := \gamma_C * \rho_m$. For every $m \in \mathbb{N}$, the function γ_C^m is convex and has strictly convex lower level sets thanks to Proposition A.2. By Sard's theorem, we can define C_m as $(\gamma_C^m)^{-1}(\{[0, r_m)\})$ with $r_m \to 1$ when $m \to +\infty$ selected such that C_m is smooth. Moreover, we can assume that there exists $r > 0$ independent of m such that $B_r(0)$ is in the interior of C_m. Then we define $\mathcal{N}_i^m$ as the gauge of C_m. Hence, by Proposition A.1 $\mathcal{N}_i^m$ is a pseudo-norm smooth outside the origin. Moreover, for every $z \neq 0$ we have that

$$\nabla\mathcal{N}_i^m(z) = \frac{\nu_{C_m}(P_m(z))}{\langle\nu_{C_m}(P_m(z)), P_m(z)\rangle} \neq 0$$

where $P_m(z)$ is the intersection between $\mathbb{R}_+ z$ and ∂C_m and ν_{C_m} is the unit outward normal vector of C_m. In order to regularize f_i we set

$$f_i^m(t) := \left(\widetilde{f}_i * \rho_m(\,\cdot\,) + \frac{1}{m}|\cdot|^2\right)\left(\left(|t|^q + \frac{1}{m}\right)^{\frac{1}{q}} - \left(\frac{1}{m}\right)^{\frac{1}{q}} + \alpha_i^m\right).$$

Here, $q \geq 6$ is chosen in order to have $f_i^m(\mathcal{N}_i^m)$ at least C^5 for the upcoming computations, α_i^m is the only point where the strictly convex and coercive function $\widetilde{f}_i * \rho_m(\,\cdot\,) + \frac{1}{m}|\cdot|^2$ attains its minimum. Finally, we set $\varphi_i^m(\,\cdot\,) := f_i^m(\mathcal{N}_i^m(\,\cdot\, - \xi_i))$. Hence, φ_i^m is a strictly convex function such that $\nabla\varphi_i^m(z) = 0 \Leftrightarrow z = \xi_i$.

Step 3. We prove that φ_i^m and $\nabla\varphi_i^m$ converge uniformly to φ_i and $\nabla\varphi_i$ on every compact set when $m \to +\infty$. For every $z \in \mathbb{R}^2\backslash\{0\}$ we have that

$$|\gamma_C(P_m(z)) - \gamma_C(P_C(z))| \leq |\gamma_C^m(P_m(z)) - \gamma_C(P_m(z))| + |\gamma_C(P_C(z)) - \gamma_C^m(P_m(z))|$$

with $P_C(z)$ the intersection of ∂C and $\mathbb{R}_+ z$. By uniform convergence of γ_C^m to γ_C on compact sets, the first term in the right-hand side converges to 0 when $m \to +\infty$ uniformly in $z \in \mathbb{R}^2\backslash\{0\}$. The second term is equal to $|r_m - 1|$ and converges also to 0 uniformly in $z \in \mathbb{R}^2\backslash\{0\}$. This means that $\gamma_C(P_m(z))$ converges uniformly to 1 on $\mathbb{R}^2\backslash\{0\}$ when $m \to +\infty$. Hence, P_m converges to P_C uniformly on $\mathbb{R}^2\backslash\{0\}$. By homogeneity of $\mathcal{N}_i^m$, we get that $\mathcal{N}_i^m(z) = \frac{|z|}{|P_m(z)|}$ for every $z \neq 0$. The convergence of P_m combined with the fact that $\mathcal{N}_i^m(0) = 0 = \mathcal{N}_i(0)$ gives that $\mathcal{N}_i^m$ converges uniformly to $\mathcal{N}_i$ on every compact sets of $\mathbb{R}^2$ when $m \to +\infty$. Thus, we obtain that φ_i^m converges uniformly to φ_i on every compact sets of $\mathbb{R}^2$ when $m \to +\infty$.

When $z = \xi_i$, $\nabla\varphi_i^m(\xi_i) = 0$ and for every $z \neq \xi_i$, we have that

$$\nabla\varphi_i^m(z) = (f_i^m)'(\mathcal{N}_i^m(z - \xi_i))\nabla\mathcal{N}_i^m(z - \xi_i).$$

Moreover, if we set $f_i^m := g_i^m(\Theta_q^m)$ with $\Theta_q^m(t) = (|t|^q + \frac{1}{m})^{\frac{1}{q}} - \left(\frac{1}{m}\right)^{\frac{1}{q}}$ then

$$(f_i^m)''(t) = (g_i^m)'(\Theta_q^m(t))(\Theta_q^m)''(t) + (g_i^m)''(\Theta_q^m(t))((\Theta_q^m)'(t))^2.$$

The fact that $(g_i^m)'(0) = 0$ and $(\Theta_q^m)''(t) \leq \frac{C}{t}$ with C independent of $m \in \mathbb{N}$ gives that the functions $(f_i^m)_{m\in\mathbb{N}}$ are uniformly in $C^{1,1}(\mathbb{R}^2)$. Hence, it only remains to check that $\nabla\mathcal{N}_i^m$ converge uniformly to $\nabla\mathcal{N}_i$ on $\mathbb{R}^2\backslash\{0\}$. For every $z \in \mathbb{R}^2\backslash\{0\}$ we have that $\nabla\mathcal{N}_i^m(z) = \frac{\nu_{C_m}(P_m(z))}{\langle\nu_{C_m}(P_m(z)), P_m(z)\rangle} \neq 0$. The function $\nu_{C_m}(P_m)$ is equal to $\frac{\nabla\gamma_C^m(P_m)}{|\nabla\gamma_C^m(P_m)|}$ that converges uniformly on $\mathbb{R}^2\backslash\{0\}$ to $\frac{\nabla\gamma_C(P_C)}{|\nabla\gamma_C(P_C)|}$ that is equal to $\nu_C(P_C)$. Since there exists a small ball $B_r(0)$ with $r > 0$ independent of $m \in \mathbb{N}$ inside every C^m the scalar product $\langle\nu_{C_m}(P_m(z)), P_m(z)\rangle$ can be bounded from below by a positive constant independent of $m \in \mathbb{N}$. Hence, $\nabla\mathcal{N}_i^m$ converges uniformly on $\mathbb{R}^2\backslash\{0\}$ to $\nabla\mathcal{N}_i$. Thus, $\nabla\varphi_i^m$ converges uniformly on every compact sets of $\mathbb{R}^2$ to $\nabla\varphi_i$.

Thanks to Proposition A.3 the sets C_m have a Lipschitz continuous normal with a Lipschitz constant independent of $m \in \mathbb{N}$. Hence, the functions

$$\nabla\varphi_i^m(z) := (f_i^m)'(\mathcal{N}_i^m(z))\nabla\mathcal{N}_i^m(z)$$

if $z \neq 0$ and $\nabla\varphi_i^m(0) = 0$ are equi-Lipschitz continuous on each compact set of $\mathbb{R}^2$.

Step 4. In order to have $(G^m)_{m\in\mathbb{N}}$ satisfying (2.1) uniformly in $m \in \mathbb{N}$ we add a term of the following form: $\nabla[(\Phi * \rho_m(|\cdot|) + \frac{1}{m}|\cdot|^2]$. We define G^m as the following: $G^m := \sum_{i=1}^n G_i^m$. Since we add the identity in G^m, we have that DG^m is invertible everywhere. Thus, G^m is a function that satisfies the assumptions of Theorem 1.5 and (2.1) uniformly in $m \in \mathbb{N}$. □

We have the following result for Theorem 1.7:

Proposition 2.4. *If G satisfies the assumptions of Theorem 1.7 then there exists a sequence of smooth functions $(G^m)_{m\in\mathbb{N}}$ converging to G uniformly on $B_L(0)$ that satisfy the same assumptions as G and* (2.1) *uniformly in $m \in \mathbb{N}$. Moreover, DG^m is invertible everywhere for every $m \in \mathbb{N}$ and for every $0 < r < L$, $\sup_{r\le t\le L}(f_2^m)''(t)$ can be bounded uniformly in $m \in \mathbb{N}$.*

Proof. Let us consider $L > \|\nabla u_0\|_{L^\infty(\Omega)}$. In the framework of Theorem 1.7, we have that $G = G_1 + G_2$ with $G_i(\cdot) = f_i'(\langle\cdot,\xi_i\rangle)\xi_i$ for $i = 1,2$. We introduce $\Phi_1(z) := (|\langle z,\xi_1\rangle| - 2L)_+^2$, $\Phi_2(z) := (|\langle z,\xi_2\rangle| - 2L)_+^2$ and $\Phi(z) := \Phi_1(z) + \Phi_2(z)$. We also introduce $\widetilde{f}_1$ and $\widetilde{f}_2$ that satisfy

$$\widetilde{f}_i(t) := \begin{cases} f_i(-2L) + f_i'(-2L)(t+2L) + (t+2L)^2 & \text{if } t < -2L \\ f_i(t) & \text{if } -2L \le t \le 2L, \\ f_i(2L) + f_i'(2L)(t-2L) + (t-2L)^2 & \text{if } t > 2L. \end{cases}$$

Hence, we set for every $m \in \mathbb{N}$, $G_i^m(\cdot) := \nabla[(\widetilde{f}_i * \rho_m)(\langle\cdot,\xi_i\rangle) + \frac{1}{m}|\langle\cdot,\xi_i\rangle|^2 + \Phi_i * \rho_m(\cdot)]$ for $i = 1,2$. Then we can define G^m as the sum of these two functions: $G^m := G_1^m + G_2^m$. Since we add $\frac{1}{m}|\langle\cdot,\xi_1\rangle|^2 + \frac{1}{m}|\langle\cdot,\xi_2\rangle|^2$ in G^m we have that DG^m is invertible everywhere.

It remains to check that around the origin where $f_2^m(\cdot) := f_2 * \rho_m(\cdot) + \frac{1}{m}|\cdot|^2 + \Phi_i * \rho_m(\cdot)$ this function has a uniform modulus of convexity ω without any dependence on $m \in \mathbb{N}$. For every $m \ge \frac{2}{r}$ and every $x, y \in (-\frac{r}{2}, \frac{r}{2})$ we have that

$$((f_2^m)'(x) - (f_2^m)'(y))(x-y) \ge \int_{B_{\frac{r}{2}}(0)} (f_2'(x-t) - f_2'(y-t))((x-t)-(y-t))\rho_m(t)\mathrm{d}t.$$

By the uniform convexity assumption made on f_2, we have that $(f_2'(x-t) - f_2'(y-t))((x-t)-(y-t)) \geq \omega(|x-y|)$ for every $x, y, t \in (-\frac{r}{2}, \frac{r}{2})$. Hence, $((f_2^m)'(x) - (f_2^m)'(y))(x-y) \geq \omega(|x-y|)$ for every $m \geq \frac{2}{r}$ and every $x, y \in (-\frac{r}{2}, \frac{r}{2})$. □

In the case of Theorem 1.10 we just approximate G by $(G * \rho_m + \nabla\Phi^m + \frac{1}{m}\,\mathrm{Id})_{m\in\mathbb{N}}$ with $\Phi(\,\cdot\,) := (|\cdot| - 2L)_+^2$.

Proposition 2.5. *If G satisfies the assumptions of Theorem 1.10 then there exists a sequence of smooth functions $(G^m)_{m\in\mathbb{N}}$ converging to G uniformly on $B_L(0)$ that satisfy (2.1) uniformly in $m \in \mathbb{N}$. Moreover, for every $r > 0$ there exists $k_r \in \mathbb{N}$ such that for every $m \geq \frac{2}{r}$ and every $z \in \mathbb{R}^N$ such that $\operatorname{dist}(z, D_G) \geq r$ we have $\frac{1}{k_r}\,\mathrm{Id} < (DG^m(z))^s < k_r Id$ with $(DG^m)^s := \frac{DG^m + (DG^m)^T}{2}$.*

Proof. We regularize G as in the proof of Proposition 2.2. More precisely, we introduce $\Phi(\,\cdot\,) := (|\cdot| - 2L)_+^2$, $\theta \in C_0^\infty(B_{4L}(0))$ such that $0 \leq \theta \leq 1$ on $\mathbb{R}^N$, $\theta \equiv 1$ on $B_{3L}(0)$. For every $m \in \mathbb{N}$, we set $G^m := \theta(G * \rho_m) + \frac{1}{m}\,\mathrm{Id} + K\nabla\Phi * \rho_m$ with $K > 0$ such that G^m is monotone. Thus, for every $r > 0$ the support of ρ_m is inside $B_{\frac{r}{2}}(0)$ for every $m \geq \frac{2}{r}$. Since $\{z \in \mathbb{R}^N$ such that $\operatorname{dist}(z, D_G) \geq \frac{r}{2}\}$ is inside $O_{k_r'}$ for a certain k_r' the conclusion follows. □

In all the four cases, when $m \to +\infty$ we have that $G^m \to G$ uniformly on $B_L(0)$ with $L > 0$ selected such that $\|\nabla u_0\|_{L^\infty(\Omega)} \leq L$. Hence, up to a modification of G outside $B_L(0)$, we can assume that $G^m \to G$ uniformly on every compact sets of $\mathbb{R}^N$ when $m \to +\infty$.

For every $m \in \mathbb{N}$, we can consider the following equation:

$$\begin{cases} \operatorname{div} G_m(\nabla v(x)) = f_m & \text{in } \Omega, \\ v = u_0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

with $f_m := f * \rho_m$ and u_0 a globally Lipschitz continuous solution of (1.4).

By [17, Theorem 6.33], the solution u_m of (2.2) is C^3 inside Ω if $q \geq 6$ in the proof of Proposition 2.3. We have that (2.1) implies the existence of $C > 0$, $C' > 0$, $D \in \mathbb{R}$ and $D' > 0$ such that

$$C|z_1 - z_2|^2 + D \leq \langle G(z_1) - G(z_2), z_1 - z_2 \rangle \leq C'|z_1 - z_2|^2 + D' \tag{2.3}$$

for every $z_1, z_2 \in \mathbb{R}^N$. By the growth assumptions of G^m, the sequence $(u_m)_{m\in\mathbb{N}}$ is uniformly bounded in $W^{1,2}(\Omega)$:

Proposition 2.6. *The sequence $(u_m)_{m\in\mathbb{N}}$ is uniformly bounded in $W_{u_0}^{1,2}(\Omega)$.*

Proof. For every $m \in \mathbb{N}$, using the fact that u_m is a solution of (2.2) we obtain:

$$\int_\Omega \langle G_m(\nabla u_m), \nabla(u_m - u_0)\rangle = -\int_\Omega f_m(u_m - u_0)$$

Thanks to the first inequality in (2.3) we get:

$$\int_\Omega \langle G_m(\nabla u_0), \nabla(u_m - u_0)\rangle + C|\nabla u_m - \nabla u_0|^2 + D \leq -\int_\Omega f_m(u_m - u_0).$$

Hence, since $u_0 \in W^{1,\infty}(\Omega)$ and $\|f_m\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)}$ we have:

$$\int_\Omega |\nabla u_m - \nabla u_0|^2 \leq C' \int_\Omega |\nabla u_m - \nabla u_0| + D'.$$

Applying Young's inequality on the first term of the right-hand side gives that $|\nabla u_m - \nabla u_0|$ is bounded in $L^2(\Omega)$. Thus, (u_m) is uniformly bounded in $W^{1,2}_{u_0}(\Omega)$. □

By (2.1), we can assume that for every $\Omega' \Subset \Omega$ there exists $L_{\Omega'}$ such that $\|\nabla u_m\|_{L^\infty(\Omega')} \leq L_{\Omega'}$ for every $m \in \mathbb{N}$.

Since the sequence $(u_m)_{m\in\mathbb{N}}$ is uniformly bounded in $W^{1,2}(\Omega)$, we can extract a subsequence that converges weakly to a function $u \in W^{1,2}_{u_0}(\Omega)$. Moreover, for every subset $\Omega' \Subset \Omega$, we can use the Ascoli theorem to extract a subsequence of $(u_m)_{m\in\mathbb{N}}$ that converges uniformly to u on Ω'. Up to a diagonal process, we can assume that the sequence $(u_m)_{m\in\mathbb{N}}$ converges locally uniformly to u on Ω.

We can prove that u is a solution of (1.4). To do so, we use the following result on Young measures:

Lemma 2.7. *There exists a family of probability measures $(\nu_x)_{x\in\Omega}$ measurable with respect to x such that for a.e. $x \in \Omega$ and for ν_x-a.e. $y \in \mathbb{R}^N$ we have:*

$$\langle G(y) - G(\nabla u(x)), y - \nabla u(x)\rangle = 0. \tag{2.4}$$

Moreover, these probability measures satisfy the following property:

$$\int_\Omega H(x, \nabla u_m(x))\mathrm{d}x \longrightarrow \int_\Omega \int_{\mathbb{R}^N} H(x, y)\mathrm{d}\nu_x(y)\mathrm{d}x \tag{2.5}$$

when $m \to +\infty$, for every bounded Carathéodory function $H : \Omega \times \mathbb{R}^N \to \mathbb{R}$.

Proof. Since for every $\Omega' \Subset \Omega$ there exists $L_{\Omega'}$ independent of $m \in \mathbb{N}$ such that $\|\nabla u_m\|_{L^\infty(\Omega')} \leq L_{\Omega'}$, we get:

$$\lim_{k\to+\infty} \sup_{m\in\mathbb{N}} |\{x \in \Omega,\ |\nabla u_m(x)\}| > k| = 0.$$

By [3, Theorem] and [3, Remark 3], the sequence $(\nabla u_m)_{m\in\mathbb{N}}$, up to an extraction, generates a family of Young measures denoted by $(\nu_x)_{x\in\Omega}$ satisfying (2.5). By weak convergence of $(\nabla u_m)_{m\in\mathbb{N}}$ to ∇u, we obtain that

$$\nabla u(x) = \int_{\mathbb{R}^N} y \mathrm{d}\nu_x(y) \tag{2.6}$$

for a.e. $x \in \Omega$. Since u_m is a solution of (2.2), for every $\theta \in C_0^\infty(\Omega)$ we have that

$$\int_\Omega \langle G^m(\nabla u_m), \nabla\theta\rangle = -\int_\Omega f_m\theta.$$

When $m \to +\infty$, $G^m \to G$ on the compact set $B_L(0)$ where L is the uniform bound of $(\|\nabla u_m\|_{L^\infty(\operatorname{supp}\theta)})$. Thus,

$$\lim_{m\to+\infty} \int_\Omega \langle G^m(\nabla u_m) - G(\nabla u_m), \nabla\theta\rangle = 0.$$

Hence, by (2.5) and the previous equation we have that

$$-\int_\Omega f\theta = \int_\Omega \langle\nabla\theta, \int_{\mathbb{R}^N} G(y)\mathrm{d}\nu_x(y)\rangle \mathrm{d}x.$$

We introduce X_θ :

$$X_\theta := \lim_{m\to+\infty} \int_\Omega H(x, \nabla u_m(x))\mathrm{d}x$$

with $H(x, y) := \theta(x)\langle G(y), y - \nabla u(x)\rangle$. Thus, by (2.5) we obtain:

$$X_\theta = \int_\Omega \theta(x) \int_{\mathbb{R}^N} \langle G(y), y - \nabla u(x)\rangle \mathrm{d}\nu_x(y)\mathrm{d}x.$$

Since u_m is solution of a (2.2), we get that X_θ is equal to:

$$\lim_{m\to+\infty} \int_\Omega \theta\langle G(\nabla u_m) - G^m(\nabla u_m), \nabla u_m - \nabla u\rangle \\ - \langle G^m(\nabla u_m), \nabla\theta\rangle(u_m - u) - f_m\theta(u_m - u)\mathrm{d}x.$$

Using the fact that $(u_m)_{m\in\mathbb{N}}$ converges uniformly to u on $\operatorname{supp}\theta$ and that $(G^m)_{m\in\mathbb{N}}$ converges uniformly to G on the compact set $B_L(0)$ when $m \to +\infty$ imply that $X_\theta = 0$. But by (2.6) we get that

$$\begin{aligned} 0 = X_\theta &= \int_\Omega \theta \int_{\mathbb{R}^N} \langle G(y), y - \nabla u\rangle \mathrm{d}\nu_x(y)\mathrm{d}x \\ &= \int_\Omega \theta \int_{\mathbb{R}^N} \langle G(y) - G(\nabla u), y - \nabla u\rangle \mathrm{d}\nu_x(y)\mathrm{d}x \end{aligned} \tag{2.7}$$

where the last equality comes from (2.6). Since (2.7) is true for every $\theta \in C_0^\infty(\Omega)$, we obtain that for a.e. $x \in \Omega$,

$$\int_{\mathbb{R}^N} \langle G(y) - G(\nabla u(x)), y - \nabla u(x)\rangle \mathrm{d}\nu_x(y) = 0$$

and the conclusion follows from the fact that $\langle G(y) - G(\nabla u(x)), y - \nabla u(x)\rangle \geq 0$. □

We can make the following observation:

Remark 2.8. For a.e. $x \in \Omega$, if G is strictly monotone at $\nabla u(x)$, namely

$$\langle G(\nabla u(x)) - G(y), \nabla u(x) - y\rangle > 0$$

for every $y \in \mathbb{R}^N \backslash \{\nabla u(x)\}$, then $\nu_x = \delta_{\nabla u(x)}$ thanks to Lemma 2.7.

With this lemma, we can show that:

Proposition 2.9. *Under the assumptions of Theorem 1.1, Theorem 1.5 or Theorem 1.7, we have $G(\nabla u_m) \to G(\nabla u)$ in $L^1(\Omega)$ when $m \to +\infty$.*

Proof. Thanks to Lemma 2.7 and (2.5), it remains to prove the following result:

$$\operatorname{supp} \nu_x \subset \{y \in \mathbb{R}^N \text{ such that } G(y) = G(\nabla u(x))\} \text{ for a.e. } x \in \Omega.$$

Since G is the sum of monotone functions $(G_i)_{1\leq i\leq n}$, we have that $\langle G(y) - G(\nabla u(x)), y - \nabla u(x)\rangle$ is the sum of n nonnegative terms : $\langle G_i(y) - G_i(\nabla u(x)), y - \nabla u(x)\rangle$ for every $1 \leq i \leq n$. For a.e. $x \in \Omega$ and ν_x-a.e. $y \in \mathbb{R}^N$, since $\langle G(y) - G(\nabla u(x)), y - \nabla u(x)\rangle = 0$ for every $1 \leq i \leq n$ we get that $\langle G_i(y) - G_i(\nabla u(x)), y - \nabla u(x)\rangle = 0$. We distinguish two cases:

(a) If the condition (A_1) is satisfied for G_i, for a.e. $x \in \Omega$ and ν_x-a.e. $y \in \mathbb{R}^N$, we have that $G_i(y) = G_i(\nabla u(x))$ at once.

(b) If G_i is the gradient of a convex function φ_i, we obtain that for a.e. $x \in \Omega$ and ν_x-a.e. $y \in \mathbb{R}^N$, $G_i(y) = G_i(\nabla u(x))$ and the conclusion follows. □

We can prove that u is a solution of (1.4):

Proposition 2.10. *Under the assumptions of Theorems 1.1, 1.5 or 1.7, the function u is a solution of*

$$\begin{cases} \operatorname{div} G(\nabla u(x)) = f & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega. \end{cases}$$

Proof. Since $(u_m)_m$ converges weakly to $u \in W^{1,2}_{u_0}(\Omega)$, we have that u satisfies the boundary condition. It remains to prove that

$$\int_\Omega \langle G(\nabla u), \nabla\theta\rangle = -\int_\Omega f\theta$$

for every $\theta \in C_0^\infty(\Omega)$. Since $\|\nabla u_m\|_{L^\infty(\operatorname{supp}\theta)}$ is uniformly bounded, $G^m(\nabla u_m) - G(\nabla u_m) \to 0$ in $L^1(\operatorname{supp}\theta)$ when $m \to +\infty$. By Remark 2.9, this implies that $G^m(\nabla u_m) \to G(\nabla u)$ in $L^1(\operatorname{supp}\theta)$. Since

$$\int_\Omega \langle G^m(\nabla u_m), \nabla\theta\rangle = -\int_\Omega f_m\theta$$

we have our desired result. □

We conclude this section with a counterpart of Proposition 2.9 in the case of Theorem 1.10:

Proposition 2.11. *Under the assumptions of Theorem 1.10, we have that*

$$\int_\Omega |\operatorname{dist}(\nabla u_m, D_G) - \operatorname{dist}(\nabla u, D_G)| \longrightarrow 0$$

and

$$\int_\Omega |\operatorname{dist}(\nabla u_m, D_G)\nabla u_m - \operatorname{dist}(\nabla u, D_G)\nabla u| \longrightarrow 0$$

when $m \to +\infty$.

Proof. For every $x \in \Omega$, such that $\nabla u(x) \notin D_G$ we have that $\langle DG(\nabla u(x))A, A\rangle > 0$ for every $A \neq 0$. Hence, $\langle G(\nabla u(x)) - G(A), \nabla u(x) - A\rangle > 0$ for every $A \in \mathbb{R}^N \backslash \{\nabla u(x)\}$. Thanks to Remark 2.8, we get that $\nu_x = \delta_{\nabla u(x)}$. Since $(x, y) \to |\operatorname{dist}(y, D_G) - \operatorname{dist}(\nabla u(x), D_G)|$ is a Carathéodory function, this implies together with (2.5) that

$$\int_\Omega |\operatorname{dist}(\nabla u_m, D_G) - \operatorname{dist}(\nabla u, D_G)| \longrightarrow \int_\Omega \int_{\mathbb{R}^N} |\operatorname{dist}(y, D_G) - \operatorname{dist}(\nabla u(x), D_G)| \mathrm{d}\nu_x(y)\mathrm{d}x = 0$$

and for the same reasons

$$\int_\Omega |\nabla u_m \times \operatorname{dist}(\nabla u_m, D_G) - \nabla u \times \operatorname{dist}(\nabla u, D_G)| \longrightarrow 0$$

when $m \to +\infty$. □

3. **Uniform estimates for Theorem 1.1 and Theorem 1.5**

In this section, we prove that $G^m(\nabla u_m)$ is continuous with a modulus of continuity independent of $m \in \mathbb{N}$ if we are under the assumptions of Theorem 1.1 or Theorem 1.5.

3.1. $W^{1,2}$ regularity of $G^m(\nabla u_m)$

In this subsection, we show that $G^m(\nabla u_m) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ with a norm uniformly bounded in $m \in \mathbb{N}$. More precisely, our goal is to prove, in the framework of Theorem 1.1 and Theorem 1.5, that for every $1 \le i \le n$ the function $G^m_i(\nabla u_m)$ is in $W^{1,2}_{\mathrm{loc}}(\Omega)$ with a norm that does not depend on $m \in \mathbb{N}$. To do this, we use the same method as in [15, Proposition 2.4] to smooth functions, namely the regularized equations.

Proposition 3.1. *Let us consider u a C^2 solution of* (1.4) *where $G \in C^1(\mathbb{R}^N)$ satisfies the assumptions of Theorem 1.1 with (A_1) replaced by (A_1') or the assumptions of Theorem 1.5. If we write $G := \sum_{i=1}^n G_i$ as in those theorems, then $G_i(\nabla u) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ for every $1 \le i \le n$ and $G(\nabla u) \in W^{1,2}_{\mathrm{loc}}(\Omega)$. Moreover, the norms of these quantities depend only on the Lipschitz constant of u, the norm of the right-hand side, the Lipschitz constant of G and C_1 from assumption (A_1').*

Proof. Let us consider $\Omega'' \Subset \Omega' \Subset \Omega$. By differentiating (1.4), for every $e \in \mathbb{S}^1$, every $\theta \in C^1_0(\Omega')$ we have that

$$\int_\Omega \langle DG(\nabla u)\nabla \partial_e u, \nabla\theta\rangle = -\int_\Omega \partial_e f\theta.$$

In this last equality, we choose the following test function: $\theta = \eta^2\partial_e u$ with $\eta \in C^\infty_0(\Omega')$ and $\eta \equiv 1$ on Ω''. Hence, we get:

$$\int_\Omega \langle DG(\nabla u)\nabla\partial_e u, \nabla\partial_e u\rangle\eta^2 = -2\int_\Omega \eta\partial_e u\langle DG(\nabla u)\nabla\partial_e u, \nabla\eta\rangle - \int_\Omega \partial_e f\eta^2\partial_e u.$$

Since $G = \sum_{i=1}^n G_i$ it can be rewritten as

$$\sum_{i=1}^n \int_\Omega \langle DG_i(\nabla u)\nabla\partial_e u, \nabla\partial_e u\rangle\eta^2 = -2\sum_{i=1}^n \int_\Omega \eta\partial_e u\langle DG_i(\nabla u)\nabla\partial_e u, \nabla\eta\rangle - \int_\Omega \partial_e f\eta^2\partial_e u. \quad (3.1)$$

By Remark 1.2 and Lemma 2.1, each G_i satisfies the assumption (A_1'). Hence, there exists $C_i := C_i(L) > 0$ with $L := \|\nabla u\|_{L^\infty(\Omega')}$ such that

$$\langle DG_i(\nabla u)\nabla\partial_e u, \nabla\partial_e u\rangle \ge C_i|DG_i(\nabla u)\nabla\partial_e u|^2.$$

We set $K := \min_{1\le i\le N}\{C_i\}$. Thanks to Young's inequality, each term of the sum in the right-hand side of (3.1) can be bounded by

$$\alpha \int_\Omega |DG_i(\nabla u)\nabla\partial_e u|^2\eta^2 + \frac{1}{\alpha}\int_\Omega |\nabla\eta|^2|\nabla u|^2 + |\nabla f|\eta^2|\nabla u|$$

with $0 < \alpha < K$.

Thus, since $\eta \equiv 1$ on Ω'' we have that

$$\sum_{i=1}^{n}\int_{\Omega''} |DG_i(\nabla u)\nabla\partial_e u|^2 \le \frac{2L(L|\Omega|n\,\mathrm{dist}(\partial\Omega',\partial\Omega'')^{-2} + \|f\|_{W^{1,1}(\Omega)})}{\alpha(G,L,C_1)K(G,L,C_1)}. \tag{3.2}$$

Here, the dependence on G in α and K is just the dependence on $\|DG_i\|_{L^\infty(B_L(0))}$. Hence, for every $e \in \mathbb{S}^1$, the function $\partial_e(G_i(\nabla u)) = DG_i(\nabla u)\nabla\partial_e u$ is in $L^2_{\mathrm{loc}}(\Omega)$. Thus, $G_i(\nabla u) \in W^{1,2}_{\mathrm{loc}}(\Omega)$, and we have an explicit estimate for the norm from (3.2). Moreover, $G(\nabla u)$ is also in $W^{1,2}_{\mathrm{loc}}(\Omega)$ as the sum of $(G_i(\nabla u))_{1\le i\le n}$. □

We apply this result to G^m and u_m to prove a uniform estimate on the Sobolev norm of $G^m_i(\nabla u_m)$.

Proposition 3.2. *If $G \in C^{0,1}(\mathbb{R}^N)$ satisfies the assumption of Theorem 1.1 or Theorem 1.5, then, $G^m(\nabla u_m) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ with a norm independent of $m \in \mathbb{N}$. Moreover, $G^m_i(\nabla u_m) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ with a norm independent of $m \in \mathbb{N}$ for every $1 \le i \le n$.*

Proof. We introduce $\Omega'' \Subset \Omega' \Subset \Omega$. In the case of Theorem 1.1 and Theorem 1.5 the functions $(G^m)_{m\in\mathbb{N}}$ satisfy uniformly the assumption (A_1'). Since the norm $\|\nabla u_m\|_{L^\infty(\Omega')}$ can be bounded uniformly in $m \in \mathbb{N}$, all the estimates of the previous proposition are independent of $m \in \mathbb{N}$ if we apply it to G^m and u_m. Hence, $\|G^m_i(\nabla u_m)\|_{W^{1,2}(\Omega'')}$ can be bounded uniformly in $m \in \mathbb{N}$. That is also the case for their sum: $G^m(\nabla u_m) \in W^{1,2}_{\mathrm{loc}}(\Omega)$. □

3.2. Continuity of $G^m(\nabla u_m)$

In this subsection, we use the $W^{1,2}$ regularity obtained earlier to prove the continuity of $G^m(\nabla u_m)$.

The following proposition is crucial to the proofs of Theorem 1.1 and Theorem 1.5. As in [15, Lemma 2.1] and [26, Theorem 3.1] our strategy to prove it relies on a maximum principle that can be found in [19, Theorem 3.1] and a theorem due to Lebesgue stated in [25, p. 388]:

Proposition 3.3. *Let $H \in W^{1,2}_{\text{loc}}(\Omega)$. Assume that for every $\epsilon > 0$ and every $x_0 \in \Omega$ there exists $C(\epsilon, x_0) > 0$ such that for every $0 < \delta < \operatorname{dist}(x_0, \partial\Omega)$:*

$$\operatorname{osc}_{B_\delta(x_0)} H \geq \epsilon \Longrightarrow \operatorname{osc}_{\partial B_\delta(x_0)} H \geq C(\epsilon, x_0).$$

Then, H is continuous. Here, $\operatorname{osc}_{B_\delta(x_0)} H := \sup_{x,y \in B_\delta(x_0)} |H(x) - H(y)|$.

Proof. We argue by contradiction. Let us assume that there exist $\epsilon > 0$ and $x_0 \in \Omega$ such that for every $0 < \delta < \frac{\operatorname{dist}(x_0,\partial\Omega)}{2}$ there are $x, y \in B_\delta(x_0)$ such that $|H(x) - H(y)| \geq \epsilon$. By assumption, there exist $x_1, x_2 \in \partial B_\delta(x_0)$ such that $|H(x_1) - H(x_2)| \geq C(\epsilon, x_0)$. Hence, there exists $e \in \mathbb{S}^1$ such that

$$C(\epsilon, x_0) \leq \langle H(x_1), e\rangle - \langle H(x_2), e\rangle.$$

For a.e. $0 < \delta < \frac{\operatorname{dist}(x_0,\partial\Omega)}{2}$, the term in the right-hand side can be bounded from above by $\int_{\partial B_\delta(x_0)} |\nabla H| \mathrm{d}\mathcal{H}^1$. By Cauchy–Schwarz inequality we obtain

$$\frac{C(\epsilon, x_0)^2}{2\pi\delta} \leq \int_{\partial B_\delta(x_0)} |\nabla H|^2 \mathrm{d}\mathcal{H}^1.$$

By integrating over δ between a certain δ_ϵ and $\frac{\operatorname{dist}(x_0,\partial\Omega)}{2}$ we have

$$\frac{C(\epsilon, x_0)^2}{2\pi} \ln \frac{\operatorname{dist}(x_0, \partial\Omega)}{2\delta_\epsilon} \leq \|H\|^2_{W^{1,2}(B_{\frac{\operatorname{dist}(x_0,\partial\Omega)}{2}}(x_0))}.$$

By taking $\delta_\epsilon > 0$ small enough we obtain a contradiction thanks to the fact that $H \in W^{1,2}_{\text{loc}}(\Omega)$. □

Thanks to Proposition 3.1 we can make the following observation:

Remark 3.4. In Proposition 3.2, we have proved that $G^m(\nabla u_m)$ is bounded in $W^{1,2}_{\text{loc}}(\Omega)$ uniformly in $m \in \mathbb{N}$. Hence, the functions $(G^m(\nabla u_m))_{m\in\mathbb{N}}$ are uniformly continuous on any compact subset of Ω with a modulus of continuity independent of $m \in \mathbb{N}$ if they satisfy the assumptions of Proposition 3.3 with a constant $C(\epsilon, x_0)$ independent of $m \in \mathbb{N}$.

It remains to prove that under the assumptions of Theorem 1.1 and Theorem 1.5, the functions $G^m(\nabla u_m)$ or $G^m_i(\nabla u_m)$ satisfy the maximum principle stated in Proposition 3.3 with C that does not depend on $m \in \mathbb{N}$.

The following lemma, instrumental for the proof of Theorem 1.1 uses the fact that $f \equiv 0$:

Lemma 3.5. *Let u_m be a C^2 solution of* (2.2) *with $f \equiv 0$. We have that* $\det(D^2 u_m) \leq 0$.

Proof. Since u_m is a solution of (2.2) with $f \equiv 0$, we have that

$$\operatorname{Tr}(DG^m(\nabla u_m) D^2 u_m) = 0. \tag{3.3}$$

Thus,

$$\mathrm{Tr}((DG^m)^s(\nabla u_m)D^2u_m) = 0 \tag{3.4}$$

where $(DG^m)^s = \frac{DG^m+(DG^m)^T}{2}$ is the symmetric part of DG^m. Thanks to Proposition 2.2, $(DG^m)^s$ is positive-definite. We work in a basis where $(DG^m)^s$ is diagonal, let us say that

$$(DG^m)^s = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

with $a, b > 0$ and

$$D^2u_m = \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix}.$$

Then, by (3.4) we have that $\alpha + \beta$ is non-positive. Hence, $\det(D^2u_m) \leq 0$. □

As a consequence of Lemma 3.5, [22, Theorem 2] implies that:

Proposition 3.6. *Let u_m be a solution of* (2.2) *with $f \equiv 0$. Then for every $\Omega' \Subset \Omega$, we have that $\partial\nabla u_m(\Omega') \subset \nabla u_m(\partial\Omega')$.*

Remark 3.7. This last result is true in any dimension, provided that the sign of $\det(D^2u_m)$ does not change. This is the case if u_m is convex everywhere or concave for instance. This proposition can be used as an improved version of [15, Lemma 3.2].

With this result we can prove the following lemma:

Lemma 3.8. *Let u_m be a solution of* (2.2) *with $f \equiv 0$. Then for every $\Omega' \Subset \Omega$ we have that $\partial\sigma(\Omega') \subset \sigma(\partial\Omega')$ where $\sigma := G^m(\nabla u_m)$.*

Proof. We consider $z \in \partial\sigma(\Omega')$, then there exists a sequence $(z_n)_{n\in\mathbb{N}}$ such that $z_n \in \sigma(\Omega')$ and $z_n \to z$ when $n \to +\infty$. Hence, there exists a sequence $(y_n)_{n\in\mathbb{N}}$ with $y_n \in \nabla u_m(\Omega')$ such that $G^m(y_n) = z_n$. Since the sequence $(y_n)_{n\in\mathbb{N}}$ is bounded by $L := \|\nabla u_m\|_{L^\infty(\Omega')}$, we can extract a subsequence converging to $y \in \overline{\nabla u_m(\Omega')}$. By continuity of G^m, $z_n \to z = G^m(y)$. Since, $DG^m(y)$ is invertible, by the inverse function theorem if $y \in \mathrm{Int}(\nabla u_m(\Omega'))$ then $z \in \mathrm{Int}(\sigma(\Omega'))$. Since this is not the case, we have that $y \in \partial\nabla u_m(\Omega')$. Thanks to Proposition 3.6, we obtain that $y \in \nabla u_m(\partial\Omega')$, thus $z = G^m(y) \in \sigma(\partial\Omega')$. □

This lemma leads to the proof of Theorem 1.1 in the regularized setting:

Proposition 3.9. *Under the assumptions of Theorem 1.1, $G^m(\nabla u_m)$ is continuous with a modulus of continuity independent of $m \in \mathbb{N}$.*

Proof. We just have to prove that $G^m(\nabla u_m)$ satisfies the assumption of Proposition 3.3 uniformly in $m \in \mathbb{N}$. Let us assume that there exist $x_1, x_2 \in B_\delta(x_0)$ such that $|G^m(\nabla u_m(x_1)) - G^m(\nabla u_m(x_2))| \geq r > 0$. By Lemma 3.8 with $\Omega' = B_\delta(x_0)$, the diameter of $G^m(\nabla u_m(B_\delta(x_0)))$ can be bounded from above by the diameter of $G^m(\nabla u_m(\partial\Omega'))$. Thus, there exist $x_3, x_4 \in \partial B_\delta(x_0)$ such that $|G^m(\nabla u_m(x_4)) - G^m(\nabla u_m(x_3))| \geq r$. Hence, thanks to Proposition 3.3, $G^m(\nabla u_m)$ is continuous with a modulus of continuity independent of the parameter of regularization $m \in \mathbb{N}$. □

In the remaining part of the section, we proceed to establish continuity estimates for $G^m(\nabla u_m)$ independent of $m \in \mathbb{N}$ under the assumptions of Theorem 1.5. We prove that for any $1 \leq i \leq n$, $G_i^m(\nabla u_m)$ is continuous.

To do so, we use the fact that ∇u_m satisfies the following classical maximum principle, see e.g. [19, Theorem 3.1]:

Proposition 3.10. *Let u_m be a solution of* (2.2) *with f_m a constant. Then for any $e \in \mathbb{S}^{N-1}$ and any open set $\Omega' \Subset \Omega$, we have that*

$$\sup_{x \in \Omega'} \partial_e u_m(x) = \sup_{x \in \partial\Omega'} \partial_e u_m(x).$$

We start with the case when $G_i^m = \nabla \varphi_i^m$ where $\varphi_i^m(\,\cdot\,) = f_i^m(\mathcal{N}_i^m(\,\cdot - \xi_i)) \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$ with f_i^m a convex function, $\mathcal{N}_i^m$ a *pseudo-norm* and $\xi_i \in \mathbb{R}^2$. The *pseudo-norm* is introduced in Definition 1.4. We denote the non-oriented angle between two vectors z_1, z_2 by $\angle(z_1, z_2) \in [0, \pi]$ with the convention that $\angle(z, 0) = 0$. We can apply the following lemma to G_i^m:

Lemma 3.11. *Let us assume that $G_i^m = \nabla \varphi_i^m$ with $\varphi_i^m(\,\cdot\,) = f_i^m(\mathcal{N}_i^m(\,\cdot - \xi_i)) \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$.*

- *For every $r > 0$ there exists $C(r) > 0$ independent of $m \in \mathbb{N}$ such that if $|\nabla\varphi_i^m(z)| \geq r$ then $|z - \xi_i| \geq C(r)$.*
- *For every $0 < \theta \leq \pi$ there exists $0 < D(\theta) \leq \pi$ independent of $m \in \mathbb{N}$ such that if $\angle(\nabla\varphi_i^m(z), \nabla\varphi_i^m(z')) \geq \theta$ then $\angle(z - \xi_i, z' - \xi_i) \geq D(\theta)$ for every $z, z' \in \mathbb{R}^2 \backslash \{\xi_i\}$.*

Moreover, $C(r) \to 0$ when $r \to 0$ and $D(\theta) \to 0$ when $\theta \to 0$.

Proof. For every $r > 0$ we introduce

$$C(r) := \inf\{|z - \xi_i|,\ z \in \mathbb{R}^2 \text{ such that } |\nabla\varphi_i^m(z)| \geq r \text{ for some } m \in \mathbb{N}\}$$

and for every $0 < \theta \leq \pi$ we introduce

$$D(\theta) := \inf\left\{\angle(z - \xi_i, z' - \xi_i),\ \begin{array}{l} z, z' \in \mathbb{R}^2 \backslash \{\xi_i\} \text{ such that} \\ \angle(\nabla\varphi_i^m(z), \nabla\varphi_i^m(z')) \geq \theta \text{ for some } m \in \mathbb{N} \end{array}\right\}.$$

In the definition of the constant $D(\theta)$ we can replace $\mathbb{R}^2\backslash\{\xi_i\}$ by $B_R(\xi_i)\backslash B_\rho(\xi_i)$ with $R > \rho > 0$ since the direction of $\nabla\varphi_i^m$ is constant on the half-lines starting at ξ_i.

Since for every $m \in \mathbb{N}$, $\nabla\varphi_i^m(z) = 0$ only when $z = \xi_i$, $C(r) \to 0$ when $r \to 0$. The fact that for each $m \in \mathbb{N}$, the range of the gradient of $\mathcal{N}_i^m$ is not in a half-line provides that $D(\theta) \to 0$ when $\theta \to 0$.

It remains to prove that $C(r) > 0$ and $D(\theta) > 0$. If $C(r) = 0$ then there exist $(z_n)_{n\in\mathbb{N}}$ and $(m_n)_{n\in\mathbb{N}}$ such that $z_n \to \xi_i$ when $n \to +\infty$ and $|\nabla\varphi_i^{m_n}(z_n)| \geq r$ for every $n \in \mathbb{N}$. We set $M := \limsup_{n\to+\infty} m_n$. If $M \in \mathbb{N}$ then up to an extraction, we can assume that $m_n \equiv M$ for n large enough. Thus, $|\nabla\varphi_i^M(z_n)| \geq r$ and $z_n \to \xi_i$ when $n \to +\infty$ which is a contradiction with the fact that $\nabla\varphi_i^M(\xi_i) = 0$. If $M = +\infty$ then we combine the fact that $\nabla\varphi_i^m$ converges to $\nabla\varphi_i$ uniformly with the fact that $\nabla\varphi_i(\xi_i) = 0$ to obtain a contradiction. Hence, $C(r) > 0$.

If $D(\theta) = 0$ then there exist $(z_n)_{n\in\mathbb{N}}$, $(z'_n)_{n\in\mathbb{N}}$ and $(m_n)_{n\in\mathbb{N}}$ such that $\angle(z_n - \xi_i, z'_n - \xi_i) \to 0$ when $n \to +\infty$ and $\angle(\nabla\varphi_i^{m_n}(z_n), \nabla\varphi_i^{m_n}(z'_n)) \geq \theta$. If $M := \limsup_{n\to+\infty} m_n < +\infty$ we use the continuity of $\nabla\varphi_i^M$ to obtain a contradiction. If $M = +\infty$ we use the fact that the sequences $(z_n)_{n\in\mathbb{N}}$ and $(z'_n)_{n\in\mathbb{N}}$ are in $B_R(\xi_i)\backslash B_\rho(\xi_i)$ to extract two converging subsequences that tend to $z \neq \xi_i$ and $z' \neq \xi_i$. By uniform convergence of $\nabla\varphi_i^m$ to $\nabla\varphi_i$ we obtain that $\angle(z - \xi_i, z' - \xi_i) = 0$ and $\angle(\nabla\varphi_i(z), \nabla\varphi_i(z')) \geq \theta$. That contradicts the fact that $\nabla\varphi_i(z)$ and $\nabla\varphi_i(z')$ are colinear when z and z' are colinear. Hence, $D(\theta) > 0$. ◻

The converse is also true:

Lemma 3.12. *Let us assume that $G_i^m = \nabla\varphi_i^m$ with $\varphi_i^m(\,\cdot\,) = f_i^m(\mathcal{N}_i^m(\,\cdot\, - \xi_i)) \in C^{1,1}_{\text{loc}}(\mathbb{R}^2)$.*

- *For every $r > 0$ there exists $C'(r) > 0$ independent of $m \in \mathbb{N}$ such that if $|z - \xi_i| \geq r$ then $|\nabla\varphi_i^m(z)| \geq C'(r)$.*
- *For every $0 < \theta \leq \pi$ there exists $0 < D'(\theta) \leq \pi$ independent of $m \in \mathbb{N}$ such that if $\angle(z - \xi_i, z' - \xi_i) \geq \theta$ then $\angle(\nabla\varphi_i^m(z), \nabla\varphi_i^m(z')) \geq D'(\theta)$ for every $z, z' \in \mathbb{R}^2\backslash\{\xi_i\}$.*

Moreover, $C'(r) \to 0$ when $r \to 0$ and $D'(\theta) \to 0$ when $\theta \to 0$.

Proof. We argue as in the proof of the previous lemma. For every $r > 0$ and every $0 < \theta \leq \pi$ we set

$$C'(r) := \inf\{|\nabla\varphi_i^m(z)|, \text{ with } m \in \mathbb{N},\ z \in \mathbb{R}^2 \text{ such that } |z - \xi_i| \geq r\}$$

and

$$D'(\theta) := \inf\left\{\angle(\nabla\varphi_i^m(z), \nabla\varphi_i^m(z')), \begin{array}{l}\text{with } m \in \mathbb{N},\ z, z' \in \mathbb{R}^2\backslash\{\xi_i\} \\ \text{such that } \angle(z - \xi_i, z' - \xi_i) \geq \theta\end{array}\right\}.$$

Since the direction of $\nabla\varphi_i^m$ is constant on the half-lines starting at ξ_i, the quantity $D'(\theta)$ is the same if we replace $\mathbb{R}^2\backslash\{\xi_i\}$ by $B_R(\xi_i)\backslash B_\rho(\xi_i)$ by $R > \rho > 0$. The continuity of $\nabla\varphi_i^m$ gives that $C'(r) \to 0$ when $r \to 0$ and $D'(\theta) \to 0$ when $\theta \to 0$.

If we assume that $C'(r) = 0$, then by uniform coercivity of $|\nabla\varphi_i^m|$ we can find $z \in \mathbb{R}^2$ and $M \in \mathbb{N} \cup \{+\infty\}$ such that $\nabla\varphi_i^M(z) = 0$ and $|z - \xi_i| \geq r$ with the convention $\varphi_i^{+\infty} = \varphi_i$. Since $\nabla\varphi_i^M(z) = 0 \Rightarrow z = \xi_i$ this is absurd. Hence, we have that $C'(r) > 0$.

If we assume that $D'(\theta) = 0$ then once again we can find $z, z' \in B_R(\xi_i)\backslash B_\rho(\xi_i)$ and $M \in \mathbb{N} \cup \{+\infty\}$ such that $\angle(\nabla\varphi_i^M(z), \nabla\varphi_i^M(z')) = 0$ and $\angle(z - \xi_i, z' - \xi_i) \geq \theta$. Using the strict convexity of the level sets of φ_i^M we obtain a contradiction. Thus, we have $D'(\theta) > 0$. □

With these two results, we can prove that:

Proposition 3.13. *We set* $\sigma^i := G_i^m(\nabla u_m)$ *with* $G_i^m = \nabla\varphi_i^m$ *where the convex function* $\varphi_i^m(\,\cdot\,) = f_i^m(\mathcal{N}_i^m(\,\cdot\, - \xi_i))$ *satisfies* (A_2). *If* $x_0 \in \Omega$ *is such that* $\sigma^i(x_0) = 0$ *then* σ^i *is continuous at* x_0 *and the modulus of continuity is independent of* $m \in \mathbb{N}$.

Proof. Thanks to Proposition 3.1 and Proposition 3.3, it remains to prove that for every $\epsilon > 0$ there exists $\widetilde{C}(\epsilon, x_0) > 0$ such that for every $\delta > 0$ if there exists $x \in B_\delta(x_0)$ such that $|\sigma^i(x)| \geq \epsilon$ then there exist $x_1, x_2 \in \partial B_\delta(x_0)$ such that $|\sigma^i(x_1) - \sigma^i(x_2)| \geq \widetilde{C}(\epsilon, x_0)$.

By the first point of Lemma 3.11, $|\nabla u_m(x) - \xi_i| \geq C(\epsilon) > 0$ with $C(\epsilon)$ that does not depend on $m \in \mathbb{N}$. We set $e := \frac{\nabla u_m(x) - \xi_i}{|\nabla u_m(x) - \xi_i|}$. By the maximum principle from Proposition 3.10, there exists $x_1 \in \partial B_\delta(x_0)$ such that

$$\langle \nabla u_m(x_1) - \xi_i, e\rangle \geq \langle \nabla u_m(x) - \xi_i, e\rangle = |\nabla u_m(x) - \xi_i| \geq C(\epsilon).$$

In particular $|\nabla u_m(x_1) - \xi_i| \geq C(\epsilon)$ and by Lemma 3.12 we obtain that $|\sigma^i(x_1)| \geq C'(C(\epsilon))$. If we set $e_1 := \frac{\nabla u_m(x_1) - \xi_i}{|\nabla u_m(x_1) - \xi_i|}$ then once again by Proposition 3.10 there exists $x_2 \in \partial B_\delta(x_0)$ such that

$$\langle \nabla u_m(x_2) - \xi_i, e_1\rangle \leq \langle \nabla u_m(x_0) - \xi_i, e_1\rangle = \langle 0, e_1\rangle = 0. \tag{3.5}$$

The last equality comes from the fact that $\sigma^i(x_0) = 0 \Rightarrow \nabla u_m(x_0) = \xi_i$. If $\nabla u_m(x_2) = \xi_i$ then $|\sigma^i(x_1) - \sigma^i(x_2)| = |\sigma^i(x_1)| \geq C'(C(\epsilon)) > 0$. Otherwise, by (3.5) we get that $\angle(\nabla u_m(x_1) - \xi_i, \nabla u_m(x_2) - \xi_i) \geq \frac{\pi}{2}$. In that case, by the second point of Lemma 3.12 we obtain that

$$\angle(\sigma^i(x_1), \sigma^i(x_2)) \geq D'\Big(\frac{\pi}{2}\Big).$$

Since $|\sigma^i(x_1)| \geq C'(C(\epsilon))$ there exists $\widetilde{C}(\epsilon) > 0$ such that $|\sigma^i(x_1) - \sigma^i(x_2)| \geq \widetilde{C}(\epsilon)$.

By taking $\widetilde{C}(\epsilon) := \min\{\widetilde{C}(\epsilon), C'(C(\epsilon))\}$ we can apply Proposition 3.3. The conclusion follows. □

In the case where $\sigma^i(x_0) \neq 0$, we have the following lemma:

Lemma 3.14. *We set* $\sigma^i := G_i^m(\nabla u_m)$ *with* $G_i^m = \nabla\varphi_i^m$ *where the convex function* $\varphi_i^m(\,\cdot\,) = f_i^m(\mathcal{N}_i^m(\,\cdot\, - \xi_i))$ *satisfies* (A_2)*. If* $\sigma^i(x_0) \neq 0$ *then for every* $r > 0$, $\theta > 0$ *there exists* $0 < \delta(r,\theta) < \frac{\mathrm{dist}(x_0,\partial\Omega)}{2}$ *independent of* $m \in \mathbb{N}$ *such that for every* $x \in B_\delta(x_0)$*, either* $|\sigma^i(x)| \leq r$ *or the angle between* $\sigma^i(x)$ *and* $\sigma^i(x_0)$ *is smaller than* θ*.*

Proof. Given $0 < \delta < \frac{\mathrm{dist}(x_0,\partial\Omega)}{2}$, let us assume that there exists $x \in B_\delta(x_0)$ such that $|\sigma^i(x)| > r$ and $\angle(\sigma^i(x), \sigma^i(x_0)) \geq \theta$. By Lemma 3.11 we have that

$$|\nabla u_m(x) - \xi_i| > C(r) \text{ and } \angle(\nabla u_m(x) - \xi_i, \nabla u_m(x_0) - \xi_i) \geq D(\theta). \tag{3.6}$$

If $\langle \nabla u_m(x) - \xi_i, \frac{\nabla u_m(x_0) - \xi_i}{|\nabla u_m(x_0) - \xi_i|}\rangle \leq -\frac{C(r)}{2}$ then by Proposition 3.10 there exist $x_1 \in \partial B_\delta(x_0)$ such that $\langle \nabla u_m(x_1) - \xi_i, \frac{\nabla u_m(x_0) - \xi_i}{|\nabla u_m(x_0) - \xi_i|}\rangle \leq -\frac{C(r)}{2}$ and $x_2 \in \partial B_\delta(x_0)$ such that

$$\left\langle \nabla u_m(x_2) - \xi_i, \frac{\nabla u_m(x_0) - \xi_i}{|\nabla u_m(x_0) - \xi_i|}\right\rangle \geq \left\langle \nabla u_m(x_0) - \xi_i, \frac{\nabla u_m(x_0) - \xi_i}{|\nabla u_m(x_0) - \xi_i|}\right\rangle > 0.$$

The last inequality comes from the assumption $\sigma^i(x_0) \neq 0$. In that case, the angle between $\nabla u_m(x_1) - \xi_i$ and $\nabla u_m(x_2) - \xi_i$ is bounded from below by a constant $0 < \theta' \leq \pi$ depending only on $C(r)$ and a Lipschitz constant of u_m on $B_{\frac{\mathrm{dist}(x_0,\partial\Omega)}{2}}(x_0)$ independent of $m \in \mathbb{N}$. By the second point of Lemma 3.12 the angle between $\sigma^i(x_1)$ and $\sigma^i(x_2)$ is bounded from below by $D'(\theta') > 0$. Since $|\nabla u_m(x_1) - \xi_i| \geq \frac{C(r)}{2}$ by the first point of Lemma 3.12 we obtain that $|\sigma^i(x_1)|$ is larger than $C'(\frac{C(r)}{2})$. Hence, there exists a constant $F(r) > 0$ such that $|\sigma^i(x_1) - \sigma^i(x_2)| \geq F(r)$.

If $\langle \nabla u_m(x) - \xi_i, \frac{\nabla u_m(x_0) - \xi_i}{|\nabla u_m(x_0) - \xi_i|}\rangle > -\frac{C(r)}{2}$ then by (3.6) there exists e a unit vector orthogonal to $\nabla u_m(x_0) - \xi_i$ and $F'(r,\theta) > 0$ such that $\langle \nabla u_m(x) - \xi_i, e\rangle \geq F'(r,\theta)$. Once again by Proposition 3.10 there exist $x_1, x_2 \in \partial B_\delta(x_0)$ such that

$$\begin{aligned}\langle \nabla u_m(x_1) - \xi_i, e\rangle &\geq \langle \nabla u_m(x) - \xi_i, e\rangle \\ &\geq F'(r,\theta) > 0 = \langle \nabla u_m(x_0) - \xi_i, e\rangle \geq \langle \nabla u_m(x_2) - \xi_i, e\rangle.\end{aligned}$$

If $\nabla u_m(x_2) = \xi_i$ then by Lemma 3.12, $|\sigma^i(x_1) - \sigma^i(x_2)| \geq C'(F'(r,\theta))$. Otherwise, the angle between $\nabla u_m(x_1) - \xi_i$ and $\nabla u_m(x_2) - \xi_i$ is bounded from below by a constant $0 < \theta' \leq \pi$ depending only on $r > 0$, $\theta > 0$ and the Lipschitz constant of u_m on $B_{\frac{\mathrm{dist}(x_0,\partial\Omega)}{2}}(x_0)$ independent of $m \in \mathbb{N}$. We conclude as in the first case.

Hence, we have proved that for every $0 < \delta < \frac{\mathrm{dist}(x_0,\partial\Omega)}{2}$ if there exists $x \in B_\delta(x_0)$ such that $|\sigma^i(x)| > r$ and the angle between $\sigma^i(x)$ and $\sigma^i(x_0)$ is larger than θ, then there exist $x_1, x_2 \in \partial B_\delta(x_0)$ and $F(r,\theta) > 0$ such that $|\sigma^i(x_1) - \sigma^i(x_2)| \geq F(r,\theta)$. We can conclude as in the proof of Proposition 3.3:

$$F(r,\theta) \leq |\sigma^i(x_1) - \sigma^i(x_2)| \leq \int_{\partial B_\delta(x_0)} |\nabla\sigma^i| \mathrm{d}\mathcal{H}^1.$$

Using the Cauchy–Schwarz inequality and integrating over δ between $\delta' > 0$ and $\frac{\text{dist}(x_0,\partial\Omega)}{2}$ we obtain

$$\frac{F(r,\theta)^2}{2\pi} \ln \frac{\text{dist}(x_0,\partial\Omega)}{2\delta'} \leq \|\sigma^i\|^2_{W^{1,2}(B_{\frac{\text{dist}(x_0,\partial\Omega)}{2}}(x_0))}.$$

The conclusion follows from the fact that $\sigma^i \in W^{1,2}_{\text{loc}}(\Omega)$ with a norm independent of $m \in \mathbb{N}$ by Proposition 3.1. □

The following lemma asserts that the component of σ^i is continuous in the direction of $\sigma^i(x_0)$.

Lemma 3.15. *We set* $\sigma^i := G^m_i(\nabla u_m)$ *with* $G^m_i = \nabla\varphi^m_i$ *where the convex function* $\varphi^m_i(\,\cdot\,) = f^m_i(\mathcal{N}^m_i(\,\cdot\, - \xi_i))$ *satisfies* (A_2). *If* $\sigma^i(x_0) \neq 0$ *then for every* $\epsilon > 0$ *there exists* $\delta > 0$ *independent of* $m \in \mathbb{N}$ *such that*

$$|\sigma^i(x_0)| - \epsilon \leq \langle \sigma^i(x), \frac{\sigma^i(x_0)}{|\sigma^i(x_0)|}\rangle \leq |\sigma^i(x_0)| + \epsilon$$

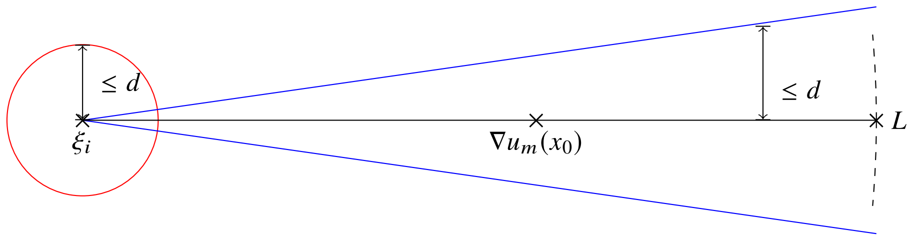
for every $x \in B_\delta(x_0)$.

Proof. Thanks to Lemma 3.14, for every $r > 0$ and $\theta > 0$ there exists $\delta(r,\theta) > 0$ such that for every $x \in B_\delta(x_0)$, $|\sigma^i(x)| \leq r$ or $\angle(\sigma^i(x), \sigma^i(x_0)) \leq \theta$. By the contrapositive statement of Lemma 3.12, there exist $\widetilde{C}(r) > 0$ and $\widetilde{D}(\theta) > 0$ such that $|\nabla u_m(x) - \xi_i| \leq \widetilde{C}(r)$ or $\angle(\nabla u_m(x) - \xi_i, \nabla u_m(x_0) - \xi_i) \leq \widetilde{D}(\theta)$ for every $x \in B_\delta(x_0)$. Moreover, we can choose them in such a way that $\widetilde{C}(r) \to 0$ when $r \to 0$ and $\widetilde{D}(\theta) \to 0$ when $\theta \to 0$.

For $\epsilon > 0$, we introduce $\eta > 0$ independent of $m \in \mathbb{N}$ such that the oscillations of G^m_i on the square of center ξ_i and sides of length 2η are smaller than $\frac{\epsilon}{2}$. Since $\widetilde{C}(r)$ goes to 0 when r goes to 0 and $\widetilde{D}(\theta)$ goes to 0 when θ goes to 0, we can choose $\delta > 0$ small enough such that for every $x \in B_\delta(x_0)$ we have

$$|\nabla u_m(x) - \xi_i| \leq \eta \text{ or } \angle(\nabla u_m(x) - \xi_i, \nabla u_m(x_0) - \xi_i) \text{ is as small as we want.} \tag{3.7}$$

We introduce p_x the projection of $\nabla u_m(x)$ on $\mathbb{R}_+(\nabla u_m(x_0) - \xi_i) + \xi_i$. Hence, for every $d > 0$, we can choose δ_0 such that for every $x \in B_{\delta_0}(x_0)$ the distance between $\nabla u_m(x)$ and p_x is smaller than d. By uniform continuity of G^m_i in $m \in \mathbb{N}$, there exists ω a modulus of continuity independent of $m \in \mathbb{N}$ such that $|G_i^m(p_x) - G_i^m(\nabla u_m(x))| \leq \omega(d)$.



Let us argue by contradiction. We assume that for every $0 < \delta < \delta_0$ there exists $x \in B_\delta(x_0)$ such that $|\sigma^i(x_0)| + \epsilon < \langle \sigma^i(x), e\rangle$ or $\langle \sigma^i(x), e\rangle < |\sigma^i(x_0)| - \epsilon$ with $e := \frac{\sigma^i(x_0)}{|\sigma^i(x_0)|}$.

Case 1. We begin with the case where $\langle \sigma^i(x), e\rangle > |\sigma^i(x_0)| + \epsilon$. Since $|G_i^m(p_x) - \sigma^i(x)| \leq \omega(d)$ and $\langle \sigma^i(x), e\rangle \geq |\sigma^i(x_0)| + \epsilon$ we obtain that

$$\langle G_i^m(p_x), e\rangle \geq |\sigma^i(x_0)| = \langle \sigma^i(x_0), e\rangle$$

when d is small enough such that $\omega(d) \leq \epsilon$ without any dependence on $m \in \mathbb{N}$.

By definition of φ_i^m, we have that $G_i^m(p_x) = (f_i^m)'(\mathcal{N}_i^m(p_x - \xi_i))\nabla\mathcal{N}_i^m(p_x - \xi_i)$ and $\sigma^i(x_0) = (f_i^m)'(\mathcal{N}_i^m(\nabla u_m(x_0) - \xi_i))\nabla\mathcal{N}_i^m(\nabla u_m(x_0) - \xi_i)$. Thus, these two vectors are positively colinear to e. This means that $(f_i^m)'(\mathcal{N}_i^m(p_x - \xi_i)) \geq (f_i^m)'(\mathcal{N}_i^m(\nabla u_m(x_0) - \xi_i))$. Thus, by strict convexity of f_i^m we have that

$$\langle \nabla u_m(x) - \nabla u_m(x_0), \nabla u_m(x_0) - \xi_i\rangle = \langle p_x - \nabla u_m(x_0), \nabla u_m(x_0) - \xi_i\rangle \geq 0.$$

Hence, by Proposition 3.10 there exists $x_1, x_2 \in \partial B_\delta(x_0)$ such that

$$\langle \nabla u_m(x_1) - \xi_i, e'\rangle \leq \langle \nabla u_m(x_0) - \xi_i, e'\rangle \leq \langle \nabla u_m(x) - \xi_i, e'\rangle \leq \langle \nabla u_m(x_2) - \xi_i, e'\rangle$$

with $e' := \frac{\nabla u_m(x_0) - \xi_i}{|\nabla u_m(x_0) - \xi_i|}$. We introduce p_1 and p_2 the projection of $\nabla u_m(x_1)$ and $\nabla u_m(x_2)$ on $\mathbb{R}_+(\nabla u_m(x_0) - \xi_i) + \xi_i$. Since $\langle p_1, e'\rangle \leq \langle \nabla u_m(x_0), e'\rangle$ the convexity of f_i^m gives that $\langle G_i^m(p_1), e\rangle \leq \langle \sigma^i(x_0), e\rangle$. For the same reasons, $\langle G_i^m(p_2), e\rangle \geq \langle G_i^m(p_x), e\rangle \geq \langle \sigma^i(x), e\rangle - \omega(d)$ where the last inequality comes from the fact that $|G_i^m(p_x) - \sigma^i(x)| \leq \omega(d)$.

We also have that $|\nabla u_m(x_1) - p_1|$ and $|\nabla u_m(x_2) - p_2|$ are smaller than d. Thus, $\langle \sigma^i(x_1), e\rangle \leq \langle G_i^m(p_1), e\rangle + \omega(d)$ and $\langle \sigma^i(x_2), e\rangle \geq \langle G_i^m(p_2), e\rangle - \omega(d)$. Hence,

$$\langle \sigma^i(x_1), e\rangle \leq \langle \sigma^i(x_0), e\rangle + \omega(d) \tag{3.8}$$

and $\langle \sigma^i(x_2), e\rangle \geq \langle \sigma^i(x), e\rangle - 2\omega(d)$. When d is sufficiently small with respect to ϵ we have that $|\sigma^i(x_2) - \sigma^i(x_1)| \geq \frac{\epsilon}{2}$.

Case 2. Let us assume that there exists $x \in B_\delta(x_0)$ such that $|\sigma^i(x_0)| - \epsilon > \left\langle \sigma^i(x), \frac{\sigma^i(x_0)}{|\sigma^i(x_0)|} \right\rangle$. Hence, if we apply Proposition 3.10 to $\langle \nabla u_m(\,\cdot\,) - \xi_i, \nabla u_m(x_0) - \xi_i \rangle$ we can show that there exists $x_1 \in \partial B_\delta(x_0)$ such that $\langle \nabla u_m(x_1) - \xi_i, \nabla u_m(x_0) - \xi_i \rangle \geq \langle \nabla u_m(x_0) - \xi_i, \nabla u_m(x_0) - \xi_i \rangle$. As in the previous case we can show that $\langle \sigma^i(x_1), e \rangle \geq |\sigma^i(x_0)| - \omega(d)$ that is an analogous result to (3.8) (up to interchanging $\leq$ in $\geq$).

It remains to find a point $x_2 \in \partial B_\delta(x_0)$ such that $\langle \nabla u_m(x_2) - \xi_i, \nabla u_m(x_0) - \xi_i \rangle$ is sufficiently small. We distinguish two subcases. We start by assuming that $\nabla u_m(x) \in B_\eta(\xi_i)$. By Proposition 3.10 there exists $x_2 \in \partial B_\delta(x_0)$ such that

$$\langle \nabla u_m(x_2) - \xi_i, \nabla u_m(x_0) - \xi_i \rangle \leq \langle \nabla u_m(x) - \xi_i, \nabla u_m(x_0) - \xi_i \rangle.$$

By (3.7) we have that $\nabla u_m(x_2) \in B_\eta(\xi_i)$. Since the oscillations of G_i^m are smaller than $\frac{\epsilon}{2}$ on that set we get that $\langle \sigma^i(x_2), e \rangle \leq \langle \sigma^i(x), e \rangle + \frac{\epsilon}{2}$. In that case we obtain

$$\langle \sigma^i(x_2), e \rangle \leq \langle \sigma^i(x), e \rangle + \frac{\epsilon}{2} \leq \langle \sigma^i(x_0), e \rangle - \frac{\epsilon}{2} \leq \langle \sigma^i(x_1), e \rangle - \frac{\epsilon}{2} + \omega(d).$$

Thus, by taking d small enough $|\sigma^i(x_1) - \sigma^i(x_2)| \geq \frac{\epsilon}{4}$.

If $\nabla u_m(x) \notin B_\eta(\xi_i)$ then $\langle p_x - \xi_i, \nabla u_m(x_0) - \xi_i \rangle > 0$ with p_x the projection of $\nabla u_m(x)$ on $\mathbb{R}_+(\nabla u_m(x_0) - \xi_i) + \xi_i$. Hence, there exists $x_2 \in \partial B_\delta(x_0)$ such that $\langle \nabla u_m(x_2), e' \rangle \leq \langle \nabla u_m(x), e' \rangle$ which implies that $\langle p_x - p_2, \nabla u_m(x_0) - \xi_i \rangle \geq 0$. Hence, by convexity of f_i^m we have that $\langle G_i^m(p_2), e \rangle \leq \langle G_i^m(p_x), e \rangle$. Thus,

$$\langle \sigma^i(x_2), e \rangle \leq \langle \sigma^i(x), e \rangle + 2\omega(d) \leq \langle \sigma^i(x_0), e \rangle - \epsilon + 2\omega(d).$$

Once again, by taking d small enough $|\sigma^i(x_1) - \sigma^i(x_2)| \geq \frac{\epsilon}{4}$.

In any case, for every $0 < \delta < \delta_0$, if there exists $x \in B_\delta(x_0)$ such that $|\sigma^i(x_0)| + \epsilon < \langle \sigma^i(x), e \rangle$ or $\langle \sigma^i(x), e \rangle < |\sigma^i(x_0)| - \epsilon$ we can find $x_1, x_2 \in \partial B_\delta(x_0)$ satisfying $|\sigma^i(x_1) - \sigma^i(x_2)| \geq \frac{\epsilon}{4}$. The conclusion follows Proposition 3.3. □

The combination of the last three results gives the following proposition:

Proposition 3.16. *Under the assumptions of Theorem 1.5, in the case where* $G_i^m = \nabla \varphi_i^m$ *with* $\varphi_i^m(\,\cdot\,) = f_i^m(\mathcal{N}_i^m(\,\cdot\, - \xi_i))$, *we have that* $G_i^m(\nabla u_m)$ *is continuous on* Ω *with a modulus of continuity that does not depend on* $m \in \mathbb{N}$.

Now, let us focus on the case where G_i satisfies the assumption (A_3).

Proposition 3.17. *Let us assume that* $G_i^m = \nabla \varphi_i^m$ *with* $\varphi_i^m(\,\cdot\,) = f_i^m(\langle \,\cdot\,, \xi_i \rangle)$ *and* $\xi_i \neq 0$. *Then if* $G_i^m(\nabla u_m) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ *we have that* $\sigma^i := G_i^m(\nabla u_m)$ *is continuous on* Ω *with a modulus of continuity that does not depend on* $m \in \mathbb{N}$.

Proof. For every $\epsilon > 0$ we have

$$|G_i^m(z_1) - G_i^m(z_2)| \geq \epsilon \Leftrightarrow |(f_i^m)'(\langle z_1, \xi_i \rangle) - (f_i^m)'(\langle z_2, \xi_i \rangle)| \geq \frac{\epsilon}{|\xi_i|} \tag{3.9}$$

for every $z_1, z_2 \in \mathbb{R}^2$.

Thus, for $\delta > 0$, if we assume that there exists $x \in B_\delta(x_0)$ such that

$$|G_i^m(\nabla u_m(x)) - G_i^m(\nabla u_m(x_0))| \geq \epsilon$$

then

$$|(f_i^m)'(\langle \nabla u_m(x), \xi_i \rangle) - (f_i^m)'(\langle \nabla u_m(x_0), \xi_i \rangle)| \geq \frac{\epsilon}{|\xi_i|}. \tag{3.10}$$

Up to a change of sign of ξ_i we can assume that $\langle \nabla u_m(x_0), \xi_i \rangle \leq \langle \nabla u_m(x), \xi_i \rangle$. By the maximum principle from Proposition 3.10 applied to $y \to \langle \nabla u_m(y), \xi_i \rangle$, there exist $x_1, x_2 \in \partial B_\delta(x_0)$ such that $\langle \nabla u_m(x_1), \xi_i \rangle \leq \langle \nabla u_m(x_0), \xi_i \rangle$ and $\langle \nabla u_m(x), \xi_i \rangle \leq \langle \nabla u_m(x_2), \xi_i \rangle$.

We use the fact that $(f_i^m)'$ is increasing with (3.10) to obtain that

$$|(f_i^m)'(\langle \nabla u_m(x_1), \xi_i \rangle) - (f_i^m)'(\langle \nabla u_m(x_2), \xi_i \rangle)| \geq \frac{\epsilon}{|\xi_i|}.$$

Thus, by (3.9), we get that $|G_i^m(\nabla u_m(x_1)) - G_i^m(\nabla u_m(x_2))| \geq \epsilon$. Once again, we can conclude thanks to Proposition 3.3. ◻

Hence, we have proved the following result:

Proposition 3.18. *Under the assumptions of Theorem 1.5, the functions $(G^m(\nabla u_m))_{m\in\mathbb{N}}$ are continuous with the same modulus of continuity on each compact subset of Ω.*

Proof. For every $m \in \mathbb{N}$ we have proved in Proposition 3.16 or in Proposition 3.17 that for every $1 \leq i \leq n$ the function $G_i^m(\nabla u_m)$ is continuous with a modulus of continuity independent of $m \in \mathbb{N}$. Hence, that is the case for their sum, namely $G^m(\nabla u_m)$. ◻

4. Uniform estimates for Theorem 1.7

In this section, we study the case when $G = G_1 + G_2$ where $G_i(z) := f_i'(\langle z, \xi_i \rangle)\xi_i$ with $f_1 \in C^{1,1}_{\mathrm{loc}}(\mathbb{R})$ and $f_2 \in C^1(\mathbb{R}) \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}\backslash\{0\})$ two convex functions. Moreover, the right-hand side of (1.4) is a constant $\lambda \in \mathbb{R}$.

We begin by the following observation on ξ_1 and ξ_2:

Proposition 4.1. *Under the assumptions of Theorem 1.7 we can assume that $\xi_1 = e_1$ and $\xi_2 = e_2$ are the two standard vectors of the canonical basis.*

Proof. Let us assume that A is an invertible linear matrix. We introduce the convex function $\varphi(z) := f_1(\langle z, \xi_1 \rangle) + f_2(\langle z, \xi_2 \rangle)$. Let us consider u a solution of (2.2), then u is a minimizer of

$$\min_{w \in W^{1,2}_u(\Omega)} \int_\Omega \varphi(\nabla w(x)) + \lambda w(x) \mathrm{d}x. \tag{4.1}$$

For every $w \in W^{1,2}(\Omega)$, we have:

$$\int_\Omega \varphi(\nabla w(x)) + \lambda w(x)\mathrm{d}x = |\det A| \int_{A^{-1}(\Omega)} \varphi(\nabla w(Ay)) + \lambda w(Ay)\mathrm{d}y.$$

If we set $v(y) := w(Ay)$ then:

$$\int_\Omega \varphi(\nabla w(x)) + \lambda w(x)\mathrm{d}x = |\det A| \int_{A^{-1}(\Omega)} \varphi((A^T)^{-1}\nabla v(y)) + \lambda v(y)\mathrm{d}y.$$

Hence, since u is a minimizer of (4.1), we obtain that $u(A\cdot)$ is a minimizer of

$$\int_{A^{-1}(\Omega)} \varphi((A^T)^{-1}\nabla v(y)) + \lambda v(y)\mathrm{d}y$$

on $W^{1,2}_{u(A\cdot)}(A^{-1}(\Omega))$. It remains to choose A such that $(A^{-1})\xi_1 = e_1$ and $(A^{-1})\xi_2 = e_2$.

Since proving that $\nabla\varphi(\nabla u) \in C^0(\Omega)$ is equivalent to proving that $\nabla\varphi(\nabla u(A\cdot)) \in C^0(A^{-1}(\Omega))$, we can assume that $\xi_1 = e_1$ and $\xi_2 = e_2$. □

Remark 4.2. This previous proposition is important because, in this section, we will consider partial derivatives in orthogonal directions e_1 and e_2 instead of differentiating our functions in any direction. By doing so, we can use the properties of f_i while differentiating with respect to e_i where $i = 1, 2$.

We want to establish continuity estimates for $G^m(\nabla u_m)$ independent of $m \in \mathbb{N}$ with u_m solution of (2.2). We start by proving the following lemma inspired by [15, Proposition 2.3]:

Lemma 4.3. *Let f_1 and f_2 be two smooth convex functions. Let u be a smooth solution of* (2.2) *with $G(z) := f_1'(\langle z, e_1\rangle)e_1 + f_2'(\langle z, e_2\rangle)e_2$. Then the function $f_1'(\partial_1 u)$ belongs to $W^{1,2}_{\mathrm{loc}}(\Omega)$. Moreover, for every $\Omega'' \Subset \Omega' \Subset \Omega$ and every $L_{\Omega'} \geq \|\nabla u\|_{L^\infty(\Omega')}$ we have*

$$\begin{aligned}&\|f_1'(\partial_1 u)\|_{W^{1,2}(\Omega'')}\\ &\quad\leq C(L'_\Omega, \|f_1'\|_{L^\infty(-L_{\Omega'},L_{\Omega'})}, \|f_2'\|_{L^\infty(-L_{\Omega'},L_{\Omega'})}, \|f_1''\|_{L^\infty(L_{\Omega'},L_{\Omega'})}, \mathrm{dist}(\partial\Omega', \partial\Omega'')).\end{aligned}$$

Proof. By local Lipschitz regularity of u, we already know that $f_1'(\partial_1 u) \in L^2_{\mathrm{loc}}(\Omega)$. Since u is a solution of (2.2) we have that:

$$\int_\Omega \langle \nabla\varphi(\nabla u), \nabla\theta\rangle = -\int_\Omega \lambda\theta$$

for every $\theta \in C_0^\infty(\Omega)$. If we differentiate the Euler–Lagrange equation in the first direction, the fact that $\lambda \in \mathbb{R}$ gives that

$$\int_\Omega \langle \nabla^2\varphi(\nabla u)\nabla\partial_1 u, \nabla\theta\rangle = 0 \tag{4.2}$$

for every $\theta \in C_0^\infty(\Omega)$. If we replace θ by $\xi^2 f_1'(\partial_1 u)$ with $\xi \in C_0^\infty(\Omega')$ and $\Omega' \Subset \Omega$ we obtain

$$\sum_{i=1,2} \int_\Omega f_i''(\partial_i u)\partial_{1i}u\xi^2 f_1''(\partial_1 u)\partial_{1i}u = -2 \sum_{i=1,2} \int_\Omega f_i''(\partial_i u)\partial_{1i}u\xi\partial_i\xi f_1'(\partial_1 u).$$

Since the terms in the left-hand side are nonnegative, we have that

$$\int_\Omega (f_1''(\partial_1 u)\partial_{11}u\xi)^2 \le -2 \sum_{i=1,2} \int_\Omega f_i''(\partial_i u)\partial_{1i}u\xi\partial_i\xi f_1'(\partial_1 u).$$

With an integration by parts on the right-hand side we get

$$\int_\Omega (\partial_1[f_1'(\partial_1 u)])^2\xi^2 \le 2 \sum_{i=1,2} \int_\Omega f_i'(\partial_i u)\partial_1(\xi\partial_i\xi) f_1'(\partial_1 u) + f_i'(\partial_i u)\xi\partial_i\xi\partial_1[f_1'(\partial_1 u)].$$

With the Young and Hölder inequalities and the fact that $\|\nabla u\|_{L^\infty(\Omega')} \le L_{\Omega'}$ we obtain that

$$\int_\Omega (\partial_1[f_1'(\partial_1 u)])^2\xi^2 \le C(L_{\Omega'}, \|f_1'\|_{L^\infty(-L_{\Omega'},L_{\Omega'})}, \|f_2'\|_{L^\infty(-L_{\Omega'},L_{\Omega'})}, \|\xi\|_{W^{1,\infty}(\Omega)}).$$

Hence, if we take $\xi \in C_0^\infty(\Omega')$ such that $\xi \equiv 1$ on Ω'' with $\Omega'' \Subset \Omega'$, then we get that $\partial_1[f_1'(\partial_1 u)]$ belongs to $L^2(\Omega'')$ and

$$\|\partial_1[f_1'(\partial_1 u)]\|_{L^2(\Omega'')} \le C(L_{\Omega'}, \|f_1'\|_{L^\infty(-L_{\Omega'},L_{\Omega'})}, \|f_2'\|_{L^\infty(-L_{\Omega'},L_{\Omega'})}, \mathrm{dist}(\Omega', \Omega'')).$$

With the same strategy, we can also prove that $\partial_2\left[f_2'(\partial_2 u)\right] \in L^2(\Omega'')$ with the same estimate.

It remains to prove that $\partial_2\left[f_1'(\partial_1 u)\right] \in L^2_{\mathrm{loc}}(\Omega)$. We proceed as in the proof of [8, Theorem 2.1]. For $0 < h < \frac{1}{4}$ and $x \in \Omega''$, we introduce $\tau(x) = \frac{\sigma(x+he_2)-\sigma(x)}{h}$ where $\sigma(x) = G(\nabla u(x))$. We set $\tau_1(x) := \frac{f_1'(\partial_1 u(x+he_2))-f_1'(\partial_1 u(x))}{h}$, we want to prove that $\|\tau_1\|_{L^2_{\mathrm{loc}}(\Omega'')}$ is bounded uniformly in h.

Since f_1' is Lipschitz continuous and increasing there exists $K > 1$ such that

$$\tau_1(x)^2 \le K\tau_1(x) \times \frac{\partial_1 u(x+he_2) - \partial_1 u(x)}{h}.$$

Using the fact that f_2 is convex we have that:

$$\tau_1(x)^2 \le K\left\langle \tau(x), \frac{\nabla u(x+he_2) - \nabla u(x)}{h} \right\rangle.$$

With $\xi \in C_0^\infty(\Omega''')$ where $\Omega''' \Subset \Omega''$ and $0 < h < \mathrm{dist}(\partial\Omega''', \partial\Omega'')$ we have

$$\|\xi\tau_1\|^2_{L^2(\Omega''')} \le K \int_{\Omega'''} \xi^2 \left\langle \tau(x), \frac{\nabla u(x+he_2) - \nabla u(x)}{h} \right\rangle.$$

Since $\operatorname{div}\tau = 0$, an integration by parts gives that

$$\|\xi\tau_1\|^2_{L^2(\Omega''')} \leq 2K \int_{\Omega'''} \left| \xi(x) \frac{u(x+he_2) - u(x)}{h} \langle \nabla\xi(x), \tau(x)\rangle \right|.$$

Thus,

$$\|\xi\tau_1\|^2_{L^2(\Omega''')} \leq C\big(L_{\Omega'}, K, \|\xi\|_{W^{1,\infty}(\Omega'')}\big) \|\xi\tau\|_{L^2(\Omega''')}.$$

We already know that $\|\tau_2\|_{L^2(\Omega''')}$ is bounded uniformly in h thanks to the fact that $\partial_2\big[f_2'(\partial_2 u)\big] \in L^2(\Omega'')$. Hence, $\|\xi\tau_1\|_{L^2(\Omega''')}$ is bounded uniformly in h. Thus, if we take $\xi \equiv 1$ on a subset of Ω''', we obtain that $\partial_2\big[f_1'(\partial_1 u)\big]$ belongs to $L^2_{\mathrm{loc}}(\Omega''')$ with an explicit estimate. □

If we apply this lemma with f_1^m and f_2^m we obtain an estimate on the Sobolev norm of $(f_1^m)'(\partial_1 u_m)$ independent of $m \in \mathbb{N}$. Hence, we can apply Proposition 3.17 to prove that $(f_1^m)'(\partial_1 u_m)$ is continuous with a modulus of continuity that does not depend on $m \in \mathbb{N}$. It remains to do the same for $(f_2^m)'(\partial_2 u_m)$. For every $r > 0$, we have the following result coming from [15, Proposition 2.4]:

Proposition 4.4. *Let f_1 and f_2 be two smooth convex functions. Let u be a smooth solution of* (2.2) *with $G(z) := f_1'(\langle z, e_1\rangle)e_1 + f_2'(\langle z, e_2\rangle)e_2$ and $f \equiv \lambda \in \mathbb{R}$. For every $r > 0$ and every $x_0 \in \Omega' \Subset \Omega$, we have*

$$\int_{\Omega'\cap U_r} |f_2''(\partial_2 u)\partial_{22} u|^2 \leq C(r, \alpha_r, \|f_1''\|_{L^\infty(B_{L_{\Omega'}})})$$

with $U_r := \{x \in \Omega,\ |\partial_2 u(x)| \geq r\}$, $L_{\Omega'} := \|\nabla u\|_{L^\infty(\Omega')}$ and $\alpha_r := \sup_{r \leq t \leq L_{\Omega'}} f_2''(t)$.

Proof. Since λ is a constant, the right-hand side of (2.2) vanishes when we differentiate the equation. By [15, Proposition 2.4], we have that

$$\int_{\Omega'\cap U_r} |f_1''(\partial_1 u)\partial_{11} u|^2 + |f_2''(\partial_2 u)\partial_{22} u|^2 \leq C(r, \alpha_r, \|f_1''\|_{L^\infty(B_{L_{\Omega'}})})$$

and the conclusion follows. □

Since this proposition allows avoiding the values of f_2'' around the origin, we can apply it with f_1^m and f_2^m. In that case, the constant $C(r, \alpha_r, \|(f_1^m)''\|_{L^\infty(B_{L_{\Omega'}})})$ can be taken independent of $m \in \mathbb{N}$.

Let us use this estimate in order to prove the continuity of $(f_2^m)'(\partial_2 u_m)$ uniformly in $m \in \mathbb{N}$. Thanks to Proposition 2.4 there exist $r > 0$, $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous function that satisfies $\omega(t) = 0 \Leftrightarrow t = 0$ and $M_r \in \mathbb{N}$ such that for every $m \geq M_r$, every $x, y \in (-\frac{r}{2}, \frac{r}{2})$ we have that

$$((f_2^m)'(x) - (f_2^m)'(y))(x-y) \geq \omega(|x-y|). \tag{4.3}$$

We prove the following alternative:

Lemma 4.5. *For every $t > 0$ there exist $0 < d_t < t$ and $\delta_0 > 0$ such that for every $m \geq M_r$, $\partial_2 u_m(x) > d_t$ for every $x \in B_{\delta_0}(x_0)$ or $\partial_2 u_m(x) < t$ for every $x \in B_{\delta_0}(x_0)$.*

Proof. By (4.3), for every $t > 0$ there exist $0 < d < t$ and C independent of $m \geq M_r$ such that $0 < C \leq (f_2^m)'(t) - (f_2^m)'(d)$. We introduce $F : \mathbb{R} \to \mathbb{R}$ a smooth increasing function such that $F(s) = 0$ for every $s < (f_2^m)'(d)$ and $F(s) = C$ for every $s > (f_2^m)'(t)$. We assume that for every $0 < \delta < \frac{\text{dist}(x_0,\partial\Omega)}{2}$ there exist x_1 and x_2 in $B_\delta(x_0)$ such that $\partial_2 u_m(x_1) < d < t < \partial_2 u_m(x_2)$. Once again, thanks to the maximum principle from Proposition 3.10, there exist x_1' and x_2' on $\partial B_\delta(x_0)$ such that $\partial_2 u_m(x_1') < d < t < \partial_2 u_m(x_2')$. Hence,

$$\begin{aligned} C &= F((f_2^m)'(\partial_2 u_m(x_2'))) - F((f_2^m)'(\partial_2 u_m(x_1'))) \\ &\leq \int_{\partial B_\delta(x_0)} |\nabla[F((f_2^m)'(\partial_2 u_m))]| \mathrm{d}\mathcal{H}^1 \\ &\leq \|\nabla F\|_{L^\infty(\mathbb{R})} \int_{\partial B_\delta(x_0)\cap U_d} |(f_2^m)''(\partial_2 u_m)\nabla\partial_2 u_m| \mathrm{d}\mathcal{H}^1 \end{aligned}$$

with $U_d = \{x \in \Omega, |\partial_2 u_m| \geq d\}$. Thus, as in the proof of Proposition 3.3, if we integrate between δ_0 and $\frac{\text{dist}(x_0,\partial\Omega)}{2}$ we obtain that:

$$\frac{C^2}{2\pi\|\nabla F\|^2_{L^\infty(\mathbb{R})}} \ln \frac{\text{dist}(x_0, \partial\Omega)}{2\delta_0} \leq \int_{B_{\frac{\text{dist}(x_0,\partial\Omega)}{2}}(x_0)\cap U_d} |(f_2^m)''(\partial_2 u_m)\nabla\partial_2 u_m|^2 \mathrm{d}\mathcal{H}^1.$$

Hence, using Proposition 4.4 there exists $\delta_0 > 0$ such that $\partial_2 u_m(x) \geq d$ for every $x \in B_{\delta_0}(x_0)$ or $\partial_2 u_m(x) \leq t$ for every $x \in B_{\delta_0}(x_0)$. □

We have the same result with $t < 0$:

Lemma 4.6. *For every $t < 0$ there exist $t < d_t' < 0$ and $\delta_0 > 0$ such that $\partial_2 u_m(x) < d_t'$ for every $x \in B_{\delta_0}(x_0)$ or $\partial_2 u_m(x) > t$ for every $x \in B_{\delta_0}(x_0)$.*

Proof. The proof is the same as the proof of Proposition 4.5 replacing $\partial_2 u$ by $-\partial_2 u$. □

Hence, we have proved:

Proposition 4.7. *For every $t > 0$ there exist $d_t > 0$ and $\delta_0 > 0$ such that for every $m \geq M_r$ we have that $-t < \partial_2 u_m(x) < t$ for every $x \in B_{\delta_0}(x_0)$ or $\partial_2 u_m(x) \in (-\infty, d'_{-t}) \cup (d_t, \infty)$ for every $x \in B_{\delta_0}(x_0)$.*

With this result, we are ready to prove the continuity of $(f_2^m)'(\partial_2 u_m)$.

Proposition 4.8. *Under the assumptions of Theorem 1.7, $(f_2^m)'(\partial_2 u_m)$ is continuous with a modulus of continuity independent of $m \in \mathbb{N}$.*

Proof. For every $\epsilon > 0$, we want to find $\delta > 0$ such that

$$(f_2^m)'(\partial_2 u_m(B_\delta(x_0))) \subset B_\epsilon((f_2^m)'(\partial_2(u_m(x_0)))).$$

We introduce $C(\epsilon) > 0$ such that if $-C(\epsilon) \le t \le C(\epsilon)$ then

$$|(f_2^m)'(t) - (f_2^m)'(0)| \le \frac{\epsilon}{2} \tag{4.4}$$

for every $m \ge M_r$. Then, thanks to Proposition 4.7, we have two options.

If $\partial_2 u_m(x) \in (-\infty, d'_{-C(\epsilon)}) \cup (d_{C(\epsilon)}, \infty)$ for every $x \in B_{\delta_0}(x_0)$, then we can assume that f_1 and f_2 are in $C^{1,1}_{\mathrm{loc}}$ and apply Theorem 1.5.

Otherwise, $-C(\epsilon) \le \partial_2 u_m(x) \le C(\epsilon)$ for every $x \in B_{\delta_0}(x_0)$. In that case we conclude thanks to (4.4). □

If we combine the results of this section, we have proved:

Proposition 4.9. *Under the assumptions of Theorem 1.7, the functions* $(G^m(\nabla u_m))_{m\in\mathbb{N}}$ *are continuous with the same modulus of continuity on each compact subset of* Ω.

5. Uniform estimate for Theorem 1.10

This section is devoted to the proof of Theorem 1.10. We assume that there exists a compact set D_G such that $G \in C^1(\mathbb{R}^N \backslash D_G)$ and $D_G = \mathbb{R}^N \backslash \bigcup_{k\in\mathbb{N}} O_k$ with

$$O_k := \left\{ z \in \mathbb{R}^N,\ \frac{1}{k}|v|^2 < \langle DG^s(z)v, v\rangle < k|v|^2 \text{ for every } v \in \mathbb{R}^N \right\}$$

where $DG^s := \frac{DG+DG^T}{2}$. For every $r > 0$, we introduce the closed r−neighborhood of a set U:

$$\overline{N}_r(U) := \{y \in \mathbb{R}^N,\ \mathrm{dist}(y, U) \le r\}.$$

We assume that there exists $t_0 > 0$ such that for every $0 \le t \le t_0$ the connected components of $\overline{N}_t(D_G)$ are simply connected.

As in the previous sections, we want to obtain uniform estimates for smooth approximations of the original problem. Therefore, we work with the smooth function G^m from Proposition 2.5 and the smooth solution u_m of (2.2). Let r_0 be the smallest distance between two connected components of D_G. We introduce $\rho_0 < \min\{\frac{r_0}{8}, t_0\}$. In this section, we prove that:

Proposition 5.1. *For every* $0 < t < \frac{\rho_0}{2}$ *and for every subset* $\Omega' \Subset \Omega$*, the functions*

$$\mathrm{dist}(\nabla u_m, \overline{N}_t(D_G)) \text{ and } \mathrm{dist}(\nabla u_m, \overline{N}_t(D_G))\nabla u_m$$

are continuous with a uniform modulus of continuity in $m \ge \frac{2}{t}$*. Moreover, for every* $x_0 \in \Omega'$*, there exists* $r > 0$ *independent of* $m \in \mathbb{N}$ *such that* $\nabla u_m(B_r(x_0))$ *encounters at* *most one connected component* D^0_G *of* D_G*. Furthermore, if* $\nabla u_m(B_r(x_0)) \cap D^0_G$ *is not empty then* $\nabla u_m(B_r(x_0)) \subset \overline{N}_{3\rho_0}(D^0_G)$.

We define

$$\widetilde{O}^t_k := \left(\bigcap_{m \geq \frac{2}{t}} \left\{ \frac{1}{k} \operatorname{Id} < (DG^m)^s < k \operatorname{Id} \right\} \right) \setminus \overline{N}_t(D_G).$$

Hence the sets $\widetilde{O}^t_k$ are independent of m when $m \geq \frac{2}{t}$.

Since every estimate of this section is independent of $m \geq \frac{2}{t}$ we can drop the subscript $m \in \mathbb{N}$ in order to simplify the notations. Moreover, since we want to prove continuity results for every $x_0 \in \Omega'$, we can replace Ω' by $B_{\operatorname{dist}(x_0,\partial\Omega)}(x_0)$. By replacing $u(\,\cdot\,)$ by $u(\operatorname{dist}(x_0, \partial\Omega) \cdot + x_0)$, we can assume that $\Omega' = B_1(0)$. We introduce the constant $L > 0$ that is a Lipschitz constant of u_m on $B_1(0)$ uniform in $m \in \mathbb{N}$. In the remaining of the section $0 < t < \frac{\rho_0}{2}$ is fixed. Hence, we do not state the dependence on t in the constants of the following results.

5.1. **Preliminary results**

In this section, we introduce two results that are adaptations of [28, Proposition 3.1, Lemma 3.2] in the case where $f \neq 0$.

We introduce the following operator:

$$L^u_G : v \longmapsto \operatorname{div}(DG(\nabla u)\nabla v).$$

Remark 5.2. We have that $L^u_G(\partial_e u) = \partial_e f$ for every $e \in \mathbb{S}^{N-1}$.

We prove the following result:

Proposition 5.3. *Let* $K \geq 0$ *and* $q > N$*. We assume that* v *is in* $W^{1,2}(B_1(0)) \cap L^\infty(B_1(0))$*,* $v \geq 0$ *and solves* $L^u_G(v) \geq g$ *in the weak sense with* $\{v > 0\} \subset \widetilde{O}^t_k$ *for some* $k \in \mathbb{N}$ *and* $g \in L^q(B_1(0))$*. Then for all* $0 < \mu \leq 1$*, there exists* $\nu := \nu(\mu, N, k)$ *such that if*

$$\frac{|\{v > 0 \cap B_{\frac{1}{2}}(0)\}|}{|B_{\frac{1}{2}}(0)|} \leq 1 - \mu$$

and

$$\|g\|_{L^q(B_1(0))} \leq \frac{\nu}{2} \max\Big\{ \sup_{B_1(0)} v, K \Big\}$$

then

$$\sup_{B_{\frac{1}{4}}(0)} v \leq (1 - \nu) \max\Big\{ \sup_{B_1(0)} v, K \Big\}.$$

Proof. Let us introduce $m := \sup_{B_1(0)} v$, then $m - v \geq 0$ and $L_G^u(m-v) \leq -g$. If we replace $DG(\nabla u)$ by the identity matrix on the set where $v = 0$, then by [19, Theorem 8.18], there exists $C_0 := C_0(N,k) > |B_1(0)|$ such that

$$2^N \int_{B_{\frac{1}{2}}(0)} m - v \leq C_0\Big(\inf\{m - v(x), x \in B_{\frac{1}{4}}(0)\} + k\|g\|_{L^q(B_1(0))}\Big).$$

We estimate the left-hand side from below by integrating over the set $[v = 0]$. Thus,

$$2^N \mu m |B_{\frac{1}{2}}(0)| \leq C_0\Big(m - \sup_{B_{\frac{1}{4}(0)}} v + k\|g\|_{L^q(B_1(0))}\Big).$$

Hence,

$$\sup_{B_{\frac{1}{4}}(0)} v \leq \left(1 - \frac{2^N \mu |B_{\frac{1}{2}}(0)|}{C_0}\right) m + k\|g\|_{L^q(B_1(0))}.$$

Since $\|g\|_{L^q(B_1(0))} \leq \frac{\nu}{2}\max\{\sup_{B_1(0)} v, K\}$, if we take $\nu = (1 + \frac{k}{2})^{-1}\left(\frac{2^N\mu|B_{\frac{1}{2}}(0)|}{C_0}\right)$, we have:

$$\sup_{B_{\frac{1}{4}}(0)} v \leq (1-\nu)\max\{\sup_{B_1(0)} v, K\}. \qquad \square$$

In the rest of the paper we are going to apply this proposition to functions of ∇u that are concave in one direction. To do so, we prove the following result:

Lemma 5.4. *Let $\widetilde{f} \in C^1(\overline{B_1(0)})$ and v a smooth solution in $B_1(0)$ of*

$$\operatorname{div}(G(\nabla v)) = \widetilde{f}. \tag{5.1}$$

Let η be a smooth function in a neighborhood of $\nabla v(B_1(0))$. We assume that

$$\nabla v(B_1(0)) \cap \{\eta > 0\} \subset \widetilde{O}_k^t.$$

Then there exists $\lambda := \lambda(k,N) > 0$ such that if in $\nabla v(B_1(0)) \cap \{\eta > 0\}$ the eigenvalues $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_N$ of $D^2\eta$ satisfy $\gamma_2 > 0$ and $0 \geq \gamma_1 \geq -\lambda\gamma_2$ then

$$L_G^v(\eta_+(\nabla v)) \geq \Big(\langle \nabla\widetilde{f}, \nabla\eta(\nabla v)\rangle - \lambda\gamma_2 k^3 \widetilde{f}^2\Big)\mathbb{1}_{\{\eta(\nabla v)>0\}}$$

in the weak sense.

Here, $\eta_+ := \max\{\eta, 0\}$.

Proof. Since v is a solution of (5.1) we have that

$$L_G^v(\eta(\nabla v)) = \mathrm{div}(DG(\nabla v)\nabla[\eta(\nabla v)])$$
$$= \sum_{i,j,s,l} D_i G_j(\nabla v) v_{js} \eta_{sl}(\nabla v) v_{li} + \langle \nabla \widetilde{f}, \nabla \eta(\nabla v)\rangle.$$

As in [28, Lemma 3.2], for $x_0 \in \{\eta(\nabla v) > 0\}$ we can choose coordinates such that $\eta_{sl}(\nabla v(x_0)) = \gamma_s \delta_{sl}$. Hence,

$$L_G^v(\eta(\nabla v))(x_0) = \sum_s \gamma_s \sum_{i,j} D_i G_j(\nabla v) v_{js} v_{si} + \langle \nabla \widetilde{f}, \nabla \eta(\nabla v)\rangle.$$

Since $\nabla v(B_1(0)) \cap \{\eta > 0\} \subset \widetilde{O}_k^t$ we obtain that

$$L_G^v(\eta(\nabla v))(x_0) \geq \gamma_1 k |\nabla v_1|^2 + \sum_{n=2}^N \gamma_n \frac{1}{k} |\nabla v_n|^2 + \langle \nabla \widetilde{f}, \nabla \eta(\nabla v)\rangle.$$

This last inequality combined with the fact that $\gamma_1 \geq -\lambda \gamma_2$ on $\{\eta(\nabla v) > 0\}$ provides that

$$\gamma_2^{-1} L_G^v(\eta(\nabla v))(x_0) \geq k^{-1} \sum_{n=2}^N |\nabla v_n(x_0)|^2 - k\lambda |\nabla v_1(x_0)|^2 + \gamma_2^{-1} \langle \nabla \widetilde{f}, \nabla \eta(\nabla v)\rangle.$$

If $L_G^v(\eta(\nabla v))(x_0) < \langle \nabla \widetilde{f}, \nabla \eta(\nabla v)\rangle - \lambda \gamma_2 k^3 \widetilde{f}^2$ then

$$k^{-1} \sum_{n=2}^N |\nabla v_n(x_0)|^2 - k\lambda |\nabla v_1(x_0)|^2 < -\lambda k^3 \widetilde{f}^2.$$

Since $\sum_{n=2}^N |\nabla v_n(x_0)|^2 \geq \frac{1}{2} \sum_{(i,j)\neq(1,1)} v_{ij}^2(x_0)$ we have that

$$\left(\frac{\lambda^{-1} k^{-2}}{2} - 1\right) \sum_{(i,j)\neq(1,1)} v_{ij}^2(x_0) + k^2 \widetilde{f}^2 < v_{11}^2(x_0). \tag{5.2}$$

We introduce $D^s G = \frac{DG + DG^T}{2}$. Since $\mathrm{Tr}(DG^s(\nabla v) D^2 v) = \mathrm{Tr}(DG(\nabla v) D^2 v) = \widetilde{f}$, we obtain that

$$D_1 G_1^s(\nabla v) v_{11} = -\sum_{(i,j)\neq(1,1)} D_i G_j^s v_{ij} + \widetilde{f}.$$

Thus, since $x_0 \in \{\eta(\nabla v) > 0\}$ and $\{\eta > 0\} \subset \widetilde{O}_k^t$ we have

$$v_{11}^2(x_0) \leq C(N,k) \sum_{(i,j)\neq(1,1)} v_{ij}^2(x_0) + k^2 \widetilde{f}^2.$$

We get a contradiction with (5.2) when $\lambda \leq \frac{k^{-2}}{2(C(N,k)+1)}$. □

5.2. When ∇u is close to the convex hull of D_G

Let C_G be the convex hull of D_G. In this section, we study the behavior of $\nabla u(B_\delta(0))$ when $\nabla u(B_\delta(0))$ is close to C_G. By [14, Proposition 4.3], we have the following result:

Proposition 5.5. *We assume that $\epsilon > 0$ and that*

$$\mathrm{div}(G(\nabla v)) = \widetilde{f}$$

in $B_1(0)$ with $\widetilde{f} \in L^q(B_1(0))$ and $q > N$. We consider p such that $B_\epsilon(p) \cap \overline{N}_t(D_G) = \emptyset$. Let $k \in \mathbb{N}$ such that $B_\epsilon(p) \subset \widetilde{O}^t_k$. Then there exist $\delta_0 > 0$ and $\mu_0 > 0$ depending on the modulus of continuity of DG in $B_{\epsilon+\frac{t}{2}}(p)$, k and ϵ such that if $\|v - l_p\|_{L^\infty(B_1(0))} \leq \delta_0$ for some affine function l_p with $\nabla l_p = p$ and $\|\widetilde{f}\|_{L^q(B_1(0))} \leq \delta_0\mu_0$, then

$$\nabla v(B_{\frac{1}{2}}(0)) \subset B_\epsilon(p).$$

We can use this result to show that:

Lemma 5.6. *For every $\epsilon > 0$ there exist $\alpha := \alpha(\rho_0, DG, \|f\|_{L^q(\Omega)}, L, N, \epsilon)$, $\kappa := \kappa(DG, L, N, \epsilon) \leq \epsilon$ and $\mu_1 := \mu_1(DG, L, N, \epsilon)$ with L the Lipschitz constant of u on $B_1(0)$ such that if*

$$\frac{|\{\nabla u \in B_\kappa(p)\} \cap B_r(0)|}{|B_r(0)|} \geq 1 - \mu_1$$

for some $p \notin \overline{N}_{t+\epsilon}(D_G)$ and $r \leq \alpha$ then

$$\nabla u(B_{\frac{r}{2}}(0)) \subset B_\epsilon(p).$$

Remark 5.7. When a dependence in DG appears in a constant it means that the constant depends only on the sets of ellipticity $\widetilde{O}^t_k$ and the modulus of continuity of DG^m outside $\overline{N}_t(D_G)$ with $m \geq \frac{2}{t}$ the parameter of regularization. Since all those quantities are independent of $m \geq \frac{2}{t}$, we can just denote this dependence by DG.

Proof. For $r > 0$, we introduce $v(x) := \frac{1}{r}u(rx)$ in $B_1(0)$. Then

$$\mathrm{div}(G(\nabla v(x))) = rf(rx)$$

in $B_1(0)$. Since $q > N$ by taking r small enough, we can assume that $\|rf(r\cdot)\|_{L^q(B_1(0))}$ is as small as we want. Hence, there exists $\alpha := \alpha(\rho_0, DG, \|f\|_{L^q(\Omega)}, L, N) > 0$ such that $\|rf(r\cdot)\|_{L^q(B_1(0))} \leq \delta_0\mu_0$ for every $r \leq \alpha$.

We show as in [14, Lemma 4.1] that there exists an affine function l_p such that $\|v - l_p\|_{L^\infty(B_1(0))} \leq \delta_0$. By Morrey's inequality, there exists a constant C_0 depending only on N such that for every $x \in B_r(0)$ and every $w \in W^{1,2N}(B_1(0))$ we have

$$\left| w(x) - \frac{1}{|B_1(0)|}\int_{B_1(0)} w(y)\mathrm{d}y \right| \leq C_0 \left(\frac{1}{|B_1(0)|}\int_{B_1(0)} |\nabla w(y)|^{2N}\mathrm{d}y \right)^{\frac{1}{2N}}.$$

We set $l_p(x) := \langle p, x\rangle + \frac{1}{|B_1(0)|}\int_{B_1(0)} v(y)\mathrm{d}y$. In that case, we obtain

$$|v(x) - l_p(x)| \leq C_0\left(\frac{1}{|B_1(0)|}\int_{B_1(0)} |\nabla v(y) - p|^{2N}\mathrm{d}y\right)^{\frac{1}{2N}}.$$

We estimate the right-hand side by splitting the integral in two sets. The first one is $X := \{x \in B_1(0), \nabla v(x) \in B_\kappa(p)\}$. A direct computation gives that

$$\int_{B_1(0)\cap X} |\nabla v(y) - p|^{2N}\mathrm{d}y \leq \kappa^{2N}|B_1(0)|.$$

Since the complement of X has a measure less than $\mu|B_1(0)|$, we have that

$$\int_{B_1(0)\backslash X} |\nabla v(y) - p|^{2N}\mathrm{d}y \leq \mu|B_1(0)|(2L)^{2N}$$

with L the Lipschitz constant of u on $B_1(0)$. Thus,

$$|v(x) - l_p(x)| \leq C_0(\kappa^{2N} + \mu(2L)^{2N})^{\frac{1}{2N}}.$$

Hence, it remains to take κ and μ small enough such that $C_0(\kappa^{2N} + \mu(2L)^{2N})^{\frac{1}{2N}} \leq \delta_0$ in order to apply Proposition 5.5 to v. The conclusion follows for u. □

In the rest of the section, we write the vectors $z \in \mathbb{R}^N$ in the following way: $z = (p, p')$ with $p \in \mathbb{R}^2$ and $p' \in \mathbb{R}^{N-2}$ and we assume that $D_G \subset \{p' = 0\}$. Now we present the main result of this subsection, which has the same conclusion as [28, Proposition 3.7].

Proposition 5.8. *For every $\epsilon > 0$, there exist $\beta := \beta(\|f\|_{W^{1,q}(\Omega)}, \kappa, \mu_1, N, DG, \epsilon) > 0$, $s_0 := s_0(\kappa, \mu_1, L, DG, \epsilon) > 0$ and $\sigma_0 := \sigma_0(\kappa, \mu_1, N, DG, \epsilon) > 0$ with κ and μ_1 from Lemma 5.6 such that if $\nabla u(B_r(0)) \subset \{|p'| < \sigma_0\}$ with $r \leq \beta$, then either $\nabla u(B_{s_0 r}(0)) \subset B_\epsilon(p)$ for some $p \notin \overline{N}_{\epsilon+t}(D_G)$ or $\nabla u(B_{s_0 r}(0)) \subset \overline{N}_{\epsilon+t}(D_G)$.*

As in [28], we need the following preliminary lemma:

Lemma 5.9. *Let $(p_0, 0) \notin \overline{N}_{\epsilon+t}(D_G)$. There exist $\sigma_0 := \sigma_0(\kappa, \mu_1, N, DG, \epsilon) > 0$, $\beta := \beta(\|f\|_{W^{1,q}(\Omega)}, \kappa, \mu_1, N, DG, \epsilon) > 0$ and $C_0 := C_0(\kappa, \mu_1, N, DG, \epsilon) > 0$ such that if*

$$\nabla u(B_r(0)) \subset \{|p'| < \sigma_0\} \cap \left\{|p - p_0| \geq \frac{\kappa}{4}\right\}$$

and

$$\frac{|\{\nabla u \in B_\kappa(p_0, 0)\} \cap B_{\frac{r}{2}}(0)|}{|B_{\frac{r}{2}}(0)|} < 1 - \mu_1,$$

for $r \leq \beta$, then

$$\nabla u(B_{\frac{r}{4}}(0)) \subset \left\{|p - p_0| \geq \frac{\kappa}{4} + C_0\right\}.$$

Proof. There exists $k \in \mathbb{N}$ such that $B_\epsilon(p_0, 0) \subset \widetilde{O}_k^t$. We follow the proof of [28, Lemma 3.8]. We set $v(x) = \frac{1}{r}u(rx)$ on $B_1(0)$ and we replace $f(x)$ by $rf(rx)$. We define

$$\eta_A(p, p') := \exp\left(\frac{A^2}{2}|p'|^2 - A|p|\right),\ \eta_{A,p_0} := \eta_A(p - p_0, p') - \exp\left(-A\frac{\kappa}{2}\right).$$

In the basis $(\frac{p^\perp}{|p|}, \frac{p}{|p|}, e_3, \ldots, e_N)$, on the set $\{|p'| < A^{-3}\}$, we have

$$(A^2\eta_A)^{-1}D^2\eta_A = \mathrm{diag}(-(A|p|)^{-1}, 1, \ldots, 1) + O(A^{-2})$$

and $\{\eta_A > \exp(-A\frac{\kappa}{2})\} \subset \{|p| < \frac{\kappa}{2} + A^{-5}\}$. Since $\nabla u(B_r(0)) \subset \{|p'| < \sigma_0\}$, for A large enough, depending only on κ, if $\sigma_0 < A^{-3}$ then $\nabla v(B_1(0)) \cap \{\eta_{A,p_0} > 0\} \subset B_L \cap B_\kappa(p_0, 0)$. Since $(p_0, 0) \notin \overline{N}_{\epsilon+t}(D_G)$ we also have $B_L(0) \cap B_\kappa(p_0, 0) \subset B_L \backslash \overline{N}_t(D_G)$. Hence, we obtain that $\nabla v(B_1(0)) \cap \{\eta_{A,p_0} > 0\} \subset B_L \cap B_\kappa(p_0, 0) \subset B_L \backslash \overline{N}_t(D_G)$.

Once again if A is large enough depending only on $\lambda(k, N)$ from Lemma 5.4 and κ then the eigenvalues $\gamma_1 \leq \cdots \leq \gamma_n$ of $D^2\eta_{A,p_0}$ satisfy $\gamma_2 > 0$ and $\gamma_1 > -\lambda\gamma_2$ in $\nabla v(B_1(0)) \cap \{\eta_{A,p_0} > 0\}$. Thanks to Lemma 5.4, the function $v_{p_0} := (\eta_{A,p_0})_+(\nabla v)$ satisfies

$$L_G(v_{p_0}) \geq \left(\langle r^2\nabla f(r\cdot), \nabla\eta_{A,p_0}(\nabla u(r\cdot))\rangle - r^2\lambda\gamma_2 k^3 f(r\cdot)^2\right)\mathbb{1}_{\{\eta_{A,p_0}(\nabla v)>0\}}. \tag{5.3}$$

Let denote by ν_1 the constant $\nu(\mu_1, N, k)$ from Proposition 5.3. We select $\sigma_0(\kappa, \nu_1, A)$ small in such a way that

$$\exp\left(-A\left(\frac{\kappa}{4}\right) - \exp\left(-A\frac{\kappa}{2}\right) > (1 - \nu_1)\left(\exp\left(\frac{A^2}{2}\sigma_0^2 - A\frac{\kappa}{4}\right) - \exp\left(-A\frac{\kappa}{2}\right)\right)\right).$$

Then, if we look at the right-hand side of (5.3) there exists a positive constant $\beta(\|f\|_{W^{1,q}(\Omega)}, \kappa, \mu_1, L, N, DG, \epsilon)$ such that when $r \leq \beta$, we can apply Proposition 5.3 to v_{p_0} with $K = \exp(\frac{A^2}{2}\sigma_0^2 - A\frac{\kappa}{4}) - \exp(-A\frac{\kappa}{2})$. Hence, we have that

$$\nabla v(B_{\frac{1}{4}}(0)) \subset \left\{\eta_{A,p_0} < (1 - \nu_1)\left(\exp\left(\frac{A^2}{2}\sigma_0^2 - A\frac{\kappa}{4}\right) - \exp\left(-A\frac{\kappa}{2}\right)\right)\right\}.$$

Thus,

$$\nabla v(B_{\frac{1}{4}}(0)) \subset \left\{\exp(-A|p - p_0|) - \exp\left(-A\frac{\kappa}{2}\right) < \exp\left(-A\frac{\kappa}{4}\right) - \exp\left(-A\frac{\kappa}{2}\right)\right\}.$$

Hence, $\nabla v(B_{\frac{1}{4}}(0)) \subset \{|p - p_0| \geq \frac{\kappa}{4} + C_0\}$ for some $C_0 := C_0(\kappa, \mu_1, N, DG, \epsilon) > 0$. The conclusion follows. □

We prove Proposition 5.8:

Proof of Proposition 5.8. We follow the proof of [28, Proposition 3.7]. We introduce α, κ and μ_1 from Lemma 5.6 and β from Lemma 5.9. Taking $r \le \min\{\alpha, \beta\}$, we use the following strategy: if there exist $(p_0, 0) \in B_{2L}(0)\backslash \overline{N}_{\epsilon+t}(D_G)$ and $n \in \mathbb{N}$ such that

$$\frac{|\{\nabla u \in B_\kappa(p_0,0)\} \cap B_{2^{-2n}r}|}{|B_{2^{-2n}r}|} \ge 1 - \mu_1 \tag{5.4}$$

then we can apply Lemma 5.6 in order to obtain that $\nabla u(B_{2^{-2n-1}r}) \subset B_\epsilon(p_0, 0)$ and the conclusion follows.

If that is not the case, then by Lemma 5.9 we obtain that for every $(p_0, 0) \in B_{2L}(0)\backslash \overline{N}_{\epsilon+t}(D_G)$ if

$$\nabla u(B_{2^{-2n+1}r}(0)) \subset \left\{|p'| < \sigma_0\right\} \cap \left\{|p - p_0| \ge \frac{\kappa}{4}\right\}$$

then

$$\nabla u(B_{2^{-2n-1}r}(0)) \subset \left\{|p - p_0| \ge \frac{\kappa}{4} + C_0\right\}. \tag{5.5}$$

The idea in [28, Proposition 3.7] is to use a covering argument with neighborhoods of lines. Unfortunately, we can not cover the set $B_{2L}(0)\backslash \overline{N}_{\epsilon+t}(D_G)$ with $\frac{\kappa}{4}$ neighborhoods of lines since $\overline{N}_{\epsilon+t}(D_G)$ is not a finite union of small balls.

Instead, since the connected components of $\overline{N}_{\epsilon+t}(D_G)$ are simply connected, we can consider a finite family of points $(x_i)_{i\in I}$ in $(B_{2L}(0)\backslash \overline{N}_{\epsilon+t}(D_G)) \cap \{p' = 0\}$ with $I := [1, I_G] \subset \mathbb{N}$ such that $B_\kappa(x_1) \cap B_L(0) = \emptyset$, $|x_i - x_{i+1}| \le C_0$ for every $i \in I\backslash\{I_G\}$ with C_0 from Lemma 5.9. Moreover, we assume that $(B_{2L}(0)\backslash \overline{N}_{\epsilon+t}(D_G)) \cap \{p' = 0\} \subset \bigcup_{i\in I} B_{\frac{\kappa}{4}}(x_i)$. By definition of L, $B_\kappa(x_1) \cap \nabla u(B_r(0)) = \emptyset$.

Thus, we can initiate the algorithm. By Lemma 5.9 we obtain that

$$\nabla u(B_{2^{-2}r}(0)) \subset \left\{|p - x_1| \ge \frac{\kappa}{4} + C_0\right\}.$$

Hence, $\nabla u(B_{2^{-2}r}(0)) \subset \{|q| < \sigma_0\} \cap \{|p - x_2| \ge \frac{\kappa}{4}\}$. If (5.4) is satisfied with $(p_0, 0) = x_2$ and $n = 3$ then we can conclude, otherwise $\nabla u(B_{2^{-4}r}(0)) \subset \{|p'| < \sigma_0\} \cap \{|p - x_3| \ge \frac{\kappa}{4}\}$. The algorithm terminates after at most I_G steps with two potential conclusions. The first one is that $\nabla u(B_{2^{-2n_0-1}r}) \subset B_\epsilon(x_{n_0})$ for a certain $n_0 \in [1, I_G]$. The second one is that $\nabla u(B_{2^{-2I_G-1}r}) \subset B_{2L}(0)\backslash \bigcup_{i\in I} B_{\frac{\kappa}{4}}(x_i)$ which is a subset of $\overline{N}_{\epsilon+t}(D_G)$. □

5.3. Reduction to the convex hull of D_G

In this subsection, we present a result that states that either $\nabla u_m(B_r(0))$ is outside C_G the convex hull of D_G or is close to it. By [14, Theorem 1.1] we have the following proposition:

Proposition 5.10. *For every $s > 0$, there exists $\delta_s := \delta_s(L, DG, s) > 0$ such that either $\nabla u_m(B_{\delta_s}) \subset B_s(\nabla u_m(0))$ or $\nabla u_m(B_{\delta_s}) \subset \overline{N}_{4s}(C_G)$ for every $m \in \mathbb{N}$.*

Proof. By [14, Theorem 1.1], if H is a continuous function on $\mathbb{R}^N$ such that $H \equiv 0$ on $\overline{N}_s(C_G)$ then $H(\nabla u_m)$ has a modulus of continuity depending on L and DG that does not depend on the parameter of regularity $m \in \mathbb{N}$. We can take $H(\,\cdot\,) := \operatorname{dist}(\,\cdot\,, \overline{N}_s(C_G))$. In that case if $\nabla u_m(0) \in \overline{N}_{3s}(C_G)$ then there exists $\delta_s := \delta_s(M, G, r) > 0$ such that $\nabla u_m(B_{\delta_s}) \subset \overline{N}_{4s}(C_G)$. If we assume that $\nabla u_m(0) \notin \overline{N}_{3s}(C_G)$ then there exists $\delta'_s := \delta'_s(M, G, s) > 0$ such that $\nabla u_m(B_{\delta'_s}) \cap \overline{N}_{2s}(C_G) = \emptyset$. By classical results on uniform elliptic equation, there exists $\delta_s := \delta_s(L, G, s) > 0$ such that $\nabla u_m(B_{\delta_s}) \subset B_s(\nabla u(0))$. □

5.4. Proof of Proposition 5.1

To finish this section, we reintroduce the subscript $m \in \mathbb{N}$. We can prove that for every $0 < t < \frac{\rho_0}{2}$, $\operatorname{dist}(\nabla u_m, \overline{N}_t(D_G))$ and $\operatorname{dist}(\nabla u_m, \overline{N}_t(D_G))\nabla u_m$ are uniformly continuous in $m \geq \frac{2}{t}$ with a similar strategy as in [28].

Proof of Proposition 5.1. We take $x_0 \in \Omega'$ and $0 < \epsilon < r_0$, we consider σ_0 from Lemma 5.9. We apply Proposition 5.10 with $s = \frac{\sigma_0}{4}$. Hence, there exists $\delta_s > 0$ such that either $\nabla u_m(B_{\delta_s}(x_0)) \subset B_{\frac{\sigma_0}{4}}(\nabla u_m(x_0))$ or $\nabla u_m(B_{\delta_s}(x_0)) \subset \mathcal{N}_{\sigma_0}(C_G)$.

In the first case for every $0 < \epsilon_1 < \frac{\sigma_0}{4}$, we can find $\delta := \delta(L, DG, \epsilon_1)$ such that $\nabla u_m(B_\delta(x_0)) \subset B_{\epsilon_1}(\nabla u_m(x_0))$ thanks to Proposition 5.10.

In the second case, we apply Proposition 5.8 with $r = \min\{\beta, \delta_s\}$. Hence, in this case, we either have $\nabla u_m(B_{s_0 r}(x_0)) \subset B_\epsilon(p)$ for some $p \notin \overline{N}_{\epsilon+t}(D_G)$ or $\nabla u_m(B_{s_0 r}(x_0)) \subset \overline{N}_{\epsilon+t}(D_G)$. Since $\nabla u_m(B_{s_0 r})$ is connected, by definition of ρ_0 the set $\nabla u_m(B_{s_0 r})$ encounters at most one connected component D_G.

Hence, for every $\epsilon > 0$ and every $t > 0$ there exists

$$\delta_\epsilon(L, DG, N, \epsilon, t, \|f\|_{W^{1,q}(\Omega)}) > 0$$

such that either $\nabla u_m(B_{\delta_\epsilon}) \subset B_\epsilon(\nabla u_m(x_0))$ or $\nabla u_m(B_{\delta_\epsilon}) \subset \overline{N}_{\epsilon+t}((D_G))$ for every $m \geq \frac{2}{t}$.

Thus, we have that $\operatorname{dist}(\nabla u_m, \overline{N}_t(D_G))$ and $\nabla u_m \times \operatorname{dist}(\nabla u_m, \overline{N}_t(D_G))$ are continuous uniformly in $m \in \mathbb{N}$ with $m \geq \frac{2}{t}$. □

6. Main proofs

We assume that G satisfies the assumptions of Theorem 1.1, Theorem 1.5 or Theorem 1.7. Before proving these three theorems, we show the following result:

Proposition 6.1. *If G satisfies the assumptions of Theorem 1.1, Theorem 1.5 or Theorem 1.7 then $G(\nabla u)$ does not depend on the choice of the solution u of* (1.4) *when the Dirichlet boundary condition is fixed.*

Proof. Let us assume that u and v are two solutions of the same equation (1.4) such that $u = v$ on $\partial\Omega$. Then we have:

$$\int_{\Omega} \langle G(\nabla u) - G(\nabla v), \nabla u - \nabla v\rangle = 0.$$

Since $\langle G(A)-G(B), A-B\rangle \geq 0$, we get that $\langle G(\nabla u(x))-G(\nabla v(x)), \nabla u(x)-\nabla v(x)\rangle = 0$ for a.e. $x \in \Omega$. The condition (A_1) gives that $G(\nabla u(x)) = G(\nabla v(x))$ for a.e. $x \in \Omega$. In the other cases, we use the convexity of φ to get the same result. □

With this proposition, we just have to show that $G(\nabla u)$ is continuous for u the solution of (1.4) obtained as the limit of $(u_m)_{m\in\mathbb{N}}$ when $m \to +\infty$:

Proof of Theorem 1.1, Theorem 1.5 and Theorem 1.7. Thanks to Proposition 3.9, Proposition 3.18 and Proposition 4.9, we have that for every compact $\Omega' \Subset \Omega$, the family $(G^m(\nabla u_m))_{m\in\mathbb{N}}$ is equicontinuous. Hence, by the Arzelà–Ascoli Theorem, $G^m(\nabla u_m)$ converges to v uniformly on Ω', up to a subsequence. Since $\|\nabla u_m\|_{L^\infty(\Omega')}$ is bounded uniformly in $m \in \mathbb{N}$, we have that $|G^m(\nabla u_m) - G(\nabla u_m)| \to 0$ in $L^1(\Omega')$. Thanks to Remark 2.9, we get that $G^m(\nabla u_m)$ converges to $G(\nabla u)$ in $L^1(\Omega')$. Thus, $v = G(\nabla u)$ is continuous for any solution of (1.4) thanks to Proposition 6.1. □

We are ready to prove Proposition 1.8.

Proof of Proposition 1.8. On the set $\sigma^{-1}(V)$ we define $F(x)$ as $G^{-1}(\sigma(x))$. Thus, the function F is continuous and $F = \nabla u_0$ a.e. on $\sigma^{-1}(V)$. Hence, ∇u_0 has a continuous representative on $\sigma^{-1}(V)$. □

Let us prove Proposition 1.9.

Proof of Proposition 1.9. We know that $G(\nabla u_0)$ has a continuous representative σ where $G = \nabla\varphi$ with φ a convex function that depends only on the Euclidean norm $|\cdot|$.

For every $t \in \mathbb{R}$, we introduce the super level set $E_t = [u_0 > t]$. By the co-area formula we obtain that for a.e. $t \in \mathbb{R}$, $D\mathbb{1}_{E_t} = \frac{\nabla u_0}{|\nabla u_0|}\mathcal{H}^{N-1} \llcorner \partial^e E_t$ where $\partial^e E_t$ is the support of the measure $D\mathbb{1}_{E_t}$. Since φ is radial $\frac{\sigma}{|\sigma|} = \frac{\nabla u_0}{|\nabla u_0|}$ a.e. on $\Omega \cap [\nabla u_0 \neq 0]$. Thus, by the co-area formula we get that $D\mathbb{1}_{E_t} = \frac{\sigma}{|\sigma|}\mathcal{H}^{N-1} \llcorner \partial^e E_t$ for a.e. $t \in \mathbb{R}$.

Hence, we get

$$\lim_{r\to 0} \frac{\int_{B_r(x)} D\mathbb{1}_{E_t}}{\int_{B_r(x)} |D\mathbb{1}_{E_t}|} = \frac{\sigma(x)}{|\sigma(x)|}$$

for every $x \in \partial^e E_t \cap [\sigma \neq 0]$. By [20, Theorem 4.11] we obtain that the connected components of $\partial^e E_t \cap [\sigma \neq 0]$ are C^1 curves.

Let $x_0 \in [\sigma \neq 0]$. By continuity of σ this set is open. Hence, there exist $r > 0$ and $C > 0$ such that for every $x \in B_r(x_0)$ we have that $\langle \sigma(x), e \rangle \geq C$ with $e := \frac{\sigma(x_0)}{|\sigma(x_0)|}$. Since, $\sigma(x) = \nabla\varphi(\nabla u_0(x))$ for a.e. $x \in \Omega$ the continuity of $\nabla\varphi$ gives that $|\nabla u_0(x)| \geq \widetilde{C} > 0$ for a.e. $x \in B_r(x_0)$. Since $\frac{\sigma}{|\sigma|} = \frac{\nabla u_0}{|\nabla u_0|}$ a.e. on $\Omega \cap [\nabla u_0 \neq 0]$, there exists C' such that for a.e. $x \in B_r(x_0)$, $\langle \nabla u_0(x), e \rangle \geq C' > 0$. By Lipschitz continuity of u_0 for a.e. $x \in B_{\frac{r}{2}}(x_0)$ and every $0 < t < \frac{r}{2}$ we have that

$$u_0(x + te) - u_0(x) = \int_0^t \langle \nabla u_0(x + se), e \rangle \mathrm{d}s \geq C't > 0.$$

By continuity of u_0, for every $x \in B_{\frac{r}{2}}(x_0)$ and every $0 < t < \frac{r}{2}$ we obtain that $u_0(x + te) - u_0(x) \geq C't > 0$. Hence, by continuity of u_0, for every $\rho > 0$ we have that $0 < |B_\rho(x_0) \cap [u_0 > u_0(x_0)]| < B_\rho(x_0)$. This means that $x_0 \in \partial^e[u > u(x_0)]$.

Finally, let $t \in \mathbb{R}$ such that $\partial^e E_t \cap [\sigma \neq 0]$ is a C^1 curve. By continuity of u_0, for every $x \in \partial^e E_t \cap [\sigma \neq 0]$ we have that $u_0(x) = t$. Moreover, for every $x \in [\sigma \neq 0]$ such that $u_0(x) = t$ we have that $x \in \partial^e E_t$. Hence, for a.e. $t \in \mathbb{R}$, $[u_0 = t] \cap [\sigma \neq 0] = \partial^e E_t \cap [\sigma \neq 0]$. Hence, the connected components of $[u_0 = t] \cap [\sigma \neq 0]$ are C^1 curves for a.e. $t \in \mathbb{R}$. □

To prove Theorem 1.10, we use the following proposition instead of Proposition 6.1:

Proposition 6.2. *Let us assume that G satisfies the assumptions of Theorem 1.10. We consider two solutions u and v of* (1.4) *such that $u = v$ on $\partial\Omega$. Then $\nabla u(x) = \nabla v(x)$ for a.e. $x \in \Omega$ such that $\nabla u(x) \notin D_G$.*

Proof. As in the proof of Proposition 6.1, for a.e. $x \in \Omega$ we have that

$$\langle G(\nabla u(x)) - G(\nabla v(x)), \nabla u(x) - \nabla v(x) \rangle = 0.$$

If for some $x \in \Omega$ we have that $\nabla u(x) \notin D_G$, then there exists $k \in \mathbb{N}$ such that $\nabla u(x) \in \mathring{O}_k := \{\frac{1}{k}Id < DG < kId\}$. Hence, for every $A \in \mathbb{R}^N \backslash \{\nabla u(x)\}$, we have that $\langle G(A) - G(\nabla v(x)), A - \nabla v(x) \rangle > 0$. Thus, $\nabla v(x) = \nabla u(x)$ for a.e. $x \in \Omega$ such that $\nabla u(x) \notin D_G$. □

Remark 6.3. Thanks to this proposition, $\mathrm{dist}(\nabla u, D_G)$ and $\nabla u \times \mathrm{dist}(\nabla u, D_G)$ do not depend on the choice of a solution of (1.4).

Finally, we prove Theorem 1.10:

Proof of Theorem 1.10. By Remark 2.9 and Remark 2.8, for every $\Omega' \Subset \Omega$ and every $t > 0$ we have:

$$\big(\mathrm{dist}(\nabla u_m, \overline{N}_t(D_G)) - \mathrm{dist}(\nabla u, \overline{N}_t(D_G))\big) \longrightarrow 0 \text{ in } L^1(\Omega')$$

and

$$\big(\nabla u_m \times \mathrm{dist}(\nabla u_m, \overline{N}_t(D_G)) - \nabla u \times \mathrm{dist}(\nabla u, \overline{N}_t(D_G))\big) \longrightarrow 0 \text{ in } L^1(\Omega')$$

when $m \to +\infty$. Thus, thanks to Proposition 5.1, the functions $\mathrm{dist}(\nabla u, \overline{N}_t(D_G))$ and $\nabla u \times \mathrm{dist}(\nabla u, \overline{N}_t(D_G))$ have a continuous representative σ_t and Σ_t respectively for every $t > 0$.

We introduce the following open subset of Ω, $\Omega_0 := \bigcup_{t>0}[\sigma_t > 0]$. Let us consider $x_0 \in \Omega_0$. Then there exist $t > 0$ and $\epsilon > 0$ such that $\sigma_t(x_0) > \epsilon$. By continuity of σ_t, there exists a neighborhood U of x_0 such that $\sigma_t \geq \frac{\epsilon}{2}$ on U. The continuity of Σ_t on U and the fact that $\sigma_t \geq \frac{\epsilon}{2}$ on U give that ∇u has a continuous representative on U. Hence, $\mathrm{dist}(\nabla u, D_G)$ has a continuous representative on U. Thus, ∇u and $\mathrm{dist}(\nabla u, D_G)$ have continuous representatives F_0 and σ_0 respectively on Ω_0. Let us extend σ_0 by 0 on $\Omega\backslash\Omega_0$. We claim that this function σ is continuous and coincides a.e. with $\mathrm{dist}(\nabla u, D_G)$. To prove the continuity, we assume that there exists $x \in \Omega\backslash\Omega_0$ and $(x_n)_{n\in\mathbb{N}}$ in Ω_0 a sequence converging to x such that $(\sigma(x_n))_{n\in\mathbb{N}}$ does not converge to 0. This means that we can extract a subsequence from $(\sigma(x_n))_{n\in\mathbb{N}}$, still denoted $(\sigma(x_n))_{n\in\mathbb{N}}$ such that $\sigma(x_n) \geq l > 0$ when n is large enough. Thus, $\sigma_{\frac{l}{2}}(x_n) \geq \frac{l}{2}$ for every n large enough. By continuity of $\sigma_{\frac{l}{4}}$, we obtain that $x \in \Omega_0$ which is a contradiction. Hence, σ is continuous. Moreover, for a.e. $x \in \Omega_0$, $\sigma(x) := \mathrm{dist}(\nabla u(x), D_G)$ and for a.e $x \in \Omega$ such that $\sigma(x) = 0$ we have that $x \in D_G$. Thus, σ is a representative of $\mathrm{dist}(\nabla u, D_G)$ and $F_0 \times \sigma$ is a continuous representative of $\nabla u \times \mathrm{dist}(\nabla u, D_G)$.

To conclude, we assume that G is constant on each connected components of D_G. By Remark 2.8, there exists a subsequence of $(\nabla u_m)_{m\in\mathbb{N}}$ still denoted $(\nabla u_m)_{m\in\mathbb{N}}$ such that $\nabla u_m \to \nabla u$ a.e. on Ω_0 when $m \to +\infty$.

Let us assume that $(x_n)_{n\in\mathbb{N}}$ is a sequence in Ω_0 converging to $x_0 \in \Omega\backslash\Omega_0$ when $n \to +\infty$. By Proposition 5.1, there exists $r > 0$ such that for every $m \in \mathbb{N}$, $\nabla u_m(B_r(x_0)) \subset \overline{N}_{3\rho_0}(D_G^m)$ with D_G^m a connected component of D_G. By convergence a.e. on Ω_0 of ∇u_m, we have that D_G^m is independent of m for m large enough. Let us call D_G^0 this connected component, by continuity of $\mathrm{dist}(\nabla u, D_G)$ we obtain that $\mathrm{dist}(\nabla u(x_n), D_G^0) \to 0$ when $n \to +\infty$. Since G is continuous on $\mathbb{R}^N$ and constant on D_G^0, we have that $G(\nabla u(x_n)) \to G(\nabla u(x_0))$ when $n \to +\infty$. □

Appendix A. Regularity of the gauge function

This section is dedicated to the convex gauge, or Minkowski functional. Most of the results presented here are classical, and the reader can find parts of the proofs in [24, Section 13] for instance. Let C be a bounded convex set of $\mathbb{R}^N$ such that its interior contains 0. We define the gauge associated to C as the following function:

$$\gamma_C(z) := \inf\left\{t > 0 \text{ such that } \frac{z}{t} \in C\right\}.$$

We have the following result about the regularity of γ_C:

Proposition A.1. *Let $k \in \mathbb{N}$ and $0 \leq \alpha \leq 1$. If C is a strictly convex bounded set of $\mathbb{R}^N$ of regularity $C^{k,\alpha}$ such that its interior contains the origin, then the gauge γ_C associated to C is in $C^{k,\alpha}_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$.*

Proof. The function γ_C is convex on $\mathbb{R}^N$. For every $z \in \mathbb{R}^N$ we compute the convex subdifferential $\partial\gamma_C(z)$ of γ_C at the point z. By definition of the subdifferential we have

$$\partial\gamma_C(z) := \{y \in \mathbb{R}^N \text{ such that } \gamma_C(z') \geq \gamma_C(z) + \langle y, z' - z\rangle \text{ for every } z' \in \mathbb{R}^N\}.$$

By homogeneity of γ_C, for every $y \in \partial\gamma_C(z)$ we get that $\gamma_C(z') \geq \langle y, z'\rangle$ for every $z' \in \mathbb{R}^N$. By taking $z' = 0$ we get that $\gamma_C(z) \leq \langle y, z\rangle$. Hence, we have that $\gamma_C(z) = \langle y, z\rangle$ for every $y \in \partial\gamma_C(z)$. Thus, for every $z \in \mathbb{R}^N$, we obtain that

$$\partial\gamma_C(z) = \left\{y \in \mathbb{R}^N \text{ such that } \gamma_C(z) = \langle y, z\rangle \text{ and } \gamma_C(z') \geq \langle y, z'\rangle \text{ for every } z' \in \mathbb{R}^N\right\}. \tag{A.1}$$

This convex set is not empty since γ_C is a convex continuous function. We claim that when $z \neq 0$, $\partial\gamma_C(z)$ is reduced to a singleton. In fact, if there exist y_1 and y_2 two different points of $\partial\gamma_C(z)$ then $\langle y_1, z\rangle = \langle y_2, z\rangle = \gamma_C(z)$ and $\langle y_1, z'\rangle \leq 1$, $\langle y_2, z'\rangle \leq 1$ for every $z' \in \partial C$. Hence, C is on one side of the hyperplane $\langle \xi, y_1\rangle = 1$, on one side of another hyperplane $\langle \xi, y_2\rangle = 1$ and $\frac{z}{\gamma_C(z)} \in \partial C$ is in their intersection. This contradicts the fact that C is at least C^1. Thus, for every $z \neq 0$, $\partial\gamma_C(z)$ contains only one vector. Hence, γ_C is differentiable at every $z \neq 0$. By homogeneity of γ_C we have that $\nabla\gamma_C(z)$ is positively colinear to $\nu_C(P_C(z))$ where ν_C is the unit outward normal vector to C and $P_C(z) := \frac{z}{\gamma_C(z)}$. By (A.1) we have that $\langle z, \nabla\gamma_C(z)\rangle = \gamma_C(z)$. Hence, $|\nabla\gamma_C(z)|\langle z, \nu_C(P_C(z))\rangle = \gamma_C(z)$ for every $z \neq 0$. Again by homogeneity of γ_C, for every $z \neq 0$ we obtain that

$$\nabla\gamma_C(z) = \frac{\nu_C(P_C(z))}{\langle \nu_C(P_C(z)), P_C(z)\rangle}. \tag{A.2}$$

This scalar product in the denominator is not 0 because C contains a small ball centered at 0, thus for every $z' \in \partial C$ the normal vector $\nu_C(z')$ cannot be orthogonal to z'.

With this expression of the gradient of γ_C we can find the regularity of γ_C. In fact, we know that ν_C is $C^{k-1,\alpha}$ continuous with $k \geq 1$. Since γ_C is Lipschitz continuous, the map P_C is locally Lipschitz continuous on $\mathbb{R}^N \setminus \{0\}$. Hence, $\nabla\gamma_C$ is $C^{0,\alpha}_{\mathrm{loc}}$ continuous on $\mathbb{R}^N \setminus \{0\}$. Thus, P_C is $C^{1,\alpha}_{\mathrm{loc}}$ continuous on $\mathbb{R}^N \setminus \{0\}$. By a bootstrap argument, we get that γ_C is $C^{k,\alpha}_{\mathrm{loc}}$ on $\mathbb{R}^N \setminus \{0\}$. □

We also prove a convexity result for the lower level sets of the convolution product of γ_C:

Proposition A.2. *Let C be a strictly convex bounded set of* $\mathbb{R}^N$ *such that its interior contains* 0. *If* $(\rho_m)_{m\in\mathbb{N}}$ *is a standard mollifying sequence, then the lower level sets of* $\gamma_C^m := \gamma_C * \rho_m$ *are strictly convex for every* $m \in \mathbb{N}$.

Proof. Let us consider $z_1 \neq z_2$ on the boundary of a lower level set of γ_C^m. By continuity of γ_C^m, $\gamma_C^m(z_1) = \gamma_C^m(z_2) =: s$. Then for every $0 < t < 1$ we have:

$$\gamma_C^m(tz_1 + (1-t)z_2) = \int_{B_{\frac{1}{m}}(0)} \gamma_C(t(z_1 - y) + (1-t)(z_2 - y))\rho_m(y)dy.$$

By convexity of γ_C, we have

$$\gamma_C(t(z_1 - y) + (1-t)(z_2 - y)) \leq \gamma_C(z_1 - y) + (1-t)\gamma_C(z_2 - y). \tag{A.3}$$

If $\gamma_C(z_1 - y) = \gamma_C(z_2 - y)$ by strict convexity of C we get that

$$\gamma_C(t(z_1 - y) + (1-t)(z_2 - y)) < t\gamma_C(z_1 - y) + (1-t)\gamma_C(z_2 - y).$$

Hence, if we have equality in (A.3) this means that $z_1 - y$ and $z_2 - y$ are colinear. Since $z_1 \neq z_2$, for a.e. $y \in B_{\frac{1}{m}}(0)$ $z_1 - y$ and $z_2 - y$ are not colinear. Thus, $\gamma_C^m(t(z_1) + (1-t)(z_2)) < t\gamma_C^m(z_1) + (1-t)\gamma_C^m(z_2)$. This provides the strict convexity of the lower level sets of γ_C^m for every $m \in \mathbb{N}$. □

We prove that the approximations of a $C^{1,1}$ strictly convex set through convolutions of γ_C are also $C^{1,1}$ with a uniform norm.

Proposition A.3. *Let C be a* $C^{1,1}$ *strictly convex bounded set of* $\mathbb{R}^N$ *such that its interior contains* 0. *We consider* $\gamma_C^m := \gamma_C * \rho_m$ *with* $(\rho_m)_{m\in\mathbb{N}}$ *a standard mollifying sequence and* $r_m \to 1$ *when* $m \to +\infty$ *such that* $C_m := (\gamma_C^m)^{-1}(\{[0, r_m)\})$ *is a smooth convex set containing* 0. *Then* C_m *is a* $C^{1,1}$ *strictly convex set of* $\mathbb{R}^N$ *and the Lipschitz constant of its outward normal vector can be bounded uniformly in* $m \in \mathbb{N}$.

Proof. By Proposition A.2 the set C_m is strictly convex. Since $(r_m)_{m\in\mathbb{N}}$ converges to 1 and $(\gamma_C^m)_{m\in\mathbb{N}}$ converges uniformly to γ_C on $\mathbb{R}^N$ when $m \to +\infty$ we have that $\lim_{m\to+\infty} \mathrm{dist}(\partial C_m, \partial C) = 0$. Thanks to this last result and the fact that 0 is in the interior of C we can find $r > 0$ such that $B_r(0)$ is in C_m for every m large enough. Since C_m is a level set of γ_C^m for every $z \in \partial C^m$ we have that $\nu_{C_m}(z) = \frac{\nabla\gamma_C^m(z)}{|\nabla\gamma_C^m(z)|}$. By Proposition A.1 and (A.2) the function $\nabla\gamma_C^m := \nabla\gamma_C * \rho_m$ is uniformly Lipschitz continuous on $\mathbb{R}^N \backslash B_r(0)$. Moreover, there exists $\kappa > 0$ such that $|\nabla\gamma_C^m(z)| \geq \kappa$ for every $m \in \mathbb{N}$ large enough according to $r > 0$ and C.

Hence, ν_{C_m} is Lipschitz continuous for every $m \in \mathbb{N}$ with a Lipschitz constant independent of $m \in \mathbb{N}$. □

Acknowledgments

I warmly thank Xavier Lamy for very interesting questions, discussions and ideas on this subject.

References

[1] Gabriele Anzellotti and Mariano Giaquinta. Convex functionals and partial regularity. *Arch. Ration. Mech. Anal.*, 102(3):243–272, 1988.

[2] Benny Avelin, Tuomo Kuusi, and Giuseppe Mingione. Nonlinear Calderón-Zygmund theory in the limiting case. *Arch. Ration. Mech. Anal.*, 227(2):663–714, 2018.

[3] John M. Ball. A version of the fundamental theorem for Young measures. In *PDEs and continuum models of phase transitions (Nice, 1988)*, volume 344 of *Lecture Notes in Physics*, pages 207–215. Springer, 1989.

[4] Verena Bögelein, Frank Duzaar, Raffaella Giova, and Antonia Passarelli di Napoli. Higher regularity in congested traffic dynamics. *Math. Ann.*, 385(3-4):1823–1878, 2023.

[5] Pierre Bousquet. Another look to the orthotropic functional in the plane. In *"Bruno Pini" Mathematical Analysis Seminar 2020*, pages 1–29. Università di Bologna, Alma Mater Studiorum, 2021.

[6] Pierre Bousquet and Lorenzo Brasco. C^1 regularity of orthotropic p-harmonic functions in the plane. *Anal. PDE*, 11(4):813–854, 2018.

[7] Lorenzo Brasco. Global L^∞ gradient estimates for solutions to a certain degenerate elliptic equation. *Nonlinear Anal., Theory Methods Appl.*, 74(2):516–531, 2011.

[8] Carsten Carstensen and Stefan Müller. Local stress regularity in scalar nonconvex variational problems. *SIAM J. Math. Anal.*, 34(2):495–509, 2002.

[9] Pietro Celada, Giovanni Cupini, and Marcello Guidorzi. Existence and regularity of minimizers of nonconvex integrals with $p-q$ growth. *ESAIM, Control Optim. Calc. Var.*, 13(2):343–358, 2007.

[10] Arrigo Cellina. Uniqueness and comparison results for functionals depending on ∇u and on u. *SIAM J. Optim.*, 18(3):711–716, 2007.

[11] Andrea Cianchi and Vladimir G. Maz'ya. Second-order two-sided estimates in nonlinear elliptic problems. *Arch. Ration. Mech. Anal.*, 229(2):569–599, 2018.

[12] Andrea Cianchi and Vladimir G. Maz'ya. Optimal second-order regularity for the p-Laplace system. *J. Math. Pures Appl.*, 132:41–78, 2019.

[13] Francis Clarke. *Functional analysis, calculus of variations and optimal control*, volume 264 of *Graduate Texts in Mathematics*. Springer, 2013.

[14] Maria Colombo and Alessio Figalli. Regularity results for very degenerate elliptic equations. *J. Math. Pures Appl.*, 101(1):94–117, 2014.

[15] Daniela de Silva and Ovidiu Savin. Minimizers of convex functionals arising in random surfaces. *Duke Math. J.*, 151(3):487–532, 2010.

[16] Luca Esposito, Giuseppe Mingione, and Cristina Trombetti. On the Lipschitz regularity for certain elliptic problems. *Forum Math.*, 18(2):263–292, 2006.

[17] Gerald B. Folland. *Introduction to partial differential equations*. Princeton University Press, second edition, 1995.

[18] Irene Fonseca, Nicola Fusco, and Paolo Marcellini. An existence result for a nonconvex variational problem via regularity. *ESAIM, Control Optim. Calc. Var.*, 7:69–95, 2002.

[19] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer, 2001. Reprint of the 1998 edition.

[20] Enrico Giusti. *Minimal surfaces and functions of bounded variation*, volume 80 of *Monographs in Mathematics*. Birkhäuser, 1984.

[21] Umberto Guarnotta and Sunra Mosconi. A general notion of uniform ellipticity and the regularity of the stress field for elliptic equations in divergence form. *Anal. PDE*, 16(8):1955–1988, 2023.

[22] Philip Hartman and Louis Nirenberg. On spherical image maps whose Jacobians do not change sign. *Am. J. Math.*, 81:901–920, 1959.

[23] Bernhard Kawohl, Jana Stara, and Gabriel Wittum. Analysis and numerical studies of a problem of shape design. *Arch. Ration. Mech. Anal.*, 114(4):349–363, 1991.

[24] Andreas Kriegl and Peter W. Michor. *The convenient setting of global analysis*, volume 53 of *Mathematical Surveys and Monographs*. American Mathematical Society, 1997.

[25] Henri Lebesgue. Sur le problème de dirichlet. *Rend. Circ. Mat. Palermo*, 24:371–402, 1907.

[26] Peter Lindqvist and Diego Ricciotti. Regularity for an anisotropic equation in the plane. *Nonlinear Anal., Theory Methods Appl.*, 177:628–636, 2018.

[27] Benjamin Lledos. A uniqueness result for a translation invariant problem in the calculus of variations. *J. Convex Anal.*, 31(1):121–130, 2024.

[28] Connor Mooney. Minimizers of convex functionals with small degeneracy set. *Calc. Var. Partial Differ. Equ.*, 59(2): article no. 74 (19 pages), 2020.

[29] Diego Ricciotti. Regularity of the derivatives of p-orthotropic functions in the plane for $1 < p < 2$. *Ann. Acad. Sci. Fenn., Math.*, 44(2):1093–1099, 2019.

[30] Filippo Santambrogio and Vincenzo Vespri. Continuity in two dimensions for a very degenerate elliptic equation. *Nonlinear Anal., Theory Methods Appl.*, 73(12):3832–3841, 2010.

Benjamin Lledos
Université de Toulouse
Institut de Mathématiques de Toulouse
CNRS UMR 5219
31062 Toulouse Cedex 9, France
benjamin.lledos@math.univ-toulouse.fr