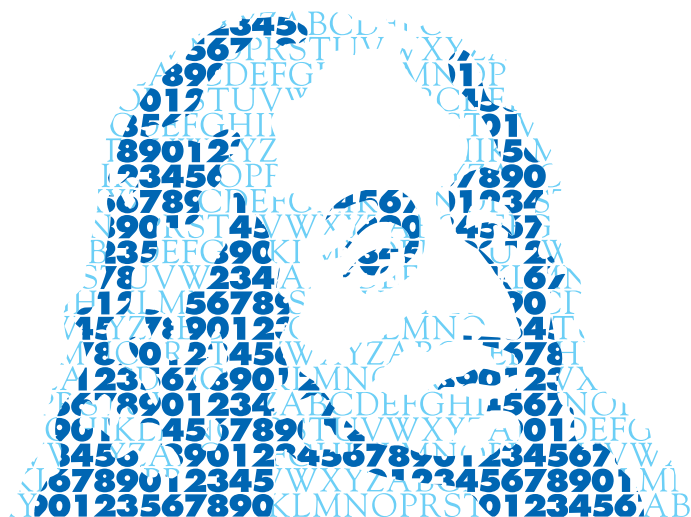


# ANNALES MATHÉMATIQUES



## BLAISE PASCAL

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**Integral foliated simplicial volume and ergodic decomposition**

Volume 31, n° 1 (2024), p. 47-64.

<https://doi.org/10.5802/ambp.425>



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*Publication éditée par le laboratoire de mathématiques Blaise Pascal  
de l'université Clermont Auvergne, UMR 6620 du CNRS  
Clermont-Ferrand — France*



Publication membre du  
Centre Mersenne pour l'édition scientifique ouverte

<http://www.centre-mersenne.org/>

e-ISSN : 2118-7436

# Integral foliated simplicial volume and ergodic decomposition

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## Abstract

We establish an integration formula for integral foliated simplicial volume along ergodic decompositions. This is analogous to the ergodic decomposition formula for the cost of groups.

## *Volume simplicial feuilleté intégral et décomposition ergodique*

## Résumé

Nous établissons une formule d'intégration pour le volume simplicial feuilleté intégral sur des décompositions ergodiques. Celle-ci est analogue à la formule de décomposition ergodique pour le coût des groupes.

## 1. Introduction

The integral foliated simplicial volume is a dynamical version of the simplicial volume of manifolds: It measures the size of fundamental cycles of a manifold  $M$  with respect to twisted coefficients in  $L^\infty(X, \mu, \mathbb{Z})$ , where  $\pi_1(M) \curvearrowright (X, \mu)$  is a probability measure preserving action on a standard Borel probability space (see Section 2 for the definitions). The integral foliated simplicial volume provides upper bounds for the  $L^2$ -Betti numbers [7, p. 305f][17] and the cost of the fundamental group [13].

In the case of a residually finite fundamental group, a dynamical system of particular interest is the profinite completion. The cost of the action on the profinite completion is the rank gradient [1]. Analogously, the integral foliated simplicial volume with respect to the profinite completion equals the stable integral simplicial volume [15, Remark 6.7]. The stable integral simplicial volume is one of the few known upper bounds for logarithmic torsion homology growth [5, Theorem 1.6]. However, as in the fixed price problem for the cost of groups, it is unknown in general whether different essentially free dynamical systems can lead to different values for the same manifold; in particular, which dynamical systems can be used for logarithmic torsion growth estimates?

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This work was supported by the CRC 1085 *Higher Invariants* (Universität Regensburg, funded by the DFG) and is partially based on GS's MSc project.

*Keywords:* Integral foliated simplicial volume, ergodic decomposition.

*2020 Mathematics Subject Classification:* 55N10, 55N35, 28D15.

Following the theory for cost [8, Corollary 10.14], it has been shown that the integral foliated simplicial volume is monotonic with respect to weak containment of dynamical systems [5, Theorem 1.5] and that for several classes of manifolds all essentially free dynamical systems lead to the same value [5, Theorem 1.9][3][14].

In the present paper, in analogy with the ergodic decomposition formula for the cost of groups [10, Proposition 18.4], we show in Section 4:

**Theorem 1.1.** *Let  $M$  be an oriented closed connected manifold with fundamental group  $\Gamma$ , let  $(\alpha, \mu): \Gamma \curvearrowright X$  be a standard probability action, and let  $\beta: X \rightarrow \text{Erg}(\alpha)$  be an ergodic decomposition of  $(\alpha, \mu)$ . Then*

$$|M|^{(\alpha, \mu)} = \int_X |M|^{(\alpha, \beta_x)} d\mu(x).$$

In particular, there exists an ergodic parameter space that realises the integral foliated simplicial volume:

**Corollary 1.2.** *Let  $M$  be an oriented closed connected manifold with fundamental group  $\Gamma$ . Then there exists an essentially free ergodic standard  $\Gamma$ -space  $(\alpha, \mu)$  with*

$$|M| = |M|^{(\alpha, \mu)}.$$

*Proof.* Taking products of standard actions shows that there is an essentially free standard probability action  $(\alpha, \mu): \Gamma \curvearrowright X$  with  $|M| = |M|^{(\alpha, \mu)}$  [15, Corollary 4.14]. Let  $\beta: X \rightarrow \text{Erg}(\alpha)$  be an ergodic decomposition for  $(\alpha, \mu)$ , as provided by the ergodic decomposition theorem [19] (Theorem 3.5). The ergodic decomposition formula for integral foliated simplicial volume (Theorem 1.1) gives

$$|M| = |M|^{(\alpha, \mu)} = \int_X |M|^{(\alpha, \beta_x)} d\mu(x).$$

Moreover, by definition,  $|M|^{(\alpha, \beta_x)} \geq |M|$  for all  $x \in X$ . Therefore, we obtain

$$|M| = |M|^{(\alpha, \beta_x)}$$

for  $\mu$ -almost every  $x \in X$ . As  $\alpha$  is essentially free, there also exists an  $x \in X$  such that the  $\Gamma$ -action on  $X$  is essentially free with respect to  $\beta_x$  and simultaneously  $|M| = |M|^{(\alpha, \beta_x)}$  is satisfied (Remark 3.7). By construction,  $\beta_x$  is ergodic.  $\square$

For the proof of Theorem 1.1, we first show that for each standard action  $\alpha: \Gamma \curvearrowright X$ , there is a countable subcomplex  $\Sigma_*(M, X; \mathbb{Z})$  of the (strict) chain complex  $B(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\tilde{M}; \mathbb{Z})$  with the following property: For every  $\Gamma$ -invariant probability measure  $\nu$  on  $X$ , we have

$$|M|^{(\alpha, \nu)} = \inf \{ |c|^{(\alpha, \nu)} \mid c \in \Sigma_*(M, X; \mathbb{Z}) \text{ is a fundamental cycle} \}.$$

The ergodic decomposition formula can then be viewed as an instance of switching this particular infimum with integration.

A weaker result relating integral foliated simplicial volume and ergodic parameter spaces was already known:

**Proposition 1.3** ([15, Proposition 4.17]). *Let  $M$  be an oriented closed connected manifold with fundamental group  $\Gamma$ . Then, for every  $\varepsilon \in \mathbb{R}_{>0}$ , there exists an ergodic standard  $\Gamma$ -space  $(\alpha, \mu)$  such that*

$$|M|^{(\alpha, \mu)} \leq |M| + \varepsilon.$$

The following terminology is borrowed from the theory of cost [6]:

**Definition 1.4** (fixed price [13, Definition 1.3]). Let  $M$  be an oriented closed connected manifold; we say that  $M$  has *fixed price* if  $|M|^{(\alpha, \mu)} = |M|^{(\alpha', \mu')}$  holds for all essentially free standard  $\Gamma$ -spaces  $(\alpha, \mu)$  and  $(\alpha', \mu')$ .

As in the case of cost the fixed price problem for manifolds is still open:

**Question 1.5** (fixed price problem [5, Question 1.13]). *Do all oriented closed connected manifolds have fixed price?*

## 1.1. Organisation of this paper

We recall the definition of integral foliated simplicial volume in Section 2; ergodic decompositions are recalled in Section 3. The proof of Theorem 1.1 is given in Section 4.

## 2. Integral foliated simplicial volume

Parametrised simplicial volume and integral foliated simplicial volume arise as a dynamical generalisation of integral and real simplicial volume: One replaces the constant integral/real coefficients with twisted coefficients of spaces of (essentially) bounded integer-valued functions [7, p. 305f][17].

### 2.1. Basic definitions

A *standard Borel space* is a measurable space that is isomorphic to a Polish space together with its Borel  $\sigma$ -algebra [9].

**Definition 2.1** (standard (probability) actions and bounded functions).

- A *standard action* is a measurable action of a countable group on a standard Borel space.

- A *standard probability action* is a pair  $(\alpha, \mu)$ , where  $\alpha: \Gamma \curvearrowright X$  is a standard action and where  $\mu$  is an  $\alpha$ -invariant probability measure on  $X$ .
- If  $(\alpha, \mu): \Gamma \curvearrowright X$  is a standard probability action, then we equip the  $\mathbb{Z}$ -module  $L^\infty(X, \mu, \mathbb{Z})$  of  $\mu$ -equivalence classes of measurable  $\mu$ -essentially bounded functions  $X \rightarrow \mathbb{Z}$  with the  $\mathbb{Z}\Gamma$ -module structure

$$\begin{aligned} L^\infty(X, \mu, \mathbb{Z}) \times \Gamma &\longrightarrow L^\infty(X, \mu, \mathbb{Z}) \\ (f, \gamma) &\longmapsto (x \mapsto f(\gamma \cdot x)). \end{aligned}$$

The resulting  $\mathbb{Z}\Gamma$ -module is denoted by  $L^\infty(\alpha, \mu, \mathbb{Z})$ .

In the following, in the notation of functions spaces etc. we will always use “ $\alpha$ ” instead of “ $X$ ” to emphasise the underlying action instead of the underlying measure space.

Let  $M$  be a connected manifold with fundamental group  $\Gamma$  and universal covering  $\pi: \tilde{M} \rightarrow M$ . If  $A$  is a right  $\mathbb{Z}\Gamma$ -module, then we denote the twisted singular chain complex and the twisted singular homology of  $M$  with coefficients in  $A$  by (where  $\Gamma$  acts by deck transformations on the singular simplices of  $\tilde{M}$ ):

$$\begin{aligned} C_*(M; A) &:= A \otimes_{\mathbb{Z}\Gamma} C_*(\tilde{M}; \mathbb{Z}) \\ H_*(M; A) &:= H_*(C_*(M; A)). \end{aligned}$$

For the constant coefficients  $\mathbb{Z}$ , the universal covering map induces a  $\mathbb{Z}$ -chain isomorphism between the untwisted singular chain complex of  $M$  with  $\mathbb{Z}$ -coefficients and  $\mathbb{Z} \otimes_{\mathbb{Z}\Gamma} C_*(\tilde{M}; \mathbb{Z})$ . We will always use this identification for  $C_*(M; \mathbb{Z})$ .

**Definition 2.2** (parametrised fundamental cycles). Let  $M$  be an oriented closed connected manifold with fundamental group  $\Gamma$ , let  $(\alpha, \mu): \Gamma \curvearrowright X$  be a standard probability action. We write

$$i_M^{(\alpha, \mu)}: C_*(M; \mathbb{Z}) \longrightarrow C_*(M; L^\infty(\alpha, \mu, \mathbb{Z}))$$

for the chain map induced by the inclusion of  $\mathbb{Z}$  into  $L^\infty(\alpha, \mu, \mathbb{Z})$  as constant functions. All cycles in  $C_*(M; L^\infty(\alpha, \mu, \mathbb{Z}))$  representing

$$[M]^{(\alpha, \mu)} := H_*(i_M^{(\alpha, \mu)})([M]_{\mathbb{Z}}) \in H_*(M; L^\infty(\alpha, \mu, \mathbb{Z}))$$

are called  $(\alpha, \mu)$ -*parametrised fundamental cycles of  $M$* .

**Definition 2.3** (integral foliated simplicial volume). Let  $M$  be an oriented closed connected manifold with fundamental group  $\Gamma$  and let  $(\alpha, \mu): \Gamma \curvearrowright X$  be a standard probability action.

- A chain  $\sum_{j=1}^m f_j \otimes \sigma_j \in C_*(M; L^\infty(\alpha, \mu, \mathbb{Z}))$  is in *reduced form* if for all  $j, k \in \{1, \dots, m\}$  with  $j \neq k$  we have that  $\pi \circ \sigma_j \neq \pi \circ \sigma_k$ . In other words the singular simplices  $\sigma_1, \dots, \sigma_m$  of  $\widetilde{M}$  arise from different simplices in  $M$ . Reduced forms are essentially unique (up to the  $\Gamma$ -action on the simplices).
- Let  $c = \sum_{j=1}^m f_j \otimes \sigma_j \in C_*(M; L^\infty(\alpha, \mu, \mathbb{Z}))$  be in reduced form. The  $(\alpha, \mu)$ -parametrised  $\ell^1$ -norm of  $c$  is

$$|c|^{(\alpha, \mu)} = \sum_{j=1}^m \int_X |f_j| \, d\mu \in \mathbb{R}_{\geq 0}.$$

- The  $(\alpha, \mu)$ -parametrised simplicial volume of  $M$  is the infimum
 
$$|M|^{(\alpha, \mu)} := \inf \left\{ |c|^{(\alpha, \mu)} \mid c \in C_*(M; L^\infty(\alpha, \mu, \mathbb{Z})) \text{ is an } (\alpha, \mu)\text{-parametrised fundamental cycle} \right\}.$$
- The *integral foliated simplicial volume*  $|M|$  of  $M$  is the infimum of the  $|M|^{(\alpha, \mu)}$  over all isomorphism classes of standard probability actions  $(\alpha, \mu) : \Gamma \curvearrowright X$ .

If  $\zeta \in H_*(M; L^\infty(\alpha, \mu, \mathbb{Z}))$ , then we denote

$$|\zeta|^{(\alpha, \mu)} := \inf \{ |c|^{(\alpha, \mu)} \mid c \in C_*(M; L^\infty(\alpha, \mu, \mathbb{Z})) \text{ and } [c] = \zeta \}$$

so that we can express the  $(\alpha, \mu)$ -parametrised simplicial volume of  $M$  as  $||M|^{(\alpha, \mu)}|^{(\alpha, \mu)}$ .

**Proposition 2.4** (comparison with integral and real simplicial volume [15, Proposition 4.6]). *Let  $M$  be an oriented closed connected manifold with fundamental group  $\Gamma$ . For every standard probability  $\Gamma$ -action  $(\alpha, \mu)$ , we have*

$$\|M\|_{\mathbb{R}} \leq |M| \leq |M|^{(\alpha, \mu)} \leq \|M\|_{\mathbb{Z}},$$

where  $\|M\|_{\mathbb{R}}$  and  $\|M\|_{\mathbb{Z}}$  denote the real and integral simplicial volume, respectively.

Computations of integral foliated simplicial volume have been performed for various oriented closed connected aspherical manifolds [2, 3, 4, 5, 14, 15, 17].

## 2.2. A strict version

For the proof of Theorem 1.1, we will need to consider chains and norms with respect to different probability measures on the same measurable action. In order to avoid complications caused by sets of measure 0 with respect to different measures, we explain how to compute the integral foliated simplicial volume via strict chains.

**Definition 2.5** (bounded functions). If  $\alpha: \Gamma \curvearrowright X$  is a standard action, we equip the  $\mathbb{Z}$ -module  $B(X, \mathbb{Z})$  of measurable bounded functions  $X \rightarrow \mathbb{Z}$  with the induced right  $\mathbb{Z}\Gamma$ -module structure given by

$$\begin{aligned} B(X, \mathbb{Z}) \times \Gamma &\longrightarrow B(X, \mathbb{Z}) \\ (f, \gamma) &\longmapsto (x \mapsto f(\gamma \cdot x)). \end{aligned}$$

This  $\mathbb{Z}\Gamma$ -module is denoted by  $B(\alpha, \mathbb{Z})$ .

If  $(\alpha, \mu): \Gamma \curvearrowright X$  is a standard probability action, then there is a canonical isomorphism  $L^\infty(\alpha, \mu, \mathbb{Z}) \cong B(\alpha, \mathbb{Z})/N(\alpha, \mu, \mathbb{Z})$  of  $\mathbb{Z}\Gamma$ -modules, where  $N(\alpha, \mu, \mathbb{Z}) \subset B(\alpha, \mathbb{Z})$  is the  $\mathbb{Z}\Gamma$ -submodule of functions that are  $\mu$ -almost everywhere 0. Moreover, as bounded functions to  $\mathbb{Z}$  only take on finitely many different values,  $B(\alpha, \mathbb{Z})$  is generated as a  $\mathbb{Z}$ -module by the set  $\{\chi_A \mid A \subset X \text{ measurable}\}$ .

**Definition 2.6** (parametrised strict fundamental cycles). Let  $M$  be an oriented closed connected manifold with fundamental group  $\Gamma$  and let  $\alpha: \Gamma \curvearrowright X$  be a standard action. We write

$$j_M^\alpha: C_*(M; \mathbb{Z}) \longrightarrow C_*(M; B(\alpha, \mathbb{Z}))$$

for the chain map induced by the inclusion of  $\mathbb{Z}$  into  $B(\alpha, \mathbb{Z})$  as constant functions. All cycles in  $C_*(M; B(\alpha, \mathbb{Z}))$  representing

$$H_*(j_M^\alpha)([M]_{\mathbb{Z}}) \in H_*(M; B(\alpha, \mathbb{Z}))$$

are called  $\alpha$ -parametrised strict fundamental cycles of  $M$ .

In the same way as for  $L^\infty$ -coefficients, we can introduce a notion of chains to be in reduced form in  $C_*(M; B(\alpha, \mathbb{Z}))$ .

**Definition 2.7** ( $\ell^1$ -norm for strict chains). Let  $M$  be an oriented closed connected manifold with fundamental group  $\Gamma$  and let  $(\alpha, \mu): \Gamma \curvearrowright X$  be a standard probability action. If  $c = \sum_{j=1}^m f_j \otimes \sigma_j \in C_*(M; B(\alpha, \mathbb{Z}))$  is in reduced form, then we define

$$|c|^{(\alpha, \mu)} := \sum_{j=1}^m \int_X |f_j| \, d\mu \in \mathbb{R}_{\geq 0}.$$

**Proposition 2.8** (integral foliated simplicial volume via strict chains). *Let  $M$  be an oriented closed connected manifold with fundamental group  $\Gamma$  and let  $(\alpha, \mu): \Gamma \curvearrowright X$  be a standard probability action. Then*

$$|M|^{(\alpha, \mu)} = \inf\{|c|^{(\alpha, \mu)} \mid c \in C_*(M; B(\alpha, \mathbb{Z})) \text{ is a strict fundamental cycle}\}.$$

*Proof.* The canonical map  $\varphi: \mathbf{B}(\alpha, \mathbb{Z}) \rightarrow L^\infty(\alpha, \mu, \mathbb{Z})$  acts as identity on constant functions. The induced chain map

$$\Phi_*: C_*(M; \mathbf{B}(\alpha, \mathbb{Z})) \longrightarrow C_*(M; L^\infty(\alpha, \mu, \mathbb{Z}))$$

satisfies  $i_M^{(\alpha, \mu)} = \Phi_* \circ j_M^\alpha$  and thus maps  $\alpha$ -parametrised strict fundamental cycles (in reduced form) to  $(\alpha, \mu)$ -parametrised fundamental cycles (in reduced form). Moreover,  $\Phi_*$  is isometric with respect to  $|\cdot|^{(\alpha, \mu)}$ . Therefore, the estimate “ $\leq$ ” of the claim holds.

For the converse estimate, we argue via the approximation of boundaries: Let  $n := \dim M$  and let  $c \in C_n(M; L^\infty(\alpha, \mu, \mathbb{Z}))$  be an  $(\alpha, \mu)$ -parametrised fundamental cycle. It suffices to find an  $\alpha$ -parametrised strict fundamental cycle having norm at most  $|c|^{(\alpha, \mu)}$ . By definition, there exist a fundamental cycle  $c_{\mathbb{Z}} \in C_*(M; \mathbb{Z})$  of  $M$  and a chain  $d \in C_{n+1}(M; L^\infty(\alpha, \mu, \mathbb{Z}))$  with

$$c = c_{\mathbb{Z}} + \partial d \in C_n(M; L^\infty(\alpha, \mu, \mathbb{Z})).$$

With  $\varphi$  also  $\Phi_*$  is surjective in every degree. Let  $\tilde{d} \in C_{n+1}(M; \mathbf{B}(\alpha, \mathbb{Z}))$  be a chain with  $\Phi_{n+1}(\tilde{d}) = d$ . For the strict chain

$$\tilde{c} := c_{\mathbb{Z}} + \partial \tilde{d} \in C_n(M; \mathbf{B}(\alpha, \mathbb{Z}))$$

we obtain the estimate

$$\begin{aligned} |\tilde{c}|^{(\alpha, \mu)} &= |\Phi_n(\tilde{c})|^{(\alpha, \mu)} && (\Phi_* \text{ is isometric}) \\ &= |c_{\mathbb{Z}} + \Phi_n(\partial \tilde{d})|^{(\alpha, \mu)} && (i_M^{(\alpha, \mu)} = \Phi_* \circ j_M^\alpha) \\ &= |c_{\mathbb{Z}} + \partial \Phi_{n+1}(\tilde{d})|^{(\alpha, \mu)} && (\Phi_* \text{ is a chain map}) \\ &= |c_{\mathbb{Z}} + \partial d|^{(\alpha, \mu)} && (\Phi_{n+1}(\tilde{d}) = d) \\ &= |c|^{(\alpha, \mu)} \end{aligned}$$

By construction,  $\tilde{c}$  is an  $\alpha$ -parametrised strict fundamental cycle of  $M$ . This completes the proof of the estimate “ $\geq$ ” of the claim.  $\square$

*Remark 2.9* (decomposition into invariant subspaces). Let  $M$  be an oriented closed connected  $n$ -manifold, let  $\Gamma := \pi_1(M)$ , and let  $\alpha: \Gamma \curvearrowright X$  be a standard action. Let  $A \subset X$  be a measurable subset with  $\Gamma \cdot A = A$  and let  $\bar{A} := X \setminus A$ ; then the restrictions  $\alpha|_A: \Gamma \curvearrowright A$  and  $\alpha|_{\bar{A}}: \Gamma \curvearrowright \bar{A}$  are standard actions. If  $c, \bar{c} \in C_*(M; \mathbf{B}(\alpha, \mathbb{Z}))$  are fundamental cycles, then also

$$\chi_A \cdot c + \chi_{\bar{A}} \cdot \bar{c} \in C_*(M; \mathbf{B}(\alpha, \mathbb{Z}))$$

is a fundamental cycle. Here,  $\chi_A \cdot c$  and  $\chi_{\bar{A}} \cdot \bar{c}$  come from the  $\mathbf{B}(\alpha, \mathbb{Z})^\Gamma$ - $\mathbb{Z}\Gamma$ -bimodule structure on  $\mathbf{B}(\alpha, \mathbb{Z})$ ; more explicitly, if  $c = \sum_{i=1}^k f_i \otimes \sigma_i$  is a strict chain, then  $\chi_A \cdot c$



denotes the strict chain  $\sum_{i=1}^k (\chi_A \cdot f_i) \otimes \sigma_i = \sum_{i=1}^k f_i|_A \otimes \sigma_i$ ; this is well-defined because  $A$  satisfies  $\Gamma \cdot A = A$ . Indeed the mutually inverse isomorphisms

$$\begin{aligned} \mathbf{B}(\alpha, \mathbb{Z}) &\longleftrightarrow \mathbf{B}(\alpha|_A, \mathbb{Z}) \oplus \mathbf{B}(\alpha|_{\bar{A}}, \mathbb{Z}) \\ f &\longmapsto (f|_A, f|_{\bar{A}}) \\ g|_A + h|_{\bar{A}} &\longleftarrow (g, h) \end{aligned}$$

of  $\mathbb{Z}\Gamma$ -modules induce an isomorphism

$$C_*(M; \mathbf{B}(\alpha, \mathbb{Z})) \cong C_*(M; \mathbf{B}(\alpha|_A, \mathbb{Z})) \oplus C_*(M; \mathbf{B}(\alpha|_{\bar{A}}, \mathbb{Z}))$$

of  $\mathbb{Z}$ -chain complexes that is compatible with the inclusions of  $C_*(M; \mathbb{Z})$  as constant chains. Hence, the  $\alpha$ -parametrised strict fundamental cycles of  $M$  correspond to pairs of  $\alpha|_A$ - and  $\alpha|_{\bar{A}}$ -parametrised strict fundamental cycles of  $M$ . In particular,  $\chi_A \cdot c + \chi_{\bar{A}} \cdot \bar{c}$  is an  $\alpha$ -parametrised strict fundamental cycle of  $M$ .

### 3. Ergodic decomposition

We quickly recall a version of the ergodic decomposition theorem, introduce notation, and collect some basic properties that will be used in Section 4.

**Definition 3.1** (ergodicity). A standard probability action  $(\alpha, \mu): \Gamma \curvearrowright X$  is *ergodic* if for every measurable subset  $A \subset X$  with  $\Gamma \cdot A = A$ , we have

$$\mu(A) = 0 \quad \text{or} \quad \mu(A) = 1.$$

**Definition 3.2** (spaces of measures). Let  $\alpha: \Gamma \curvearrowright X$  be a standard action;

- (1) We denote the set of probability measures on  $X$  by  $\text{Prob}(X)$ ;
- (2) We write  $\text{Prob}(\alpha) \subset \text{Prob}(X)$  for the subset of all probability measures  $\mu$  on  $X$  for which the action  $\alpha$  is  $\mu$ -preserving.
- (3) We write  $\text{Erg}(\alpha) \subset \text{Prob}(\alpha)$  for the subset of all  $\alpha$ -invariant probability measures that are ergodic.

**Definition 3.3** (ergodic decomposition). An *ergodic decomposition* of a standard probability action  $(\alpha, \mu): \Gamma \curvearrowright X$  is a map  $\beta: X \rightarrow \text{Erg}(\alpha)$  (and we will write  $\beta_x$  for  $\beta(x)$ ) with the following properties:

- (1) For every measurable subset  $A \subset X$  the evaluation map

$$\begin{aligned} X &\longrightarrow [0, 1] \\ x &\longmapsto \beta_x(A) \end{aligned}$$

is measurable;

- (2) For every measurable subset  $A \subset X$ , we have  $\mu(A) = \int_X \beta_x(A) \, d\mu(x)$ ;
- (3) For all  $x \in X$  and for all  $\gamma \in \Gamma$ , we have  $\beta_{\gamma \cdot x} = \beta_x$ ;
- (4) For every  $\nu \in \text{Erg}(\alpha)$ , the preimage  $X_\nu := \beta^{-1}(\nu)$  is measurable and  $\nu(X_\nu) = 1$ .

**Proposition 3.4** (integrals and ergodic decomposition). *Let  $(\alpha, \mu): \Gamma \curvearrowright X$  be a standard probability action and let  $\beta: X \rightarrow \text{Erg}(\alpha)$  be an ergodic decomposition of  $(\alpha, \mu)$ . For every  $f \in B(\alpha, \mathbb{Z})$ , the function*

$$\begin{aligned} X &\longrightarrow \mathbb{R} \\ x &\longmapsto \int_X f \, d\beta_x \end{aligned}$$

is measurable and

$$\int_X f \, d\mu = \int_X \left( \int_X f \, d\beta_x \right) d\mu(x).$$

*Proof.* Members of  $B(\alpha, \mathbb{Z})$  are  $\mathbb{Z}$ -linear combinations of characteristic functions of measurable subsets of  $X$ . Therefore, linearity of integration reduces the claim to the definition of ergodic decomposition.  $\square$

A standard probability action always admits an ergodic decomposition. We can even get a stronger existence result: If a standard action  $\alpha: \Gamma \curvearrowright X$  admits at least one invariant probability measure (i.e.,  $\text{Prob}(\alpha) \neq \emptyset$ ), then there exists a *universal ergodic decomposition*, namely a function  $\beta: X \rightarrow \text{Erg}(\alpha)$  that is an ergodic decomposition for every standard probability action  $(\alpha, \mu): \Gamma \curvearrowright X$ .

**Theorem 3.5** (ergodic decomposition theorem [19, Section 4]). *Let  $\alpha: \Gamma \curvearrowright X$  be a standard action and let us assume that the standard Borel space  $X$  admits at least one  $\Gamma$ -invariant probability measure. Then there exists a map  $\beta: X \rightarrow \text{Erg}(\alpha)$  that for every  $\mu \in \text{Prob}(\alpha)$  is an ergodic decomposition of  $(\alpha, \mu): \Gamma \curvearrowright X$ .*

*Remark 3.6* ([19, Lemma 4.1]). Let  $\alpha: \Gamma \curvearrowright X$  be a measurable action of a countable group on a standard Borel space, let  $\mu \in \text{Prob}(\alpha)$ , and let  $\beta: X \rightarrow \text{Erg}(\alpha)$  be an ergodic decomposition of  $\Gamma \curvearrowright (X, \mu)$ .

Let  $A \subset X$  be a measurable subset that is  $\beta$ -compatible in the sense that

$$\forall_{x, y \in X} \beta_x = \beta_y \implies (x \in A \iff y \in A).$$

Then, we have

$$\forall_{x \in X} x \in A \iff \beta_x(A) = 1.$$

Indeed, let  $x \in A$ . From  $\beta$ -compatibility, we obtain that  $X_{\beta_x} \subset A$ , and so  $\beta_x(A) \geq \beta_x(X_{\beta_x}) = 1$ . Thus,  $\beta_x(A) = 1$ . If  $x \in X \setminus A$ , then we can apply the same argument (with  $A$  also  $X \setminus A$  is  $\beta$ -compatible) to show that  $\beta_x(X \setminus A) = 1$ , whence  $\beta_x(A) = 0$ .

*Remark 3.7.* Let  $\alpha: \Gamma \curvearrowright X$  be a standard action, let  $\mu \in \text{Prob}(\alpha)$ , and let  $\beta: X \rightarrow \text{Erg}(\alpha)$  be an ergodic decomposition of  $(\alpha, \mu): \Gamma \curvearrowright X$ . If the given  $\Gamma$ -action  $\alpha$  is essentially free with respect to  $\mu$ , then for  $\mu$ -almost every  $x \in X$ , it is also essentially free with respect to  $\beta_x$ : Indeed, if  $\alpha: \Gamma \curvearrowright X$  is essentially free, then  $A := \{x \in X \mid \Gamma_x \neq 1\}$  is measurable and  $\mu(A) = 0$ . The identity

$$0 = \mu(A) = \int_X \beta_x(A) \, d\mu(x)$$

yields that  $\beta_x(A) = 0$  for  $\mu$ -almost every  $x \in X$ .

**Proposition 3.8.** *Let  $M$  be an oriented closed connected manifold with fundamental group  $\Gamma$ . Let  $(\alpha, \mu): \Gamma \curvearrowright X$  be a standard probability action and let  $\beta: X \rightarrow \text{Erg}(\alpha)$  be an ergodic decomposition of  $(\alpha, \mu)$ . Then, for every strict chain  $c \in C_*(M; \mathbb{B}(\alpha, \mathbb{Z}))$ , the map*

$$\begin{aligned} F_c: X &\longrightarrow \mathbb{R}_{\geq 0} \\ x &\longmapsto |c|^{(\alpha, \beta_x)} \end{aligned}$$

is measurable and

$$|c|^{(\alpha, \mu)} = \int_X |c|^{(\alpha, \beta_x)} \, d\mu(x).$$

*Proof.* Measurability of the function  $x \mapsto |c|^{(\alpha, \beta_x)}$  follows from Proposition 3.4. Let  $c = \sum_{j=1}^m f_j \otimes \sigma_j \in C_*(M; \mathbb{B}(\alpha, \mathbb{Z}))$  be in reduced form. Then

$$\begin{aligned} |c|^{(\alpha, \mu)} &= \sum_{j=1}^m \int_X |f_j| \, d\mu && \text{(definition)} \\ &= \sum_{j=1}^m \int_X \left( \int_X |f_j| \, d\beta_x \right) d\mu(x) && \text{(Proposition 3.4)} \\ &= \int_X \left( \sum_{j=1}^m \int_X |f_j| \, d\beta_x \right) d\mu(x) && \text{(linearity)} \\ &= \int_X |c|^{(\alpha, \beta_x)} \, d\mu(x), \end{aligned}$$

as claimed. □

## 4. Proof of Theorem 1.1

Regarding the proof of Theorem 1.1, we have shown that the parametrised simplicial volume can be computed via strict fundamental cycles. The strict chain complex  $B(\alpha, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\tilde{M}; \mathbb{Z})$  is, in general, not countable; our goal is now to reduce to a countable subcomplex  $\Sigma_*(M, \alpha; \mathbb{Z})$  that works uniformly for all probability measures on the given standard action. We proceed in two steps: We reduce the singular simplices and the measurable function spaces separately and then combine both reductions through the tensor product.

### 4.1. Reduction: countably many simplices

We first show that we need only countably many singular simplices:

**Proposition 4.1.** *Let  $M$  be a compact connected manifold with fundamental group  $\Gamma$ . Then there exists a countable  $\mathbb{Z}\Gamma$ -subcomplex  $C_*$  of  $C_*(\tilde{M}; \mathbb{Z})$  such that the inclusion  $C_* \hookrightarrow C_*(\tilde{M}; \mathbb{Z})$  is a  $\mathbb{Z}\Gamma$ -chain homotopy equivalence and such that there exists a chain homotopy inverse of norm at most 1.*

*Proof.* We proceed by the standard inductive simplices selection procedure, similar to the non-equivariant case of smooth simplices [12, Lemma 18.9]: As a compact manifold,  $M$  is homotopy equivalent to a countable (even finite) simplicial complex  $M'$  [11, 16, 18]. The chain complex  $C_*(\tilde{M}; \mathbb{Z})$  is thus  $\mathbb{Z}\Gamma$ -chain homotopy equivalent to  $C_*(\tilde{M}'; \mathbb{Z})$ , where the chain homotopies in both directions may be chosen of norm at most 1. Therefore, we may and will assume that  $M$  itself is a countable simplicial complex.

Let  $S$  be the set of all singular simplices of  $\tilde{M}$  that are (geometric realisations of) simplicial maps to  $\tilde{M}$ , defined on iterated barycentric subdivisions of the standard simplices. Using inductive (relative) simplicial approximation we can thus find a family  $(h_\sigma : \Delta^{\dim \sigma} \times [0, 1] \rightarrow \tilde{M})_{\sigma \in \text{map}(\Delta^*, \tilde{M})}$  of continuous maps with the following properties:

- The set  $S$  is closed under taking faces and under the deck transformation action.
- For all  $\sigma \in \text{map}(\Delta^*, \tilde{M})$ , we have  $h_\sigma(\cdot, 1) \in S$ .
- If  $\sigma \in S$ , then  $h_\sigma$  is the constant homotopy from  $\sigma$  to itself.
- For all  $\sigma \in \text{map}(\Delta^*, \tilde{M})$  and all  $\gamma \in \Gamma$ , we have  $h_{\gamma \cdot \sigma} = \gamma \cdot h_\sigma$ .
- For all  $n \in \mathbb{N}$ , all  $j \in \{0, \dots, n\}$ , and every singular  $n$ -simplex  $\sigma \in \text{map}(\Delta^n, \tilde{M})$ , we have  $h_\sigma \circ (\iota_j \times \text{id}_{[0,1]}) = h_{\sigma \circ \iota_j}$  where  $\iota_j : \Delta^{n-1} \rightarrow \Delta^n$  denotes the inclusion of the  $j$ -th face.

We define  $C_*$  to be the subcomplex of  $C_*(\widetilde{M}; \mathbb{Z})$  spanned by  $S$ , which is countable. The inclusion  $i_* : C_* \hookrightarrow C_*(\widetilde{M}; \mathbb{Z})$  is a  $\mathbb{Z}\Gamma$ -chain map. Conversely, we define  $\Phi_* : C_*(\widetilde{M}; \mathbb{Z}) \rightarrow C_*$  as the  $\mathbb{Z}\Gamma$ -linear extension of

$$\forall_{\sigma: \Delta^* \rightarrow \widetilde{M}} \Phi_*(\sigma) := h_\sigma(\cdot, 1);$$

this indeed is a chain map. Moreover,  $\Phi_* \circ i_* = \text{id}_{C_*}$  and the standard prism decomposition of  $\Delta^* \times [0, 1]$  shows that  $i_* \circ \Phi_* \simeq_{\mathbb{Z}\Gamma} \text{id}_{C_*(\widetilde{M}; \mathbb{Z})}$ . By construction,  $\|\Phi_*\| \leq 1$ .  $\square$

**Corollary 4.2.** *Let  $M$  be an oriented closed connected manifold with fundamental group  $\Gamma$  and let  $C_*$  be as provided by Proposition 4.1. Then  $\mathbb{Z} \otimes_{\mathbb{Z}\Gamma} C_*$  contains an integral fundamental cycle of  $M$ .*

*Proof.* Let  $c \in \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M}; \mathbb{Z})$  be an integral fundamental cycle of  $M$  and let  $\Phi_*$  be a chain homotopy inverse as provided by Proposition 4.1. Then  $(\text{id}_{\mathbb{Z}} \otimes_{\mathbb{Z}\Gamma} \Phi_*)(c)$  is an integral fundamental cycle of  $M$  that lies in the subcomplex  $\mathbb{Z} \otimes_{\mathbb{Z}\Gamma} C_*$ .  $\square$

## 4.2. Reduction: countably many functions

Standard Borel spaces have the following uniform regularity property for probability measures:

**Proposition 4.3.** *Let  $X$  be a standard Borel space. Then there exists a countable subalgebra  $\Sigma$  of the Borel  $\sigma$ -algebra of  $X$  that is dense with respect to every probability measure on  $X$ . I.e.: For every probability measure  $\mu$ , for every measurable subset  $A$  and every  $\varepsilon \in \mathbb{R}_{>0}$ , there exists an  $A' \in \Sigma$  with*

$$\mu(A \Delta A') < \varepsilon.$$

*Proof.* As  $X$  is a standard Borel space, we may assume without loss of generality that  $X$  is the Borel space associated with a separable metric space  $Y$ . Let  $Y' \subset Y$  be a countable dense subset. Then the algebra  $\Sigma$  generated by  $\{U_r(y) \mid y \in Y', r \in \mathbb{Q}_{>0}\}$  has the claimed property (here,  $U_r(y)$  denotes the open ball of radius  $r$  and centre in  $y$  with respect to the metric of  $Y$ ):

Indeed, let  $\mu$  be a probability measure on  $X$  and let  $A \subset Y$  be measurable. By regularity on standard Borel spaces [9, Theorem 17.10], we have

$$\mu(A) = \inf\{\mu(U) \mid U \subset X \text{ is open and } A \subset U\}.$$

Hence, it suffices to prove the claim if  $A$  is open. If  $A$  is open, then  $A = \bigcup_{n \in \mathbb{N}} U_n$ , where each of the  $U_n$  is of the form  $U_r(y)$  with  $y \in Y'$  and  $r \in \mathbb{Q}_{>0}$ , because  $\{U_r(y) \mid y \in$

$Y'$ ,  $r \in \mathbb{Q}_{>0}$  is a basis for the topology on  $Y$ . Then

$$\mu(A) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{j=0}^n U_j\right).$$

So, for every  $\varepsilon \in \mathbb{R}_{>0}$ , if  $n \in \mathbb{N}$  is large enough, then  $U := \bigcup_{j=0}^n U_j$  satisfies  $\mu(A \triangle U) = \mu(A \setminus U) < \varepsilon$ ; moreover, by construction,  $U \in \Sigma$ .  $\square$

**Corollary 4.4.** *Let  $\Gamma \curvearrowright X$  be a standard action. Then there exists a  $\Gamma$ -invariant countable subalgebra  $\Sigma$  of the Borel  $\sigma$ -algebra on  $X$  that is dense with respect to every probability measure on  $X$ .*

*Proof.* Let  $\Sigma$  be a subalgebra as provided by Proposition 4.3. Then the algebra generated by  $\{\gamma \cdot A \mid A \in \Sigma, \gamma \in \Gamma\}$  has the claimed property.  $\square$

### 4.3. Reduction: countably many parametrised chains

We now combine the geometric and the dynamical reduction steps:

**Proposition 4.5.** *Let  $M$  be an oriented closed connected  $n$ -manifold, let  $\Gamma := \pi_1(M)$ , and let  $C_*$  be as provided by Proposition 4.1. Moreover, let  $\Gamma \curvearrowright X$  be a standard action, let  $\Sigma$  be as in Corollary 4.4, and let  $B_\Sigma(\alpha, \mathbb{Z}) := \text{Span}_{\mathbb{Z}}\{\chi_A \mid A \in \Sigma\} \subset B(\alpha, \mathbb{Z})$ . Then the chain complex*

$$\Sigma_*(M, \alpha; \mathbb{Z}) := B_\Sigma(\alpha, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*$$

*has the following property: For every  $\alpha$ -invariant probability measure  $\nu$  on  $X$ , we have*

$$|M|^{(\alpha, \nu)} = \inf\{|c|^{(\alpha, \nu)} \mid c \in \Sigma_*(M, \alpha; \mathbb{Z}) \text{ is a strict fundamental cycle}\}.$$

*Proof.* By Proposition 2.8, we know that  $|M|^{(\alpha, \nu)}$  can be computed by fundamental cycles in  $B(\alpha, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\tilde{M}; \mathbb{Z})$ . We split the argument into the following steps:

$$\Sigma_*(M, \alpha; \mathbb{Z}) \hookrightarrow B(\alpha, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_* \hookrightarrow B(\alpha, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\tilde{M}; \mathbb{Z}).$$

First, we have

$$|M|^{(\alpha, \nu)} = \inf\{|c|^{(\alpha, \nu)} \mid c \in B(\alpha, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_* \text{ is a fundamental cycle}\},$$

because: Clearly, we have “ $\leq$ ”. Conversely, let  $\Phi_*: C_*(\tilde{M}; \mathbb{Z}) \rightarrow C_*$  be a  $\mathbb{Z}\Gamma$ -chain homotopy inverse of the inclusion  $C_* \hookrightarrow C_*(\tilde{M}; \mathbb{Z})$  with  $\|\Phi_*\| \leq 1$ , as provided by Proposition 4.1. Then

$$\text{id}_{B(\alpha, \mathbb{Z})} \otimes_{\mathbb{Z}\Gamma} \Phi_*: B(\alpha, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\tilde{M}; \mathbb{Z}) \longrightarrow B(\alpha, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*$$

is a chain homotopy inverse of  $B(\alpha, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_* \hookrightarrow B(\alpha, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\tilde{M}; \mathbb{Z})$  that is compatible with integral chains and thus maps fundamental cycles to fundamental cycles. Moreover,  $\|\text{id}_{B(\alpha, \mathbb{Z})} \otimes_{\mathbb{Z}\Gamma} \Phi_*\| \leq 1$ . Therefore, “ $\geq$ ” holds as well.

Second, we have

$$|M|^{(\alpha, \nu)} = \inf\{|c|^{(\alpha, \nu)} \mid c \in \Sigma_*(M, \alpha; \mathbb{Z}) \text{ is a fundamental cycle}\},$$

because: Again, “ $\leq$ ” is clear. For the converse estimate, let  $c \in B(\alpha, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_n$  be a fundamental cycle and let  $\varepsilon \in \mathbb{R}_{>0}$ . We construct a fundamental cycle  $c' \in \Sigma_n(M, \alpha; \mathbb{Z})$  with  $|c' - c|^{(\alpha, \nu)} \leq \varepsilon$ . As  $C_*$  is a chain complex and  $c$  is a fundamental cycle,  $c$  is of the form  $c = c_{\mathbb{Z}} + \partial d$  for some integral fundamental cycle  $c_{\mathbb{Z}} \in \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} C_n \hookrightarrow B(\alpha, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_n$  and some  $d \in B(\alpha, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_{n+1}$ . Using the density result of Proposition 4.3, we obtain that  $B_{\Sigma}(\alpha, \mathbb{Z})$  is dense in  $B(\alpha, \mathbb{Z})$  with respect to the  $L^1$ -norm induced by  $\nu$ . Hence, also  $B_{\Sigma}(\alpha, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*$  is dense in  $B(\alpha, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*$ . In particular, there exists a chain  $d' \in B_{\Sigma}(\alpha, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_{n+1}$  with  $|d' - d|^{(\alpha, \nu)} \leq \varepsilon/(n+2)$ . Then

$$c' := c_{\mathbb{Z}} + \partial d'$$

is a fundamental cycle in  $\Sigma_n(M, \alpha; \mathbb{Z})$  and

$$|c' - c|^{(\alpha, \nu)} = |\partial(d' - d)|^{(\alpha, \nu)} \leq (n+2) \cdot |d' - d|^{(\alpha, \nu)} \leq \varepsilon,$$

as desired. □

#### 4.4. Proof of the ergodic decomposition formula

We prove Theorem 1.1 using the countable setting in Proposition 4.5. We first establish notation:

Let  $M$  be an oriented closed connected  $n$ -manifold, let  $\Gamma := \pi_1(M)$  and let  $C_*$  be as provided by Proposition 4.1. Let  $\alpha: \Gamma \curvearrowright X$  be a standard action and let  $\Sigma_*(M, \alpha; \mathbb{Z})$  be the chain complex provided by Proposition 4.5. Let  $\text{FC}(M, \alpha) \subset \Sigma_n(M, \alpha; \mathbb{Z})$  be the set of all fundamental cycles of  $M$  in  $\Sigma_*(M, \alpha; \mathbb{Z})$ .

We use an integral fundamental cycle as baseline: There exists an integral fundamental cycle  $c_{\mathbb{Z}} \in \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} C_*$  (Corollary 4.2). As  $B_{\Sigma}(\alpha, \mathbb{Z})$  contains all constant  $\mathbb{Z}$ -valued functions, we can view  $c_{\mathbb{Z}}$  as a fundamental cycle in  $\Sigma_*(M, \alpha; \mathbb{Z})$ . We set  $\nu := |c_{\mathbb{Z}}|_1$ .

Let  $\mu$  be a  $\Gamma$ -invariant probability measure on  $X$  and let  $\beta: X \rightarrow \text{Erg}(\alpha)$  be an ergodic decomposition of  $\Gamma \curvearrowright (X, \mu)$ . For  $c \in \Sigma_*(M, \alpha; \mathbb{Z})$ , we recall that

$$\begin{aligned} F_c: X &\longrightarrow \mathbb{R}_{\geq 0} \\ x &\longmapsto |c|^{(\alpha, \beta_x)} \end{aligned}$$

is an integrable function (Proposition 3.8). By Proposition 4.5, for all  $x \in X$ , we have

$$|M|^{(\alpha, \beta_x)} = \inf_{c \in \text{FC}(M, \alpha)} F_c(x).$$

We emphasise that  $\text{FC}(M, \alpha)$  is countable. As a countable infimum of integrable functions the function

$$F: X \longrightarrow \mathbb{R}_{\geq 0}$$

$$x \longmapsto \inf_{c \in \text{FC}(M, \alpha)} F_c(x) = |M|^{(\alpha, \beta_x)}$$

is measurable and bounded by  $\|M\|_{\mathbb{Z}}$ ; hence,  $F$  is  $\mu$ -integrable and

$$\int_X |M|^{(\alpha, \beta_x)} d\mu(x) = \int_X F d\mu \quad (\text{definition of } F)$$

$$= |M|^{(\alpha, \mu)}. \quad (\text{Lemma 4.6 below})$$

It remains to show the second equality, i.e., that we can indeed swap taking this specific infimum with integration.

**Lemma 4.6.** *In the situation above, we have*

$$\int_X F d\mu = |M|^{(\alpha, \mu)}.$$

*Proof.* On the one hand, by definition,  $F \leq F_c$  for all  $c \in \text{FC}(M, \alpha)$ . In particular, monotonicity of the integral gives

$$\int_X F d\mu \leq \inf_{c \in \text{FC}(M, \alpha)} \int_X F_c d\mu.$$

$$= \inf_{c \in \text{FC}(M, \alpha)} |c|^{(\alpha, \mu)} \quad (\text{Proposition 3.8})$$

$$= |M|^{(\alpha, \mu)}. \quad (\text{Proposition 4.5})$$

For the converse inequality, we use the “self-referentiality” of the construction to produce fundamental cycles in  $C_*(M; \mathbb{B}(\alpha, \mathbb{Z}))$  with “small” norm: For notational convenience, we enumerate the countable set  $\text{FC}(M, \alpha)$  as  $c_0, c_1, \dots$ . Let  $\varepsilon \in \mathbb{R}_{>0}$ . For  $n \in \mathbb{N}$ , we consider the set

$$A_n := \{x \in X \mid |c_n|^{(\alpha, \beta_x)} \leq |M|^{(\alpha, \beta_x)} + \varepsilon\}.$$

Then,  $A_n$  is measurable and  $\Gamma \cdot A_n = A_n$ . Moreover,  $\bigcup_{n \in \mathbb{N}} A_n = X$  by Proposition 4.5. Let  $N \in \mathbb{N}$  be so large that  $A := \bigcup_{n=0}^N A_n$  satisfies  $\mu(X \setminus A) \leq \varepsilon$ . We then consider the pairwise disjoint family  $(\bar{A}_n)_{n \in \{0, \dots, N\}}$  given by  $\bar{A}_0 := A_0$  and

$$\forall n \in \{0, \dots, N-1\} \quad \bar{A}_{n+1} := A_{n+1} \setminus \bigcup_{j=0}^n \bar{A}_j.$$



Then, also the  $\bar{A}_n$  are measurable and satisfy  $\Gamma \cdot \bar{A}_n = \bar{A}_n$ . By construction,  $A = \bigsqcup_{n=0}^N \bar{A}_n$  and

$$c := \sum_{n=0}^N \chi_{\bar{A}_n} \cdot c_n + \chi_{X \setminus A} \cdot c_{\mathbb{Z}} \in C_*(M; \mathbf{B}(\alpha, \mathbb{Z}))$$

is a strict fundamental cycle of  $M$  (Remark 2.9). We show that  $|c|^{(\alpha, \mu)}$  is small enough: By construction, the sets  $\bar{A}_n$  are  $\beta$ -compatible. For all  $n \in \{0, \dots, N\}$  and all  $x \in \bar{A}_n$ , we thus have  $\beta_x(\bar{A}_n) = 1$  (Remark 3.6) and so

$$F_c(x) = |c_n|^{(\alpha, \beta_x)} \leq |M|^{(\alpha, \beta_x)} + \varepsilon \leq F(x) + \varepsilon;$$

furthermore,  $F_c|_{X \setminus A} = |c_{\mathbb{Z}}|_1$ . We conclude that

$$\begin{aligned} |M|^{(\alpha, \mu)} &\leq |c|^{(\alpha, \mu)} = \int_X F_c \, d\mu = \sum_{n=0}^N \int_{\bar{A}_n} F_c \, d\mu + \int_{X \setminus A} F_c \, d\mu \\ &\leq \sum_{n=0}^N \int_{\bar{A}_n} F + \varepsilon \, d\mu + \int_{X \setminus A} v \, d\mu \\ &\leq \sum_{n=0}^N \int_{\bar{A}_n} F \, d\mu + \sum_{n=0}^N \mu(\bar{A}_n) \cdot \varepsilon + \mu(X \setminus A) \cdot v \\ &\leq \int_X F \, d\mu + \mu(A) \cdot \varepsilon + \varepsilon \cdot v \leq \int_X F \, d\mu + \varepsilon \cdot (1 + v). \end{aligned}$$

Therefore, taking  $\varepsilon \rightarrow 0$  shows that  $|M|^{(\alpha, \mu)} \leq \int_X F \, d\mu$ . □

This finishes the proof of Theorem 1.1.

Finally, we mention two straightforward directions of generalisations:

*Remark 4.7* (general spaces). The following version of Theorem 1.1 also holds: Let  $M$  be a path-connected topological space that has the homotopy type of a countable CW-complex and that admits a universal covering. Let  $\Gamma := \pi_1(M)$ , let  $\alpha: \Gamma \curvearrowright X$  be a standard action, and let  $\mu \in \text{Prob}(\alpha)$ . If  $\zeta \in H_*(M; L^\infty(\alpha, \mu, \mathbb{Z}))$  is an integral homology class (i.e., coming from  $H_*(M; \mathbb{Z})$ ) and  $\beta: X \rightarrow \text{Erg}(\alpha)$  is an ergodic decomposition of  $(\alpha, \mu): \Gamma \curvearrowright X$ , then

$$|\zeta|^{(\alpha, \mu)} = \int_X |\zeta|^{(\alpha, \beta_x)} \, d\mu(x).$$

Indeed, all of our proofs work on this level of generality. We restricted ourselves to the manifold/fundamental class case to keep the notation somewhat lighter.

*Remark 4.8* (general decompositions). Moreover, decomposition formulas as in Theorem 1.1 and Remark 4.7 also hold for other decompositions of the base measure (as

in Definition 3.3, but with maps to  $\text{Prob}(\alpha)$  instead of  $\text{Erg}(\alpha)$ ), not only for the ergodic decomposition. Indeed, we never used the fact that the measures occurring in the decomposition are ergodic.

## Acknowledgements

We would like to thank the anonymous referee for the constructive report and for catching several inaccuracies and a major omission in the main argument.

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