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Geodesic covers and Erdős distinct distances in hyperbolic surfaces

Zhipeng Lu
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Abstract

In this paper, we introduce the notion of "geodesic cover" for Fuchsian groups, which summons copies of fundamental polygons in the hyperbolic plane to cover pairs of representatives realizing distances in the corresponding hyperbolic surface. Then we use estimates of geodesic-covering numbers to study the distinct distances problem in hyperbolic surfaces. Especially, for Y from a large class of hyperbolic surfaces, we establish the nearly optimal bound $\geq c(Y)N/\log N$ for distinct distances determined by any N points in Y, where $c(Y) > 0$ is some constant depending only on Y. In particular, for Y being modular surface or standard regular of genus $g \geq 2$, we evaluate $c(Y)$ explicitly in terms of g.

1. Introduction

1.1. Distinct distances problem in hyperbolic surfaces

In 1946, Erdős [5] posed the distinct distances problem which asks for the least number of distinct distances among any N points in the Euclidean plane, and conjectured that it is in the order of $N/\sqrt{\log N}$. Guth–Katz [9] obtained the nearly optimal bound $\gtrsim N/\log N$ (we use the notation $f \gtrsim g$ to mean that there is an absolute constant $C > 0$ such that $f \geq Cg$). Erdős also considered the higher dimensional generalization of the problem in $\mathbb{R}^d$ ($d \geq 3$) and conjectured the lower bound $\gtrsim N^{2/d}$. For $d \geq 3$, Solymosi–Vu [29] obtained the lower bound $\gtrsim N^{2/d-2/d(d+2)}$ by an induction on the dimension with the best known lower bound in the plane as the base case. Combining the Guth–Katz bound with the induction of Solymosi–Vu, one may improve the lower bounds of Solymosi–Vu for higher dimensional Euclidean spaces. For example when $d = 3$, it gives the lower bound $\gtrsim N^{3/5-\epsilon}$ for any $\epsilon > 0$, see Sheffer [27] for details. There is also a continuous analogue of the problem in geometric measure theory, i.e. the Falconer's conjecture, which asks about the lower bound of Hausdorff dimension of the sets in $\mathbb{R}^d$ for which the difference set has positive Lebesgue measure. Interested readers may check [6], [8], [14] etc. In addition to the Euclidean space, Erdős–Falconer type problems have also been studied in vector spaces over finite fields and other spaces, see e.g. Bourgain–Katz–Tao [3], Iosevich–Rudnev [15],

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Hart–Iosevich–Koh–Rudnev [11], Rudnev–Selig [25], and Sheffer–Zahl [28], Tao's blog[1] etc.

In the present paper, we establish lower bounds of distinct distances for a large class of hyperbolic surfaces. Hyperbolic surfaces as quotients of the hyperbolic plane $\mathbb{H}^2$ by the action of Fuchsian groups, are locally isometric to $\mathbb{H}^2$. Noting that geodesics in $\mathbb{H}^2$ may be complicatedly folded by the quotient of a Fuchsian group, it is not clear whether the nearly optimal lower bound as of Guth–Katz [9] still holds for general hyperbolic surfaces. By studying actions of Fuchsian groups relatively explicitly and excavating a general notion of "geodesic covering", we establish

Theorem 1.1. *Assume Y is the modular surface or a surface whose fundamental group is co-compact as a Fuchsian group. Then any set of N points in Y determines $\geq c(Y)N/\log N$ distinct distances for some constant $c(Y) > 0$ depending only on Y.*

In particular for the standard regular surface of genus $g \geq 2$, denoted by Y_g, whose fundamental domain in the upper half plane $\mathbb{H}^2$ can be chosen as a standard regular $4g$-gon, we are able to estimate $c(Y_g)$ explicitly and get the following theorem.

Theorem 1.2. *For Y_g being standard regular of genus $g \geq 2$, the lower bound of distinct distances among any N points in Y_g is $\geq c\frac{N}{g^{18}(\log N+\log g)}$ for some absolute constant $c > 0$.*

Here the asymptotic stands with respect to both g and N, which is not trivial only when $N \gtrsim g^{18}$. There is a parallel question on how many points there can be with pairwise equal distance in a surface of genus g. See Section 1.3 for more discussions.

More generally, we also derive a lower bound for the number of distinct distances between points of any two finite sets P_1 and P_2 in hyperbolic surfaces with finite geodesic-covering numbers.

Theorem 1.3. *Let $P_1, P_2 \subset Y$ be any finite sets in a hyperbolic surface Y with finite geodesic-covering number. Then we have*

$$\left|\{d_Y(p_1, p_2) : p_1 \in P_1, p_2 \in P_2\}\right| \gtrsim_Y \frac{|P_1|^2|P_2|^2}{|P_1 \cup P_2|^3 \log |P_1 \cup P_2|}.$$

Remark 1.4. When P_1 and P_2 are roughly the same size, this lower bound is sharp up to a factor of log.

1.2. Geodesic cover and sketch of proofs

In order to deal with various hyperbolic surfaces, we propose the concept of "geodesic cover" of a hyperbolic surface, which itself can be of independent interest.

[1]http://terrytao.wordpress.com/2011/03/05/lines-in-the-euclidean-group-se2/

For any surface Y with universal cover $\mathbb{H}^2$, its fundamental group is isomorphic to a Fuchsian group $\Gamma_Y \leq \mathrm{PSL}_2(\mathbb{R})$. Note that Γ_Y acts on $\mathbb{H}^2$ by Möbius transformation, we have $Y \simeq \Gamma_Y \backslash \mathbb{H}^2$ endowed the hyperbolic metric from $\mathbb{H}^2$. For any points $p, q \in Y$, we pick two representatives (still denoted by p, q) in a fundamental domain F of Γ_Y. Then $d_Y(p,q) = \min_{\gamma \in \Gamma_Y} d_{\mathbb{H}^2}(p, \gamma \cdot q)$. We want to find a subset $\Gamma_0 \subset \Gamma_Y$ such that $\forall\ p, q \in F$, $d_Y(p,q) = d_{\mathbb{H}^2}(p, \gamma \cdot q)$ for some $\gamma \in \Gamma_0$. We call $\cup_{\gamma \in \Gamma_0} \gamma(F)$ a *geodesic cover* of Y and call the smallest $|\Gamma_0|$ (among choices of F), denoted by K_Y (or K_{Γ_Y}), the *geodesic-covering number* of Y (or Γ_Y).

In Section 2 we show that co-compact Fuchsian groups have finite geodesic-covering numbers. If a Fuchsian group Γ is co-compact, its fundamental domain is a closed region without ideal points as vertices. This is equivalent to $\Gamma \backslash \mathbb{H}^2$ has finite hyperbolic area and Γ contains no parabolic elements, see Corollary 4.2.7 of [17]. In particular, closed hyperbolic surfaces of genus $g \geq 2$ belong to this case. Moreover, Proposition 3.1 establishes the estimate $K_{Y_g} \lesssim g^6$ for Y_g being standard regular of genus $g \geq 2$.

For groups which are not co-compact, we show by explicit analysis that the modular group has finite geodesic-covering number. More specifically, Proposition 3.4 establishes the estimate $K_{\mathrm{PSL}_2(\mathbb{Z})} \leq 10$.

Now we briefly sketch the strategy of proving Theorem 1.1, which is a consequence of Theorem 2.3 together with Propositions 2.1 and 3.4. Given any N-point set $P \subset Y$, if $K_Y < \infty$ we duplicate the points to be $\widetilde{P} = \cup_{\gamma \in \Gamma_0} \gamma(P) \subset \mathbb{H}^2$ on a geodesic cover $\cup_{\gamma \in \Gamma_0} \gamma(F)$ of Y with $|\Gamma_0| = K_Y$. By definition, the distances among points of P in Y all belong to the distances among points of $\widetilde{P}$ in $\mathbb{H}^2$. However, we are not allowed to apply the lower bound for the hyperbolic plane to points of $\widetilde{P}$ directly, since we have more number of points now and the inequality actually goes to the wrong direction. Instead, we resort to counting of distance quadruples of $\widetilde{P} \subset \mathbb{H}^2$, where the factor with K_Y would appear, to establish Theorem 1.1. See Theorem 2.3 for details. The proof of Theorem 1.3 follows the same strategy of duplicating points and similar counting of distance quadruples, see Section 2.2 for details.

In the case of Theorem 1.2, to evaluate $c(Y_g)$ explicitly for standard regular Y_g, we rely on hyperbolic trigonometry and connect it with the hyperbolic circle problem. As a natural analogue of the Gauss circle problem in $\mathbb{H}^2$, the hyperbolic circle problem asks for the asymptotics of $\#\{\gamma \in \Gamma : d_{\mathbb{H}^2}(z_0, \gamma \cdot z_0) \leq Q\}$ for discrete subgroups $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ and $Q > 0$. This problem and its generalizations have been widely studied by various authors including Delsarte [4], Huber [12, 13], Selberg [26], Margulis [20], Patterson [23], Iwaniec [16], Phillips–Rudnick [24], Boca–Zaharescu [2], Kontorovich [18] etc. The application to our case needs certain uniformity of lattice counting over surfaces of genus g, specifically in the form of (3.5) as in the proof of Proposition 3.1.

Remark 1.5. For completeness we include the case of flat tori, i.e. $g = 1$. We may similarly define K_Γ for any discrete subgroup Γ of the rigid motion group of $\mathbb{R}^2$. For flat tori which correspond to $\Gamma \simeq \mathbb{Z}^2$, we immediately see that $K_\Gamma < \infty$. Thus by the result of Guth–Katz [9], the number of distinct distances among N points on any flat torus is $\gtrsim N/\log N$.

Remark 1.6. There is also an analogue of the unit distance problem in hyperbolic surfaces. Borrowing the arguments from Section 7.6 of [7] based on estimates of crossing numbers, one may establish the Spencer–Szemerédi–Trotter bound to $\mathbb{H}^2$, i.e. the number of pairs with unit (or equal) distance among any N points in $\mathbb{H}^2$ is $\lesssim N^{4/3}$. It is also and direct implication of Pach–Sharir theorem [22] applied to hyperbolic circles. For any set of N points in a hyperbolic surface Y with K_Y finite, we lift it to a set of $K_Y N$ points on a geodesic cover of Y. By Spencer–Szemerédi–Trotter one may bound the number of unit (or equal) distances among any N points on Y by $\lesssim (K_Y N)^{4/3}$. In particular for standard regular surfaces Y_g of genus $g \geq 2$, by Proposition 3.1, the upper bound becomes $\lesssim g^8 N^{4/3}$.

1.3. Equilateral dimension and sharpness of Theorem 1.2

In order to analyze the sharpness of Theorem 1.2, we connect it with the equilateral dimension of hyperbolic surfaces. The equilateral dimension of a metric space is defined to be the maximal number of points with pairwise equal distance. For the simplest example the equilateral dimension of the Euclidean space $\mathbb{E}^d$ is always $d+1$. The equilateral dimensions of various spaces have been studied by Alon–Milman [1], Guy [10], Koolen [19] etc. We are not aware of any non-trivial bound of equilateral dimension on hyperbolic surfaces in literature. We observe that our results can be applied to the equilateral dimension problem on hyperbolic surfaces. And in converse, the results for equilateral dimensions could also help us to analyze the sharpness of Theorem 1.2.

We claim that Theorem 1.2 implies equilateral dimension of standard regular surfaces Y_g of genus g is $\lesssim g^{18+\epsilon}$. Suppose to the contrary for infinitely many g, the surface Y_g has equilateral dimension $\geq Cg^{18+\epsilon}$ for some constant $C > 0$. Then for each such g there exists a set of $M_g = Cg^{18+\epsilon}$ points in Y_g with pairwise equal distance. Hence its number of distinct distances is 1. On the other hand, by Theorem 1.2, the number of distinct distances for any set of M_g points is $\gtrsim \frac{M_g}{g^{18}\log(gM_g)} \gtrsim g^\epsilon$ which would approach infinity as $g \to \infty$. Contradiction.

However, from another approach one may show that the equilateral dimension of Y_g is actually $\lesssim g$. Suppose there are N_g points in Y_g with pairwise equal distance $r > 0$. Choosing a fundamental domain F of Y_g, we draw a circle of radius r in $\mathbb{H}^2$ centered at one representative of the N_g points, say p_0. By definition, each point has a representative lying on the circle with distance at least r from each other. We order these representatives by $p_i, i = 1, \dots, N_g - 1$. For adjacent p_i, p_j, let α_{ij} be the smaller positive angle between geodesics connecting p_0, p_i and p_0, p_j. By hyperbolic trigonometry, since $d_{\mathbb{H}^2}(p_i, p_j) \geq r$,

$$\sin(\alpha_{ij}/2) = \frac{\sinh(d_{\mathbb{H}^2}(p_i, p_j)/2)}{\sinh(r)} \geq \frac{\sinh(r/2)}{\sinh(r)} = \frac{1}{2\cosh(r/2)}.$$

In the proof of Proposition 3.1, we get the upper bound $\cosh r \lesssim g^2$, hence $\alpha_{ij} \gtrsim 1/g$. This shows that $N_g \lesssim g$ and hence the equilateral dimension of Y_g is $\lesssim g$.

The above comparison of different estimates on equilateral dimensions may suggest that Theorem 1.2 is far from being optimal. Also, the lower bound for the number of distinct distances among any N points should be better than trivial in the range $g \lesssim N \lesssim g^{18+\epsilon}$.

One possible approach to improve Theorem 1.2 is trying to get a better bound for geodesic-covering number K_{Y_g}. One may modify the definition of geodesic cover a little bit, to choose a set $\Gamma_1 \subset \Gamma_Y$ for a surface Y such that for any $p, q \in F$, a fundamental domain of Y,

$$d_Y(p, q) = \min_{\gamma_1, \gamma_2 \in \Gamma_1} d_{\mathbb{H}^2}(\gamma_1 \cdot p, \gamma_2 \cdot q) = \min_{\gamma_1, \gamma_2 \in \Gamma_1} d_{\mathbb{H}^2}(p, \gamma_1^{-1}\gamma_2 \cdot q). \tag{1.1}$$

The set Γ_1 may be called a *geodesic pre-cover* of Y and the smallest size of such the *geodesic pre-covering number* of Y, denoted by K'_Y. Clearly, $\Gamma_0 = \Gamma_1^{-1}\Gamma_1$ is a geodesic cover in the original definition. It seems that $|\Gamma_1|$ may be expected to be $\sim |\Gamma_0|^{1/2}$ in many cases. However this definition appears not as convenient for computation.

1.4. **Further research**

We observe that for rectangle tori, the four fundamental polygons around a vertex patched together gives a geodesic pre-cover. Furthermore, we are boldly tempted to conjecture that the fundamental polygons around one vertex may also work for the hyperbolic case.

Conjecture 1. For standard regular surfaces Y_g of genus $g \geq 2$, the geodesic pre-covering number K'_{Y_g} is $\lesssim g$.

In addition, since for any finite index subgroup Γ' of a Fuchsian group Γ, its fundamental domain is the union of finitely many fundamental domains of Γ. If K_Γ is finite, one may expect that $K_{\Gamma'}$ is also finite. We further make the conjecture below.

Conjecture 2. For any subgroup $\Gamma \leq \mathrm{PSL}_2(\mathbb{Z})$ of finite index, its geodesic-covering number is finite.

There are more Fuchsian groups with finite geodesic-covering number. For example, the translation group

$$\left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$$

has the strip $\{x + iy : 0 < x \leq 1, y > 0\}$ as fundamental domain and its geodesic-covering number is ≤ 3. However it is an infinite index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. Other simple examples include finite subgroups of $\mathrm{PSL}_2(\mathbb{R})$. We further make the following more general conjecture.

Conjecture 3. For any discrete subgroup $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ whose fundamental domain has finitely many sides (geometrically finite), its geodesic-covering number is finite.

There are more questions that could be asked. For instance, if a Fuchsian group has finite geodesic-covering number, does any of its finite indexed subgroup also have finite geodesic-covering number? If true, then by Poincaré's theorem, for Conjecture 3 we only need to focus on groups without elliptic elements. In general, how does geodesic-covering number relate to the signature of a Fuchsian group? Moreover, how does it relate to Fenchel–Nielsen coordinates in the Teichmüller space?

Remark 1.7. The second author has a follow-up paper [21] published ahead, in which the above conjectures are treated especially for finite-indexed subgroups of $\mathrm{PSL}_2(\mathbb{Z})$.

Notation. Throughout this paper we use the notation $f \gtrsim g$ to mean that there is an absolute constant $C > 0$ such that $f \geq Cg$, and we use $f' \lesssim g'$ to mean that $|f'| \leq C'g'$ for some absolute constant $C' > 0$. We use $f \asymp g$ to mean that $f \lesssim g$ and also $f \gtrsim g$.

2. Geodesic-covering number and distinct distances

We propose the concept of *geodesic-covering number* for discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$ then use its estimates to deal with the distinct distances problem in closed hyperbolic surfaces and the modular surface.

2.1. Geodesic-covering number

Generally for any discrete subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$, let Y be the hyperbolic surface associated with Γ and F be a fundamental domain of Y, we propose the question of finding a subset $\Gamma_0 \subset \Gamma$ such that

$$d_Y(p,q) = \min_{\gamma \in \Gamma_0} d_{\mathbb{H}^2}(p, \gamma(q)), \ \forall\, p, q \in F. \tag{2.1}$$

We call the patched region of fundamental domains $U = \cup_{\gamma \in \Gamma_0} \gamma(F)$ a *geodesic cover* of Y. Noting that U depends on the choice of F, but what we do care is actually how small U

can be regardless of the choice of F. We say U is *minimal* if the cardinality of Γ_0 attains the minimum over all choices of fundamental domains. We denote by K_Γ the smallest $|\Gamma_0|$ and call it the *geodesic-covering number* of Γ.

We expect the geodesic-covering number is finite for many discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$. First we prove

Proposition 2.1. *For any co-compact discrete subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$, its geodesic-covering number K_Γ is finite.*

Proof. If Γ is co-compact, we may choose a closed fundamental domain $F \subset \mathbb{H}^2$ without ideal points as vertices. Denote $Y := \Gamma\backslash\mathbb{H}^2$. Then its diameter

$$\mathrm{diam}(Y) := \sup_{x,y\in F} \min_{\gamma\in\Gamma} d_{\mathbb{H}^2}(x, \gamma(y))$$

is finite. Let $U \subset \mathbb{H}^2$ be

$$U := \{z \in \mathbb{H}^2 \mid d_{\mathbb{H}^2}(z, F) \leq \mathrm{diam}(Y)\}$$

and $\widetilde{U} \supset U$ be

$$\widetilde{U} := \cup\{\gamma(F) \mid \gamma \in \Gamma, \gamma(F) \cap U \neq \emptyset\}.$$

For any $p, q \in Y$, choose two representatives $x, y \in F$. We claim that there is some representative $y' \in \widetilde{U}$ of q such that $d_{\mathbb{H}^2}(x, y') = d_Y(p, q)$. Otherwise, $d_Y(p, q) = d_{\mathbb{H}^2}(x, \gamma(y))$ for some $\gamma \in \Gamma$ with $\gamma(F) \cap U = \emptyset$, so that $d_{\mathbb{H}^2}(x, \gamma(y)) > \mathrm{diam}(Y) \geq d_Y(p, q)$, a contradiction.

Now for each fundamental domain $F' \subset \widetilde{U}$, we choose a $\gamma' \in \Gamma$ such that $F' = \gamma'(F)$. The set Γ_0 consisting of these isometries satisfies (2.1). The number of fundamental domains $F' \subset \widetilde{U}$ is finite and so $K_\Gamma \leq |\Gamma_0| < \infty$. □

Let $\mathbb{H}^2$ be the hyperbolic plane and $G = \mathrm{PSL}_2(\mathbb{R})$ be its isometry group which acts on $\mathbb{H}^2$ by Möbius transformations:

$$z \mapsto \gamma \cdot z = \frac{az+b}{cz+d}, \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R}), z \in \mathbb{H}^2.$$

Let $P \subset \mathbb{H}^2$ be a set of N points and define the set of *distance quadruples*

$$Q(P) := \{(p_1, p_2; p_3, p_4) \in P^4 : d(p_1, p_2) = d(p_3, p_4) \neq 0\}, \tag{2.2}$$

where $d(\cdot, \cdot)$ denotes the hyperbolic metric. Denote the distance set by

$$d(P) := \{d(p_1, p_2) : p_1, p_2 \in P\}.$$

Then we have a close relation between $d(P)$ and $Q(P)$ as follows. Suppose $d(P) = \{d_i : 1 \leq i \leq m\}$ and n_i is the number of pairs of points in P with distance d_i. So

$|Q(P)| = \sum_{i=1}^m n_i^2$. Since $\sum_{i=1}^m n_i = 2\binom{N}{2} = N^2 - N$, by Cauchy–Schwarz inequality we get

$$(N^2 - N)^2 = \left(\sum_{i=1}^m n_i\right)^2 \le \left(\sum_{i=1}^m n_i^2\right) m = |Q(P)||d(P)|.$$

Rearranging the inequality gives

$$|d(P)| \ge \frac{N^4 - 2N^3}{|Q(P)|}. \tag{2.3}$$

Tao gave an argument in his aforementioned blog then later fulfilled by [25] with further details using Klein quadric to derive the following result.

Lemma 2.2.

$$|Q(P)| \lesssim N^3 \log N. \tag{2.4}$$

We are also able to prove this result by working explicitly with isometries of $\mathbb{H}^2$. Combining this Lemma with (2.3), one derives a lower bound for distinct distances in the hyperbolic plane.

Now we connect the geodesic-covering number with distinct distances problem on any hyperbolic surface Y with corresponding fundamental group $\Gamma \subset G$.

Theorem 2.3. *Assume Y is a hyperbolic surface with fundamental group Γ and K_Γ is finite. Then a set of N points on Y determines $\gtrsim \frac{N}{K_\Gamma^3 \log(K_\Gamma N)}$ distinct distances.*

Proof. For any set P of N points on Y, we choose a minimal geodesic cover $\Gamma_0 \subset \Gamma$ with $|\Gamma_0| = K_\Gamma$ such that

$$d_Y(P) := \{d_Y(p,q) : p, q \in P\} \subset d_{\mathbb{H}^2}(\cup_{\gamma\in\Gamma_0}\gamma(P)).$$

Then

$$\begin{aligned} Q_Y(P) :=& \{(p_1, p_2; p_3, p_4) \in P^4 : d_Y(p_1, p_2) = d_Y(p_3, p_4) \ne 0\} \\ \subset& Q(\cup_{\gamma\in\Gamma_0}\gamma(P)), \end{aligned}$$

where $Q(P)$ is defined in (2.2). Since $|\cup_{\gamma\in\Gamma_0}\gamma(P))| \le K_\Gamma|P| = K_\Gamma N$, by Lemma 2.2 we get

$$|Q_Y(P)| \le |Q(\cup_{\gamma\in\Gamma_0}\gamma(P))| \lesssim (K_\Gamma N)^3 \log(K_\Gamma N). \tag{2.5}$$

Similar to (2.3), by the Cauchy–Schwarz inequality, we have

$$|d_Y(P)| \ge \frac{N^4 - 2N^3}{|Q_Y(P)|} \gtrsim \frac{N}{K_\Gamma^3 \log(K_\Gamma N)}.$$

We get the desired lower bound. ◻

2.2. Distinct distances between two sets in hyperbolic surfaces

This section contributes to the proof of Theorem 1.3.

Let $P_1, P_2 \subset Y$ be finite sets in a hyperbolic surface Y with geodesic-covering number $K_Y < \infty$. Choose a geodesic cover $\Gamma_0 \subset \Gamma_Y$ for the associated Fuchsian group Γ_Y of Y and $|\Gamma_0| = K_Y$, and duplicate P_1, P_2 to be $\widetilde{P_j} = \cup_{\gamma \in \Gamma_0} \gamma \cdot P_j$, $j = 1, 2$. Define

$$d_Y(P_1, P_2) := \{d_Y(p_1, p_2) : p_1 \in P_1, p_2 \in P_2\},$$
$$Q_Y(P_1, P_2) := \{(p_1, p_2; q_1, q_2) : p_1, q_1 \in P_1, p_2, q_2 \in P_2, d_Y(p_1, p_2) = d_Y(q_1, q_2) \neq 0\}.$$

and

$$Q_{\mathbb{H}^2}(\widetilde{P_1}, \widetilde{P_2}) := \{(p_1, p_2; q_1, q_2) : p_1, q_1 \in \widetilde{P_1}, p_2, q_2 \in \widetilde{P_2}, d_{\mathbb{H}^2}(p_1, p_2) = d_{\mathbb{H}^2}(q_1, q_2) \neq 0\}.$$

Certainly $Q_Y(P_1, P_2) \subset Q_{\mathbb{H}^2}(\widetilde{P_1}, \widetilde{P_2})$. Suppose $d_Y(P_1, P_2) = \{d_1, \ldots, d_m\}$ and n_k is the number of pairs (p_1, p_2) for $p_1 \in P_1, p_2 \in P_2$ with $d_Y(p_1, p_2) = d_k$. We see that $|P_1||P_2| - |P_1 \cap P_2| = \sum_{k=1}^m n_k$ and $|Q_Y(P_1, P_2)| = \sum_{k=1}^m n_k^2$. Then by the Cauchy–Schwarz inequality we get

$$|P_1|^2 |P_2|^2 \lesssim (|P_1||P_2| - |P_1 \cap P_2|)^2 \leq m \sum_{k=1}^m n_k^2 = m |Q_Y(P_1, P_2)|.$$

By Lemma 2.2, we have

$$|Q_Y(P_1, P_2)| \lesssim |Q(\widetilde{P_1} \cup \widetilde{P_2})| \lesssim K_Y^3 |P_1 \cup P_2|^3 \log(K_Y |P_1 \cup P_2|),$$

where $Q(P)$ is defined in (2.2), and consequently

$$|d_Y(P_1, P_2)| \gtrsim_Y \frac{|P_1|^2 |P_2|^2}{|P_1 \cup P_2|^3 \log |P_1 \cup P_2|}.$$

This finishes the proof of Theorem 1.3.

We may replace $|P_1 \cup P_2|$ by $\max\{|P_1|, |P_2|\}$ in the above inequality. If $|P_1|^2 \leq |P_2|$, the inequality gives a trivial lower bound.

3. Distinct distances in hyperbolic surfaces

In this section, we give explicit estimates for geodesic-covering numbers of standard regular hyperbolic surfaces and the modular surface.

3.1. Standard regular hyperbolic surfaces of genus $g \geq 2$

For the standard regular surfaces, we estimate their geodesic-covering numbers concretely as follows.

Proposition 3.1. *For the surface of genus g with standard regular fundamental $4g$-gon of inner angle $\frac{\pi}{2g}$, we have $K_{Y_g} \lesssim g^6$.*

Proof. Let $\Gamma_g \subset G$ be the corresponding surface group. For a standard regular geodesic $4g$-gon $F \subset \mathbb{H}^2$ centered at i (denote by O) serving as a fundamental domain of Y_g, we estimate its diameter as follows.

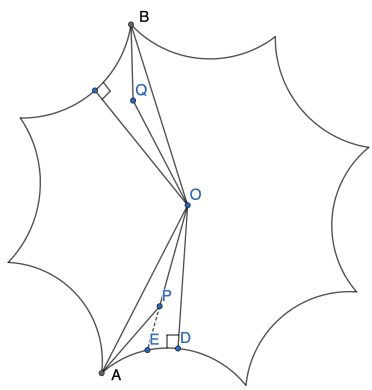

FIGURE 3.1. Distance between P and Q

First we determine a bound for $\operatorname{diam}(Y_g)$. For any $P, Q \in F$ (see Figure 3.1), choose two vertices A and B of F that are closest to P and Q correspondingly. Since there exists $\gamma \in \Gamma_g$ such that $\gamma(A) = B$, we have ($d = d_{\mathbb{H}^2}$) by triangle inequality

$$d_{Y_g}(P,Q) \le d(O,Q) + d(O,P), \ d_{Y_g}(P,Q) \le d(P,A) + d(Q,B),$$

hence

$$2d_{Y_g}(P,Q) \le d(O,P) + d(P,A) + d(O,Q) + d(Q,B).$$

We claim that $d(O,P)+d(P,A) \le d(O,D)+d(A,D)$. Indeed, if we extend the geodesic between O and P to E, by triangle inequality we have $d(O,E) = d(O,P) + d(P,E) \le d(O,D) + d(D,E)$ and $d(P,A) \le d(P,E) + d(E,A)$, so that

$$\begin{aligned} d(O,P) + d(P,A) &= d(O,E) - d(P,E) + d(P,A) \\ &\le d(O,D) + d(D,E) - d(P,E) + d(P,E) + d(E,A) \\ &= d(O,D) + d(D,E) + d(E,A) \\ &= d(O,D) + d(A,D). \end{aligned}$$

Since F is regular and P, Q are arbitrary, we have

$$\operatorname{diam}(Y_g) \leq d(O, D) + d(A, D).$$

Noting that (O, D, A) forms a right triangle and $\angle AOD = \angle OAD = \frac{\pi}{4g} =: \beta$, hyperbolic trigonometry gives

$$\cosh(d(O, D)) = \cosh(d(D, A)) = \cot\beta, \tag{3.1}$$

thus

$$\cosh(\operatorname{diam}(Y_g)) \leq \cosh(d(O, D) + d(D, A)) = \cosh(2d(O, D)) = 2\cot^2(\beta) - 1.$$

Also, we have

$$\cosh(d(O, A)) = \cot^2(\beta). \tag{3.2}$$

By construction of the geodesic cover $\widetilde{U}$ as in the proof of Proposition 2.1, we only need to choose $\gamma \in \Gamma_g$ such that the distance between $\gamma(F)$ and F is $\leq \operatorname{diam}(Y_g)$. Thus it suffices to consider $d(\gamma(i), i) \leq 2d(O, A) + \operatorname{diam}(Y_g)$ since the distance between the border of $\gamma(F)$ and $\gamma(i)$ is no more than $d(O, A)$ by trigonometry. Since for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{R})$,

$$2\cosh(d(\gamma(i), i)) = \|\gamma\|^2 = a^2 + b^2 + c^2 + d^2,$$

we get

$$K_{Y_g} \leq \#\{\gamma \in \Gamma_g : \|\gamma\|^2 \leq 2\cosh(2d(O, A) + \operatorname{diam}(Y_g))\}. \tag{3.3}$$

By the sum of arguments formula,

$$\begin{aligned}
&\cosh(2d(O, A) + \operatorname{diam}(Y_g)) \\
&\qquad = \cosh(2d(O, A))\cosh(\operatorname{diam}(Y_g)) + \sinh(2d(O, A))\sinh(\operatorname{diam}(Y_g)) \\
&\qquad = (2\cot^4(\beta) - 1)(2\cot^2(\beta) - 1) \\
&\qquad\qquad + 2\cot^2(\beta)\sqrt{\cot^4(\beta) - 1}\sqrt{(2\cot^2(\beta) - 1)^2 - 1} \\
&\qquad \lesssim g^4 \cdot g^2 + g^2 \cdot g^2 \cdot g^2 \lesssim g^6.
\end{aligned} \tag{3.4}$$

By the result of counting hyperbolic lattices inside a circle (see [2] or [18]), we have asymptotically

$$\#\{\gamma \in \Gamma : \|\gamma\| \leq R\} \sim \frac{\pi}{\operatorname{Area}(\Gamma\backslash\mathbb{H}^2)} R^2, \text{ as } R \to \infty.$$

This is the so-called *hyperbolic circle problem* which has been widely studied in various prominent works. However, we cannot apply the above formula directly to (3.3) since we also need certain uniformity among the surfaces of genus g.

Here we borrow an idea of Huber [13] (sections 3.1 and 3.2) to show that, for Y_g being standard regular with fundamental group Γ_g,

$$N_{\Gamma_g}(R) := \#\{\gamma \in \Gamma_g : \|\gamma\| \leq R\} \leq CR^2 \tag{3.5}$$

for some absolute constant C independent of g.

Now we prove (3.5). Since for any id $\neq \gamma \in \Gamma_g$, it maps F to another fundamental domain centered at $\gamma(i)$, the smallest distance between i and $\gamma(i)$ is at least $2|OD|$ (see Figure 3.1). Let $\mathcal{D}(p,r) := \{x : d_{\mathbb{H}^2}(x,p) < r\}$ be the disk of radius r centered at p. Then the disks $\mathcal{D}(\gamma(i), |OD|)$ are disjoint for distinct γ. Thus we get

$$\bigcup_{\|\gamma\|\leq R} \mathcal{D}(\gamma(i), |OD|) \subset \mathcal{D}(i, Q + |OD|),$$

where $Q = \operatorname{arccosh}(R^2/2)$, which implies that

$$\operatorname{Area}(\mathcal{D}(i, |OD|)) \cdot N_{\Gamma_g}(R) = \sum_{\|\gamma\|\leq R} \operatorname{Area}(\mathcal{D}(\gamma(i), |OD|)) \leq \operatorname{Area}(\mathcal{D}(i, Q + |OD|)).$$

By the hyperbolic area formula ([13, 2.10])

$$\operatorname{Area}(\mathcal{D}(p,r)) = 2\pi(\cosh r - 1),$$

together with (3.1), we get

$$N_{\Gamma_g}(R) \leq \frac{\cosh(Q + |OD|) - 1}{\cosh(|OD|) - 1} \leq C_0 \cosh(Q) = \frac{C_0}{2} R^2$$

for some absolute constant C_0 independent of g.

Finally, applying (3.5) to (3.3) and by (3.4), we deduce that $K_{Y_g} \lesssim g^6$ and finish the proof. $\square$

Remark 3.2. Note that $\operatorname{diam}(Y_g) \geq d_{Y_g}(O, A) = d_{\mathbb{H}^2}(O, A)$ in Figure 3.1, we have

$$\cosh(\operatorname{diam}(Y_g)) \geq \cosh(d(O, A)) = \cot^2(\beta) \gtrsim g^2.$$

Thus by (3.4) the estimate of the diameter is tight. But our estimate of other factors, say as pointed out by the anonymous referee, $N_{\Gamma_g}(R)$ may be slightly improved, probably resulting in $\lesssim g^{5.5}$ at last.

Remark 3.3. One may also use the spectral decomposition and Weyl's law about the density of exceptional eigenvalues to get a uniform bound for the counting of hyperbolic lattices.

3.2. **Modular surface**

The key to prove Proposition 2.1 is that co-compact Fuchsian groups have fundamental domains of finite diameter. For other Fuchsian groups Γ whose fundamental domain is not of finite diameter, the geodesic-covering number K_Γ may still exist. In particular for the modular group $\mathrm{PSL}_2(\mathbb{Z})$ we have the following result.

Proposition 3.4. *For modular surface, we have $K_{\mathrm{PSL}_2(\mathbb{Z})} \leq 10$ and the number of distinct distances among N points on the modular surface X is $\gtrsim N/\log N$.*

Proof. Let F be the standard fundamental domain

$$F := \left\{ z \in \mathbb{H}^2 \,\middle|\, -\frac{1}{2} < \Re(z) < \frac{1}{2},\ |z| > 1 \right\}.$$

For any $z_1 = x_1 + y_1 i, z_2 = x_2 + y_2 i \in F$, it is immediately to verify the Möbius transformation

$$z_j = \begin{pmatrix} \sqrt{y_j} & \frac{x_j}{\sqrt{y_j}} \\ 0 & \frac{1}{\sqrt{y_j}} \end{pmatrix} \cdot i = \gamma_j(i),\ j = 1, 2.$$

Then for each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we get

$$2\cosh(d_{\mathbb{H}^2}(z_1, \gamma(z_2))) = 2\cosh(d_{\mathbb{H}^2}(i, \gamma_1^{-1}\gamma\gamma_2(i))) = \|\gamma_1^{-1}\gamma\gamma_2\|^2.$$

Note that $\cosh(x)$ is monotonic for $x > 0$, we see that

$$d_X(z_1, z_2) = \operatorname{arccosh}\left(\min_{\gamma \in \mathrm{SL}_2(\mathbb{Z})} \|\gamma_1^{-1}\gamma\gamma_2\|/2 \right).$$

By computation,

$$\gamma_1^{-1}\gamma\gamma_2 = \begin{pmatrix} \sqrt{\frac{y_2}{y_1}}(a - x_1 c) & \frac{x_2 a + b - x_1 x_2 c - x_1 d}{\sqrt{y_1 y_2}} \\ \sqrt{y_1 y_2} c & \sqrt{\frac{y_1}{y_2}}(x_2 c + d) \end{pmatrix},$$

whence

$$\|\gamma_1^{-1}\gamma\gamma_2\|^2 = \frac{y_2}{y_1}(a - x_1 c)^2 + \frac{1}{y_1 y_2}(x_2 a + b - x_1 x_2 c - x_1 d)^2 + y_1 y_2 c^2 + \frac{y_1}{y_2}(x_2 c + d)^2.$$

If $c = 0$, then $\gamma = \left(\begin{smallmatrix} \pm 1 & b \\ 0 & \pm 1 \end{smallmatrix}\right)$ and

$$\|\gamma_1^{-1}\gamma\gamma_2\|^2 = \frac{y_2}{y_1} + \frac{1}{y_1 y_2}(x_2 - x_1 \pm b)^2 + \frac{y_1}{y_2}.$$

Note that $-\frac{1}{2} < x_1, x_2 < \frac{1}{2}$, the module $\|\gamma_1^{-1}\gamma\gamma_2\|^2$ attains minimum at $|b| \leq 1$ whose value is $< \frac{y_2}{y_1} + \frac{1}{4 y_1 y_2} + \frac{y_1}{y_2} := U(0)$.

If $c \neq 0$, we have $\frac{a}{c}\frac{d}{c} - \frac{1}{c^2} = \frac{b}{c}$ and

$$\begin{aligned}
&\|\gamma_1^{-1}\gamma\gamma_2\|^2 \\
&= c^2\left[\frac{y_2}{y_1}\left(\frac{a}{c}-x_1\right)^2 + \frac{1}{y_1y_2}\left(\frac{a}{c}x_2 + \frac{b}{c} - x_1x_2 - x_1\frac{d}{c}\right)^2 + y_1y_2 + \frac{y_1}{y_2}\left(\frac{d}{c}+x_2\right)^2\right] \\
&= c^2\left[\frac{y_2}{y_1}\left(\frac{a}{c}-x_1\right)^2 + \frac{1}{y_1y_2}\left(\left(\frac{a}{c}-x_1\right)\left(\frac{d}{c}+x_2\right) - \frac{1}{c^2}\right)^2 + y_1y_2 + \frac{y_1}{y_2}\left(\frac{d}{c}+x_2\right)^2\right] \\
&\geq c^2y_1y_2.
\end{aligned} \tag{3.6}$$

Comparing it with $U(0)$ and note that $-1/2 < x_2, x_1 < 1/2$ we get

$$\begin{aligned}
\|\gamma_1^{-1}\gamma\gamma_2\|^2 - U(0) &\geq c^2y_1y_2 - \frac{y_2}{y_1} - \frac{1}{4y_1y_2} - \frac{y_1}{y_2} \\
&> \frac{c^2y_1^2y_2^2 - y_1^2 - y_2^2 - \frac{1}{4}}{y_1y_2} \\
&= \frac{\left(|c|y_1^2 - \frac{1}{|c|}\right)\left(|c|y_2^2 - \frac{1}{|c|}\right) - \frac{1}{c^2} - \frac{1}{4}}{y_1y_2}.
\end{aligned}$$

For $|c| \geq 2$ we have

$$\begin{aligned}
\|\gamma_1^{-1}\gamma\gamma_2\|^2 - U(0) &\geq \frac{\left(2y_1^2 - \frac{1}{2}\right)\left(2y_2^2 - \frac{1}{2}\right) - \frac{1}{2}}{y_1y_2} \\
&> \frac{\left(\frac{3}{2} - \frac{1}{2}\right)^2 - \frac{1}{2}}{y_1y_2} > 0,
\end{aligned}$$

since $y_j > \sqrt{3}/2$, $j = 1, 2$. Thus in order to choose for Γ_0 as in (2.1), we only need $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ with $|c| \leq 1$.

For $|c| = 1$, we have $ad \pm b = 1$ (so that a and d can be chosen arbitrarily). We claim that in this case, (3.6) attains minimum when $|a| \leq 1, |d| \leq 1$. By choosing $\gamma \in \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ we may assume $c = 1$. Let $t_1 = a - x_1$ and $t_2 = d + x_2$, then (3.6) becomes

$$\|\gamma_1^{-1}\gamma\gamma_2\|^2 = \frac{y_2}{y_1}t_1^2 + \frac{1}{y_1y_2}(t_1t_2 - 1)^2 + \frac{y_1}{y_2}t_2^2 + y_1y_2. \tag{3.7}$$

We prove the claim by refuting the contradictory cases: (i) if $c = 1, |a| \geq 2, |d| \geq 2$, note that $|x_1| \leq 1/2, |x_2| \leq 1/2$, then $|t_1| \geq 3/2, |t_2| \geq 3/2$ and (3.7) becomes

$$\|\gamma_1^{-1}\gamma\gamma_2\|^2 \geq \frac{y_2}{y_1}\cdot\frac{9}{4} + \frac{1}{y_1y_2}\cdot\frac{25}{16} + \frac{y_1}{y_2}\cdot\frac{9}{4} + y_1y_2 > U(0);$$

(ii) if $c = 1, |a| \leq 1, |d| \geq 2$, then $|t_2| \geq 3/2$ and we take the difference (t_1 or a fixed)

$$
\begin{aligned}
&\min_{c=1,|d|\geq 2} \|\gamma_1^{-1}\gamma\gamma_2\|^2 - \min_{c=1,|d|\leq 1} \|\gamma_1^{-1}\gamma\gamma_2\|^2 \\
&\geq \frac{1}{y_1y_2}(|t_1|\cdot\frac{3}{2}-1)^2 + \frac{y_1}{y_2}\cdot\frac{9}{4} - \min_{|d|\leq 1}\left\{\frac{1}{y_1y_2}(|t_1||d+x_2|-1)^2 + \frac{y_1}{y_2}(d+x_2)^2\right\} \\
&\geq \min_{|d|\leq 1}\left\{\frac{1}{y_1y_2}\left[\left(\frac{9}{4}-(d+x_2)^2\right)t_1^2 + (2|d+x_2|-3)|t_1|\right] + \frac{y_1}{y_2}\cdot\left(\frac{9}{4}-(d+x_2)^2\right)\right\} \\
&\geq \min_{|d|\leq 1}\left\{\frac{1}{y_1y_2}\left[2t_1^2 + (2|d+x_2|-3)|t_1| + 2y_1^2\right]\right\} \\
&\geq \frac{1}{y_1y_2}\left[2\left(|t_1|-\frac{3}{4}\right)^2 - \frac{9}{8} + 2\cdot\frac{3}{4}\right] > 0,
\end{aligned}
$$

noting that $|y_1| \geq \sqrt{3}/2$; (iii) for $c = 1, |a| \geq 2, |d| \leq 1$, the above difference (for t_2 fixed) stays positive if symmetrically the roles of t_1, d are replaced by t_2, a. Thus the claim is proved.

In conclusion, we may choose $\Gamma_0 \subset \mathrm{PSL}_2(\mathbb{Z})$ consisting of

$$
1, \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & \pm 1 \end{pmatrix}, \begin{pmatrix} \pm 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
$$

Thus $K_{\mathrm{PSL}_2(\mathbb{Z})} \leq 10$. Then by Theorem 2.3 we get the desired lower bound for distinct distances on modular surface. □

Here the geodesic cover $\cup_{\gamma\in\Gamma_0}\gamma(\mathbb{F})$ is F together with the nine neighbouring fundamental domains on $\mathbb{H}^2$. Actually we may only choose the geodesic cover in the sense of (1.1) (for *geodesic pre-cover*) as

$$
\Gamma_1 = \left\{1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}
$$

since $\Gamma_1^{-1}\Gamma_1 = \Gamma_0$.

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