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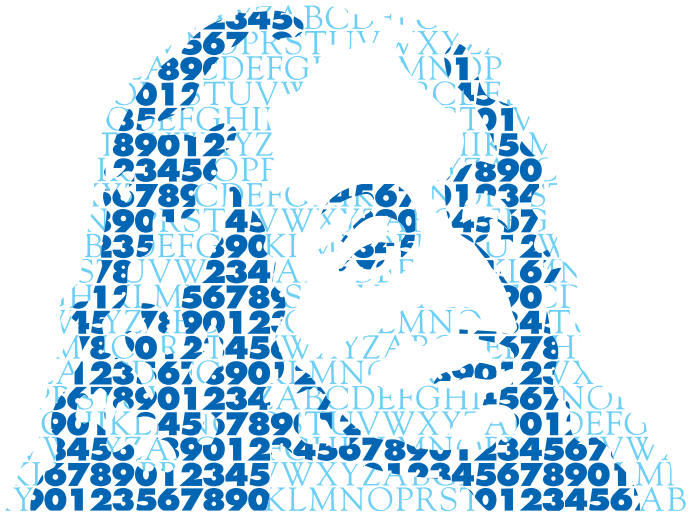

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Geodesic cover of Fuchsian groups

Zhipeng Lu

Abstract

We study unions of fundamental domains of a Fuchsian group, especially those with hyperbolic plane metric realizing the metric of the corresponding hyperbolic surface. We call these unions the *geodesic covers* of the Fuchsian group or the hyperbolic surface. The paper contributes to showing that finiteness of geodesic covers is basically another characterization of geometrically finiteness. The resolution of geometrically finite case is based on Shimizu's lemma.

1. Introduction

In the work of Lu–Meng [6], a notion called *geodesic cover* of Fuchsian groups is introduced to deal with the Erdős distinct distances problem in hyperbolic surfaces. There we initiate the investigation of quantitative aspects of the notion for special Fuchsian groups including the modular group and standard regular surface groups. In this paper, we conduct an overall qualitative study and give a relatively complete description of geodesic covers for general Fuchsian groups.

We recall the definition as follows. Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian group acting on the upper half plane $\mathbb{H}^2$ by Möbius transformation, and $F \subset \mathbb{H}^2$ be a fundamental domain of Γ whose interior contains no two points in the same orbit of the Möbius transformation by Γ. A subset $\Gamma_0 \subset \Gamma$, which we always assume contains the identity, is called a *geodesic cover* of Γ corresponding to F if for any $p, q \in F$, we have

$$\min_{\gamma_1,\gamma_2\in\Gamma} d_{\mathbb{H}^2}(\gamma_1 \cdot p, \gamma_2 \cdot q) = \min_{\gamma_1,\gamma_2\in\Gamma_0} d_{\mathbb{H}^2}(\gamma_1 \cdot p, \gamma_2 \cdot q),$$

where $\gamma \cdot p$ denotes the Möbius transformation and $d_{\mathbb{H}^2}$ denotes the hyperbolic metric in $\mathbb{H}^2$. A weaker requirement gives rise to another definition: $\Gamma_1 \subset \Gamma$ is a geodesic cover of Γ if

$$\min_{\gamma\in\Gamma} d_{\mathbb{H}^2}(p, \gamma \cdot q) = \min_{\gamma\in\Gamma_0} d_{\mathbb{H}^2}(p, \gamma \cdot q).$$

We may call the former definition the *first geodesic cover* and the latter the *second geodesic cover*. Clearly if Γ_0 is a first geodesic cover of Γ, then $\Gamma_0^{-1}\Gamma_0 := \{\gamma_1^{-1}\gamma_2 \mid \gamma_1, \gamma_2 \in \Gamma_0\}$ is a second geodesic cover of Γ. We focus on the second geodesic cover since it appears

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more convenient for estimation. If Γ has finite geodesic covers, we call the smallest size of (first or second) geodesic covers the (first or second) *geodesic covering number* of Γ.

The definitions arises equivalently in the corresponding hyperbolic surface $S_\Gamma := \Gamma\backslash\mathbb{H}^2$ endowed the hyperbolic metric from $\mathbb{H}^2$. For $p, q \in S_\Gamma$, we can pick two representatives (still denoted by p, q) in F to realize their distance by $d_{S_\Gamma}(p, q) = \min_{\gamma\in\Gamma_Y} d_{\mathbb{H}^2}(p, \gamma \cdot q)$. Thus sometimes we may also refer the same notion to the corresponding hyperbolic surface of a Fuchsian group. Note that although geodesic covers may depend on the choice of fundamental domains, the geodesic covering number only depend on the Fuchsian group. A problem may arise that, since in general two fundamental domains may not be commensurable, we are not sure if any two minimal geodesic covers Γ_1, Γ_2 corresponding to different fundamental domains are *commensurable*, i.e. there are constants $c, C > 0$ (depending only on Γ) such that $c|\Gamma_1| \le |\Gamma_2| \le C|\Gamma_1|$.

In [6], the geodesic covering number of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ is precisely estimated to be 4 in the first definition and 10 in the second. Also, those of standard regular hyperbolic surfaces of genus $g \ge 2$ are estimated to establish lower bounds of Erdős distinct distances problem in such surfaces, see Theorem 1.2 and Proposition 3.1 of [6].

In this paper, we extend our scope of estimation and prove

Theorem 1.1. *Geometrically finite Fuchsian groups have finite geodesic covers.*

This is the combination of Theorem 3.11 and Theorem 3.13, which addresses Conjecture 3 of [6]. Our proof is based on Shimizu's lemma introduced by Shimizu [7].

Similar to the argument for Theorem 1.1 of [6], this in turn establishes Guth–Katz type lower bound of Erdős distinct distances problem in hyperbolic surfaces corresponding to geometrically finite Fuchsian groups. More specifically, to enumerate $d(P) = |\{d_Y(p, q) \mid p, q \in P\}|$ for any finite set $P \subset Y$ and Y a geometrically finite hyperbolic surface corresponding to the Fuchsian group Γ, we estimate for $Q(P) = |\{(p_1, q_1, p_2, q_2) \in P^4 \mid d_Y(p_1, q_1) = d_Y(p_2, q_2)\}|$, since by Cauchy–Schwarz inequality $d(P) \ge \frac{|P|^4}{Q(P)}$. By alleviating P onto $\widetilde{P} \subset \mathbb{H}^2$ by a finite geodesic cover Γ_1 with $|\widetilde{P}| \le |\Gamma_1||P|$, we can estimate $Q(\widetilde{P})$ in $\mathbb{H}^2$. Due to the similar estimate as to Guth–Katz as in [6], we have $Q(\widetilde{P}) \le c|\widetilde{P}|^3 \log|\widetilde{P}| \le c|\Gamma_1|^3|P|^3 \log(|\Gamma_1||P|)$. Hence Theorem 1.1 directly implies

Corollary 1.2. *A set of N points in a hyperbolic surface Y corresponding to a geometrically finite Fuchsian group determines $\ge c\frac{N}{K_Y^3 \log(K_Y N)}$ distinct distances, where K_Y denotes the geodesic covering number of Y and $c > 0$ is some absolute constant.*

To the complementary, in Section 2.2 we show that

Theorem 1.3. *Infinitely generated Fuchsian groups (without elliptic elements) can not have finite geodesic covers.*

In the case of being infinitely generated, we wonder what the lower bound of Erdős distinct distances problem in the corresponding hyperbolic surface would look like.

It will be interesting to see more precise computational results on geodesic covering numbers of geometrically finite groups and their applications to various geometric and combinatorial problems. The conjectured relation between those numbers and signatures of Fuchsian groups or Fenchel–Nielsen coordinates in the Techmüller space in [6] still calls for more studies, to which the current paper may have implications. We also look forward to generalizing the notion and the quantitative and qualitative results to Klein groups and cases of higher rank discrete groups where potential substitutes of Shimizu's lemma may hold, see [4] for instances.

2. **Generals on geodesic covers**

In this section, we give a construction of geodesic covers for general Fuchsian groups using Dirichlet domains, and establish some preliminary results.

2.1. **Construction of geodesic cover**

For a general Fuchsian group $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ and any $z_0 \in \mathbb{H}^2$, we define the *Dirichlet domain* for Γ *centered* at z_0 to be

$$D_\Gamma(z_0) := \{z \in \mathbb{H}^2 \mid d_{\mathbb{H}^2}(z, z_0) \leq d_{\mathbb{H}^2}(z, \gamma \cdot z_0), \forall\, \gamma \in \Gamma\}.$$

Geometrically, $D_\Gamma(z_0)$ is the intersection of hyperbolic half-planes bounded by perpendicular bisectors of the geodesic segments between z_0 and $\gamma \cdot z_0$ for all $\gamma \in \Gamma$. If z_0 is not fixed by any non-identity element of Γ, then $D_\Gamma(z_0)$ becomes a connected fundamental domain for Γ. Otherwise, z_0 may only be fixed by an elliptic element of Γ, in which case z_0 is an *elliptic point* of Γ. In the case, $D_\Gamma(z_0)$ is not a fundamental domain, but we have

Lemma 2.1. *For any Fuchsian group Γ and an elliptic point z_0 of Γ, $D_\Gamma(z_0)$ meets each orbit $\Gamma \cdot z$ at least $|\mathrm{Stab}_\Gamma(z_0)|$ times.*

Proof. For any orbit $\Gamma \cdot z$, there is a $\gamma_1 \in \Gamma$ such that $d_{\mathbb{H}^2}(z_0, \gamma_1 \cdot z) = d_{\mathbb{H}^2}(z_0, \Gamma \cdot z)$ by the discrete action of Γ, hence $\gamma_1 \cdot z \in D_\Gamma(z_0)$. Clearly for any $z \in D_\Gamma(z_0)$ and $1 \neq \gamma \in \mathrm{Stab}_\Gamma(z_0)$, $d(z_0, z) = d(z_0, \gamma \cdot z)$. Hence $\gamma \cdot z \in D_\Gamma(z_0)$ and $\gamma \cdot z \neq z$, otherwise $\gamma = 1$. Thus $\mathrm{Stab}_\Gamma(z_0) \subset D_\Gamma(z_0) \cap \Gamma \cdot z$ which contains at least $|\mathrm{Stab}_\Gamma(z_0)|$ points. □

For regular z, $\mathrm{Stab}_\Gamma(z) = \{1\}$ so that the lemma also applies.

For any $z_0 \in \mathbb{H}^2$ not fixed by any non-identity elements of Γ, define $B_\Gamma(z_0) := \bigcup_{z \in D_\Gamma(z_0)} D_\Gamma(z)$, and

$$U_\Gamma(z_0) := \{\gamma \in \Gamma \mid \gamma \cdot D_\Gamma(z_0) \cap B_\Gamma(z_0) \neq \emptyset\}. \tag{2.1}$$

Then we simply have

Proposition 2.2. *$U_\Gamma(z_0)$ is a geodesic cover of Γ corresponding to $D_\Gamma(z_0)$.*

Proof. For any $z_1, z_2 \in D_\Gamma(z_0)$, by Lemma 2.1, there exists $z_2' \in D_\Gamma(z_1)$ such that $z_2' = \gamma' \cdot z_2$ for some $\gamma' \in \Gamma$, and $d_{\mathbb{H}^2}(z_1, z_2') \leq d_{\mathbb{H}^2}(z_1, \gamma \cdot z_2)$ for any $\gamma \in \Gamma$. Then $z_2' \in \gamma' \cdot D_\Gamma(z_0) \cap B_\Gamma(z_0) \neq \emptyset$. Hence $\gamma' \in U_\Gamma(z_0)$. □

For Γ co-compact, i.e. some fundamental Dirichlet domain $D_\Gamma(z_0)$ is compact, $B_\Gamma(z_0)$ is compact by local finiteness of Dirichlet tessellations (Theorem 3.5.1 of [5]). In turn, further by local finiteness, $U_\Gamma(z_0)$ contains only finitely many elements of Γ, which accounts for Proposition 2.1 of [6].

2.2. **Geodesic covers and finite index subgroups**

Now we establish two basic results, one relating geodesic covers and finite index subgroups of Fuchsian groups, and one concerning groups without elliptic elements. Especially the latter one shows that generally infinitely generated Fuchsian groups can not have finite geodesic covers.

Proposition 2.3. *Let $H \leq \Gamma$ be any finite index subgroup of a Fuchsian group Γ. If H has finite geodesic covers, then so does Γ and the geodesic covering number of Γ is no more than $[\Gamma : H]$ times that of H.*

Proof. Let $\Gamma = Hg_1 \cup \cdot \cup Hg_n$ be the right coset decomposition. If F is a fundamental domain of Γ, then $g_1 \cdot F \cup \cdots \cup g_n \cdot F$ is a fundamental domain of H, see Theorem 3.12 of [5]. We may choose F such that there is a corresponding finite geodesic cover $H_0 \subset H$. For any $z_1 \neq z_2 \in F$, we have

$$\begin{aligned}\min_{\gamma\in\Gamma} d_{\mathbb{H}^2}(z_1, \gamma \cdot z_2) &= \min_{1\leq i\leq n} \min_{\gamma\in H} d_{\mathbb{H}^2}(z_1, \gamma \cdot (g_i \cdot z_2)) \\ &= \min_{1\leq i\leq n} \min_{\gamma\in H_0} d_{\mathbb{H}^2}(z_1, \gamma g_i \cdot z_2) = \min_{\gamma\in H_0 g_1\cup\cdots\cup H_0 g_n} d_{\mathbb{H}^2}(z_1, \gamma \cdot z_2).\end{aligned}$$

Hence $H_0 g_1 \cup \cdots \cup H_0 g_n$ is a geodesic cover of Γ which implies the lemma. □

If we can show that some well-behaved finite index subgroup of a Fuchsian group has finite geodesic cover, then so does the bigger group. We will use this fact to deal with geometrically finite groups in the next section.

Next, we prove the following result based on geometric considerations.

Proposition 2.4. *Let Γ be any Fuchsian group without elliptic elements, and Γ_0 be its geodesic cover corresponding to a fundamental domain F of Γ. Then $\Gamma_0 \supset \{\gamma \in \Gamma \mid \gamma \cdot F \cap F \neq \emptyset\} := \Gamma(F)$.*

Proof. Our proof is motivated by the proof of Theorem 9.3.3 of Beardon [1]. For any $1 \neq \gamma \in \Gamma(F)$, suppose $w \in \gamma \cdot F \cap F$. Then there is an open disc N with center w and elements γ_0 (= 1), $\gamma_1, \ldots, \gamma_t \in \Gamma$ such that $\gamma_j = \gamma$ for some j, and

$$w \in \bigcap_{i=0}^{t} \gamma_i \cdot F, \quad N \subset \bigcup_{i=0}^{t} \gamma_i \cdot F.$$

N can be chosen small enough so that no other vertices of any $\gamma_i \cdot F$ falls inside N. More features are shown in Figure 9.3.1 of [1] on p. 220.

Now that Γ has no elliptic elements, for N small enough, any two points in N are not in the same orbit of Γ by local finiteness. Choose any point $z \in \gamma \cdot F \cap N$, and its conjugate $z' = \gamma^{-1} \cdot z \in F$. By definition, there is some $\gamma_0 \in \Gamma_0$ such that

$$d_{\mathbb{H}^2}(w, z) = \min_{\gamma \in \Gamma} d_{\mathbb{H}^2}(w, \gamma \cdot z') = d_{\mathbb{H}^2}(w, \gamma_0 \cdot z').$$

Then $z = \gamma_0 \cdot z' = \gamma_0 \gamma^{-1} \cdot z$, which implies $\gamma = \gamma_0$ since there are no elliptic elements in Γ. This shows that any elements of $\Gamma(F)$ belongs to Γ_0. □

Remark 2.5. Proposition 3.1 of [6] says the smaller set $\{\gamma \in \Gamma \mid \gamma \cdot D_\Gamma(z_0) \cap D_\Gamma(z_0) \neq \emptyset\}$ is a geodesic cover for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. It is hard to tell if this set consists in a geodesic cover in general from the above proof.

The proposition reveals the following basic fact:

Corollary 2.6. *Any infinitely generated Fuchsian group without elliptic elements can not have finite geodesic covers.*

Proof. Note that in general, $\Gamma(F)$ generates Γ, see Theorem 9.2.7 of [1]. With the notations in Proposition 2.4, if some Γ_0 is finite, so is the corresponding $\Gamma(F)$, then $\Gamma = \langle \Gamma(F) \rangle$ becomes finitely generated. □

Note that in general any Fuchsian group, has a finite index subgroup without elliptic elements, which in the case of geometrically finite groups is accounted as Nielsen–Fenchel–Fox theorem, see Proposition 3.10 later. Thus the corollary exhibits groups with only infinite geodesic covers for general infinitely generated Fuchsian groups. To conclude the scenario, we will show that finitely generated Fuchsian groups have finite geodesic covers in the next section.

3. Geodesic covers of geometrically finite Fuchsian groups

3.1. Geometry of geometrically finite Fuchsian groups

This section contributes to introducing preliminary notions on geometrically finite Fuchsian groups following Katok [5].

Call a Fuchsian group $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ *geometrically finite* if there exists one (hence every) geodesically convex fundamental domain of Γ having finitely many sides. Equivalently, Γ is finitely generated. Also co-finite Fuchsian groups, i.e. with fundamental domains of finite hyperbolic area, are precisely geometrically finite of first kind.

We describe some basic facts on the geometry of boundary of a fundamental Dirichlet domain $D := D_\Gamma(z_0)$ for an arbitrary Fuchsian group Γ. By definition, the boundary of D is a collection of geodesic segments or segments of the real axis (free sides), called *sides*. An intersection point of two sides is called a *vertex*. Two points of D are congruent, i.e. in the same orbit of Γ, if and only if they belong to the boundary ∂D. In this case, they are of same distance to z_0 (see Theorem 9.4.3 of [1]).

Let us consider the vertices in congruence, each class of which is called a *cycle*. By local finiteness, a cycle contains only finitely many points. If a vertex $v \in \mathbb{H}^2$ of D is an elliptic point fixed by $\gamma\Gamma$, it must lie on ∂D, and every elliptic point of Γ is congruent to one on ∂D. Also the *elliptic cycle* of v is fixed by elliptic elements congruent to γ in Γ. This reveals the following

Proposition 3.1 (Theorem 3.5.2 of [5])**.** *With the above notations, the elliptic cycles of ∂D 1-1 correspond to the congruence classes of elliptic points or maximal elliptic subgroups of Γ.*

If v is an elliptic vertex of D, fixed by $\gamma \in \Gamma$ say with order k, then γ maps edges to edges and the inner angle at v is at most $2\pi/k$. If $k = 2$, then γ flips the edge containing v around v. Moreover, we have

Proposition 3.2 (Theorem 3.5.3 of [5])**.** *With the above notations, let $\theta_1, \ldots, \theta_t$ be the internal angles at all vertices in an elliptic cycle and m be the order of the elliptic elements fixing these vertices. Then $\theta_1 + \cdots + \theta_t = 2\pi/m$.*

If a cycle has no fixed points, we may set $m = 1$ and then $\theta_1 + \cdots + \theta_t = 2\pi$. Regarded as infinite order elliptic elements, each parabolic element of Γ has a unique fixed point on $\widehat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, which are usually called *cusps* of Γ. For co-finite groups, we have

Proposition 3.3 (Theorem 4.2.5 of [5]). *Suppose* Γ *has a non-compact Dirichlet fundamental domain* D *of finite hyperbolic area. Then*

(1) *each vertex of* D *on* $\widehat{\mathbb{R}}$ *is a cusp;*

(2) *each cusp is congruent to a vertex of* D *on* $\widehat{\mathbb{R}}$.

Similarly, congruence classes of cusps of Γ 1-1 correspond to maximal parabolic subgroups of Γ.

Note that co-finite Fuchsian groups are geometrically finite of first kind. As to the remaining geometrically finite groups Γ of second kind, according to their limit sets $\Lambda(\Gamma) := \{z \in \widehat{\mathbb{R}} \mid z \in \overline{\Gamma \cdot a} \text{ for some } a \in \mathbb{H}^2\}$, we classify them as follows. If $|\Lambda(\Gamma)| \leq 2$, then Γ is *elementary* (Exercise 3.8 of [5]), i.e. has finite orbits in $\mathbb{H}^2 \cup \widehat{\mathbb{R}}$, which can be described as follows.

Proposition 3.4 (Theorem 2.4.3 of [5]). *Any elementary Fuchsian group is either cyclic or is conjugate in* $\mathrm{PSL}_2(\mathbb{R})$ *to a group generated by* $g_k = \left(\begin{smallmatrix} k & 0 \\ 0 & 1/k \end{smallmatrix}\right)$ *(*$k > 1$*) and* $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

If Γ is finite cyclic, then it is elliptic and $\Lambda(\Gamma) = \emptyset$; if Γ is infinite cyclic, then it is parabolic and $|\Lambda(\Gamma)| = 1$, or it is hyperbolic congruent to $\langle g_k \rangle$ for some $k > 0$, in which case $|\Lambda(\Gamma)| = 2$. For Γ conjugate to $\langle g_k, S \rangle$, similarly $|\Lambda(\Gamma)| = 2$. For example, $\Lambda\left(\langle g_k, S \rangle\right) = \{0, \infty\}$ for any $k > 1$.

If $|\Lambda(\Gamma)| > 2$, then Γ is non-elementary and $\Lambda(\Gamma)$ is a perfect nowhere dense subset of $\widehat{\mathbb{R}}$ (hence uncountably infinite) by Theorem 3.4.6 of [5]. In this case, the complement of $\Lambda(\Gamma)$ in $\widehat{\mathbb{R}}$ is a union of countably many open intervals $\{I_j\}_{j=1}^{\infty}$ that are mutually disjoint. Let L_j be the geodesic striding over I_j and connecting its two end points in $\mathbb{H}^2$, and let H_j be the open half-plane bounded by L_j away from I_j. Now we introduce the notion of *Nielsen region* following 8.4 of [1] as

$$N_\Gamma := \bigcap_{j=1}^{\infty} H_j. \tag{3.1}$$

Note that $\{I_j\}$ is Γ-invariant, so is $\{H_j\}$. Therefore N_Γ is a Γ-invariant geodesically convex subset of $\mathbb{H}^2$. Actually, it is the smallest such non-empty set, see Theorem 8.5.2 of [1]. The following fact reveals its significance:

Proposition 3.5 (Theorem 10.1.2 of [1]). *A non-elementary Fuchsian group* Γ *is finitely generated if and only if for any convex fundamental domain* D *of* Γ*, the hyperbolic area of* $D \cap N_\Gamma$ *is finite.*

One key observation is that the free sides (segments of $\mathbb{R}$) are each contained in some I_j, then $D \cap N_\Gamma$ is an polygon in $\mathbb{H}^2$ which has finite area by Gauss–Bonnet formula, see the proof on p. 254 of [1].

3.2. **Geodesic cover of co-finite groups**

Now we try to establish the seemingly surprising fact that, the construction of (2.1) gives a finite geodesic cover for geometrically finite Fuchsian groups of first kind, i.e. co-finite groups, with a technicality assumption that they contain no elliptic elements.

To proceed we introduce two useful corollaries of Shimizu's lemma which first appeared in [7]. For a given Fuchsian group Γ, denote by $\widetilde{\mathbb{H}}^2$ the union of $\mathbb{H}^2$ and the cusps of Γ.

Proposition 3.6 (Lemma 1.26 of Shimura [8])**.** *For every cusp s of a Fuchsian group Γ, there exists a neighborhood $U \subset \widetilde{\mathbb{H}}^2$ of s such that* $\mathrm{Stab}_\Gamma(s) = \{\gamma \in \Gamma \mid \gamma \cdot U \cap U \neq \emptyset\}$.

Proposition 3.7 (Theorem 9.2.8(ii) of [1])**.** *Let D be any locally finite fundamental domain for a Fuchsian group Γ, $\gamma \in \Gamma$ be a parabolic element, and U be a horocyclic region with $\gamma \cdot U = U$. Then D meets some finitely many distinct translates $\gamma' \cdot U, \gamma' \in \Gamma$.*

Here a *horocyclic region* of a parabolic element γ is a neighborhood of the fixed point of γ congruent to $\{z \in \mathbb{H}^2 \mid \mathrm{Im}(z) > t\}$ for some $t > 0$. Beardon's proof uses Jørgensen's inequality, which may be seen as a generalized version of Shimizu's lemma.

Now we modify the construction of (2.1) using a *horocyclic surgery* on Dirichlet domains resorting to the above two results. Assume Γ is a co-finite Fuchsian group without elliptic elements (with cusps otherwise co-compact), so that for any $z \in \mathbb{H}^2$, $D_\Gamma(z)$ is a fundamental Dirichlet domain. Fix a point $z_0 \in \mathbb{H}^2$ and let $D := D_\Gamma(z_0)$. Suppose $s_1, \ldots, s_t$ are all the cusp vertices on ∂D. According to Proposition 3.6 and 3.7, we may choose horocyclic neighborhood U_i of each s_i such that

$$\mathrm{Stab}_\Gamma(s_i) = \{\gamma \in \Gamma \mid \gamma \cdot U_i \cap U_i \neq \emptyset\}, \quad \forall\, i = 1, \ldots, t, \tag{3.2}$$

and D meets only finitely many translates of U_i's. Now define

$$\widetilde{D} = D \setminus \left(\bigcup_{i=1}^{t} U_i \right) \tag{3.3}$$

to be the *horocyclic truncation* of D which becomes compact.

For any $z \in \mathbb{H}^2$, by Proposition 3.3, each cusp of $D_\Gamma(z)$ is congruent to some s_i in Γ. We may list its cusps as $s_1(z) = \gamma_{1,z} \cdot s_{i_1}, \ldots, s_{t_z}(z) = \gamma_{t_z,z} \cdot s_{i_{t_z}}$ for some $\gamma_{1,z}, \ldots, \gamma_{t_z,z} \in \Gamma$. Note that for each j, there are infinitely many $\gamma_{j,z}$ such that $s_j(z) = \gamma_{j,z} s_{i_j}$. But we can restrict the translations to a bounded number by requiring that $\gamma_{j,z} \cdot (D \cap U_{i_j}) \subset D_\gamma(z)$. Similar to (3.2),

$$\mathrm{Stab}_\Gamma(s_j(z)) = \gamma_{j,z}\, \mathrm{Stab}_\Gamma(s_{i_j}) \gamma_{j,z}^{-1} = \{\gamma \in \Gamma \mid \gamma \cdot (\gamma_{j,z} \cdot U_{i_j}) \cap (\gamma_{j,z} \cdot U_{i_j}) \neq \emptyset\},$$

and similar to (3.3) we define the horocyclic truncation

$$\widetilde{D}_\Gamma(z) = D_\Gamma(z) \setminus \left(\bigcup_{j=1}^{t_z} \gamma_{j,z} \cdot U_{i_j} \right).$$

After the above surgery we modify our previous construction of (2.1) by

$$\widetilde{B}_\Gamma(z_0) := \bigcup_{z \in \tilde{D}} \widetilde{D}_\Gamma(z),\ \widetilde{U}_\Gamma(z_0) := \{\gamma \in \Gamma \mid \gamma \cdot D \cap \widetilde{B}_\Gamma(z_0) \neq \emptyset\}. \tag{3.4}$$

The following two facts account for the significance of our construction.

Lemma 3.8. *With the above notations as* (3.4)*, $\widetilde{B}_\Gamma(z_0)$ is compact.*

Proof. For each $z \in \widetilde{D}$ and each cusp $s_j(z)$ of $D_\Gamma(z)$, we have the intersection of two closed sets $\gamma_{j,z} \cdot D \cap D_\Gamma(z)$ contains some horocyclic region $\gamma_{j,z} \cdot U_{i_j}$. Then $\gamma_{j,z}^{-1} \cdot D \cap \widetilde{D} \neq \emptyset$. Since $\widetilde{D}$ is compact, the set $\{\gamma_{j,z} \mid z \in \widetilde{D}, j \leq t_z\}$ is finite by local finiteness. Thus $\widetilde{D}_\Gamma(z)$ is uniformly bounded for $z \in \widetilde{D}$ and their union, i.e. $\widetilde{B}_\Gamma(z_0)$ is compact. □

Lemma 3.9. *With the notations of Lemma 3.8, $U_\Gamma(z_0) \setminus \widetilde{U}_\Gamma(z_0)$ is a finite set.*

Proof. Clearly $\widetilde{U}_\Gamma(z_0) \subset U_\Gamma(z_0)$ by definition. Suppose $\gamma \in U_\Gamma(z_0)$. Then $\gamma \cdot D \cap D_\Gamma(z) \neq \emptyset$ for some $z \in D$. If $\gamma \notin \widetilde{U}_\Gamma(z_0)$, then $\gamma \cdot D \cap \gamma_{j,z} U_{i_j} \neq \emptyset$ for some horocyclic region U_{i_j} of D, or equivalently $D \cap \gamma^{-1}\gamma_{j,z} U_{i_j} \neq \emptyset$. By Proposition 3.7 and our restriction on the $\gamma_{j,z}$'s, there are finitely many elements, say $\gamma_1, \ldots, \gamma_n \in \Gamma$, such that $\gamma^{-1}\gamma_{j,z} = \gamma_k$ for some k. Then $\gamma \in \{\gamma_{j,z}\gamma_k^{-1} \mid z \in D, j \leq t_z, k \leq n\}$, a finite set by the last proof. □

To eliminate the restriction of having no elliptic elements, we resort to

Proposition 3.10 (Nielsen–Fenchel–Fox)**.** *Any finitely generated Fuchsian group contains a subgroup of finite index without elliptic elements.*

This was the Fenchel's conjecture proved by Fox [3], the proof of which contains an error later fixed by Chau [2]. Note that further by Poincaré's lemma in group theory, there exists a normal subgroup of finite index without elliptic elements although we do not need this latter fact.

Together with Proposition 2.3 we are ready to prove the following

Theorem 3.11. *Any co-finite Fuchsian group has finite geodesic covers.*

Proof. By Proposition 3.10, choose a finite index subgroup $\Gamma' \leq \Gamma$ which contains no elliptic elements. Fix any $z_0 \in \mathbb{H}^2$, then $\widetilde{B}_{\Gamma'}(z_0)$ is compact by Lemma 3.8, which implies that $\widetilde{U}_{\Gamma'}(z_0)$ is finite by local finiteness. Further by Lemma 3.9 and Proposition 2.2, $\widetilde{U}_{\Gamma'}(z_0)$ is finite geodesic cover of Γ'. Finally, by Proposition 2.3, Γ has a finite geodesic of size $\leq [\Gamma : \Gamma']|\widetilde{U}_{\Gamma'}(z_0)|$. □

3.3. **Geodesic cover of geometrically finite groups of second kind**

For the remaining case where Γ is a geometrically finite Fuchsian group of second kind, we modify our construction through a bisection using Nielsen region N_Γ introduced as (3.1).

We assume Γ is non-elementary with no elliptic elements. Then for each $z \in \mathbb{H}^2$, the convex fundamental Dirichlet domain $D_\Gamma(z)$ intersects N_Γ at a polygon with finitely many sides in $\mathbb{H}^2$. Moreover, each free side of $D_\Gamma(z)$ is contained in an open interval in $\widehat{\mathbb{R}} \smallsetminus \Lambda(\Gamma)$.Write $\widehat{D}_\Gamma(z) := D_\Gamma(z) \cap N_\Gamma$. Now for a point $z_0 \in \mathbb{H}^2$, form the set $\widehat{B}_\Gamma(z_0) := \bigcup_{z\in D} \widehat{D}_\Gamma(z)$ and construct

$$\widehat{U}_\Gamma(z_0) := \{\gamma \in \Gamma \mid \gamma \cdot D \cap \widehat{B}_\Gamma(z_0) \neq \emptyset\}.$$

Since each $\widehat{D}_\Gamma(z)$ is a finite-sided polygon possible with cusps of Γ, we may perform the same horocyclic surgery as in the last subsection on $\widehat{D}_\Gamma(z)$.

Similar to the proof of Lemma 3.8, $\widehat{B}_\Gamma(z_0)$ has finitely many cusps hence $B_\Gamma(z_0)$ has finitely many free sides, say contained in the intervals $I_1, \ldots, I_n \subset \widehat{R} \smallsetminus \Lambda(\Gamma)$. Also, similarly by Lemma 3.8 $\widehat{U}_\Gamma(z_0)$ is finite. For any $\gamma \in U_\Gamma(z_0) \smallsetminus \widehat{U}_\Gamma(z_0)$, $\gamma \cdot D$ intersects some of the interval I_j. Then it has a free side contained in I_j and must intersect a *hypercyclic region* U_j having same end points (fixed points of some hyperbolic elements of Γ) with I_j. Here a hypercyclic region is congruent to a set of the form $\{z \in \mathbb{H}^2 \mid |\arg(z) - \pi/2| < \theta\}$. However, such γ if restricted to having intersection with a Dirichlet region contained in $B_\Gamma(z_0)$ with a free side in I_j, amount to be finite due to

Proposition 3.12 (Theorem 9.8.2 (iii) of [1])**.** *Let D be any locally finite fundamental domain for Γ and U be a hypercyclic region. Then D meets some finitely many translates $\gamma \cdot U, \gamma \in \Gamma$.*

Now that there are finitely many I_j's, similar to Lemma 3.9 we see that

$$|U_\Gamma(z_0) \smallsetminus \widehat{U}_\Gamma(z_0)| < \infty.$$

Covering the general case of non-elementary geometrically finite Fuchsian groups of second kind using Proposition 3.10 and 2.3, we have proved

Theorem 3.13. *Any non-elementary geometrically finite Fuchsian group of second kind has finite geodesic covers.*

Wrapping up the elementary groups resorting to the classification by Proposition 3.4 which are easily shown to have finite geodesic covers by hand, we conclude the proof of Theorem 1.1, i.e. all geometrically finite Fuchsian groups have finite geodesic covers.

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