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On the Calogero–Moser space associated with dihedral groups II. The equal parameter case

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Abstract

We continue the study of Calogero–Moser spaces associated with dihedral groups by investigating in more details the equal parameter case: we obtain explicit equations, some informations about the Poisson bracket, the structure of the Lie algebra associated with the cuspidal point and the action of $\mathbf{SL}_2(\mathbb{C})$.

Sur l'espace de Calogero–Moser associé aux groupes diédraux II. Le cas des paramètres égaux

Résumé

Nous continuons ici l'étude des espaces de Calogero–Moser associés aux groupes diédraux en se focalisant sur le cas des paramètres égaux : nous obtenons des équations explicites, des informations sur le crochet de Poisson, la structure de l'algèbre de Lie associée au point cuspidal et l'action de $\mathbf{SL}_2(\mathbb{C})$.

1. Introduction

In this paper, we continue the study of Calogero–Moser space $\mathcal{Z}_c$ associated with the dihedral group W of order $2d$ started in [5], from which we keep the notation. We mainly focus on the equal parameter case (i.e. the case where c is constant or, i.e. if $a = b$ with the notation of [5, §3.4])[1]. In this case, the main results of this paper are the following:

- We describe explicit equations for $\mathcal{Z}_c$.
- We obtain informations about the Poisson bracket that allow to determine the structure of the Lie algebra associated with the cuspidal point.
- We describe the action of $\mathbf{SL}_2(\mathbb{C})$ on the generators of Z_c and explain how the presentation of Z_0 can be interpreted in terms of Hermite's reciprocity law[2] (see for instance [9, Cor. 2.2]).

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[1]Recall that, if d is odd, then we have necessarily $a = b$.

[2]We wish to thank warmly Pierre-Louis Montagard for his enlighting explanations.

- If τ denotes the diagram automorphism of W, then τ acts on $\mathcal{Z}_c$ because we are in the equal parameter case, and we prove that the irreducible components of $\mathcal{Z}_c^\tau$ are also Calogero–Moser spaces associated with other reflection groups. This confirms [7, Conj. FIX] (or [6, Conj. B]) in this small case.

These results are used by G. Bellamy, B. Fu, D. Juteau, P. Levy, E. Sommers and the author in [3], where it is shown that, for $d \geqslant 5$, the symplectic singularity of $\mathcal{Z}_c$ at its cuspidal point is a new family of isolated symplectic singularities whose local fundamental group is trivial [3], answering an old question of Beauville [1].

These computations are based on a first paper of the author on Calogero–Moser spaces associated with dihedral groups [5] and on the above mentioned algorithm developed by Thiel and the author [8]. Explicit computer computations in small cases (i.e. $d \in \{4, 5, 6, 7\}$) were necessary to find the general pattern. So, even though this does not appear in this paper, it is fair to say that the above results owe their existence to Magma.

Recollection of notation from [5]

We will use the notation of the first part [5] and we recall here some of them, the most important ones. We set $V = \mathbb{C}^2$ and (x, y) denotes its canonical basis while (X, Y) is the dual basis of V^*. We identify $\mathbf{GL}_{\mathbb{C}}(V)$ with $\mathbf{GL}_2(\mathbb{C})$. We also fix a non-zero natural number d, as well as a primitive d-th root of unity $\zeta \in \mathbb{C}^\times$. If $i \in \mathbb{Z}$ or $\mathbb{Z}/d\mathbb{Z}$, we set

$$s_i = \begin{pmatrix} 0 & \zeta^i \\ \zeta^{-i} & 0 \end{pmatrix},$$

$s = s_0$, $t = s_1$ and $W = \langle s, t\rangle$: it is the dihedral group of order $2d$. In particular,

$$s_i(x) = \zeta^{-i} y, \quad s_i(y) = \zeta^i x, \quad s_i(X) = \zeta^i Y \quad \text{and} \quad s_i(Y) = \zeta^{-i} X.$$

The set $\mathrm{Ref}(W)$ of reflections of W (for its action on V) is $\{s_i \mid i \in \mathbb{Z}/d\mathbb{Z}\}$. Finally, let w_0 denote the longest element of W (we have $w_0 = t(st)^{(d-1)/2}$ if d is odd and $w_0 = (st)^{d/2}$ if d is even): this notation was used in the first part [5, Rem. 6.4] but we had forgotten to define it! It will be used here in Section 5.

We set $q = xy$, $Q = XY$, $r = x^d + y^d$, $R = X^d + Y^d$ and, if $0 \leqslant i \leqslant d$,

$$\mathbf{a}_{i,0} = x^{d-i} Y^i + y^{d-i} Y^i.$$

In this second part, we will not use the notation r or R as $r = \mathbf{a}_{0,0}$ and $R = \mathbf{a}_{d,0}$: we prefer this second notation. If $i \geqslant 0$, we set

$$\mathbf{eu}_0^{(i)} = (xX)^i + (yY)^i$$

and we set $\mathbf{eu}_0 = \mathbf{eu}_0^{(1)} = xX + yY$. Recall from [5, Thm. 2.1] that

$$\mathbb{C}[V \times V^*]^W = \mathbb{C}[q, Q, \mathbf{eu}_0, \mathbf{a}_{0,0}, \mathbf{a}_{1,0}, \dots, \mathbf{a}_{d,0}].$$

We fix a map $c : \mathrm{Ref}(W) \to \mathbb{C}$ and we set $a = c_s$ and $b = c_t$. We denote by $\mathbf{H}_c$ the rational Cherednik algebra at $t = 0$, with parameter c, whose presentation is given in [5, (3.2)]: it is defined as the quotient of $\mathrm{T}(V \oplus V^*) \rtimes W$ by the following relations (here, $\mathrm{T}(V \oplus V^*)$ is the tensor algebra of $V \oplus V^*$ over $\mathbb{C}$):

$$\begin{cases} [u, u'] = [U, U'] = 0, \\ [u, U] = -2 \sum_{i \in \mathbb{Z}/d\mathbb{Z}} c_{s_i} \frac{\langle u, \alpha_i \rangle \cdot \langle \alpha_i^\vee, U \rangle}{\langle \alpha_i^\vee, \alpha_i \rangle} s_i, \end{cases} \tag{$*$}$$

for $U, U' \in V^*$ and $u, u' \in V$. Note that we have followed the convention of [7].

Recall from [11] that $\mathbf{H}_c$ contains naturally the algebras $\mathbb{C}[V]$, $\mathbb{C}[V^*]$ and $\mathbb{C}W$ as subalgebras and that the multiplication map

$$\mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*] \longrightarrow \mathbf{H}_c$$

is an isomorphism of vector spaces. The center of $\mathbf{H}_c$ is denoted by Z_c and we denote by $\mathcal{Z}_c$ the affine variety whose algebra of regular functions $\mathbb{C}[\mathcal{Z}_c]$ is precisely Z_c.

We denote by Trunc_c the $\mathbb{C}$-linear map

$$\mathrm{Trunc}_c : \mathbf{H}_c \longrightarrow \mathbb{C}[V \times V^*]$$

such that, if $f \in \mathbb{C}[V \times V^*]$ and $w \in W$, then

$$\mathrm{Trunc}_c(fw) = \begin{cases} f & \text{if } w = 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is the map induced by the map Trunc defined in [5, §3.4]. Its restriction $\mathrm{Trunc}_c : Z_c \to \mathbb{C}[V \times V^*]^W$ is an isomorphism of $\mathbb{Z}$-graded vector spaces [5, Lem. 3.5]. Recall that it is $P_\bullet$-linear, where $P_\bullet = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W = \mathbb{C}[q, Q, \mathbf{a}_{0,0}, \mathbf{a}_{d,0}]$.

We add a further notation which will be useful in this second part, namely, we set

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

so that (e, h, f) is the standard basis of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

Hypothesis. *All along this paper, together with the above notation, we make the additional assumption that $a = b$, which justifies the title of this paper. Recall that it is automatically satisfied if d is odd.*

2. Back to $\mathcal{Z}_0 = (V \times V^*)/W$

2.1. Some polynomial identities

If $i \geqslant 0$, let $\mathbf{eu}_0^{[i]}$ denote the element

$$\mathbf{eu}_0^{[i]} = \frac{(xX)^{i+1} - (yY)^{i+1}}{xX - yY} = \sum_{j=0}^{i} (xX)^{i-j}(yY)^j$$

of $Z_0 = \mathbb{C}[V \times V^*]^W$. In other words, with the notation of [5, §2],

$$\mathbf{eu}_0^{[i]} = \sum_{0 \leqslant j < i/2} (qQ)^j \mathbf{eu}_0^{(i-2j)} + \boldsymbol{\delta}_{i \text{ is even}},$$

where $\boldsymbol{\delta}_{i \text{ is even}}$ is equal to 1 (resp. 0) if i is even (resp. odd). Hence, using the inversion formula [5, (2.1)], one gets

$$\mathbf{eu}_0^{[i]} = \sum_{0 \leqslant j < i/2} \left((qQ)^j \sum_{0 \leqslant k \leqslant 1/2 - j} n_{i-2j,k} (qQ)^k \mathbf{eu}_0^{i-2j-2k} \right) + \boldsymbol{\delta}_{i \text{ is even}},$$

which can be rewritten

$$\mathbf{eu}_0^{[i]} = \sum_{0 \leqslant j \leqslant i/2} m_{i,j} (qQ)^j \mathbf{eu}_0^{i-2j}, \tag{2.1}$$

for some uniquely defined elements $m_{i,j} \in \mathbb{Z}$ (note that $m_{i,0} = 1$). Let $\Psi_i(T, T', T'')$ denote the polynomial in three indeterminates equal to $\sum_{0 \leqslant j \leqslant i/2} m_{i,j} (T'T'')^j T^{i-2j}$. It is homogeneous of degree i for the natural graduation of $\mathbb{C}[T, T', T'']$ and, as a polynomial in T with coefficients in $\mathbb{C}[T', T'']$, it is monic. If we denote by $\mathbb{C}[T, T', T'']_k$ the homogeneous component of $\mathbb{C}[T, T', T'']$ of degree k, then the unitriangularity of the formula (2.1) shows that

$$(T'^{k-j} T''^{i} \Psi_{j-i})_{0 \leqslant i \leqslant j \leqslant k} \text{ is a basis of } \mathbb{C}[T, T', T'']_k. \tag{2.2}$$

By construction, Ψ_i is the unique polynomial satisfying the following identity:

$$\Psi_i(\mathbf{eu}_0, q, Q) = \mathbf{eu}_0^{[i]} = \frac{(xX)^{i+1} - (yY)^{i+1}}{xX - yY}. \tag{2.3}$$

The unicity comes from the fact that $\mathbf{eu}_0$, q and Q are algebraically independent. Note that $\Psi_0 = 1$ and $\Psi_1 = T$. Now the sequence $(\Psi_i)_{i \geqslant 0}$ is easily determined by the following recursive formula: if $i \geqslant 1$, then

$$\Psi_{i+1} = T\Psi_i - T'T''\Psi_{i-1}. \tag{2.4}$$

Indeed, this follows from the fact that $(xX)^{i+2}-(yY)^{i+2} = (xX+yY)((xX)^{i+1}-(yY)^{i+1}) - xyXY((xX)^i - (yY)^i)$. Note also for future reference the following two relations: if $i \geqslant 1$, then

$$\begin{cases} 2T'\dfrac{\partial \Psi_i}{\partial T} + T\dfrac{\partial \Psi_i}{\partial T''} = (i+1)T'\Psi_{i-1}, \\ 2T''\dfrac{\partial \Psi_i}{\partial T} + T\dfrac{\partial \Psi_i}{\partial T'} = (i+1)T''\Psi_{i-1}. \end{cases} \tag{2.5}$$

Proof of (2.5). We prove only the first identity, the second one being obtained by exchanging the roles of (x,y) and (X,Y). Let us consider the two identities obtained by applying $\partial/\partial X$ and $\partial/\partial Y$ to (2.3):

$$\begin{cases} x\dfrac{\partial \Psi_i}{\partial T}(\mathbf{eu}_0,q,Q)+Y\dfrac{\partial \Psi_i}{\partial T''}(\mathbf{eu}_0,q,Q) = \dfrac{(i+1)x^{i+1}X^i(xX-yY)-x((xX)^{i+1}-(yY)^{i+1})}{(xX-yY)^2}, \\ y\dfrac{\partial \Psi_i}{\partial T}(\mathbf{eu}_0,q,Q)+X\dfrac{\partial \Psi_i}{\partial T''}(\mathbf{eu}_0,q,Q) = \dfrac{-(i+1)y^{i+1}Y^i(xX-yY)+y((xX)^{i+1}-(yY)^{i+1})}{(xX-yY)^2}. \end{cases}$$

Multiplying the first equality by y, the second by x, and adding the results yields exactly

$$2q\frac{\partial \Psi_i}{\partial T}(\mathbf{eu}_0,q,Q) + \mathbf{eu}_0\frac{\partial \Psi_i}{\partial T''}(\mathbf{eu}_0,q,Q) = (i+1)q\Psi_{i-1}(\mathbf{eu}_0,q,Q),$$

as expected. □

2.2. Presentation

We rewrite slightly differently the presentation of $Z_0 = \mathbb{C}[V\times V^*]^W$ obtained in [5, Thm. 2.1] according to our needs. A straightforward computation shows that, if $1 \leqslant i \leqslant j \leqslant d-1$, then

$$\mathbf{a}_{i-1,0}\mathbf{a}_{j+1,0} - \mathbf{a}_{i,0}\mathbf{a}_{j,0} = (\mathbf{eu}_0^2 - 4qQ)q^{d-j-1}Q^{i-1}\mathbf{eu}_0^{[j-i]}.$$

Using (2.1), this gives

$$\mathbf{a}_{i-1,0}\mathbf{a}_{j+1,0} - \mathbf{a}_{i,0}\mathbf{a}_{j,0} = (\mathbf{eu}_0^2 - 4qQ)q^{d-j-1}Q^{i-1}\Psi_{j-i}(\mathbf{eu}_0,q,Q) \tag{$\mathfrak{Z}^0_{i,j}$}$$

This equation can also be obtained by substracting the equation $(\mathrm{Z}^0_{i,j})$ to the equation $(\mathrm{Z}^0_{i-1,j+1})$ (with the notation of [5, §2]). Consequently, the presentation given in [5, Thm. 2.1] can be rewritten as follows:

Theorem 2.1. *The algebra of invariants* $\mathbb{C}[V \times V^*]^W$ *admits the following presentation:*

- *Generators:* q, Q, $\mathbf{eu}_0$, $\mathbf{a}_{0,0}$, $\mathbf{a}_{1,0}$, $\mathbf{a}_{2,0}$, $\dots$, $\mathbf{a}_{d,0}$.
- *Relations:*

$$\begin{cases} \mathbf{eu}_0 \mathbf{a}_{i,0} = q\mathbf{a}_{i+1,0} + Q\mathbf{a}_{i-1,0} & \textit{for } 1 \leqslant i \leqslant d-1, \\ \mathbf{a}_{i-1,0}\mathbf{a}_{j+1,0} - \mathbf{a}_{i,0}\mathbf{a}_{j,0} & \\ \qquad = (\mathbf{eu}_0^2 - 4qQ)q^{d-j-1}Q^{i-1}\Psi_{j-i}(\mathbf{eu}_0, q, Q) & \textit{for } 1 \leqslant i \leqslant j \leqslant d-1. \end{cases}$$

2.3. Poisson bracket

The Poisson bracket on $\mathbb{C}[V \times V^*]^W$ is obtained by restriction of the natural one on $\mathbb{C}[V \times V^*]$, which is completely determined by the following rules:

$$\{x, X\} = \{y, Y\} = 1 \qquad \text{and} \qquad \{x, y\} = \{X, Y\} = \{x, Y\} = \{y, X\} = 0.$$

Therefore, a straightforward computation shows that the Poisson bracket between the generators of $\mathbb{C}[V \times V^*]^W$ is given by:

$$\begin{cases} \{q, Q\} = \mathbf{eu}_0, \\ \{\mathbf{eu}_0, q\} = -2q, \\ \{\mathbf{eu}_0, Q\} = 2Q, \\ \{\mathbf{eu}_0, \mathbf{a}_{i,0}\} = (2i - d)\mathbf{a}_{i,0}, \\ \{q, \mathbf{a}_{i,0}\} = i\mathbf{a}_{i-1,0} \\ \{Q, \mathbf{a}_{i,0}\} = (i - d)\mathbf{a}_{i+1,0} \\ \{\mathbf{a}_{i,0}, \mathbf{a}_{j,0}\} = j(d-i)q^{d-j}Q^i\mathbf{eu}_0^{(j-i-1)} - i(d-j)q^{d-j-1}Q^{i-1}\mathbf{eu}_0^{(j-i+1)}, \end{cases} \tag{2.6}$$

where the last equality only holds if $0 \leqslant i < j \leqslant d$. In particular, $(Q, \mathbf{eu}_0, -q)$ is an $\mathfrak{sl}_2$-triple (for the Lie algebra structure on $\mathbb{C}[V \times V^*]^W$ induced by the Poisson bracket). Note that

$$\{Q, \mathbf{eu}_0^2 - 4qQ\} = \{q, \mathbf{eu}_0^2 - 4qQ\} = \{\mathbf{eu}_0, \mathbf{eu}_0^2 - 4qQ\} = 0. \tag{2.7}$$

2.4. Action of $\mathbf{GL}_2(\mathbb{C})$

Since W is a Coxeter group, the $\mathbb{C}W$-modules V and V^* are isomorphic. In our situation, the map

$$\begin{array}{rccc} \Phi : & V & \longrightarrow & V^* \\ & \alpha x + \beta y & \longmapsto & \beta X + \alpha Y \end{array}$$

is an isomorphism of $\mathbb{C}W$-modules. One then gets an action of $\mathbf{GL}_2(\mathbb{C})$ on $V \times V^*$ as follows:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot (u, U) = (\alpha u + \beta \Phi^{-1}(U), \gamma \Phi(u) + \delta U).$$

By construction, this action commutes with the action of W, so induces an action of $\mathbf{GL}_2(\mathbb{C})$ on the $\mathbb{C}$-algebras $\mathbb{C}[V \times V^*]$, $\mathbb{C}[V \times V^*] \rtimes W$ and $\mathbb{C}[V \times V^*]^W$. This induces an action of the Lie algebra $\mathfrak{gl}_2(\mathbb{C})$ by derivations on $\mathbb{C}[V \times V^*]$ and $\mathbb{C}[V \times V^*]^W$. For conventional reasons, if $\varphi \in \mathbb{C}[V \times V^*]$ and $\xi \in \mathfrak{sl}_2(\mathbb{C})$, we denote by $\xi \cdot \varphi$ the image of φ under the action of $-{}^t\xi$. It is easily checked on the generators x, y, X, Y of $\mathbb{C}[V \times V^*]$ that

$$e \cdot \varphi = \{Q, \varphi\}, \qquad h \cdot \varphi = \{\mathbf{eu}_0, \varphi\} \qquad \text{and} \qquad f \cdot \varphi = \{-q, \varphi\} \tag{2.8}$$

for all $\varphi \in \mathbb{C}[V \times V^*]$.

3. Calogero–Moser space at equal parameters

Notation. We denote by q, Q, $\boldsymbol{eu}$, $\boldsymbol{a}_0$, $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d$ the respective preimages of q, Q, $\mathbf{eu}_0$, $\mathbf{a}_{0,0}$, $\mathbf{a}_{1,0}, \ldots, \mathbf{a}_{d,0}$ under the isomorphism of vector spaces $\mathrm{Trunc}_c : Z_c \to \mathbb{C}[V \times V^*]^W$.

By definition, q, Q, $\boldsymbol{eu}$, $\boldsymbol{a}_0$, $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d$ are the respective images, by specializing at the parameter c, of the elements that were denoted by q, Q, $\mathbf{eu}$, $\mathbf{a}_0$, $\mathbf{a}_1, \ldots, \mathbf{a}_d$ in [5, §3.4].

Note the following formulas:

$$\begin{cases} [x, X] = -a \displaystyle\sum_{i \in \mathbb{Z}/d\mathbb{Z}} s_i, \\ [x, Y] = a \displaystyle\sum_{i \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-i} s_i, \\ [y, X] = a \displaystyle\sum_{i \in \mathbb{Z}/d\mathbb{Z}} \zeta^{i} s_i, \\ [y, Y] = -a \displaystyle\sum_{i \in \mathbb{Z}/d\mathbb{Z}} s_i. \end{cases} \tag{3.1}$$

Note also the following formula, which follows from [12, §3.6]: if $P \in \mathbb{C}[X, Y]$, then

$$[x, P] = -a \sum_{i \in \mathbb{Z}/d\mathbb{Z}} \frac{P - {}^{s_i}P}{X - \zeta^i Y} s_i = -a \sum_{i \in \mathbb{Z}/d\mathbb{Z}} s_i \frac{P - {}^{s_i}P}{X - \zeta^i Y}. \tag{3.2}$$

3.1. Explicit form of the generators

Recall from [7, §3.3 and §4.1] that

$$\boldsymbol{eu} = xX + yY + a \sum_{i \in \mathbb{Z}/d\mathbb{Z}} s_i. \tag{3.3}$$

An important feature of the equal parameter case is that the elements $\boldsymbol{a}_j$ have a reasonably simple form:

Proposition 3.1. *If* $0 \leqslant j \leqslant d$, *then*

$$\begin{aligned}\boldsymbol{a}_j &= x^{d-j}Y^j + y^{d-j}X^j - a \sum_{i\in\mathbb{Z}/d\mathbb{Z}} \zeta^{-ij}\,\frac{x^{d-j}-\zeta^{ij}y^{d-j}}{x-\zeta^{-i}y}\cdot\frac{X^j-\zeta^{ij}Y^j}{X-\zeta^i Y}\,s_i\\ &= x^{d-j}Y^j + y^{d-j}X^j - a \sum_{i\in\mathbb{Z}/d\mathbb{Z}} \zeta^{-ij}\,\frac{x^{d-j}-\zeta^{ij}y^{d-j}}{x-\zeta^{-i}y}\;s_i\;\frac{X^j-\zeta^{ij}Y^j}{X-\zeta^i Y}\end{aligned}$$

Notation. For future use of the above formula, we set

$$\gamma_{i,j} = \frac{x^{d-j}-\zeta^{ij}y^{d-j}}{x-\zeta^{-i}y} \qquad \text{and} \qquad \Gamma_{i,j} = \frac{X^j-\zeta^{ij}Y^j}{X-\zeta^i Y}$$

for $i \in \mathbb{Z}/d\mathbb{Z}$ and $0 \leqslant j \leqslant d$. Note that $\gamma_{i,d} = \Gamma_{i,0} = 0$ and that $\gamma_{i,d-1} = \Gamma_{i,1} = 1$.

Proof. Let $\boldsymbol{b}_j \in \mathbf{H}_c$ denote the right-hand side of the equation of the proposition. Since Trunc_c induces an isomorphism $Z_c \xrightarrow{\sim} \mathbb{C}[V \times V^*]^W$ and $\mathrm{Trunc}_c(\boldsymbol{a}_j) = \mathrm{Trunc}_c(\boldsymbol{b}_j)$, it is sufficient to check that $\boldsymbol{b}_j \in Z_c$. First, an easy computation shows that $\boldsymbol{b}_j$ commutes with $s = s_0$ and $t = s_1$. Now, by (3.2), we have

$$\begin{aligned}[x,\boldsymbol{b}_j] &= x^{d-j}\Big(a\sum_{i\in\mathbb{Z}/d\mathbb{Z}}\zeta^{-ij}s_i\frac{X^j-\zeta^{ij}Y^j}{X-\zeta^iY}\Big) + y^{d-j}\Big(-a\sum_{i\in\mathbb{Z}/d\mathbb{Z}}s_i\frac{X^j-\zeta^{ij}Y^j}{X-\zeta^iY}\Big)\\ &\qquad - a\sum_{i\in\mathbb{Z}/d\mathbb{Z}}\zeta^{-ij}\frac{x^{d-j}-\zeta^{ij}y^{d-j}}{x-\zeta^{-i}y}\Big[x, s_i\frac{X^j-\zeta^{ij}Y^j}{X-\zeta^iY}\Big]\\ &= a\sum_{i\in\mathbb{Z}/d\mathbb{Z}}(\zeta^{-ij}x^{d-j}-y^{d-j})s_i\frac{X^j-\zeta^{ij}Y^j}{X-\zeta^iY}\\ &\qquad - a\sum_{i\in\mathbb{Z}/d\mathbb{Z}}\frac{\zeta^{-ij}x^{d-j}-y^{d-j}}{x-\zeta^{-i}y}[x,s_i]\frac{X^j-\zeta^{ij}Y^j}{X-\zeta^iY}\\ &\qquad - a\sum_{i\in\mathbb{Z}/d\mathbb{Z}}\zeta^{-ij}\frac{x^{d-j}-\zeta^{ij}y^{d-j}}{x-\zeta^{-i}y}s_i\Big[x,\frac{X^j-\zeta^{ij}Y^j}{X-\zeta^iY}\Big]\end{aligned}$$

Now, the first two lines of this last equation compensate each other and it remains

$$\begin{aligned}[x,\boldsymbol{b}_j] &= -a\sum_{i\in\mathbb{Z}/d\mathbb{Z}}\zeta^{-ij}\frac{x^{d-j}-\zeta^{ij}y^{d-j}}{x-\zeta^{-i}y}s_i\Big[x,\frac{X^j-\zeta^{ij}Y^j}{X-\zeta^iY}\Big]\\ &= a^2\sum_{i,i'\in\mathbb{Z}/d\mathbb{Z}}\zeta^{-ij}\frac{x^{d-j}-\zeta^{ij}y^{d-j}}{x-\zeta^{-i}y}s_is_{i'}\Big(\frac{X^j-\zeta^{ij}Y^j}{X-\zeta^iY} - \frac{\zeta^{i'j}X^j-\zeta^{(i-i')j}Y^j}{\zeta^{i'}X-\zeta^{i-i'}Y}\Big),\end{aligned}$$

again by using (3.2). But $s_i s_{i'} = c^{i-i'}$, where $c = ts = \operatorname{diag}(\zeta, \zeta^{-1})$ so, if we set $k = i - i'$, we can rewrite the above formula as follows:

$$\begin{aligned}
[x, \boldsymbol{b}_j] &= a^2 \sum_{i,k \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-ij} \frac{x^{d-j} - \zeta^{ij} y^{d-j}}{x - \zeta^{-i} y} c^k \left(\frac{X^j - \zeta^{ij} Y^j}{X - \zeta^i Y} - \frac{\zeta^{(i-k)j} X^j - \zeta^{kj} Y^j}{\zeta^{i-k} X - \zeta^k Y} \right) \\
&= a^2 \sum_{i,k \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-ij} \frac{x^{d-j} - \zeta^{ij} y^{d-j}}{x - \zeta^{-i} y} \left(\frac{\zeta^{-kj} X^j - \zeta^{(i+k)j} Y^j}{\zeta^{-k} X - \zeta^{i+k} Y} - \frac{X^j - \zeta^{ij} Y^j}{X - \zeta^i Y} \right) c^k \\
&= a^2 \sum_{k \in \mathbb{Z}/d\mathbb{Z}} \Theta_{j,k} c^k,
\end{aligned}$$

where

$$\Theta_{j,k} = \sum_{i \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-ij} \frac{x^{d-j} - \zeta^{ij} y^{d-j}}{x - \zeta^{-i} y} \left(\frac{\zeta^{-kj} X^j - \zeta^{(i+k)j} Y^j}{\zeta^{-k} X - \zeta^{i+k} Y} - \frac{X^j - \zeta^{ij} Y^j}{X - \zeta^i Y} \right) \in \mathbb{C}[x, y] \otimes \mathbb{C}[X, Y].$$

This formula implies that $\Theta_{j,k}$ is a linear combination of (non-commutative) monomials of the form $x^l y^{d-1-l} X^m Y^{j-1-m}$, where $0 \leqslant l \leqslant d - j - 1$ and $0 \leqslant m \leqslant j - 1$, and the coefficient $\theta_{j,k,l,m}$ of this monomial in $\Theta_{j,k}$ is equal to

$$\begin{aligned}
\theta_{j,k,l,m} &= \sum_{i \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-ij} \zeta^{-i(d-j-1-l)} (\zeta^{-km} \zeta^{(i+k)(j-1-m)} - \zeta^{i(j-1-m)}) \\
&= \sum_{i \in \mathbb{Z}/d\mathbb{Z}} \zeta^{i(l+j-m)} (\zeta^{k(j-1-2m)} - 1).
\end{aligned}$$

But $j \leqslant l + j \leqslant d - 1$ and $0 \leqslant m \leqslant j - 1$, so $l + j \not\equiv m \bmod d$. This implies in particular that $\sum_{i \in \mathbb{Z}/d\mathbb{Z}} \zeta^{i(l+j-m)} = 0$, and so $\theta_{j,k,l,m} = 0$. This shows that $[x, \boldsymbol{b}_j] = 0$.

A similar computation shows that $[X, \boldsymbol{b}_j] = 0$ and so $\boldsymbol{b}_j$ commutes with s, t, x, X, $sxs^{-1} = y$ and $sXs^{-1} = Y$, so it is central in $\mathbf{H}_c$. This completes the proof of the proposition. □

This has the following consequence, that will be used for obtaining a presentation of the algebra Z_c.

Corollary 3.2. *If* $1 \leqslant i \leqslant j \leqslant d - 1$, *then*

$$\begin{aligned}
&\operatorname{Trunc}_c(\boldsymbol{a}_{i-1} \boldsymbol{a}_{j+1} - \boldsymbol{a}_i \boldsymbol{a}_j) \\
&\quad = q^{d-j-1} Q^{i-1} (x^{j-i+2} X^{j-i+2} + y^{j-i+2} Y^{j-i+2}) - q^{d-j} Q^i (x^{j-i} X^{j-i} + y^{j-i} Y^{j-i}) \\
&\qquad + d(1 + j - i - d) a^2 \sum_{M=i-1}^{j-1} x^{M+d-i-j} y^{d-2-M} X^M Y^{i+j-2-M}.
\end{aligned}$$

Proof. Assume first that $0 \leqslant i \leqslant j \leqslant d$. Since $\boldsymbol{a}_j$ is central, we get

$$\begin{aligned}
\boldsymbol{a}_i\boldsymbol{a}_j &= x^{d-i}\boldsymbol{a}_j Y^i + y^{d-i}\boldsymbol{a}_j X^i - a\sum_{k\in\mathbb{Z}/d\mathbb{Z}} \zeta^{-ki}\gamma_{k,i}\boldsymbol{a}_j s_k\Gamma_{k,i} \\
&= x^{d-i}x^{d-j}Y^jY^i + x^{d-i}y^{d-j}X^jY^i + y^{d-i}x^{d-j}Y^jX^i + y^{d-i}y^{d-j}X^jX^i \\
&\quad - a\sum_{k\in\mathbb{Z}/d\mathbb{Z}} x^{d-i}\gamma_{k,j}s_k\Gamma_{k,j}Y^i - a\sum_{k\in\mathbb{Z}/d\mathbb{Z}} y^{d-i}\gamma_{k,j}s_k\Gamma_{k,j}X^i \\
&\quad - a\sum_{k\in\mathbb{Z}/d\mathbb{Z}} \zeta^{-ki}\gamma_{k,i}(x^{d-j}Y^j + y^{d-j}X^j)\Gamma_{k,i}s_k \\
&\quad + a^2\sum_{k,l\in\mathbb{Z}/d\mathbb{Z}} \zeta^{-ki}\zeta^{-kj}\gamma_{k,i}\gamma_{l,j}\Gamma_{l,j}s_l s_k\Gamma_{k,i}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{Trunc}_c(\boldsymbol{a}_i\boldsymbol{a}_j) = x^{2d-i-j}Y^{i+j} + y^{2d-i-j}X^{i+j} + q^{d-j}Q^i(x^{j-i}X^{j-i} + y^{j-i}Y^{j-i}) \\
+ a^2\sum_{k\in\mathbb{Z}/d\mathbb{Z}} \zeta^{-k(i+j)}\gamma_{k,i}\gamma_{k,j}\Gamma_{k,j}\Gamma_{k,i}.
\end{aligned}$$

Expanding the product $\gamma_{k,i}\gamma_{k,j}\Gamma_{k,j}\Gamma_{k,i}$ gives

$$\begin{aligned}
&\mathrm{Trunc}_c(\boldsymbol{a}_i\boldsymbol{a}_j) = x^{2d-i-j}Y^{i+j} + y^{2d-i-j}X^{i+j} + q^{d-j}Q^i(x^{j-i}X^{j-i} + y^{j-i}Y^{j-i}) \\
&+ a^2\sum_{k\in\mathbb{Z}/d\mathbb{Z}}\sum_{l=0}^{d-i-1}\sum_{l'=0}^{d-j-1}\sum_{m=0}^{i-1}\sum_{m'=0}^{j-1} \zeta^{-k(i+j+l+l'-m-m')}x^{l+l'}y^{2d-i-j-2-l-l'}X^{m+m'}Y^{i+j-2-m-m'}.
\end{aligned}$$

If $0 \leqslant L \leqslant 2d-i-j-2$ (resp. $0 \leqslant M \leqslant i+j-2$), let $\mathcal{L}_{i,j}(L)$ (resp. $\mathcal{M}_{i,j}(M)$) denote the set of pairs (l, l') (resp. (m, m')) such that $l + l' = L$ (resp. $m + m' = M$) and $0 \leqslant l \leqslant d-i-1$ and $0 \leqslant l' \leqslant d-j-1$ (resp. $0 \leqslant m \leqslant i-1$ and $0 \leqslant m' \leqslant j-1$). Then the above equality might rewritten

$$\begin{aligned}
\mathrm{Trunc}_c(\boldsymbol{a}_i\boldsymbol{a}_j) = x^{2d-i-j}Y^{i+j} + y^{2d-i-j}X^{i+j} + q^{d-j}Q^i(x^{j-i}X^{j-i} + y^{j-i}Y^{j-i}) \\
+ a^2\sum_{k\in\mathbb{Z}/d\mathbb{Z}}\sum_{L=0}^{2d-i-j-2}\sum_{M=0}^{i+j-2} |\mathcal{L}_{i,j}(L)|\cdot|\mathcal{M}_{i,j}(M)| \\
\cdot\,\zeta^{-k(i+j+L-M)}x^L y^{2d-i-j-2-L}X^M Y^{i+j-2-M}.
\end{aligned}$$

Now, if $1 \leqslant i \leqslant j \leqslant d-1$, applying the above formula by replacing i by $i-1$ and j by $j+1$ yields

$$\begin{aligned}
&\mathrm{Trunc}_c(\boldsymbol{a}_{i-1}\boldsymbol{a}_{j+1} - \boldsymbol{a}_i\boldsymbol{a}_j) \\
&\quad = q^{d-j-1}Q^{i-1}(x^{j-i+2}X^{j-i+2} + y^{j-i+2}Y^{j-i+2}) - q^{d-j}Q^i(x^{j-i}X^{j-i} + y^{j-i}Y^{j-i}) \\
&\qquad + a^2 \sum_{L=0}^{2d-i-j-2} \sum_{M=0}^{i+j-2} \Big(\sum_{k \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-k(i+j+L-M)} \Big) \\
&\qquad\qquad \Big(|\mathcal{L}_{i-1,j+1}(L)| \cdot |\mathcal{M}_{i-1,j+1}(M)| - |\mathcal{L}_{i,j}(L)| \cdot |\mathcal{M}_{i,j}(M)| \Big) \\
&\qquad\qquad\qquad x^L y^{2d-i-j-2-L} X^M Y^{i+j-2-M}.
\end{aligned}$$

So, the coefficient of $x^L y^{2d-i-j-2-L} X^M Y^{i+j-2-M}$ is non-zero if and only if $i+j+L \equiv M \bmod d$ and $|\mathcal{L}_{i-1,j+1}(L)| \cdot |\mathcal{M}_{i-1,j+1}(M)| \neq |\mathcal{L}_{i,j}(L)| \cdot |\mathcal{M}_{i,j}(M)|$. Since $i \leqslant j$, we have

$$|\mathcal{L}_{i,j}(L)| = \begin{cases} 1+L & \text{if } 0 \leqslant L \leqslant d-j-1, \\ d-j & \text{if } d-j-1 \leqslant L \leqslant d-i-1, \\ 2d-i-j-1-L & \text{if } d-i-1 \leqslant L \leqslant 2d-i-j-2, \end{cases}$$

$$\text{and} \quad |\mathcal{M}_{i,j}(M)| = \begin{cases} 1+M & \text{if } 0 \leqslant M \leqslant i-1, \\ i & \text{if } i-1 \leqslant M \leqslant j-1, \\ i+j-1-M & \text{if } j-1 \leqslant M \leqslant i+j-2. \end{cases}$$

So $|\mathcal{L}_{i-1,j+1}(L)| \cdot |\mathcal{M}_{i-1,j+1}(M)| \neq |\mathcal{L}_{i,j}(L)| \cdot |\mathcal{M}_{i,j}(M)|$ if and only if $d-j-1 \leqslant L \leqslant d-i-1$ or $i-1 \leqslant M \leqslant j-1$. Combined with the fact that $i+j+L \equiv M \bmod d$ to obtain a non-zero coefficient for $x^L y^{2d-i-j-2-L} X^M Y^{i+j-2-M}$, this forces $i+j+L = M+d$ and so

$$\begin{aligned}
&\mathrm{Trunc}_c(\boldsymbol{a}_{i-1}\boldsymbol{a}_{j+1} - \boldsymbol{a}_i\boldsymbol{a}_j) \\
&\quad = q^{d-j-1}Q^{i-1}(x^{j-i+2}X^{j-i+2} + y^{j-i+2}Y^{j-i+2}) - q^{d-j}Q^i(x^{j-i}X^{j-i} + y^{j-i}Y^{j-i}) \\
&\qquad + da^2 \sum_{M=i-1}^{j-1} \underbrace{((d-j-1)(i-1) - (d-j)i)}_{=1+j-i-d} x^{M+d-i-j} y^{d-2-M} X^M Y^{i+j-2-M},
\end{aligned}$$

as expected. □

3.2. Poisson bracket

We determine here part of the Poisson bracket between the generators:

Proposition 3.3. *We have*

$$\{q, Q\} = \boldsymbol{eu}, \quad \{\boldsymbol{eu}, q\} = -2q \quad \textit{and} \quad \{\boldsymbol{eu}, Q\} = 2Q.$$

Moreover, if $0 \leqslant j \leqslant d$, *then*

$$\{q, \boldsymbol{a}_j\} = j\boldsymbol{a}_{j-1}, \quad \{\boldsymbol{eu}, \boldsymbol{a}_j\} = (2j-d)\boldsymbol{a}_j \quad \textit{and} \quad \{Q, \boldsymbol{a}_j\} = (j-d)\boldsymbol{a}_{j+1},$$

with the convention that $\boldsymbol{a}_{-1} = \boldsymbol{a}_{d+1} = 0$.

Proof. First, note that the Poisson bracket on Z_c is in fact the restriction of a Poisson bracket $\{\cdot,\cdot\} : \mathbf{H}_c \times Z_c \longrightarrow \mathbf{H}_c$. This Poisson bracket satisfies the following property: if $z = \sum_{w \in W} f_w w F_w \in Z_c$, with $f_w \in \mathbb{C}[x, y]$ and $F_w \in \mathbb{C}[X, Y]$, then

$$\{x, z\} = \sum_{w \in W} f_w w \frac{\partial F_w}{\partial X} \qquad \text{and} \qquad \{y, z\} = \sum_{w \in W} f_w w \frac{\partial F_w}{\partial Y}. \tag{3.4}$$

The first three equalities of the proposition are standard and hold for any Coxeter group (see [10, §4] or [4, §3]) and can easily be checked in this case by a little computation. Similarly, the fact that $\{\boldsymbol{eu}, \boldsymbol{a}_j\} = (2j-d)\boldsymbol{a}_j$ follows from the general fact that, if $h \in \mathbf{H}_c$ is homogeneous of degree k, then $\{\boldsymbol{eu}, h\} = kh$ (see for instance [7, Prop. 3.3.3]). We now prove that $\{q, \boldsymbol{a}_j\} = j\boldsymbol{a}_{j-1}$, the last equality being proved similarly. From the formula given for $\boldsymbol{a}_j$ in Proposition 3.1, we get

$$\begin{aligned}\{q, \boldsymbol{a}_j\} = \{yx, \boldsymbol{a}_j\} = j x^{d-j} Y^{j-1} x + j y^{d-j+1} X^{j-1} \\ - a \sum_{i \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-ij} \gamma_{i,j} s_i \frac{\partial \Gamma_{i,j}}{\partial Y} x - a \sum_{i \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-ij} y \gamma_{i,j} s_i \frac{\partial \Gamma_{i,j}}{\partial X}.\end{aligned}$$

In order to prove the proposition, it is sufficient to check that $\mathrm{Trunc}_c(\{q, \boldsymbol{a}_j\}) = j\mathbf{a}_{j-1}$. But, from the above formula and from (3.2), one gets

$$\mathrm{Trunc}_c(\{q, \boldsymbol{a}_j\}) = j\mathbf{a}_{j-1} - a\,\mathrm{Trunc}_c\Bigg(\sum_{i \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-ij} \gamma_{i,j} s_i \frac{\partial \Gamma_{i,j}}{\partial Y} x\Bigg).$$

Since

$$\sum_{i \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-ij} \gamma_{i,j} s_i \frac{\partial \Gamma_{i,j}}{\partial Y} x = \sum_{i \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-ij} \gamma_{i,j} s_i \Bigg(x \frac{\partial \Gamma_{i,j}}{\partial Y} - \Bigg[x, \frac{\partial \Gamma_{i,j}}{\partial Y} \Bigg] \Bigg),$$

it follows from (3.2) that

$$\mathrm{Trunc}_c(\{q, \boldsymbol{a}_j\}) = j\mathbf{a}_{j-1} + a^2 \sum_{i \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-ij} \gamma_{i,j} \frac{\frac{\partial \Gamma_{i,j}}{\partial Y} - {}^{s_i}\Big(\frac{\partial \Gamma_{i,j}}{\partial Y}\Big)}{X - \zeta^i Y}.$$

So it remains to prove that

$$\sum_{i\in\mathbb{Z}/d\mathbb{Z}} \zeta^{-ij}\gamma_{i,j}\,\frac{\frac{\partial\Gamma_{i,j}}{\partial Y} - {}^{s_i}\!\left(\frac{\partial\Gamma_{i,j}}{\partial Y}\right)}{X-\zeta^i Y} = 0. \tag{$\sharp$}$$

Let us compute the big fraction in the above formula. First,

$$\Gamma_{i,j} = \sum_{k=0}^{j-1} \zeta^{ik} X^{j-1-k} Y^k,$$

so

$$\frac{\partial\Gamma_{i,j}}{\partial Y} = \sum_{k=0}^{j-1} k\zeta^{ik} X^{j-1-k} Y^{k-1} = \sum_{k=0}^{j-2} (k+1)\zeta^{i(k+1)} X^{j-2-k} Y^k.$$

Therefore,

$${}^{s_i}\!\left(\frac{\partial\Gamma_{i,j}}{\partial Y}\right) = \sum_{k=0}^{j-2} (k+1)\zeta^{i(k+1)} (\zeta^i Y)^{j-2-k} (\zeta^{-i} X)^k.$$

Simplifying and using the change of variable $k \mapsto j-2-k$, one gets

$${}^{s_i}\!\left(\frac{\partial\Gamma_{i,j}}{\partial Y}\right) = \sum_{k=0}^{j-2} (j-1-k)\zeta^{i(k+1)} X^{j-2-k} Y^k.$$

We deduce that

$$\frac{\partial\Gamma_{i,j}}{\partial Y} - {}^{s_i}\!\left(\frac{\partial\Gamma_{i,j}}{\partial Y}\right) = \sum_{k=0}^{j-2} (2k+2-j)\zeta^{i(k+1)} X^{j-2-k} Y^k.$$

But $\sum_{k=0}^{j-2}(2k+2-j) = 0$, so

$$\frac{\partial\Gamma_{i,j}}{\partial Y} - {}^{s_i}\!\left(\frac{\partial\Gamma_{i,j}}{\partial Y}\right) = \sum_{k=0}^{j-2} (2k+2-j)\zeta^{i(k+1)} (X^{j-2-k} - \zeta^{i(j-2-k)} Y^{j-2-k}) Y^k.$$

Since the term corresponding to $k = j - 2$ vanishes, this implies that

$$\begin{aligned}
\frac{\frac{\partial \Gamma_{i,j}}{\partial Y} - {}^{s_i}\!\left(\frac{\partial \Gamma_{i,j}}{\partial Y}\right)}{X - \zeta^i Y} &= \sum_{k=0}^{j-3} \sum_{k'=0}^{j-3-k} (2k+2-j)\zeta^{i(k+1)} X^{j-3-k-k'} (\zeta^i Y)^{k'} Y^k \\
&= \sum_{k=0}^{j-3} \sum_{k'=0}^{j-3-k} (2k+2-j)\zeta^{i(k+k'+1)} X^{j-3-k-k'} Y^{k+k'} \\
&= \sum_{k=0}^{j-3} \sum_{k'=k}^{j-3} (2k+2-j)\zeta^{i(k'+1)} X^{j-3-k'} Y^{k'} \\
&= \sum_{k'=0}^{j-3} \left(\sum_{k=0}^{k'} (2k+2-j) \right) \zeta^{i(k'+1)} X^{j-3-k'} Y^{k'} \\
&= \sum_{k'=0}^{j-3} (k'+2-j)(k'+1)\zeta^{i(k'+1)} X^{j-3-k'} Y^{k'}.
\end{aligned}$$

Therefore, the left-hand side of the formula (♯) is a linear combination of monomials of the form $x^{d-j-1-l} y^l X^{j-3-m} Y^m$, where $0 \leqslant l \leqslant d-j-1$ and $0 \leqslant m \leqslant j-3$, and the coefficient of this monomial is

$$\sum_{i \in \mathbb{Z}/d\mathbb{Z}} (m+2-j)(m+1)\zeta^{-ij}\zeta^{-il}\zeta^{i(m+1)} = (m+2-j)(m+1) \sum_{i \in \mathbb{Z}/d\mathbb{Z}} \zeta^{i(m+1-j-l)}.$$

But $j \leqslant l + j \leqslant d - 1$ and $1 \leqslant m + 1 \leqslant j - 2$, so this coefficient is 0 and the equality (♯) is proved. □

Corollary 3.4. *We have*

$$\{q, \boldsymbol{eu}^2 - 4qQ\} = \{\boldsymbol{eu}, \boldsymbol{eu}^2 - 4qQ\} = \{Q, \boldsymbol{eu}^2 - 4qQ\} = 0.$$

3.3. Presentation

The main result of this paper is the following:

Theorem 3.5. *If $a = b$, then the algebra Z_c admits the following presentation:*

- *Generators: q, Q, $\boldsymbol{eu}$, $\boldsymbol{a}_0$, $\boldsymbol{a}_1, \dots, \boldsymbol{a}_d$.*
- *Relations:*

$$\begin{cases}
\boldsymbol{eu}\,\boldsymbol{a}_i = q\boldsymbol{a}_{i+1} + Q\boldsymbol{a}_{i-1} & \textit{for } 1 \leqslant i \leqslant d-1, \\
\boldsymbol{a}_{i-1}\boldsymbol{a}_{j+1} - \boldsymbol{a}_i\boldsymbol{a}_j & \\
\qquad = (\boldsymbol{eu}^2 - 4qQ - d^2a^2)q^{d-j-1}Q^{i-1}\Psi_{j-i}(\boldsymbol{eu}, q, Q) & \textit{for } 1 \leqslant i \leqslant j \leqslant d-1.
\end{cases}$$

Proof. By [8], a presentation of Z_c is obtained by deforming the generators of $Z_0 = \mathbb{C}[V \times V^*]^W$ and deforming the relations. Therefore, in order to prove the theorem, it is sufficient to check that the relations given in the statement are satisfied. So let $1 \leqslant i \leqslant j \leqslant d-1$.

Let us first prove that

$$\boldsymbol{eu}\, \boldsymbol{a}_i = q\boldsymbol{a}_{i+1} + Q\boldsymbol{a}_{i-1}. \qquad (\mathfrak{Z}_i)$$

For this, it is sufficient to prove that $\mathrm{Trunc}_c(\boldsymbol{eu}\, \boldsymbol{a}_i) = \mathrm{Trunc}_c(q\boldsymbol{a}_{i+1} + Q\boldsymbol{a}_{i-1})$. But the map Trunc_c is $P_\bullet$-linear so it is sufficient to prove that

$$\mathrm{Trunc}_c(\boldsymbol{eu}\, \boldsymbol{a}_i) = \mathbf{eu}_0\, \mathbf{a}_{i,0}. \qquad (\natural)$$

Since $\boldsymbol{a}_i$ is central, it follows from (3.3) and Proposition 3.1 that

$$\begin{aligned}\boldsymbol{eu}\, \boldsymbol{a}_i &= x\boldsymbol{a}_i X + y\boldsymbol{a}_i Y + a \sum_{k \in \mathbb{Z}/d\mathbb{Z}} \boldsymbol{a}_i s_k \\ &= x(x^{d-i}Y^i + y^{d-i}X^i)X + y(x^{d-i}Y^i + y^{d-i}X^i)Y \\ &\quad - a \sum_{k' \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-ik'} x\gamma_{k',i} s_{k'} \Gamma_{k',i} X - a \sum_{k' \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-ik'} y\gamma_{k',i} s_{k'} \Gamma_{k',i} Y \\ &\quad + a \sum_{k \in \mathbb{Z}/d\mathbb{Z}} (x^{d-i}Y^i + y^{d-i}X^i) s_k - a^2 \sum_{k,k' \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-ik'} \gamma_{k',i} \Gamma_{k',i} s_{k'} s_k.\end{aligned}$$

But $s_{k'} s_k = 1$ if and only if $k' = k$, so

$$\mathrm{Trunc}_c(\boldsymbol{eu}\, \boldsymbol{a}_i) = \mathbf{eu}_0 \mathbf{a}_{i,0} - a^2 \sum_{k \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-ik} \gamma_{k,i} \Gamma_{k,i}.$$

The element $\sum_{k \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-ik} \gamma_{k,i} \Gamma_{k,i}$ of $\mathbb{C}[V \times V^*]^W$ is a linear combination of monomials of the form $x^{d-i-1-l} y^l X^{i-1-m} Y^m$ where $0 \leqslant l \leqslant d-i-1$ and $0 \leqslant m \leqslant i-1$, and the coefficient of this monomial is equal to

$$\sum_{k \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-ki} \zeta^{-kl} \zeta^{km} = \sum_{k \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-k(i+l-m)}.$$

But $i \leqslant i+l \leqslant d-1$ and $0 \leqslant m \leqslant i-1$, so $i+l-m \not\equiv 0 \bmod d$. This shows that the above sum is zero, and this completes the proof of ($\natural$).

Let us now prove that

$$\boldsymbol{a}_{i-1}\boldsymbol{a}_{j+1} - \boldsymbol{a}_i \boldsymbol{a}_j = (\boldsymbol{eu}^2 - 4qQ - d^2a^2) q^{d-j-1} Q^{i-1} \Psi_{j-i}(\boldsymbol{eu}, q, Q). \qquad (\mathfrak{Z}_{i,j})$$

This will be proved by induction on $j-i$. So let us first consider the case where $j-i=0$, i.e. where $j=i$. Again, it is sufficient to prove the equality after applying the map Trunc_c. We deduce from Corollary 3.2 that

$$\mathrm{Trunc}_c(\boldsymbol{a}_{i-1}\boldsymbol{a}_{i+1} - \boldsymbol{a}_i^2) = q^{d-i-1} Q^{i-1} (x^2X^2 + y^2Y^2 - 2qQ - d(d-1)a^2).$$

Since Trunc_c is $P_\bullet$-linear and $\Psi_0 = 1$, proving $(\mathfrak{Z}_{i,j})$ is equivalent to proving that

$$\mathrm{Trunc}_c(\boldsymbol{eu}^2 - 4qQ - d^2a^2) = x^2X^2 + y^2Y^2 - 2qQ - d(d-1)a^2,$$

or, equivalently, that

$$\mathrm{Trunc}_c(\boldsymbol{eu}^2) = x^2X^2 + y^2Y^2 + 2qQ + da^2. \qquad (\mathcal{E})$$

But

$$\begin{aligned}\boldsymbol{eu}^2 &= x\boldsymbol{eu}X + y\boldsymbol{eu}Y + a\sum_{k\in\mathbb{Z}/d\mathbb{Z}} \boldsymbol{eu}s_k \\ &= x^2X^2 + xyYX + a\sum_{k\in\mathbb{Z}/d\mathbb{Z}} xs_kX + yxXY + y^2Y^2 + a\sum_{k\in\mathbb{Z}/d\mathbb{Z}} ys_kY \\ &\quad + a\sum_{k\in\mathbb{Z}/d\mathbb{Z}}(xX+yY)s_k + a^2\sum_{k,l\in\mathbb{Z}/d\mathbb{Z}} s_ls_k.\end{aligned}$$

It follows directly that

$$\mathrm{Trunc}_c(\boldsymbol{eu}^2) = x^2X^2 + y^2Y^2 + 2qQ + a^2\sum_{k\in\mathbb{Z}/d\mathbb{Z}} 1,$$

as desired.

Assume now that $j - i \geqslant 1$ and that $(\mathfrak{Z}_{i',j'})$ holds if $j' - i' < j - i$. Then, by the induction hypothesis, we have

$$\boldsymbol{a}_i\boldsymbol{a}_{j+1} - \boldsymbol{a}_{i+1}\boldsymbol{a}_j = (\boldsymbol{eu}^2 - 4qQ - d^2a^2)q^{d-j-2}Q^i\Psi_{j-i-1}(\boldsymbol{eu}, q, Q).$$

Applying $\{q, -\}$ to this equality, and using Proposition 3.3 and Corollary 3.4, one gets:

$$\begin{aligned}i(\boldsymbol{a}_{i-1}\boldsymbol{a}_{j+1} - \boldsymbol{a}_i\boldsymbol{a}_j) + j(\boldsymbol{a}_i\boldsymbol{a}_j - \boldsymbol{a}_{i+1}\boldsymbol{a}_{j-1}) &= (\boldsymbol{eu}^2 - 4qQ - d^2a^2)q^{d-j-1} \\ \times\Big(iQ^{i-1}\boldsymbol{eu}\Psi_{j-i-1}(\boldsymbol{eu},q,Q) + 2qQ^i\frac{\partial\Psi_{j-i-1}}{\partial T}(\boldsymbol{eu},q,Q) &+ Q^i\boldsymbol{eu}\frac{\partial\Psi_{j-i-1}}{\partial T''}(\boldsymbol{eu},q,Q)\Big).\end{aligned}$$

But, by (2.5),

$$2q\frac{\partial\Psi_{j-i-1}}{\partial T}(\boldsymbol{eu},q,Q) + \boldsymbol{eu}\frac{\partial\Psi_{j-i-1}}{\partial T''}(\boldsymbol{eu},q,Q) = (j-i)q\Psi_{j-i-2}(\boldsymbol{eu},q,Q)$$

and, by (2.4),

$$\boldsymbol{eu}\Psi_{j-i-1}(\boldsymbol{eu},q,Q) - qQ\Psi_{j-i-2}(\boldsymbol{eu},q,Q) = \Psi_{j-i}(\boldsymbol{eu},q,Q).$$

Therefore,

$$\begin{aligned}&i(\boldsymbol{a}_{i-1}\boldsymbol{a}_{j+1} - \boldsymbol{a}_i\boldsymbol{a}_j) + j(\boldsymbol{a}_i\boldsymbol{a}_j - \boldsymbol{a}_{i+1}\boldsymbol{a}_{j-1}) \\ &\quad = (\boldsymbol{eu}^2 - 4qQ - d^2a^2)q^{d-j-1}Q^{j-1}(i\Psi_{j-i}(\boldsymbol{eu},q,Q) + jqQ\Psi_{j-i-2}(\boldsymbol{eu},q,Q)).\end{aligned}$$

Since the induction hypothesis implies that

$$(\boldsymbol{a}_i\boldsymbol{a}_j - \boldsymbol{a}_{i+1}\boldsymbol{a}_{j-1}) = (\boldsymbol{eu}^2 - 4qQ - d^2a^2)q^{d-j}Q^i\Psi_{j-i-2}(\boldsymbol{eu}, q, Q),$$

the result follows. □

3.4. Back to the Poisson bracket

In Proposition 3.3, we did not determine the Poisson brackets $\{\boldsymbol{a}_i, \boldsymbol{a}_j\}$. This was only determined for $a = 0$ in (2.6): it is proven that there exists a polynomial $\Pi_{i,j} \in \mathbb{C}[T, T', T'']$, which is homogeneous of degree $d - 1$, such that

$$\{\mathbf{a}_{i,0}, \mathbf{a}_{j,0}\} = \Pi_{i,j}(\mathbf{eu}_0, q, Q).$$

This will be deformed to the $a \neq 0$ case as follows:

Proposition 3.6. *If $0 \leqslant i < j \leqslant d$, there exists a polynomial $\Phi_{i,j} \in \mathbb{C}[T, T', T'']$, homogeneous of degree $d - 3$, such that*

$$\{\boldsymbol{a}_i, \boldsymbol{a}_j\} = \Pi_{i,j}(\boldsymbol{eu}, q, Q) + a^2\Phi_{i,j}(\boldsymbol{eu}, q, Q).$$

Proof. We will prove that there exist polynomials $\Phi^\circ_{i,j}, \Phi_{i,j} \in \mathbb{C}[T, T', T'']$, homogeneous of degree $d - 1$ and $d - 3$ respectively, such that

$$\{\boldsymbol{a}_i, \boldsymbol{a}_j\} = \Phi^\circ_{i,j}(\boldsymbol{eu}, q, Q) + a^2\Phi_{i,j}(\boldsymbol{eu}, q, Q). \qquad (\wp_{i,j})$$

This is sufficient because, by specializing a to 0, one gets that $\Phi^\circ_{i,j} = \Pi_{i,j}$.

Let us first assume that $i = 0$. To make an induction argument on j work, we will prove a slightly stronger result, namely that

$$\{\boldsymbol{a}_0, \boldsymbol{a}_j\} = q^{d-j}(\varphi_j(\boldsymbol{eu}, q, Q) + a^2\theta_j(\boldsymbol{eu}, q, Q)). \qquad (\wp^+_{0,j})$$

where $\varphi_j, \theta_j \in \mathbb{C}[T, T', T'']$ are homogeneous of degree $j - 1$ and $j - 3$ respectively. For this, let us apply $\{\boldsymbol{a}_0, -\}$ to the following two relations given by Theorem 3.5

$$Q\boldsymbol{a}_0 - \boldsymbol{eu}\,\boldsymbol{a}_1 + q\boldsymbol{a}_2 = 0, \qquad (\mathfrak{Z}_1)$$

$$\boldsymbol{a}_0\boldsymbol{a}_2 - \boldsymbol{a}_1^2 = q^{d-2}(\boldsymbol{eu}^2 - 4qQ - d^2a^2). \qquad (\mathfrak{Z}_{1,1})$$

Using Proposition 3.3, this gives

$$\begin{cases} -\boldsymbol{eu}\{\boldsymbol{a}_0, \boldsymbol{a}_1\} + q\{\boldsymbol{a}_0, \boldsymbol{a}_2\} = 0, \\ q\boldsymbol{a}_0\{\boldsymbol{a}_0, \boldsymbol{a}_2\} - 2q\boldsymbol{a}_1\{\boldsymbol{a}_0, \boldsymbol{a}_1\} = q^{d-1}\{\boldsymbol{a}_0, \boldsymbol{eu}^2 - 4qQ\} = q^{d-1}(2d\boldsymbol{a}_0\boldsymbol{eu} - 4dq\boldsymbol{a}_1). \end{cases}$$

Thanks to the first equality, we can replace the term $q\{\boldsymbol{a}_0, \boldsymbol{a}_2\}$ in the second equation by $\boldsymbol{eu}\{\boldsymbol{a}_0, \boldsymbol{a}_1\}$, and this yields

$$(\boldsymbol{a}_0\boldsymbol{eu} - 2q\boldsymbol{a}_1)\{\boldsymbol{a}_0, \boldsymbol{a}_1\} = 2dq^{d-1}(\boldsymbol{a}_0\boldsymbol{eu} - 2q\boldsymbol{a}_1).$$

Since $\boldsymbol{a}_0\boldsymbol{eu} - 2q\boldsymbol{a}_1 \neq 0$ (by computing its image by Trunc_c) and since Z_c is an integral domain, we get

$$\{\boldsymbol{a}_0, \boldsymbol{a}_1\} = 2dq^{d-1},$$

which proves $(\wp^+_{0,1})$. We also deduce that

$$\{\boldsymbol{a}_0, \boldsymbol{a}_2\} = 2dq^{d-2}\boldsymbol{eu},$$

which proves $(\wp^+_{0,2})$.

Now, assume that $j \geqslant 3$ and that $(\wp^+_{0,j'})$ holds for $j' < j$. Applying $\{\boldsymbol{a}_0, -\}$ to

$$Q\boldsymbol{a}_{j-2} - \boldsymbol{eu}\boldsymbol{a}_{j-1} + q\boldsymbol{a}_j = 0 \qquad (\mathfrak{Z}_{j-1})$$

yields, thanks to Proposition 3.3,

$$d\boldsymbol{a}_1\boldsymbol{a}_{j-2} + Q\{\boldsymbol{a}_0, \boldsymbol{a}_{j-2}\} - d\boldsymbol{a}_0\boldsymbol{a}_{j-1} - \boldsymbol{eu}\{\boldsymbol{a}_0, \boldsymbol{a}_{j-1}\} + q\{\boldsymbol{a}_0, \boldsymbol{a}_j\} = 0.$$

But

$$\boldsymbol{a}_0\boldsymbol{a}_{j-1} - \boldsymbol{a}_1\boldsymbol{a}_{j-2} = (\boldsymbol{eu}^2 - 4qQ - d^2a^2)q^{d-j+1}\Psi_{j-3}(\boldsymbol{eu}, q, Q) \qquad (\mathfrak{Z}_{1,j-2})$$

by Theorem 3.5 and

$$\begin{aligned}\{\boldsymbol{a}_0, \boldsymbol{a}_{j-2}\} &= q^{d-j+2}\big(\varphi_{j-2}(\boldsymbol{eu}, q, Q) + a^2\theta_{j-2}(\boldsymbol{eu}, q, Q)\big),\\ \{\boldsymbol{a}_0, \boldsymbol{a}_{j-1}\} &= q^{d-j+1}\big(\varphi_{j-1}(\boldsymbol{eu}, q, Q) + a^2\theta_{j-1}(\boldsymbol{eu}, q, Q)\big)\end{aligned}$$

by the induction hypothesis. This gives

$$\begin{aligned}\{\boldsymbol{a}_0, \boldsymbol{a}_j\} = d(\boldsymbol{eu}^2 - 4qQ - d^2a^2)&q^{d-j}\Psi_{j-3}(\boldsymbol{eu}, q, Q)\\ &- q^{d-j+1}Q\big(\varphi_{j-2}(\boldsymbol{eu}, q, Q) + a^2\theta_{j-2}(\boldsymbol{eu}, q, Q)\big)\\ &+ q^{d-j}\boldsymbol{eu}\big(\varphi_{j-1}(\boldsymbol{eu}, q, Q) + a^2\theta_{j-1}(\boldsymbol{eu}, q, Q)\big),\end{aligned}$$

which proves that $(\wp^+_{0,j})$ holds.

We will now prove that $(\wp_{i,j})$ holds by induction on i. The case $i = 0$ has just been treated, so assume that $i \geqslant 1$ and that $(\wp_{i-1,j'})$ holds for all j'. Then $(i - 1 - d)\boldsymbol{a}_i = \{Q, \boldsymbol{a}_{i-1}\}$ and $i - 1 - d \neq 0$. By the Jacobi identity, we get

$$\begin{aligned}(i - 1 - d)\{\boldsymbol{a}_i, \boldsymbol{a}_j\} &= \{\{Q, \boldsymbol{a}_{i-1}\}, \boldsymbol{a}_j\}\\ &= \{Q, \{\boldsymbol{a}_{i-1}, \boldsymbol{a}_j\}\} - \{\boldsymbol{a}_{i-1}, \{Q, \boldsymbol{a}_j\}\}\\ &= \{Q, \{\boldsymbol{a}_{i-1}, \boldsymbol{a}_j\}\} - (j - d)\{\boldsymbol{a}_{i-1}, \boldsymbol{a}_{j+1}\}.\end{aligned}$$

So the result follows from the induction hypothesis because, if $\Theta \in \mathbb{C}[T, T', T'']$ is an homogeneous polynomial of degree k, then

$$\{Q, \Theta(\boldsymbol{eu}, q, Q)\} = -2Q\frac{\partial\Theta}{\partial T}(\boldsymbol{eu}, q, Q) - \boldsymbol{eu}\frac{\partial\Theta}{\partial T'}(\boldsymbol{eu}, q, Q)$$

is of the form $\Theta^{\#}(\boldsymbol{eu}, q, Q)$ where $\Theta^{\#}$ is homogeneous of degree k. The proof of the proposition is complete. □

3.5. Lie algebra structure at the cuspidal point

By Theorem 3.5, the affine variety $\mathcal{Z}_c$ might be described as

$$\mathcal{Z}_c = \left\{ \begin{array}{r} (\mathfrak{q}, \mathfrak{Q}, \mathfrak{e}, a_0, a_1, \dots, a_d) \\ \in \mathbb{C}^{d+4} \end{array} \middle| \begin{array}{l} \forall\, 1 \leqslant i \leqslant j \leqslant d-1, \\ \begin{cases} \mathfrak{e} a_i = \mathfrak{q} a_{i+1} + \mathfrak{Q} a_{i-1}, \\ a_{i-1}a_{j+1} - a_i a_j \\ \quad = (\mathfrak{e}^2 - 4\mathfrak{q}\mathfrak{Q} - d^2 a^2)\mathfrak{q}^{d-j-1}\mathfrak{Q}^{i-1}\Psi_{j-i}(\mathfrak{e}, \mathfrak{q}, \mathfrak{Q}) \end{cases} \end{array} \right\}.$$

If $d = 3$ and $a \neq 0$, then $\mathcal{Z}_c$ is smooth. So assume from now on that $d \geqslant 4$ and $a \neq 0$. Then the homogeneous component of minimal degree of all the above equations is equal to 2, so the point $0 = (0, \dots, 0) \in \mathcal{Z}_c$ is singular and the tangent space of $\mathcal{Z}_c$ at 0 has dimension $d + 4$. It is the only singular point and it is a cuspidal point in the sense of [2] (see [5, §5.2]). This means that the corresponding maximal ideal $\mathfrak{m}_0$ of Z_c is a Poisson ideal (since $\mathfrak{m}_0 = \langle q, Q, \boldsymbol{eu}, \boldsymbol{a}_0, \boldsymbol{a}_1, \dots, \boldsymbol{a}_d\rangle$, this can also be checked thanks to Proposition 3.6). This implies that the cotangent space $\mathfrak{m}_0/\mathfrak{m}_0^2$ of $\mathcal{Z}_c$ at 0 inherits a Lie algebra structure from the Poisson bracket: we denote by $\mathfrak{Lie}_0(\mathcal{Z}_c)$ the vector space $\mathfrak{m}_0/\mathfrak{m}_0^2$ endowed with its Lie algebra structure. It has been proved in [5, Prop. 8.4] that

$$\text{If } d = 4, \text{ then } \mathfrak{Lie}_0(\mathcal{Z}_c) \simeq \mathfrak{sl}_3(\mathbb{C}). \tag{3.5}$$

We now determine $\mathfrak{Lie}_0(\mathcal{Z}_c)$ in the remaining cases:

Proposition 3.7. *If $d \geqslant 5$, then*

$$\mathfrak{Lie}_0(\mathcal{Z}_c) = \mathfrak{sl}_2(\mathbb{C}) \oplus S_d,$$

where S_d is a commutative ideal of $\mathfrak{Lie}_0(\mathcal{Z}_c)$ of dimension $d+1$ on which $\mathfrak{sl}_2(\mathbb{C})$ acts irreducibly (i.e. $S_d \simeq \mathrm{Sym}^d(\mathbb{C}^2)$ as an $\mathfrak{sl}_2(\mathbb{C})$-module).

Proof. If $m \in \mathfrak{m}_0$, we denote by $\dot{m}$ its image in $\mathfrak{Lie}_0(\mathcal{Z}_c)$. Then $(\dot{q}, \dot{Q}, \dot{\boldsymbol{eu}}, \dot{\boldsymbol{a}}_0, \dot{\boldsymbol{a}}_1, \dots, \dot{\boldsymbol{a}}_d)$ is a basis of $\mathfrak{Lie}_0(\mathcal{Z}_c)$. We set

$$\mathfrak{g} = \mathbb{C}\dot{Q} \oplus \mathbb{C}\dot{\boldsymbol{eu}} \oplus \mathbb{C}\dot{q} \quad \text{and} \quad S_d = \bigoplus_{j=0}^{d} \mathbb{C}\dot{\boldsymbol{a}}_j.$$

It follows from Proposition 3.3 that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{Lie}_0(\mathcal{Z}_c)$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, and that S_d is normalized by $\mathfrak{g}$ and is isomorphic to $\mathrm{Sym}^d(\mathbb{C}^2)$ as an $\mathfrak{sl}_2(\mathbb{C})$-module.

Since $d \geqslant 5$ (and so $d - 3 \geqslant 2$), we get from Proposition 3.6 that $\{\boldsymbol{a}_i, \boldsymbol{a}_j\} \in \mathfrak{m}_0^2$ and so $[\dot{\boldsymbol{a}}_i, \dot{\boldsymbol{a}}_j] = 0$. This completes the proof of the proposition. □

4. Action of $\mathbf{SL}_2(\mathbb{C})$

The action of $\mathbf{GL}_2(\mathbb{C})$ on $\mathbb{C}[V \times V^*] \rtimes W$ does not deform to an action on $\mathbf{H}_c$ by automorphisms of algebras, but the action of the subgroup $\mathbf{SL}_2(\mathbb{C})$ does, as explained for instance in [7, §3.6]. This action commutes with W and is given on elements of V and V^* by the same formula as in Section 2.4.

We investigate here several of its properties (Poisson structure, existence of an equivariant morphism $\mathcal{Z}_c \to \mathfrak{sl}_2(\mathbb{C})$, interpretation of the presentation of $\mathcal{Z}_c$ in terms of Hermite's reciprocity law). The equivariant morphism to $\mathfrak{sl}_2(\mathbb{C})$ is for instance used in [3] to prove that the local fundamental group of $\mathcal{Z}_c$ at its singular point is trivial.

4.1. Action and Poisson structure

This $\mathbf{SL}_2(\mathbb{C})$-action induces an action of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ on $\mathbf{H}_c$ by derivations: as in Section 2.4, if $\xi \in \mathfrak{sl}_2(\mathbb{C})$ and $\varphi \in \mathbf{H}_c$, we denote by $\xi \cdot h$ the action of $-{}^t\xi$ on h. It is related to the Poisson bracket through the same formulas as in Section 2.4:

$$e \cdot \varphi = \{Q, \varphi\}, \qquad h \cdot \varphi = \{\mathbf{eu}_0, \varphi\} \qquad \text{and} \qquad f \cdot \varphi = \{-q, \varphi\}. \tag{4.1}$$

4.2. Map to $\mathfrak{sl}_2(\mathbb{C})$

If $(\mathfrak{q}, \mathfrak{Q}, \mathfrak{e}) \in \mathbb{C}^3$, we denote by $M(\mathfrak{q}, \mathfrak{Q}, \mathfrak{e})$ the matrix

$$M(\mathfrak{q}, \mathfrak{Q}, \mathfrak{e}) = \begin{pmatrix} \mathfrak{e} & \mathfrak{Q} \\ -\mathfrak{q} & -\mathfrak{e} \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C}).$$

We identify $\mathfrak{sl}_2(\mathbb{C})$ with the subspace of Z_c equal to $\mathbb{C}q \oplus \mathbb{C}Q \oplus \mathbb{C}\boldsymbol{eu}$ by sending (e, h, f) to $(Q, \boldsymbol{eu}, -q)$: by Proposition 3.3, this identification carries the Lie bracket on $\mathfrak{sl}_2(\mathbb{C})$ to the Poisson bracket on $\mathbb{C}q \oplus \mathbb{C}Q \oplus \mathbb{C}\boldsymbol{eu}$. This gives an identification $\mathbb{C}[q, Q, \boldsymbol{eu}] \simeq \mathrm{Sym}(\mathfrak{sl}_2(\mathbb{C}))$ and the inclusion $\mathbb{C}[q, Q, \boldsymbol{eu}] \subset Z_c$ gives an $\mathbf{SL}_2(\mathbb{C})$-equivariant Poisson map

$$\mu^* : \mathcal{Z}_c \longrightarrow \mathfrak{sl}_2(\mathbb{C})^*$$

(the equivariance follows from (4.1)). Identifying $\mathfrak{sl}_2(\mathbb{C})$ with its dual thanks to the trace map endows $\mathfrak{sl}_2(\mathbb{C})$ with a Poisson structure and gives an $\mathbf{SL}_2(\mathbb{C})$-equivariant Poisson map

$$\mu : \mathcal{Z}_c \longrightarrow \mathfrak{sl}_2(\mathbb{C}).$$

The map μ can be explicitly described by the following formula

$$\mu(\mathfrak{q}, \mathfrak{Q}, \mathfrak{e}, a_0, a_1, \ldots, a_d) = M(\mathfrak{q}, \mathfrak{Q}, \mathfrak{e}).$$

4.3. **Hermite's reciprocity law**

Let $E = E^\sharp \oplus E_d$ denote the vector space

$$E = \underbrace{\mathbb{C}Q \oplus \mathbb{C}\boldsymbol{eu} \oplus \mathbb{C}q}_{E^\sharp} \oplus \underbrace{\mathbb{C}\boldsymbol{a}_0 \oplus \mathbb{C}\boldsymbol{a}_1 \oplus \cdots \oplus \boldsymbol{a}_d}_{E_d}.$$

Theorem 3.5 shows that the natural morphism of algebras $\sigma : \mathrm{Sym}(E) \to Z_c$ is surjective and it describes its kernel. For avoiding the confusion between multiplication in Z_c and multiplication in $\mathrm{Sym}(E)$, we will denote by $\star$ the multiplication in $\mathrm{Sym}(E)$. For instance, $\boldsymbol{a}_0\star\boldsymbol{a}_2-\boldsymbol{a}_1^{\star 2}$ is an element of $\mathrm{Sym}(E)$ whereas $\boldsymbol{a}_0\boldsymbol{a}_2-\boldsymbol{a}_1^2$ is an element of Z_c, which is equal to $\sigma(\boldsymbol{a}_0 \star \boldsymbol{a}_2 - \boldsymbol{a}_1^{\star 2})$. Similarly, if $e_1, \dots, e_n$ are elements of E and if $\Psi \in \mathbb{C}[T_1, \dots, T_n]$ is a polynomial in n indeterminates, we denote by $\Psi^\star(e_1, \dots, e_n)$ the evaluation of Ψ at $(e_1, \dots, e_n)$ *inside the algebra* $\mathrm{Sym}(E)$ whereas $\Psi(e_1, \dots, e_n)$ denotes the evaluation of Ψ *inside the algebra* Z_c: they satisfy the equality $\sigma(\Psi^\star(e_1, \dots, e_n)) = \Psi(e_1, \dots, e_n)$.

Proposition 3.3 and (4.1) imply that E is an $\mathbf{SL}_2(\mathbb{C})$-stable subspace of Z_c, so that σ is $\mathbf{SL}_2(\mathbb{C})$-equivariant. Let us denote by $V_2 \simeq \mathbb{C}^2$ another copy of $\mathbb{C}^2$ viewed as the standard representation of $\mathbf{SL}_2(\mathbb{C})$ (or $\mathfrak{sl}_2(\mathbb{C})$), and we denote by (u_1, u_2) its canonical basis. We then have two isomorphisms of vector spaces

$$\sigma^\sharp : \mathrm{Sym}^2(V_2) \longrightarrow E^\sharp \qquad \text{and} \qquad \sigma_d : \mathrm{Sym}^d(V_2) \longrightarrow E_d$$

which are defined by

$$\sigma^\sharp(u_1^2) = 2q, \quad \sigma^\sharp(u_1u_2) = \boldsymbol{eu}, \quad \sigma^\sharp(u_2^2) = 2Q, \quad \text{and} \quad \sigma_d(u_1^{d-i}u_2^i) = \boldsymbol{a}_i \quad \text{for } 0 \leqslant i \leqslant d.$$

Proposition 3.3 and (4.1) imply that $\sigma^\sharp$ and σ_d are $\mathbf{SL}_2(\mathbb{C})$-equivariant and we will identify $E^\sharp$ and E_d with $\mathrm{Sym}^2(V_2)$ and $\mathrm{Sym}^d(V_2)$ through these isomorphisms.

Let us first interprete the equations $(\mathfrak{Z}_i)_{1\leqslant i\leqslant d-1}$. Note that

$$\mathrm{Sym}^2(E) = \mathrm{Sym}^2(\mathrm{Sym}^2(V_2)) \ \oplus\ \mathrm{Sym}^2(V_2) \otimes \mathrm{Sym}^d(V_2) \ \oplus\ \mathrm{Sym}^2(\mathrm{Sym}^d(V_2))$$

and that we have a natural morphism

$$\mu_{2,d} : \mathrm{Sym}^2(V_2) \otimes \mathrm{Sym}^d(V_2) \longrightarrow \mathrm{Sym}^{d+2}(V_2)$$

given by multiplication. We denote by $\mathrm{Der}(\mathrm{Sym}(V_2))$ the $\mathrm{Sym}(V_2)$-module of derivations $\mathrm{Sym}(V_2) \to \mathrm{Sym}(V_2)$. If $D \in \mathrm{Der}(\mathrm{Sym}(V_2))$, we denote by $D^{(2)}$ the map $\mathrm{Sym}^2(V_2) \otimes \mathrm{Sym}^d(V_2) \to \mathrm{Sym}(V_2)$, $\varphi \otimes \psi \longmapsto D(\varphi)\psi$. Then it is easily checked that

$$\mathrm{Ker}(\mu_{2,d}) \cap \bigcap_{D\in\mathrm{Der}(\mathrm{Sym}(V_2))} \mathrm{Ker}(D^{(2)}) = \bigoplus_{i=1}^{d-1} \mathbb{C}(Q\star\boldsymbol{a}_{i-1} - \boldsymbol{eu}\star\boldsymbol{a}_i + q\star\boldsymbol{a}_{i+1}) \subset \mathrm{Sym}^2(E).$$

So the family of equations $(\mathfrak{Z}_i)_{1\leqslant i\leqslant d-1}$ can be summarized by

$$\operatorname{Ker}(\mu_{2,d}) \cap \bigcap_{D\in \operatorname{Der}(\operatorname{Sym}(V_2))} \operatorname{Ker}(D^{(2)}) \text{ is contained in } \operatorname{Ker}(\sigma). \tag{4.2}$$

Note that $\operatorname{Ker}(\mu_{2,d}) \cap \bigcap_{D\in \operatorname{Der}(\operatorname{Sym}(V_2))} \operatorname{Ker}(D^{(2)})$ is $\mathbf{SL}_2(\mathbb{C})$-stable, as the construction is canonical.

The interpretation of the equations $(\mathfrak{Z}_{i,j})_{1\leqslant i\leqslant j\leqslant d-1}$ is somewhat more subtle and is related with Hermite's reciprocity law (see the upcoming Remark 4.4). First, evaluation induces a surjective morphism of $\mathbf{SL}_2(\mathbb{C})$-modules

$$\begin{array}{rccc} \varepsilon_{m,n}: & \operatorname{Sym}^m(\operatorname{Sym}^n(V_2)) & \longrightarrow & \operatorname{Sym}^{mn}(V_2) \\ & v_1 \star \cdots \star v_m & \longmapsto & v_1 \cdots v_m. \end{array}$$

In the special case where $m = 2$ and $n = d$, then:

Lemma 4.1. *The family* $(\boldsymbol{a}_{i-1} \star \boldsymbol{a}_{j+1} - \boldsymbol{a}_i \star \boldsymbol{a}_j)_{1\leqslant i\leqslant j\leqslant d-1}$ *of elements of* $\operatorname{Sym}(E)$ *is a basis of* $\operatorname{Ker}(\varepsilon_{2,d}) \subset \operatorname{Sym}^2(\operatorname{Sym}^d(V_2)) \simeq \operatorname{Sym}^2(E_d)$.

In fact, the family $(\boldsymbol{a}_{i-1} \star \boldsymbol{a}_{j+1} - \boldsymbol{a}_i \star \boldsymbol{a}_j)_{1\leqslant i\leqslant j\leqslant d-1}$ generates the ideal equal to the kernel of the natural morphism $\varepsilon_{\bullet,d} : \operatorname{Sym}(\operatorname{Sym}^d(V_2)) \to \operatorname{Sym}(V_2)$. On the other hand, it follows from (2.2) that:

Lemma 4.2. *The family* $(q^{\star d-j-1} \star Q^{\star i-1} \star \Psi^\star_{j-i}(\boldsymbol{eu}, q, Q))_{1\leqslant i\leqslant j\leqslant d-1}$ *of elements of* $\operatorname{Sym}(E)$ *is a basis of* $\operatorname{Sym}^{d-2}(\operatorname{Sym}^2(V_2)) \simeq \operatorname{Sym}^{d-2}(E^\sharp)$.

Lemmas 4.1 and 4.2 allow to define a linear map

$$\rho_d : \operatorname{Ker}(\varepsilon_{2,d}) \longrightarrow \operatorname{Sym}^{d-2}(E^\sharp)$$

by the formula

$$\rho_d(\boldsymbol{a}_{i-1} \star \boldsymbol{a}_{j+1} - \boldsymbol{a}_i \star \boldsymbol{a}_j) = q^{\star d-j-1} \star Q^{\star i-1} \star \Psi^\star_{j-i}(\boldsymbol{eu}, q, Q)$$

for $1 \leqslant i \leqslant j \leqslant d$. It is an isomorphism of vector spaces but an important fact is the following:

Lemma 4.3. *The map* $\rho_d : \operatorname{Ker}(\varepsilon_{2,d}) \to \operatorname{Sym}^{d-2}(E^\sharp)$ *is an isomorphism of* $\mathbf{SL}_2(\mathbb{C})$-*modules.*

Proof. This is more or less the computation done in the end of the proof of Theorem 3.6. It is sufficient to prove that it is an isomorphism of $\mathfrak{sl}_2(\mathbb{C})$-modules. By (4.1), Proposition 3.3, we have

$$\begin{aligned} f \cdot (\boldsymbol{a}_{i-1} \star \boldsymbol{a}_{j+1} - \boldsymbol{a}_i \star \boldsymbol{a}_j) & \\ &= (i-1)\boldsymbol{a}_{i-2} \star \boldsymbol{a}_{j+1} + (j+1)\boldsymbol{a}_{i-1} \star \boldsymbol{a}_j - i\boldsymbol{a}_{i-1} \star \boldsymbol{a}_j - j\boldsymbol{a}_i \star \boldsymbol{a}_{j-1} \\ &= (i-1)(\boldsymbol{a}_{i-2} \star \boldsymbol{a}_{j+1} - \boldsymbol{a}_{i-1} \star \boldsymbol{a}_j) + j(\boldsymbol{a}_{i-1} \star \boldsymbol{a}_j - \boldsymbol{a}_i \star \boldsymbol{a}_{j-1}). \end{aligned}$$

Therefore,

$$\begin{aligned}&\rho_d(f\cdot(\boldsymbol{a}_{i-1}\star\boldsymbol{a}_{j+1}-\boldsymbol{a}_i\star\boldsymbol{a}_j)\\&\quad=(i-1)q^{\star d-j-1}\star Q^{\star i-2}\star\Psi^{\star}_{j-i+1}(\boldsymbol{eu},q,Q)+jq^{\star d-j}\star Q^{\star i-1}\star\Psi^{\star}_{j-i-1}(\boldsymbol{eu},q,Q),\end{aligned}$$

and so one gets

$$\begin{aligned}&\rho_d(f\cdot(\boldsymbol{a}_{i-1}\star\boldsymbol{a}_{j+1}-\boldsymbol{a}_i\star\boldsymbol{a}_j))\\&\quad=q^{\star d-j-1}\star Q^{\star i-2}\star((i-1)\Psi^{\star}_{j-i+1}(\boldsymbol{eu},q,Q)+jq\star Q\star\Psi^{\star}_{j-i-1}(\boldsymbol{eu},q,Q))\\&\quad=q^{\star d-j-1}\star Q^{\star i-2}\star((i-1)\boldsymbol{eu}\star\Psi^{\star}_{j-i}(\boldsymbol{eu},q,Q)+(j-i+1)q\star Q\star\Psi^{\star}_{j-i-1}(\boldsymbol{eu},q,Q)),\end{aligned}$$

where the last equality follows from (2.4). Applying now (2.5) yields

$$\begin{aligned}&(i-1)\boldsymbol{eu}\star\Psi^{\star}_{j-i}(\boldsymbol{eu},q,Q)+(j-i+1)q\star Q\star\Psi^{\star}_{j-i-1}(\boldsymbol{eu},q,Q)\\&\quad=(i-1)\boldsymbol{eu}\star\Psi^{\star}_{j-i}(\boldsymbol{eu},q,Q)+2q\star Q\star\left(\frac{\partial\Psi_{j-i}}{\partial T}\right)^{\star}(\boldsymbol{eu},q,Q)+q\star\boldsymbol{eu}\left(\frac{\partial\Psi_{j-i}}{\partial T'}\right)^{\star}(\boldsymbol{eu},q,Q).\end{aligned}$$

Putting things together and using again (4.1) and Proposition 3.3 yields

$$\rho_d(f\cdot(\boldsymbol{a}_{i-1}\star\boldsymbol{a}_{j+1}-\boldsymbol{a}_i\star\boldsymbol{a}_j))=f\cdot\rho_d(\boldsymbol{a}_{i-1}\star\boldsymbol{a}_{j+1}-\boldsymbol{a}_i\star\boldsymbol{a}_j),$$

as desired. The fact that

$$\rho_d(e\cdot(\boldsymbol{a}_{i-1}\star\boldsymbol{a}_{j+1}-\boldsymbol{a}_i\star\boldsymbol{a}_j))=e\cdot\rho_d(\boldsymbol{a}_{i-1}\star\boldsymbol{a}_{j+1}-\boldsymbol{a}_i\star\boldsymbol{a}_j),$$

follows from a similar computation and this completes the proof of the Lemma. □

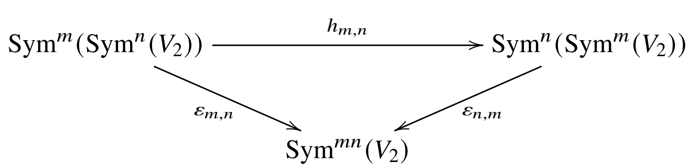
Using the isomorphism of $\mathbf{SL}_2(\mathbb{C})$-modules ρ_d, the family of equations $(\mathfrak{Z}_{i,j})$ can be rewritten as follows:

$$\forall\ \varphi\in\operatorname{Ker}(\varepsilon_{2,d}),\ \varphi-\rho_d(\varphi)\star(\boldsymbol{eu}^{\star 2}-4q\star Q-d^2a^2)\in\operatorname{Ker}(\sigma). \tag{4.3}$$

Remark 4.4. The existence of such an isomorphism of $\mathbf{SL}_2(\mathbb{C})$-modules $\operatorname{Ker}(\varepsilon_{2,d})\xrightarrow{\sim}\operatorname{Sym}^{d-2}(E^{\sharp})$ is a consequence of Hermite's reciprocity law, as it has been explained to us by Pierre-Louis Montagard. Indeed, Hermite's reciprocity law (see for instance [9, Cor. 2.2]) says that we have an isomorphism of $\mathbf{SL}_2(\mathbb{C})$-modules

$$h_{m,n}:\operatorname{Sym}^m(\operatorname{Sym}^n(V_2))\xrightarrow{\sim}\operatorname{Sym}^n(\operatorname{Sym}^m(V_2))$$

making the diagram



commutative. In particular, $h_{m,n}$ induces an isomorphism, still denoted by $h_{m,n}$, between $\mathrm{Ker}(\varepsilon_{m,n})$ and $\mathrm{Ker}(\varepsilon_{n,m})$.

In the particular case where $m = 2$ and $n = d$, the kernel of the evaluation map $\varepsilon_{\bullet,2} : \mathrm{Sym}(\mathrm{Sym}^2(V_2)) = \mathrm{Sym}(E^\sharp) \to \mathrm{Sym}(V_2)$ is the principal ideal generated by $\boldsymbol{eu}^{\star 2} - 4q \star Q$ so that the map

$$\begin{array}{ccc} \mathrm{Sym}^{d-2}(V_2) & \longrightarrow & \mathrm{Ker}(\varepsilon_{d,2}) \\ \varphi & \longmapsto & (\boldsymbol{eu}^{\star 2} - 4q \star Q) \star \varphi \end{array}$$

is an isomorphism of $\mathbf{SL}_2(\mathbb{C})$-modules. Composing the inverse of this isomorphism with $h_{2,d}$ gives an isomorphism $\mathrm{Ker}(\varepsilon_{2,d}) \xrightarrow{\sim} \mathrm{Sym}^{d-2}(E^\sharp)$.

Remark 4.5. Since $\boldsymbol{eu}^{\star 2} - 4q \star Q \in \mathrm{Sym}(E^\sharp)^{\mathbf{SL}_2(\mathbb{C})}$ (in fact, it even generates this invariant algebra) we can define, for any polynomial P in one variable, a variety $\mathcal{Z}^P$ by the following equations:

$$\mathcal{Z}^P = \left\{ \begin{array}{r|l} (\mathfrak{q}, \mathfrak{Q}, \mathfrak{e}, a_0, a_1, \dots, a_d) & \forall\, 1 \leqslant i \leqslant j \leqslant d-1, \\ \in \mathbb{C}^{d+4} & \begin{cases} \mathfrak{e} a_i = \mathfrak{q} a_{i+1} + \mathfrak{Q} a_{i-1}, \\ a_{i-1}a_{j+1} - a_i a_j \\ \quad = P(\mathfrak{e}^2 - 4\mathfrak{q}\mathfrak{Q})\mathfrak{q}^{d-j-1}\mathfrak{Q}^{i-1}\Psi_{j-i}(\mathfrak{e}, \mathfrak{q}, \mathfrak{Q}) \end{cases} \end{array} \right\}.$$

By (4.2) and (4.3), the variety $\mathcal{Z}^P$ can we rewritten as follows:

$$\mathcal{Z}^P = \left\{ \begin{array}{r||l} (\mathfrak{q}, \mathfrak{Q}, \mathfrak{e}, a_0, a_1, \dots, a_d) & \begin{cases} \forall\, \varphi \in \mathrm{Ker}(\mu_{2,d}) \cap \bigcap_{D \in \mathrm{Der}(\mathrm{Sym}(V_2))} \mathrm{Ker}(D^{(2)}), \\ \quad \varphi(\mathfrak{q}, \mathfrak{Q}, \mathfrak{e}, a_0, a_1, \dots, a_d) = 0, \\ \forall\, \varphi \in \mathrm{Ker}(\varepsilon_{2,d}), \\ \quad \varphi(a_0, a_1, \dots, a_d) = P(\mathfrak{e}^2 - 4\mathfrak{q}\mathfrak{Q})\rho_d(\varphi)(\mathfrak{e}, \mathfrak{q}, \mathfrak{Q}) \end{cases} \\ \in \mathbb{C}^{d+4} & \end{array} \right\}.$$

This shows that $\mathcal{Z}^P$ is an $\mathbf{SL}_2(\mathbb{C})$-stable subvariety of $\mathbb{C}^{d+4} \simeq E^*$.

5. Fixed points under diagram automorphism

Let $\sqrt{\zeta}$ be a primitive $2d$-th root of unity such that $(\sqrt{\zeta})^2 = \zeta$ and let $\tau = \begin{pmatrix} 0 & \sqrt{\zeta} \\ \sqrt{\zeta}^{-1} & 0 \end{pmatrix}$. Then $\tau s \tau^{-1} = t$ and $\tau t \tau^{-1} = s$. So τ normalizes W and, since $c_s = c_t$, τ acts on Z_c and so on $\mathcal{Z}_c$ by [7]. The action on the generators of Z_c given in Theorem 3.5 is easily computed:

$$^\tau q = q, \quad ^\tau Q = Q, \quad ^\tau \boldsymbol{eu} = \boldsymbol{eu} \quad \text{and} \quad ^\tau \boldsymbol{a}_i = -\boldsymbol{a}_i \tag{5.1}$$

for $0 \leqslant i \leqslant d$.

Using the description of $\mathcal{Z}_c$ as a closed subvariety of $\mathbb{C}^{d+4}$ as in Section 3.5 thanks to Theorem 3.5, one gets:

$$\mathcal{Z}_c^\tau = \{(\mathfrak{q}, \mathfrak{Q}, e, a_0, a_1, \dots, a_d) \in \mathcal{Z}_c \mid a_0 = a_1 = \cdots = a_d = 0\}.$$

Therefore,

$$\mathcal{Z}_c^\tau \simeq \left\{ (\mathfrak{q}, \mathfrak{Q}, \mathfrak{e}) \in \mathbb{C}^3 \;\middle|\; \begin{array}{l} \forall\, 1 \leqslant i \leqslant j \leqslant d-1, \\ (\mathfrak{e}^2 - \mathfrak{q}\mathfrak{Q} - d^2a^2)\mathfrak{q}^{d-j-1}\mathfrak{Q}^{i-1}\Psi_{j-i}(e, \mathfrak{q}, \mathfrak{Q}) = 0 \end{array} \right\}.$$

Let $(\mathfrak{e}, \mathfrak{q}, \mathfrak{Q}) \in \mathcal{Z}_c^\tau$. If $\mathfrak{q} \neq 0$, then the above equation with $i = j = 1$ gives $\mathfrak{e}^2 - \mathfrak{q}\mathfrak{Q} - d^2a^2 = 0$. Similarly, if $\mathfrak{Q} \neq 0$, the above equation with $i = j = d-1$ gives $\mathfrak{e}^2 - \mathfrak{q}\mathfrak{Q} - d^2a^2 = 0$. So assume now that $\mathfrak{q} = \mathfrak{Q} = 0$. Then the above equation with $i = 1$ and $j = d-1$ gives $(\mathfrak{e}^2 - d^2a^2)\Psi_{d-2}(e, 0, 0) = 0$. But an easy induction on k shows that $\Psi_k(T, 0, 0) = T^k$ for all k, so this gives $(\mathfrak{e}^2 - d^2a^2)\mathfrak{e}^{d-2} = 0$. This discussion shows that

$$\mathcal{Z}_c^\tau \simeq \{(0,0,0)\} \cup \{(\mathfrak{q}, \mathfrak{Q}, \mathfrak{e}) \in \mathbb{C}^3 \mid (\mathfrak{e} - da)(\mathfrak{e} + da) = \mathfrak{q}\mathfrak{Q}\}. \tag{5.2}$$

So the 0-dimensional irreducible component is of course isomorphic to the Calogero–Moser space associated with the trivial group(!), and the 2-dimensional irreducible component is isomorphic to the Calogero–Moser spaces associated with the pair (V^τ, W^τ) and parameter $da/2$: indeed, $\dim V^\tau = 1$, $W^\tau = \langle w_0 \rangle \simeq \boldsymbol{\mu}_2$ and equations for Calogero–Moser spaces associated with cyclic groups are given for instance in [7, Thm. 18.2.4]. Moreover, Proposition 3.3 shows that this isomorphism respect the Poisson bracket. So we have proved the following result, which confirms [7, Conj. FIX] (or [6, Conj. B]):

Proposition 5.1. *The unique* 2*-dimensional irreducible component of* $\mathcal{Z}_c^\tau$ *is isomorphic, as a Poisson variety endowed with a* $\mathbb{C}^\times$*-action, to the Calogero–Moser space associated with the pair* $(V^\tau, W^\tau) \simeq (\mathbb{C}, \boldsymbol{\mu}_2)$ *and the parameter map* $\mathrm{Ref}(\boldsymbol{\mu}_2) = \{-1\} \to \mathbb{C}$, $-1 \mapsto da$.

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