ANNALES MATHÉMATIQUES

BLAISE PASCAL

Mohamed Benelmekki & Salah Najib

A note on the spectrum of a rational function

Volume 30, nº 2 (2023), p. 107-114.

https://doi.org/10.5802/ambp.418

Publication éditée par le laboratoire de mathématiques Blaise Pascal
de l'université Clermont Auvergne, UMR 6620 du CNRS
Clermont-Ferrand — France

CENTRE MERSENNE

Publication membre du
Centre Mersenne pour l'édition scientifique ouverte
http://www.centre-mersenne.org/
e-ISSN : 2118-7436

Annales mathématiques Blaise Pascal **30**, 107-114 (2023)

A note on the spectrum of a rational function

Mohamed Benelmekki
Salah Najib

Abstract

In 2008, Bodin provided an alternative approach for bounding the total reducibility order of a non-composite rational function. His proof used some properties of jacobian derivation. In this note, we revisit this proof and eliminate the jacobian derivation aspect. The new ingredient in our presentation is a version of Lüroth's theorem.

Une note sur le spectre d'une fonction rationnelle

Résumé

En 2008, Bodin a fourni une approche alternative pour borner l'ordre total de réductibilité d'une fonction rationnelle indécomposable. Sa preuve a utilisé certaines propriétés de dérivation jacobienne. Dans cette note, nous revisitons cette preuve et éliminer l'aspect de dérivation jacobienne. Le nouvel ingrédient de notre présentation est une version du théorème de Lüroth.

1. **Introduction**

Let K be a field. Let $f = \frac{p}{q} \in K(\underline{x})$ be a rational function, where $\underline{x} = (x_1, \dots, x_n)$, $n \geq 2$ and $p, q \in K[\underline{x}]$ such that $\gcd(p, q) = 1$. We recall that $\deg f = \max\{\deg p, \deg q\}$; and if it is non-constant, the rational function f is said to be *composite* if there exist $u(T) \in K(T)$, $H(\underline{x}) \in K(\underline{x})$ such that $f = u(H)$ with $\deg u \geq 2$.

We associate to the rational function $f = \frac{p}{q} \in K(\underline{x})$, the pencil $p - \lambda q$, $\lambda \in \widehat{K} = K \cup \{\infty\}$ and by convention $\lambda = \infty$ whenever $p - \lambda q = q$. The set of $\lambda \in \widehat{K}$ for which $p - \lambda q$ is reducible is called the *spectrum* of f and is denoted by $\sigma(f)$, i.e.

$$\sigma(f) = \left\{\lambda \in \widehat{K} : p - \lambda q \text{ is reducible over } K\right\}.$$

For each $\lambda \in \sigma(f)$, write

$$p(\underline{x}) - \lambda q(\underline{x}) = \prod_{i=1}^{n(\lambda)} F_{\lambda,i}^{k_{\lambda,i}}(\underline{x})$$

Keywords: Irreducible polynomials, Indecomposable rational function, Spectrum of a rational function, Lüroth's theorem.
2020 *Mathematics Subject Classification*: 12E05, 12F20, 11C08.

where $F_{\lambda,i}$ is irreducible over K and $k_{\lambda,i} \in \mathbb{Z}_+$ for all $i = 1, \ldots, n(\lambda)$. The number $\rho_\lambda(f) = n(\lambda) - 1$ is called the *reducibility order of f* at λ, and the number $\rho(f) = \sum_{\lambda\in\sigma(f)} \rho_\lambda(f)$ is called the *total reducibility order of f*.

By an irreducible polynomial over K we mean a non-constant polynomial $F \in K[\underline{x}]$ such that $F = GH$ implies that either $G \in K^* = K \setminus \{0\}$ or $H \in K^*$; otherwise we say that F is reducible over K.

It is well-known that $\sigma(f)$ is finite if and only if f is non-composite; see for example [5, Corollary 15], [1, Corollary 2.3] and references therein. Therefore, it is natural to look for a bound on the cardinality of $\sigma(f)$ when f is non-composite. In this context, Ruppert [16] proved that there are at most $(\deg f)^2 - 1$ reducible curves in the pencil $p - \lambda q$, $\lambda \in \mathbb{C} \cup \{\infty\}$ when the generic curve in this pencil is irreducible of degree $\deg f$. For $f \in K(X,Y)$, where K is an algebraically closed field of any characteristic, Lorenzini [10] showed, under some geometric hypotheses on the pencil $p - \lambda q$, that $\rho(f) < (\deg f)^2$. This has been extended by Vistoli [20] to a pencil in several variables for an algebraically closed field of characteristic zero. The polynomial case ($q = 1$) has been considered, in particular by Cygan [6], Kaliman [8], Najib [11, 12] and Stein [19]. It has been proved that, for an algebraically closed field K of characteristic zero: *if $P \in K[X,Y]$ is a non-composite polynomial, then $\rho(P) \leq \deg P - 1$.* Later Lorenzini [10] showed (for an algebraically closed field of any characteristic and for two variables) that:

$$\rho(P) \leq \min_{\lambda\in\sigma(P)} \left\{ \sum_{i=1}^{n(\lambda)} \deg(f_{\lambda,i}) \right\} - 1 \leq \deg P - 1,$$

where the $f_{\lambda,i}$ are the irreducible distinct factors of $P - \lambda$ for $\lambda \in \sigma(P)$. This inequality has been generalized by Najib [13] for an arbitrary field of any characteristic and for any number of variables.

Following the works of Lorenzini and Ruppert, in 2008, Bodin [1], by using some results about the kernel of the jacobian derivations, provided an alternative approach leading to the following bound.

Theorem 1.1. *Let K be an algebraically closed field of characteristic 0. If $f \in K(X,Y)$ is non-composite then $\rho(f) < (\deg f)^2 + \deg f$.*

Some variants of these results have been extensively studied; see for example [2], [3], [4, Section 5.5], [5], [14] and [15].

In this note, we revisit the proof of Bodin and eliminate the jacobian derivation aspect.

2. **Preliminary Lemmas**

Our proof of Theorem 1.1 uses this new ingredient; it is a variant of Lüroth's theorem (for $n = 2$).

Lemma 2.1. *Let K be an algebraically closed field of characteristic* 0.
If $K \subset L \subset K(X,Y)$ and $\operatorname{tr.deg}_K L = 1$*, then $L = K(g)$ where $g \in K(X,Y)$.*

Proof. We adapt the proof of Theorem 3 [17, Section 1.2] in our case $n = 2$ and with K infinite of characteristic 0.

By Theorem 1 of [17, Section 1.2], there exist $g_1, g_2 \in K(X,Y)$ such that $L = K(g_1, g_2)$. Since $K \subset L \subset K(X,Y)$, then

$$K(Y) \subset L(Y) \subset K(X,Y).$$

By Lüroth's theorem (see [17, Section 1.1; Theorem 2])

$$L(Y) = K(h,Y) \text{ where } h \in K(X,Y) \setminus K(Y).$$

Thus $K(h,Y) = K(g_1, g_2, Y)$, and hence

$$g_1 = f_1(h,Y), \quad g_2 = f_2(h,Y) \quad \text{where } f_1, f_2 \in K(T_1, T_2),$$

and $h = \phi(g_1, g_2, Y)$ where $\phi \in K(T_1, T_2, Y)$. It follows that

$$g_1 = f_1(\phi(g_1, g_2, Y), Y), \quad g_2 = f_2(\phi(g_1, g_2, Y), Y).$$

Since K is infinite, we may choose a value $y_0 \in K$ such that, after substitution $Y = y_0$, the rational functions $f_1(\phi(g_1, g_2, y_0), y_0)$ and $f_2(\phi(g_1, g_2, y_0), y_0)$ make sense. Since

$$g_i = f_i(\phi(g_1, g_2, y_0), y_0), \quad \text{for } i = 1, 2,$$

we conclude that the rational function $\phi(g_1, g_2, y_0)$ generates the extension L/K. □

Let K be an algebraically closed field of characteristic 0. Let $f = \frac{p}{q} \in K(\underline{x})$ be a non-constant rational function and $\lambda_1, \ldots, \lambda_m \in \widehat{K}$. We denote by $G(f; \lambda_1, \ldots, \lambda_m)$ the multiplicative group generated by all non-associate divisors of the polynomials $p - \lambda_i q$, $(i = 1, \ldots, m)$. Let $d(f) = (\deg f)^2 + \deg f$.

Our proof of Theorem 1.1 also uses the following lemma, which states a sufficient condition for the algebraic dependence of f and some element of $G(f; \lambda_1, \ldots, \lambda_m)$.

Lemma 2.2. *Let $F_1, \ldots, F_r \in G(f; \lambda_1, \ldots, \lambda_m)$. If $r \geq d(f)$ then there exists a non-trivial collection of integers $m_1, \ldots, m_r$, such that the rational function $\prod_{i=1}^{r} F_i^{m_i}$ and f are algebraically dependent over K.*

This statement is the same as [1, Lemma 3.3], but here the conclusion *"the rational function $\prod_{i=1}^{r} F_i^{m_i}$ and f are algebraically dependent over K"* replaces the equivalent one "$g = \prod_{i=1}^{r} F_i^{m_i} \in C_f$" there [1]. The proof of Bodin is based on other stataments which use some properties of jacobian derivations and their kernels. For a self-contained argument and for the convenience of the reader, in the last section, we give a proof that does not use the jacobian derivation aspect.

3. **Proof of Theorem 1.1**

Proof. Let $f = \frac{p}{q}$ where $p, q \in K[X, Y]$ be coprime. Assume that $f(X, Y)$ is non-composite and let $\lambda_1, \ldots, \lambda_r$ be the spectral values of f. Now, we assume that $\rho(f) = \sum_{j=1}^{r} \rho_{\lambda_j}(f) \geq d(f)$. For all $j = 1, \ldots, r$, let $p - \lambda_j q = \prod_{i=1}^{n_j} F_{j,i}^{k_{j,i}}$ the decomposition of $p - \lambda_j q$ into the product of irreducible factors, where n_j denotes $n(\lambda_j)$. Consider the collection of polynomials:

$$\{F_{1,1}, \ldots, F_{1,n_1-1}, \ldots, F_{r,1}, \ldots, F_{r,n_r-1}\}.$$

Note that all these polynomials are elements of $G(f; \lambda_1, \ldots, \lambda_r)$ and their number is $\sum_{j=1}^{r}(n_j - 1) = \sum_{j=1}^{r} \rho_{\lambda_j}(f) \geq d(f)$. Hence, by Lemma 2.2, there exists a non-trivial collection of integers

$$\{m_{1,1}, \ldots, m_{1,n_1-1}, \ldots, m_{r,1}, \ldots, m_{r,n_r-1}\}$$

such that f and the rational function

$$g = \prod_{j=1}^{r} \prod_{i=1}^{n_j-1} F_{j,i}^{m_{j,i}} \tag{3.1}$$

are algebraically dependent over K.

By Lemma 2.1, for $L = K(f, g)$, there exists $h \in K(X, Y) \setminus K$ such that $L = K(h)$. Thus $f, g \in K(h)$. Since the rational function f is non-composite, it follows that $f = \frac{\alpha_1 h + \beta_1}{\alpha_2 h + \beta_2}$, with $\alpha_1, \beta_1, \alpha_2, \beta_2 \in K$ and $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$. Thus $K(h) = K(f)$, and therefore $g \in K(f)$.

Now, we write $g = \frac{u(f)}{v(f)}$, where $u, v \in K[T]$ are such that $\gcd(u, v) = 1$. Let $\mu_1, \ldots, \mu_k$ be the roots of u and $\mu_{k+1}, \ldots, \mu_\ell$ those of v. Hence

$$g = \frac{u\left(\frac{p}{q}\right)}{v\left(\frac{p}{q}\right)} = \eta . \frac{\prod_{i=1}^{k}\left(\frac{p}{q} - \mu_i\right)}{\prod_{i=k+1}^{\ell}\left(\frac{p}{q} - \mu_i\right)}, \quad \eta \in K \setminus \{0\},$$

[1]where C_f is the kernel of the jacobian derivation of $K(X, Y)$ associated to f.

thus

$$g = \eta . q^{2k-\ell} \frac{\prod_{i=1}^{k} (p - \mu_i q)}{\prod_{i=k+1}^{\ell} (p - \mu_i q)} \tag{3.2}$$

Note that, for every (j_0, i_0) such that $m_{j_0,i_0} \neq 0$, we obtain that the factor F_{j_0,i_0} divides one of the polynomials $p - \mu_i q,\ i = 1, \ldots, \ell$ (by comparing the decompositions (3.1) and (3.2)). Since F_{j_0,i_0} divides $p - \lambda_{j_0} q$, we deduce that $\lambda_{j_0} \in \{\mu_1, \ldots, \mu_\ell\}$ and hence, by (3.2), that $p - \lambda_{j_0} q$ is a factor of the numerator or the denominator of g. Thus for $(j_0, i_0) = (j_0, n_{j_0})$, we obtain that the irreducible divisor $F_{j_0,n_{j_0}}$ of $p - \lambda_{j_0} q$ should appear in the decomposition (3.1), which contradicts our choice of the collection:

$$\{F_{1,1}, \ldots, F_{1,n_1-1}, \ldots, F_{r,1}, \ldots, F_{r,n_r-1}\}.$$

Consequently, $\rho(f) < d(f)$. □

4. **Proof of Lemma 2.2**

First part. We show that it suffices to prove the existence of an r−tuple $(m_1, \ldots, m_r)$ of integers (not all equal to zero) such that the rational function $g = \prod_{i=1}^{r} F_i^{m_i}$ is constant on infinitely many irreducible components of the curves $\{p - \lambda q = 0\},\ \lambda \in \widehat{K}$.

Indeed, let $(V_n)_{n>0}$ be an infinite sequence of irreducible components of the curves $\{p - \lambda q = 0\}$ $(\lambda \in \widehat{K})$ such that the function g is constant on them. We denote by $\{p - \lambda_n q = 0\}$ the curve of which V_n is a component. As g is constant on V_n then the determinant $\mathrm{Jac}(f, g)$ of the jacobian matrix of f and g is zero on V_n: this follows, for example, by writing $K(V_n) = K(X, y)$ with y a solution of $p(X, y) - \lambda_n q(X, y) = 0$, by developing $\frac{\partial}{\partial X}(g(X, y)) = 0$ and using $\frac{\mathrm{d}y}{\mathrm{d}X} = -\frac{\partial f}{\partial X} / \frac{\partial f}{\partial y}$.

Now let $\mathrm{Jac}(f, g) = \frac{N(X,Y)}{D(X,Y)}$ where N, D are coprime polynomials in $K[X, Y]$.

The Zariski closed set $\mathcal{Z}(N) = \{(x, y) \in K^2 /\ N(x, y) = 0\}$ contains infinitely many irreducible disjoint components V_n. So N is the zero polynomial and $\mathrm{Jac}(f, g) = 0$.

Moreover by a classical result (see for Example 11.4 of [9, p. 12], applied with $\Phi = K(X, Y), m = n = 2, \beta_1 = f$ and $\beta_2 = g$), it follows that the two rational functions f and g are algebraically dependent over K.

Second part. This part largely reuses the ideas of [1, proof of Lemma 3.3]. Here *we prove that there exists an r-tuple $(m_1, \ldots, m_r)$ of integers (not all equal to zero) such that the function $g = \prod_{i=1}^{r} F_i^{m_i}$ is constant on infinitely many irreducible components of the curves $\{p - \lambda q = 0\},\ \lambda \in \widehat{K}$.*

Let $\lambda \notin \{\lambda_1, \ldots, \lambda_m\}$, S be an irreducible component of $\{p - \lambda q = 0\}$ and $\bar{S}$ be the projective closure of S.

The zeroes and poles of each F_i $(i = 1, \dots, r)$ restricted to $\bar{S}$ are among the points at infinity of S or in the intersection $S \cap \mathcal{Z}(F_i) \subset \mathcal{Z}(p) \cap \mathcal{Z}(q)$.

Let $n : S^\nu \to \bar{S}$ be a normalization of $\bar{S}$; the curve S^ν is a smooth projective model of S. Denote by $t_1, \dots, t_\ell$ the points in the inverse image $n^{-1}(\bar{S} \setminus S)$. Their number satisfies $\ell \leq \deg(S) \leq \deg(f)$.

At a point $t \in \mathcal{Z}(p) \cap \mathcal{Z}(q)$, the number of points of $n^{-1}(t)$ is the number of local branches of S at t, so is less than or equal to the order (or multiplicity) $\mathrm{ord}_t(S)$ of S at point t (see for example [18, Chapter II, Section 5.3]). Consequently

$$\begin{aligned} \# n^{-1}(t) &\leq \mathrm{ord}_t(S) \leq \mathrm{ord}_t\, \mathcal{Z}(p - \lambda q) \leq \mathrm{ord}_t\, \mathcal{Z}(p - \lambda q).\, \mathrm{ord}_t\, \mathcal{Z}(p) \\ &\leq \mathrm{mult}_t(p - \lambda q, p) = \mathrm{mult}_t(p, q), \end{aligned}$$

where $\mathrm{mult}_t(p, q)$ is the multiplicity of the intersection of the two curves defined by p and q at point t (see for example [7, Chapter 3]). Then by Bézout theorem, we obtain:

$$\sum_{t \in \mathcal{Z}(p) \cap \mathcal{Z}(q)} \# n^{-1}(t) \leq \sum_{t \in \mathcal{Z}(p) \cap \mathcal{Z}(q)} \mathrm{mult}_t(p, q) \leq \deg p . \deg q \leq (\deg(f))^2 .$$

Denote by $t_{\ell+1}, \dots, t_k$ the points in the inverse image $n^{-1}(\bigcup_{i=1}^r S \cap \mathcal{Z}(F_i))$. The above inequality shows that the number $k - \ell$ of these points is less than or equal $(\deg(f))^2$. Notice that $k \leq \deg(f) + (\deg(f))^2 = d(f)$.

Let $\nu_{i,j}$ be the order of F_i at point t_j (for $i = 1, \dots, r$ and $j = 1, \dots, k$) and consider the matrix $\mathcal{M} = (\nu_{i,j})$. As the degree of the divisor (F_i) is zero (seen over S^ν) then $\sum_{j=1}^k \nu_{i,j} = 0$ for all $i = 1, \dots, r$, which means that the columns of $\mathcal{M}$ are linearly dependent. Thus $\mathrm{rg}(\mathcal{M}) < k \leq d(f)$. Moreover by hypothesis $r \geq d(f)$, the rows of $\mathcal{M}$ are linearly dependent. Then there exists an $r-$tuple $(m_1(\lambda, S), \dots, m_r(\lambda, S))$ of integers (not all equal to zero) such that

$$\sum_{i=1}^r m_i(\lambda, S) \nu_{i,j} = 0, \ \ j = 1, \dots, k.$$

Consider now the function $g_{\lambda,S} = \prod_{i=1}^r F_i^{m_i(\lambda,S)}$.

This function is regular and does not have zeroes on S. Moreover it does not have zeroes or poles at the points $t_1, \dots, t_k$, since $\sum_{i=1}^r m_i(\lambda, S) \nu_{i,j} = 0$ (for all $j = 1, \dots, k$). Then the function $g_{\lambda,S}$ is constant on S.

Conclusion. for any choice of t and S as above, there exists $(m_1(\lambda, S), \dots, m_r(\lambda, S)) \in \mathbb{Z}^r \setminus \{(0, \dots, 0)\}$ such that the rational function $g_{\lambda,S} = \prod_{i=1}^r F_i^{m_i(\lambda,S)}$ is constant on the irreducible component S. Since, the field K is supposed uncountable, there exists infinitely many (λ, S) for which the $r-$tuple $(m_1(\lambda, S), \dots, m_r(\lambda, S))$ takes the same value $(m_1, \dots, m_r)$. Consequently, the rational function $g = \prod_{i=1}^r F_i^{m_i}$ is constant on the corresponding components. $\square$

Remark 4.1. We note that the case of several variables ($n \geq 3$) is explained in [1, Section 5]. This case is based on a result which claims that the irreducibility and the degree of a family of polynomials remain constant after a generic linear change of coordinates; see [13, Proposition 1].

Acknowledgment

We want to thank the anonymous referee for his important comments and suggestions.

The final version of this paper was written when the second author was a guest at the Laboratoire Paul Painlevé of Lille University. He is very grateful to Pierre Dèbes and Arnaud Bodin for many hours of inspiring discussion about irreducible polynomials and related subjects.

References

[1] Arnaud Bodin. Reducibility of rational functions in several variables. *Isr. J. Math.*, 164:333–348, 2008.

[2] Arnaud Bodin, Pierre Dèbes, and Salah Najib. Irreducibility of hypersurfaces. *Commun. Algebra*, 37(6):1884–1900, 2009.

[3] Arnaud Bodin, Pierre Dèbes, and Salah Najib. Families of polynomials and their specializations. *J. Number Theory*, 170:390–408, 2017.

[4] Arnaud Bodin, Pierre Dèbes, and Salah Najib. The Schinzel Hypothesis for Polynomials. *Trans. Am. Math. Soc.*, 373(12):8339–8364, 2020.

[5] Laurent Busé, Guillaume Chèze, and Salah Najib. Noether's forms for the study of non-composite rational functions and their spectrum. *Acta Arith.*, 147(3):217–231, 2011.

[6] Ewa Cygan. Factorization of polynomials. *Bull. Pol. Acad. Sci., Math.*, 40(1):45–52, 1992.

[7] William Fulton. *Algebraic curves. An introduction to algebraic geometry*. Addison-Wesley Publishing Group, new edition, 1989.

[8] Shulim Kaliman. Two remarks on polynomials in two variables. *Pac. J. Math.*, 154(2):285–295, 1992.

[9] Solomon Lefshetz. *Algebraic Geometry*, volume 18 of *Princeton Mathematical Series*. Princeton University Press, 1953.

[10] Dino Lorenzini. Reducibility of polynomials in two variables. *J. Algebra*, 156(1):65–75, 1993.

[11] Salah Najib. Sur le spectre d'un polynôme à plusieurs variables. *Acta Arith.*, 114:169–181, 2004.

[12] Salah Najib. *Factorisation des polynômes* $P(X_1, ..., X_n) - \lambda$ *et théorème de Stein*. PhD thesis, University of Lille, 2005.

[13] Salah Najib. Une généralisation de l'inégalité de Stein-Lorenzini. *J. Algebra*, 292(2):566–573, 2005.

[14] Salah Najib. The spectrum of a rational function. *Algebra Colloq.*, 27(3):477–482, 2020.

[15] Anatoliy P. Petravchuk and Oleksandr G. Iena. On closed rational functions in several variables. *Algebra Discrete Math.*, 6(2):115–124, 2007.

[16] Wolfgang Ruppert. Reduzibilität ebener Kurven. *J. Reine Angew. Math.*, 369:167–191, 1986.

[17] Andrzej Schinzel. *Polynomials with special regard to reducibility*. Cambridge University Press, 2000.

[18] Igor R. Shafarevich. *Basic algebraic geometry 1*. Springer, 1994.

[19] Yosef Stein. The total reducibility order of a polynomial in two variables. *Isr. J. Math.*, 68(1):109–122, 1989.

[20] Angelo Vistoli. The number of reducible hypersurfaces in a pencil. *Invent. Math.*, 112:247–262, 1993.

Mohamed Benelmekki
Université Sultan Moulay Slimane
Laboratoire de Mathématiques
et Applications, FST
Campus Mghilla, BP 523, 23000 Béni Mellal
MAROC
med.benelmekki@gmail.com

Salah Najib
Université Sultan Moulay Slimane,
Faculté Polydisciplinaire de Khouribga
Laboratoire multidisciplinaire de
recherche et d'innovation
BP 145, Hay Ezzaytoune, 25000 Khouribga
MAROC
slhnajib@gmail.com