ANNALES MATHÉMATIQUES

BLAISE PASCAL

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Volume 28, n° 1 (2021), p. 71-116.

<http://ambp.centre-mersenne.org/item?id=AMBP_2021__28_1_71_0>

Publication éditée par le laboratoire de mathématiques Blaise Pascal
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Clermont-Ferrand — France

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http://www.centre-mersenne.org/

Annales mathématiques Blaise Pascal **28**, 71-116 (2021)

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Abstract

We generalize a Cheeger–Müller type theorem for flat, unitary bundles on infinite covering spaces over manifolds with boundary, proven by Burghelea, Friedlander and Kappeller. Employing recent anomaly results by Brüning, Ma and Zhang, we prove an analogous statement for a general flat bundle that is only required to have a unimodular restriction to the boundary.

1. Introduction and statement of the main results

For any flat bundle E over a compact, triangulated manifold M (briefly denoted by $E \downarrow M$ throughout this paper), one can construct the classical Reidemeister torsion, see for example [24]. In [14], Chapman showed that for acyclic bundles, this torsion is independent of the chosen triangulation, thereby also suggesting that there must be an alternative way to define it.

With the aid of a Riemannian metric g on M, Ray and Singer defined in [30] the analytic torsion for unitary bundles $E \downarrow M$ and showed that it does not depend on the choice of metric g in case that $\partial M = \emptyset$. Furthermore, they conjectured that this analytic torsion must be equal to the Reidemeister torsion. This result was then independently proven by Müller [25] and Cheeger [15] in the case $\partial M = \emptyset$. Later, Müller defined analytic torsion in the setting of a unimodular bundle $E \downarrow M$ and extended his earlier result [26]. At about the same time, Bismut and Zhang formulated a Cheeger–Müller type theorem for arbitrary flat bundles $E \downarrow M$ [6], generalizing the notion of analytic and Reidemeister torsion in the same process.

The L^2-versions of Reidemeister and analytic torsion first appeared in [13], respectively [23], and were first only defined for compact manifolds that are L^2-acyclic. Burghelea, Friedlander, Kappeler and MacDonald later extended these definitions to unitary bundles $E \downarrow M$ without any assumption on L^2-acyclicity [12], and showed that both invariants are in fact equal. In [35], adapting the methods he earlier co-developed in [6], Zhang extended this result even further to arbitrary flat bundles, providing an explicit formula of the anomaly between L^2-Reidemeister and L^2-analytic torsion in this case and strengthening an earlier result [11] in the same vein by Burghelea, Friedlander and Kappeler. Instead of Reidemeister torsion, the authors of [6, 11, 12, 35] used the so-called Morse–Smale torsion (see Section 3.1), which is defined using triangulations derived from a given

This work is funded by the DFG (Deutsche Forschungsgemeinschaft) – 28186985.

Morse function $f\colon M \to \mathbb{R}$. While additionally requiring a Hermitian bundle metric h of E to be defined, the smooth data involved in its construction makes it applicable for the Witten-deformation technique that plays a key part in the comparsion with analytic torsion. Moreover, although not explicitly written down anywhere, it is folklore knowledge that the Morse–Smale torsion coincides with the Reidemeister torsion whenever the volume form induced by h is flat. An explicit proof of this and other related statements will be subject of a separate paper from the author.

Now assume that $\partial M \neq \emptyset$. Under the assumption that the Hermitian metric h is flat and the Riemannian metric g is a product near ∂M, the difference between Reidemeister and analytic torsion has been made explicit by Lück [19], Vishik [33]. After various different generalizations of this particular result with relaxed assumptions on the metrics g and h, most notable of which is the result by Hassell [18] who assumed g to be cylindrical near ∂M, the general case without any further assumptions on g or h has been studied by Brüning and Ma in [8, 9], who were able to prove an anomaly formula [9, Theorem 0.1] entirely extending the result of Bismut and Zhang [6, Theorem 0.2] to manifolds with boundary.

Adapting the techniques of their original result in the closed manifold case, the relation between L^2-Reidemeister torsion and L^2-analytic torsion on manifolds with boundary was studied by Burghelea, Friedlander and Kappeller [10] under the assumption that h is flat and g is a product near ∂M. In [21], Lück and Schick showed that anomaly of the L^2-analytic torsion is created when g is deformed near ∂M. This anomaly was made explicit by Ma and Zhang [22], showing that it in fact equals the anomaly of ordinary analytic torsion. Making use of all the results mentioned so far, our main result, Theorem 4.8, will be a Cheeger–Müller type theorem for L^2-acyclic unimodular bundles $E \downarrow M$ on manifolds with boundary satisfying $\chi(M) = 0$.

In order to state the result, we fix a flat bundle $E \downarrow M$ as above, along with a Riemannian metric g on M and an Hermitian metric h on E. Additionally, we choose a Morse function $f\colon M \to \mathbb{R}$, whose critical points lie in the interior of M and that is constant along ∂M, together with some Riemannian metric g' so that the pair (f, g') satisfies the *Smale-transversality* conditions, cf. Definition 3.1. We denote by $\nabla_{g'} f$ the corresponding gradient vector field and call the quadruple $(E \downarrow M, g, h, \nabla_{g'} f)$ a *type II Morse–Smale system*. We say that $(E \downarrow M, g, h, \nabla_{g'} f)$ is *of product form* if both g and h are products near ∂M, see Definition 3.2. Provided that the bundle $E \downarrow M$ is of *determinant class* (Definitions 3.3, 3.5 and Theorem 6.4), a Ray–Singer analytic L^2-torsion

$$T^{RS}_{(2)}(E \downarrow M, g, h, \nabla_{g'} f) \in \mathbb{R}_{>0}, \tag{1.1}$$

as well as a Morse–Smale L^2-torsion

$$T^{MS}_{(2)}(E \downarrow M, h, \nabla_{g'} f) \in \mathbb{R}_{>0} \tag{1.2}$$

can be defined, see Definition 3.3 and Equation (3.40). To make precise the anomaly between the two L^2-torsions, two quantities need to be introduced. The first of these is given by the 1-form

$$\theta(h) \in \Omega^1(M), \tag{1.3}$$

derived from h, see Equation (4.2). Roughly stated, it measures the change along M of the volume form induced by h, and vanishes precisely when h is unimodular, i.e. when the volume form induced by h is flat. The second one is the so-called *Mathai–Quillen current*

$$\Psi(TM, g) \in \Omega^{n-1}(TM \setminus M, \mathcal{O}_{TM}) \tag{1.4}$$

derived from g [6, Definition 3.6], where $\mathcal{O}_{TM}$ denotes the orientation line bundle over the tangent bundle TM. Since the gradient field $\nabla_{g'} f$ can also be regarded as a smooth embedding from $M \setminus \mathrm{Cr}(f)$ into $TM \setminus M$ (where $M \subseteq TM$ is identified with the zero section), we obtain via pullback a density

$$\nabla_{g'} f^* \Psi(TM, g) \in \Omega^{n-1}(M \setminus \mathrm{Cr}(f), \mathcal{O}_M). \tag{1.5}$$

With all these objects introduced, our first main result can be stated as follows:

Theorem 1.1 (Theorem 4.5)**.** *Let $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$ be a type II Morse–Smale system of product form, where M is an odd-dimensional compact manifold and $h|_{\partial M}$ is unimodular. Further, assume that both $E \downarrow M$ and $E|_{\partial M} \downarrow \partial M$ are of determinant class. Then*

$$\log\left(\frac{T^{RS}_{(2)}(E \downarrow M, g, h, \nabla_{g'} f)}{T^{MS}_{(2)}(E \downarrow M, h, \nabla_{g'} f)}\right) = -\frac{\log 2}{4}\chi(\partial M)\dim(E) - \frac{1}{2}\int_M \theta(h) \wedge \nabla_{g'} f^* \Psi(TM, g). \tag{1.6}$$

This result can be viewed as a strict generalization of the main result of [10], where the authors made the more restrictive assumption that the metric h is globally flat.

In order to state the second main result of this paper, we suppose that the bundle $E \downarrow M$ is unimodular, i.e. associated with a unimodular representation of $\pi_1(M)$. Then, assuming that $E \downarrow M$ is L^2-acyclic and of determinant class, one can define a *topological L^2-torsion* $T^{Top}_{(2)}(M, E) \in \mathbb{R}_{>0}$. It can be defined similarly like $T^{MS}_{(2)}(E \downarrow M, h, \nabla_{g'} f)$, with the aid of *any* given CW-structure on M and any fixed inner product on V, see [34, Definition 5.2.5, Theorem 5.3.12], and coincides with $T^{MS}_{(2)}(E \downarrow M, h, \nabla_{g'} f)$ whenever h is unimodular.

Theorem 1.2 (Theorem 4.8)**.** *Let (M, g) be a compact, connected, odd-dimensional Riemannian manifold. Then, there exists a density $B(g) \in \Omega^{n-1}(\partial M, O_{\partial M})$ with $B(g) \equiv 0$ when g is product-like near ∂M, such that the following holds:*

Let $E \downarrow M$ be a flat, finite-dimensional complex vector bundle, such that

(a) *E is unimodular,*

(b) *the pair (M, E) is L^2-acyclic and of determinant class,*

(c) *the restriction $(\partial M, E|_{\partial M})$ is of determinant class.*

Then, for any choice of unimodular metric h on E, one has

$$\log\left(\frac{T^{RS}_{(2)}(E \downarrow M, g, h, \nabla_{g'} f)}{T^{Top}_{(2)}(M, E)}\right) = \frac{1}{2}\dim_{\mathbb{C}}(E)\int_{\partial M} B(g). \tag{1.7}$$

In a forthcoming paper, Theorem 1.2 will be used to generalize the main result of [21] by Lück and Schick, in which we will show the equality of Ray–Singer analytic L^2-torsion and topological L^2-torsion for a large class of flat, unimodular bundles over finite-volume, hyperbolic manifolds, which are studied by several other authors as well [1, 4, 27, 28].

The rest of this paper is subdivided into six sections, which are structured as follows: In Section 2, we briefly review the abstract theory of Hilbert $\mathcal{N}(\Gamma)$-modules that is necessary to define the Novikov–Shubin invariants, the determinant class condition and the general L^2-torsion which are studied in the rest of the paper. In Section 3, we introduce the central objects of this paper: Morse–Smale systems, their Morse–Smale L^2-torsion and analytic L^2-torsion, as well as the derived metric L^2-torsion and relative torsion. In Section 4, we state our main results, Theorems 4.5 and 4.8 and give a proof of the latter. In Section 5, we present product and anomaly formulas for the different L^2-torsions. In Section 6, we will review the techniques employed by Burghelea et al. in their original proof for unitary bundles: Witten-Deformation, the splitting of the de Rham complex into the small and large subcomplex and the asymptotic expansions of the respective L^2-torsions. In Section 7, we give a proof of Theorem 4.5.

Acknowledgments

The paper is based on parts of the author's dissertation [34]. I thank my advisor, Prof. R. Sauer, for his constant support and numerous fruitful discussions.

2. L^2-torsion of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes

We start by recollecting the objects and theory of Hilbert $\mathcal{N}(\Gamma)$-modules that are relevant for this paper. The well-acquainted reader may skip this section.

Throughout, we fix a countable group Γ. We define $L^2(\Gamma)$ to be the complex Hilbert space generated over the set Γ. Note that multiplying group elements of Γ from the left naturally determines a left, linear, isometric Γ-action on $L^2(\Gamma)$. More generally, a complex Hilbert space $\mathcal{H}$ is called a *Hilbert $\mathcal{N}(\Gamma)$-module* if it comes equipped with a left, linear, isometric Γ-action, so that there exists a Γ-linear, isometric embedding of $\mathcal{H}$ into $L^2(\Gamma) \mathbin{\widehat{\otimes}} H$ for some Hilbert space H. Here, $L^2(\Gamma) \mathbin{\widehat{\otimes}} H$ denotes the *Hilbert space tensor product* of $L^2(\Gamma)$ and H, with isometric Γ-action given by the action on the left factor. If one can choose H to be finite-dimensional over $\mathbb{C}$, we call the Hilbert $\mathcal{N}(\Gamma)$-module $\mathcal{H}$ *finitely generated*.

A Γ-linear, closed and densely defined operator $f : \mathcal{H} \to \mathcal{H}'$ between two Hilbert $\mathcal{N}(\Gamma)$-modules $\mathcal{H}$ and $\mathcal{H}'$ is called a *morphism of Hilbert $\mathcal{N}(\Gamma)$-modules*. Since Γ is fixed and implicit throughout this section, we will simply refer to such f as a morphism. Any *positive*, bounded endomorphism $f : \mathcal{H} \to \mathcal{H}$ admits a natural *von Neumann trace*

$$\mathrm{tr}_\Gamma(f) \in [0, \infty], \tag{2.1}$$

[20, Definition 1.8] which satisfies $\mathrm{tr}_\Gamma(f) < \infty$ whenever $\mathcal{H}$ is finitely generated. With this, we define the *von Neumann dimension* of $\mathcal{H}$ by

$$\dim_{\mathcal{N}(\Gamma)}(\mathcal{H}) \coloneqq \mathrm{tr}_\Gamma(\mathbb{1}_{\mathcal{H}}). \tag{2.2}$$

The adjoint f^*, the self-adjoint composition $f^* f$, as well as the absolute value $|f| \coloneqq \sqrt{f^* f}$ of a morphism f are again morphisms. Similarly, if $E^{|f|}$ is the spectral measure associated with the positive, self-adjoint operator $|f|$ and if p is a positive, essentially bounded Borel function defined over the spectrum $\sigma(|f|)$ of $|f|$, then

$$p(f) \coloneqq \int_{-\infty}^{\infty} p \, \mathrm{d}E^{|f|}(\lambda) \tag{2.3}$$

is a positive, bounded endomorpism, which is why $\mathrm{tr}_\Gamma(p(f)) \in [0, \infty]$ is always well-defined. In particular, the family $\{\chi_{[0,\lambda]}(f)\}_{\lambda \in \mathbb{R}_{\geq 0}}$ of spectral projections associated to f further gives rise to a non-decreasing, right-continuous function

$$F_f(\lambda) \coloneqq \mathrm{tr}_\Gamma(\chi_{[0,\lambda]}(f)) \in [0, \infty] \tag{2.4}$$

in $\lambda \geq 0$, called the *spectral density function* of f. We say that f is *Fredholm* if $F_f(\lambda) < \infty$ for all $\lambda \geq 0$. As a quantitative measurement of the spectral behaviour near 0, the *Novikov–Shubin invariant* $\alpha(f) \in [0, \infty] \cup \{\infty^+\}$ of a Fredholm morphism f is defined as

$$\alpha(f) := \begin{cases} \liminf_{\lambda\to 0^+} \dfrac{\ln\left(F_f(\lambda) - F_f(0)\right)}{\ln(\lambda)} & \text{if } F_f(\lambda) > F_f(0) \quad \forall\, \lambda > 0, \\ \infty^+ & \text{else.} \end{cases} \tag{2.5}$$

$\alpha(f)$ equals the (purely formal) symbol ∞^+ precisely when $|f|$ has a spectral gap at 0.

Moreover, if f is Fredholm, its spectral density determines a Borel measure $\mathrm{d}F_f$ on $\mathbb{R}_{\geq 0}$ in the canonical fashion. A Fredholm morphism f is said to be *of determinant class* if

$$\int_{0+}^{1} \log(\lambda)\mathrm{d}F_f(\lambda) > -\infty. \tag{2.6}$$

A morphism f with $\alpha(f) > 0$ is always of determinant class, although the converse need not hold. If f is a *bounded* morphism of determinant class, we can define its *Fuglede–Kadison determinant* $\det_\Gamma(f) \in \mathbb{R}_{>0}$ as

$$\log(\det_\Gamma(f)) := \int_{0+}^{\|f\|} \log(\lambda)\mathrm{d}F_f(\lambda). \tag{2.7}$$

A cochain complex

$$(C^*, c^*)\colon 0 \to C^0 \xrightarrow{c^0} C^1 \xrightarrow{c^1} C^2 \xrightarrow{c^2} C_3 \xrightarrow{c^3} \ldots, \tag{2.8}$$

with each C^i a Hilbert $\mathcal{N}(\Gamma)$-module and each c^i a (not necessarily bounded) morphism of Hilbert $\mathcal{N}(\Gamma)$-modules is called a *Hilbert $\mathcal{N}(\Gamma)$-cochain complex*. If all but finitely many of the C^i's are trivial, each C^i is finitely generated and each c^i is bounded, then (C^*, c^*) is *of finite type*. A family $f^*\colon C^* \to D^*\colon (f^k\colon C^k \to D^k)_{k\in\mathbb{N}}$ of *bounded* morphisms is called a *morphism between the Hilbert $\mathcal{N}(\Gamma)$-cochain complexes* (C^*, c^*) and (D^*, d^*) if it additionally satisfies $f^*(\mathrm{dom}(c^*)) \subseteq \mathrm{dom}(d^*)$ and $f^{*+1} \circ c^* = d^* \circ f^*$ on $\mathrm{dom}(c^*)$. f^* is called an *isomorphism* if each f^k is an isomorphism.

We say that two morphisms $f^*, g^* : C^* \to D^*$ between Hilbert $\mathcal{N}(\Gamma)$-cochain complexes are *chain homotopic* (written $f \simeq g$) if there exists a collection of bounded morphisms $K^* : C^* \to D^{*-1}$, satisfying

$$K^*(\mathrm{dom}(c^*)) \subseteq \mathrm{dom}(d^{*-1}),$$
$$f^* - g^* = K^{*+1}c^* + d^{*-1}K^* \quad \text{on } \mathrm{dom}(c^*).$$

K^* is called an *chain homotopy* between f^* and g^*. Two Hilbert $\mathcal{N}(\Gamma)$-cochain complexes (C^*, c^*) and (D^*, d^*) are called *chain homotopy equivalent* (written $C^* \sim D^*$) if there exists morphisms $f^* : C^* \to D^*$ and $g^* : D^* \to C^*$ such that $f^*g^* \simeq \mathbb{1}_{D^*}$ and $g^*f^* \simeq \mathbb{1}_{C^*}$. f^* is called a *chain homotopy equivalence* between C^* and D^* with *chain homotopy inverse* g^*.

The (full) L^2-coholomology of a Hilbert $\mathcal{N}(\Gamma)$-cochain complex is the graded Hilbert $\mathcal{N}(\Gamma)$-module defined as

$$H^*(C^*) := \bigoplus_{k=0}^{\infty} H^k(C^*), \quad H^k(C^*) := \ker(c^k)/\mathrm{clos}(\mathrm{im}(c^{k-1})). \tag{2.9}$$

A cochain complex (C^*, c^*) is *Fredholm* if all of the restricted morphisms $c^k|_{\mathrm{im}(c^{k-1})^\perp}$, $k \in \mathbb{N}_0$, are Fredholm. Observe that a complex (C^*, c^*) of finite type is automatically Fredholm. For a Fredholm complex, we define its k-th Novikov–Shubin invariant $\alpha_k(C^*) \in [0, \infty] \cup \{\infty+\}$ as

$$\alpha_k(C^*) := \alpha(c^k|_{\mathrm{im}(c^{k-1})^\perp}). \tag{2.10}$$

A Fredholm complex is said to be *of determinant class* if all of the restricted morphisms $c^k|_{\mathrm{im}(c^{k-1})^\perp}$ are of determinant class. If C^* is a determinant class *and* of finite type, we define its L^2-Torsion $T^{(2)}(C^*) \in \mathbb{R}_{>0}$ as

$$\log(T^{(2)}(C^*)) := \sum_{k=0}^{\infty} (-1)^k \log(\det_\Gamma(c^k)). \tag{2.11}$$

Proposition 2.1 ([17, Proposition 4.1])**.** *Let (C^*, c^*) and (D^*, d^*) be two cochain complexes of Hilbert $\mathcal{N}(\Gamma)$-modules and $f^*\colon C^* \to D^*$ a chain homotopy equivalence between them. Then, f^* descends to an isomorphism of L^2-cohomologies*

$$H^*(f^*)\colon H^*(C^*) \to H^*(D^*). \tag{2.12}$$

Additionally, if both C^* and D^* are Fredholm, we have

(1) *$\alpha_k(C^*) = \alpha_k(D^*)$ for each $k \in \mathbb{N}_0$.*

(2) *C^* is of determinant class if and only if D^* is of determinant class.*

Proposition 2.2 ([20, Lemma 3.44])**.** *Let (C^*, c^*) and (D^*, d^*) be two cochain complexes of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes, both of finite type and of determinant class. Further, let $f^*\colon C^* \to D^*$ be a chain isomorphism between them. Then*

$$\log(T^{(2)}(C^*)) - \log(T^{(2)}(D^*)) = \sum_{k=0}^{\infty} (-1)^k \log(\det_\Gamma(f^k)) - \sum_{k=0}^{\infty} (-1)^k \log\left(\det_\Gamma(H^k(f^k))\right). \tag{2.13}$$

3. **Relative torsion**

We commence by introducing in order the main objects of this paper: Morse–Smale systems and their Morse–Smale, analytic, metric and relative L^2-torsion.

By a *system* $\mathcal{D} = (E \downarrow M, g, h, X)$, we will always mean a set of data consisting of a flat, complex vector bundle $E \downarrow M$ over a smooth manifold M, along with a Riemannian metric g on M, a Hermitian form h on E and X either a vector field or a complex-valued function over M.

Given a uniform lattice $\Gamma < \mathrm{Isom}(M, g)$, such a system $\mathcal{D}$ is called Γ-*invariant* if in addition, the isometric action of Γ on (M, g) leaves X invariant and extends to an action of bundle isometries on the metric bundle $(E, h) \downarrow (M, g)$. Observe that Γ-invariant systems on M are precisely the lifts of systems defined over the compact quotient M/Γ.

Throughout this chapter, we will frequently form *products* of systems: Given for $i = 1, 2$ two systems $(E_i \downarrow M_i, g_i, h_i, X_i)$ with X_i either both vector fields or functions, one obtains a new system $(E_1 \mathbin{\widehat{\otimes}} E_2 \downarrow M_1 \times M_2, g_1 \oplus g_2, h_1 \mathbin{\widehat{\otimes}} h_2, X_1 + X_2)$, where $M_1 \times M_2$ is the product manifold equipped with the (direct) sum metric $g_1 \oplus g_2$, $X_1 + X_2$ is the sum of the two vector fields or functions, and

- $E_1 \mathbin{\widehat{\otimes}} E_2 \downarrow M_1 \times M_2$ is defined to be the flat tensor product bundle $\pi_1^* E_1 \otimes \pi_2^* E_2 \downarrow M_1 \times M_2$, where $\pi_i : M_1 \times M_2 \to M_i$ denotes the projection onto the i-th factor. Here, the flat structure we choose is the canonical one induced by its flat factors $\pi_i^* E_i$. Moreover,

- $h_1 \mathbin{\widehat{\otimes}} h_2 \coloneqq \pi_1^* h_1 \otimes \pi_2^* h_2$ is the tensor product of the respective pullback Hermitian forms.

The main focus of our attention will be *Morse–Smale systems*, which are by definition systems $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$ with (f, g') a Morse–Smale pair, the latter of which we are now going to define: First of all, a pair (f, g') with $f : M \to \mathbb{R}$ a Morse function and g' a Riemannian metric on M is called a *Morse pair*. Let $\nabla_{g'} f \in \Gamma(TM)$ be the gradient vector field constructed from f and g' and let ψ_t be the flow associated to the differential equation

$$\frac{\partial y}{\partial t} = -\nabla_{g'} f(y). \tag{3.1}$$

Provided that both f and g' are lifted from a compact quotient, which we will assume throughout, it follows that ψ_t is globally defined, i.e. for all $t \in \mathbb{R}$. With $\mathrm{Cr}(f) \subset M$ denoting the set of critical points of f, define for each $p \in \mathrm{Cr}(f)$ the *stable*, respectively *unstable manifolds*

$$W^-(p) := \Big\{x \in M : \lim_{t\to-\infty} \psi_t(x) = p\Big\}, \tag{3.2}$$

$$W^+(p) := \Big\{x \in M : \lim_{t\to+\infty} \psi_t(x) = p\Big\}. \tag{3.3}$$

Both $W^+(p)$ and $W^-(p)$ are smooth submanifolds of M, the latter being diffeomorphic to $\mathbb{R}^{\mathrm{ind}(p)}$. Here, as everywhere else, $0 \le \mathrm{ind}(p) \le n$ denotes the *index* of the critical point p.

Definition 3.1. A Morse pair (f, g') on M is called a *Morse–Smale pair*, if all of the following conditions are satisfied:

(1) For each pair $p, q \in \mathrm{Cr}(f)$, the manifolds $W^-(p)$ and $W^+(q)$ intersect transversally.

(2) (f, g') is *locally trivial* at $\mathrm{Cr}(f)$. This means that:

(a) For any $0 \le k \le n$ and any $p \in \mathrm{Cr}(f)$, there exists (pairwise disjoint) coordinate neighborhoods

$$\phi_p : U_p \to \begin{cases} \mathbb{R}^n & \text{if } p \notin \partial M \\ \mathbb{R}^n_{x_n\ge 0} & \text{if } p \in \partial M \end{cases}$$

of p with $\phi_p(p) = 0$ and such that we have

$$(f \circ \phi_p^{-1})(x_1, \dots, x_n) = f(p) - \frac{1}{2}(x_1^2 + \dots x_{\mathrm{ind}(p)}^2) + \frac{1}{2}(x_{\mathrm{ind}(p)+1}^2 + \dots + x_n^2).$$

(b) The pullback $\phi_p^*(g_{\mathbb{R}^n})$ of the standard Euclidean metric on $\mathbb{R}^n$ equals $g'|_{U_p}$.

If $\partial M \ne \emptyset$, we additionally assume that there exists $\kappa > 0$, along with a collar neighborhood U of ∂M and a diffeomorphism $\psi_{g'} \colon \partial M \times [0, \kappa) \to U$ coming from the normal exponential map induced by g', so that either of the following two (mutually exclusive) conditions hold:

(i) One has $(f \circ \psi_{g'})(p, t) = f|_{\partial M}(p) + t^2$ (in particular, $f|_{\partial M}$ is a Morse function on ∂M with $\mathrm{Cr}(f|_{\partial M}) = \mathrm{Cr}(f) \cap \partial M$). In this case, we say that (f, g') is *of type I*.

(ii) One has $(f \circ \psi_{g'})(p, t) = b - t$ with $b = \max(f) \in \mathbb{Z}$ (in particular, $\mathrm{Cr}(f) \cap \partial M = \emptyset$). In this case, we say that (f, g') is *of type II*.

It is a classic result that any compact manifold admits Morse–Smale pairs (f, g'), both of type I and of type II, see e.g. [3, Theorem 6.6]. In fact, we will almost exclusively focus on type II Morse–Smale pairs. That is because the methods employed to prove Theorem 4.5 require that the critical points of a given Morse function are all interior, and thus only work for type II Morse–Smale pairs. Conversely, the techniques used in other papers, whose results play an essential role in the proof of Theorem 4.8, only work for type I Morse–Smale pairs, which is why we have included them in the above definition.

Definition 3.2. A Morse–Smale system of the form $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$ will be called a *type II Morse–Smale system* if (f, g') is a type II Morse–Smale pair. A type II Morse–Smale system is *of product form*, if

(P_1) *g is a product near ∂M*: There exists a collar neighborhood V of ∂M that is the diffeomorphic image of the normal exponential map $\psi_g : \partial M \times [0, \epsilon) \to V$ induced by g, such that $\psi_g^*(g|_V) = g|_{\partial M} \oplus \mathrm{d}t^2$, where $\mathrm{d}t^2$ denotes the standard Euclidean metric on the half-open interval $[0, \epsilon)$.

(P_2) The isometry ψ_g further extends to a flat bundle isometry

$$\Psi : (E|_{\partial M} \mathbin{\widehat{\otimes}} E_{\mathbb{C}} \downarrow \partial M \times [0, \epsilon), h|_{\partial M} \mathbin{\widehat{\otimes}} 1_{\mathbb{C}}) \to (E|_V \downarrow V, h_V).$$

Here, $E_{\mathbb{C}} \downarrow [0, \epsilon)$ is the trivial 1-dimensional vector bundle over $[0, 1)$ (with trivial connection), $E|_{\partial M} \mathbin{\widehat{\otimes}} E_{\mathbb{C}} \downarrow \partial M \times [0, \epsilon)$ denotes the flat, complex product bundle as introduced in the previous paragraph and $1_{\mathbb{C}}$ denotes the canonical constant Hermitian form on $E_{\mathbb{C}} \downarrow [0, \epsilon)$.

A type II Morse–Smale system of product form is called *weakly admissible*, if

(A_1) M is compact.

(A_2) One has $g \equiv g'$ near $\mathrm{Cr}(f)$ and outside from a neighborhood of ∂M.

(A_3) The metric h is parallel (see Definition 4.2) in a neighborhood of $\mathrm{Cr}(f)$.

Finally, a weakly admissible system $\mathcal{D}$ is called *admissible* if the following extra compatibility condition is satisfied:

(A_4) the restriction $h|_{\partial M}$ of h to ∂M is unimodular (see Definition 4.2).

A Γ-invariant system $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$ that is the lift of an admissible, respectively weakly admissible system on the compact quotient M/Γ is called Γ-*admissible*, respectively *weakly* Γ-*admissible*.

Observe that a weakly admissible system is a Morse–Smale system on a compact manifold M with special *local* conditions on the Riemannian metric g and Hermitian form h near ∂M and the critical points of f, while for an admissible system, we additionally demand a *global* condition on $h|_{\partial M}$. In particular, any flat bundle $E \downarrow M$ over a compact manifold fits into some weakly admissible system $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$ (by choosing an appropriate partition of unity), which can be chosen admissible if and only if the restriction bundle $E|_{\partial M} \downarrow \partial M$ is unimodular.

3.1. The Morse–Smale L^2-torsion $T^{MS}_{(2)}(E \downarrow M, h, \nabla_{g'} f)$

Let $\mathcal{D} = (M, E, g, h, \nabla_{g'} f)$ be a Morse–Smale system with M connected, $\widetilde{M}$ the universal cover of M and $\widetilde{\mathcal{D}} = (\widetilde{M}, \widetilde{E}, \widetilde{g}, \widetilde{h}, \nabla_{\widetilde{g}'} f)$ the corresponding lifted system over $\widetilde{M}$. With $\Gamma := \pi_1(M)$, it follows that $\widetilde{\mathcal{D}}$ is a Γ-invariant system. Let $\rho : \Gamma \to \mathrm{GL}(V)$ be the complex, finite-dimensional representation associated to the flat bundle $E \downarrow M$. Then, as a Γ-equivariant flat bundle, $\widetilde{E}$ is isomorphic to the trivial flat bundle $\widetilde{M} \times V$ with diagonal Γ-action given by $\gamma.(x, v) = (\gamma.x, \rho(\gamma)v)$. We fix one such isomorphism throughout.

As before, denote for each $p \in \mathrm{Cr}(\widetilde{f})$ by $W^-(p)$ and $W^+(p)$ the unstable, respectively stable manifold at p. Observe that we have $\gamma.W^-(p) = W^-(\gamma.p) \cong \mathbb{R}^{\mathrm{ind}(p)}$ for each $\gamma \in \Gamma$, which allows us to fix a global orientation O_p on each unstable manifold $W^-(p)$ in a Γ-invariant way. Together with the fact that $W^-(p)$ and $W^+(q)$ intersect transversely for each pair $p, q \in \mathrm{Cr}(\widetilde{f})$, we can construct as in [29, Theorem 3.6] integers $n(p, q) \in \mathbb{Z}$ whenever $\mathrm{ind}(q) = \mathrm{ind}(p) + 1$, which satisfy

$$n(p,q) = n(\gamma.p, \gamma.q) \quad \forall\, \gamma \in \Gamma, \tag{MS1}$$

$$\forall\, p \in \mathrm{Cr}(\widetilde{f}) \colon \#\{q \in \mathrm{Cr}(\widetilde{f}) : \mathrm{ind}(q) = \mathrm{ind}(p) + 1 \wedge n(p,q) \neq 0\} < \infty, \tag{MS2}$$

$$\forall\, q \in \mathrm{Cr}(\widetilde{f}) \colon \#\{p \in \mathrm{Cr}(\widetilde{f}) : \mathrm{ind}(p) = \mathrm{ind}(q) - 1 \wedge n(p,q) \neq 0\} < \infty, \tag{MS3}$$

$$\forall\, p \in \mathrm{Cr}(\widetilde{f}) \text{ and } \forall\, q \in \mathrm{Cr}(\widetilde{f}) \text{ with } \mathrm{ind}(q) = \mathrm{ind}(p) + 2 : \quad \sum_{\mathrm{ind}(r) = \mathrm{ind}(p)+1} n(p,r)n(r,q) = 0. \tag{MS4}$$

In fact, under the conditions imposed on the pair (f, g'), it follows from [29, Theorems 3.8, 3.9] (see also [34, Theorem 5.4.10, Corollary 5.4.12]) that

(1) the set $\{W^-(p) \colon p \in \mathrm{Cr}(\widetilde{f})\}$ is the collection of open cells of a Γ-CW-complex $X \subseteq \widetilde{M}$, so that

(2) the inclusion $X \hookrightarrow \widetilde{M}$ is a simple Γ-homotopy equivalence. Moreover,

(3) the integer $n(p,q)$ is precisely the degree of the attaching map of the cell $W^-(q)$ to the cell $W^-(p)$.

Define $[O_p]$ to be the complex line generated by O_p and the cochain complex of vector spaces

$$C^*(\widetilde{M},\nabla_{\widetilde{g}'}\widetilde{f},\widetilde{E}) := \bigoplus_{p\in \mathrm{Cr}(\widetilde{f})} [O_p]\otimes_{\mathbb{C}} V\ ,\ C^k(\widetilde{M},\nabla_{\widetilde{g}'}\widetilde{f},\widetilde{E}) := \bigoplus_{\mathrm{ind}(p)=k} [O_p]\otimes_{\mathbb{C}} V \tag{3.4}$$

with boundary map

$$\partial^*_{MS} : C^*(\widetilde{M},\nabla_{\widetilde{g}'}\widetilde{f},\widetilde{E}) \to C^{*+1}(\widetilde{M},\nabla_{\widetilde{g}'}\widetilde{f},\widetilde{E})$$

being the unique $\mathbb{C}$-linear extension of the assignment

$$\partial^*_{MS}([O_p]\otimes v) := \sum_{\mathrm{ind}(q)=\mathrm{ind}(p)+1} n(p,q)\cdot [O_q]\otimes v. \tag{3.5}$$

By (MS2)–(MS4), ∂^*_{MS} is well-defined and satisfies $\partial^{k+1}_{MS}\circ\partial^k_{MS}=0$ for each $0\le k\le n-1$. Furthermore, the respective Γ-actions on $\widetilde{M}$ and V intertwine to produce a Γ-action on $C^*(\widetilde{M},\nabla_{\widetilde{g}'}\widetilde{f},\widetilde{E})$ given by

$$\gamma.([O_p]\otimes v) := [O_{\gamma.p}]\otimes \rho(\gamma)v. \tag{3.6}$$

Due to (MS1), it follows that ∂^*_{MS} is Γ-equivariant. Now recall the Γ-equivariant Hermitian form $\widetilde{h}$, which is part of the system $\widetilde{\mathcal{D}}$. Equipping the total space $C^*(\widetilde{M},\nabla_{\widetilde{g}'}\widetilde{f},\widetilde{E})$ with the inner product structure given by the direct sum of inner products induced by $\widetilde{h}$ at each fiber, the Γ-action (3.6) becomes an action by isometries. Taking the corresponding L^2-completion, one obtains a Hilbert $\mathcal{N}(\Gamma)$-cochain complex of finite type, which we will denote by $C^*_{(2)}(\widetilde{M},\nabla_{\widetilde{g}'}\widetilde{f},\widetilde{E},\widetilde{h})$. In fact, each module $C^k_{(2)}(\widetilde{M},\nabla_{\widetilde{g}'}\widetilde{f},\widetilde{E},\widetilde{h})$ is isomorphic to $L^2(\Gamma)^{m_k}\otimes_{\mathbb{C}} V$, where $m_k\in\mathbb{N}$ is the number of Γ-cosets of the set $\{p\in \mathrm{Cr}(\widetilde{f})\colon\ \mathrm{ind}(p)=k\}$.

Definition 3.3. $C^*_{(2)}(\widetilde{M},\nabla_{\widetilde{g}'}\widetilde{f},\widetilde{E},\widetilde{h})$ is called the *L^2-Morse–Smale cochain complex* induced by the system $\mathcal{D}$. For $0\le k\le n$, we define the k-th L^2-Morse–Smale cohomology

$$H^k_{(2)}(M,\nabla_{g'}f,E,h) := H^k\left(C^*_{(2)}(\widetilde{M},\nabla_{\widetilde{g}'}\widetilde{f},\widetilde{E},\widetilde{h})\right) \tag{3.7}$$

and the *c-L^2-Betti number* of the pair (M,E)

$$\mathfrak{b}^{(2)}_k(M,E) := \dim_{\mathcal{N}(\Gamma)}\left(H^k_{(2)}(M,\nabla_{g'}f,E,h)\right)\in[0,\infty) \tag{3.8}$$

as the von Neumann dimension of the L^2-Morse–Smale cohomology (throughout, the prefix "c" stands for *combinatorial*) and similarly the k-th *c-Novikov–Shubin invariant*

$$\alpha^{Top}_k(M,E) := \alpha_k\left(C^*_{(2)}(\widetilde{M},\nabla_{\widetilde{g}'}\widetilde{f},\widetilde{E},\widetilde{h})\right). \tag{3.9}$$

We say that (M, E) is *c-L^2-acyclic* if $\mathfrak{b}_k^{(2)}(M, \rho) = 0$ for all $0 \leq k \leq n$. We say that (M, E) is *of c-determinant class* if the complex $C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}'}\widetilde{f}, \widetilde{E}, \widetilde{h})$ is of determinant class. If (M, E) is of c-determinant class, we define the *L^2-Morse–Smale torsion of the system $\mathcal{D}$* as

$$T^{MS}_{(2)}(E \downarrow M, h, \nabla_{g'} f) := T\left(C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}'}\widetilde{f}, \widetilde{E}, \widetilde{h})\right) = \prod_{k=0}^{n} \det{}_\Gamma(\partial^k_{MS})^{(-1)^{k+1}} \in \mathbb{R}_{>0}\,. \tag{3.10}$$

As mentioned previously, the Morse–Smale cochain complexes $C^*(\widetilde{M}, \nabla_{\widetilde{g}'_1}\widetilde{f}_1, \widetilde{E})$ and $C^*(\widetilde{M}, \nabla_{\widetilde{g}'_2}\widetilde{f}_2, \widetilde{E})$ coming from two distinct Morse–Smale systems $\mathcal{D}_1 = (M, E, g_1, h_1, \nabla_{g'_1} f_1)$ and $\mathcal{D}' = (M, E, g_2, h_2\nabla_{g'_2} f_2)$ defined over a fixed pair (M, E) are the cellular cochain complexes of two Γ-homotopy equivalent subcomplexes of $\widetilde{M}$. By picking a cellular approximation of an explicit homotopy equivalence, one can easily show that the L^2-Morse–Smale complexes $C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}'_1}\widetilde{f}_1, \widetilde{E}, \widetilde{h}_1)$ and $C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}'_2}\widetilde{f}_2, \widetilde{E}, \widetilde{h}_2)$ are chain homotopy equivalent. By Proposition 2.1, it follows that the c-L^2-betti numbers $\mathfrak{b}_k^{(2)}(M, E)$, the c-Novikov–Shubin invariants $\alpha_k^{Top}(M, E)$, as well as the c-determinant class condition do not depend on the explicit choices of metrics and Morse Smale function.

On the other hand, the L^2-Morse–Smale torsion $T^{MS}_{(2)}(E \downarrow M, h, \nabla_{g'} f)$ *does* in general depend on the choices of Hermitian forms and Morse–Smale pairs (although it is entirely independent of the Riemannian metric on M). However, under the assumption that $E \downarrow M$ is a unimodular bundle, $\chi(M) = 0$, and that (M, E) is c-L^2-acyclic and of c-determinant class defined as above, there exists a *topological L^2-torsion*

$$T^{Top}_{(2)}(M, E) \in \mathbb{R}_{\geq 0}\,. \tag{3.11}$$

It can be defined similarly like $T^{MS}_{(2)}(E \downarrow M, h, \nabla_{g'} f)$, with the aid of *any* given CW-structure on M and any fixed inner product on V, see [34, Definition 5.2.5, Theorem 5.3.12]. The following key result establishes a connection between the *a priori* different Morse–Smale torsions that come from distinct Morse–Smale systems and $T^{Top}_{(2)}(M, E)$.

Theorem 3.4 ([34, Theorem 5.4.15])**.** *Assume that $E \downarrow M$ is a unimodular bundle over a compact manifold and that $\chi(M) = 0$. Let $\mathcal{D} = (M, E, g, h, \nabla_{g'} f)$ be an associated Morse–Smale system with h unimodular and assume that $E \downarrow M$ is c-L^2-acyclic and of c-determinant class. Then, one has*

$$T^{Top}_{(2)}(M, E) = T^{MS}_{(2)}(E \downarrow M, h, \nabla_{g'} f).$$

3.2. The analytic L^2-torsion $T^{An}_{(2)}(E \downarrow M, g, h)$

For a Morse–Smale system $\mathcal{D}$ as above, we now explain the construction of the L^2-de Rham complex $\Omega^*_{(2)}(\widetilde{M}, \widetilde{E}, \widetilde{g}, \widetilde{h})$ as well as the computation of the L^2-analytic torsion $T^{An}_{(2)}(E \downarrow M, g, h)$. None of these considerations will take the Morse–Smale pair (f, g') into account.

To begin with, let

$$\Omega^*(\widetilde{M}, \widetilde{E}) \cong \Omega^*(\widetilde{M}) \otimes_{\mathbb{C}} \Gamma(\widetilde{E}) \tag{3.12}$$

be the twisted de Rham complex of $\widetilde{E}$-valued forms, with differential

$$d^*\colon \Omega^*(\widetilde{M}, \widetilde{E}) \to \Omega^{*+1}(\widetilde{M}, \widetilde{E}) \tag{3.13}$$

induced by the flat connection on $\widetilde{E}$. Notice that the (fixed) flat identification $\widetilde{E} \cong \widetilde{M} \times V$ allows us to naturally identify $\Omega^*(\widetilde{M}, \widetilde{E})$ with $\Omega^*(\widetilde{M}) \otimes_{\mathbb{C}} C^\infty(\widetilde{M}, V)$. The canonical Γ-action on $\Omega^*(\widetilde{M})$ given by pullbacks and the natural Γ-action on $\Gamma(\widetilde{E}) \cong C^\infty(\widetilde{M}, V)$ induced by the linear representation $\rho : \Gamma \to \mathrm{GL}(V)$ intertwine to produce a Γ-action on $\Omega^*(\widetilde{M}, \widetilde{E})$, with respect to which d^* becomes Γ-equivariant. For $x \in \mathcal{M}$, denote by $\langle \cdot, \cdot \rangle_x$ the inner product at the fiber vector space $(\Lambda^* T^* \widetilde{M} \otimes \widetilde{E})_x$ naturally derived from the pair g and h. Let $\mu_g \in \Omega^n(\widetilde{M})$ be the volume form induced by $\widetilde{g}$. Restricting to the Γ-invariant subspace $\Omega^*_c(\widetilde{M}, \widetilde{E})$ of compactly supported forms, the integration over the pointwise inner product

$$\langle \cdot, \cdot \rangle\colon \Omega^*_c(\widetilde{M}, \widetilde{E}) \times \Omega^*_c(\widetilde{M}, \widetilde{E}) \to \mathbb{C}, \tag{3.14}$$

$$\langle f, g \rangle := \int_{\widetilde{M}} \langle f(x), g(x) \rangle_x \mathrm{d}\mu_g(x) \tag{3.15}$$

determines itself an inner product on $\Omega^*_c(\widetilde{M}, \widetilde{E})$, with respect to which the Γ-action on $\Omega^*_c(\widetilde{M}, \widetilde{E})$ is by isometries.

Let $T^*\partial\widetilde{M}$ the cotangent bundle over the boundary $\partial\widetilde{M}$. As usual, the Riemannian metric g induces an *orthogonal* decomposition of the restricted cotangent bundle $T^*\widetilde{M}|_{\partial\widetilde{M}} = T^*\partial\widetilde{M} \oplus N^*\partial\widetilde{M}$, where $N^*\partial\widetilde{M} \downarrow \partial\widetilde{M}$ denotes the 1-dimensional conormal bundle over $\widetilde{M}$. For each $x \in \partial\widetilde{M}$, each $0 \le k \le n$ and each $\omega \in \Omega^k(\widetilde{M}, \widetilde{E})$, the vector $\omega(x) \in (\Lambda^k T^* \widetilde{M} \otimes \widetilde{E})_x$ consequently decomposes orthogonally into a *tangential* and a *normal* part:

$$\omega(x) = \vec{t}\omega(x) + \vec{n}\omega(x) \in (\Lambda^k T^*\partial\widetilde{M} \otimes \widetilde{E})_x \oplus (\Lambda^{k-1} T^*\partial\widetilde{M} \otimes N^*\partial\widetilde{M} \otimes \widetilde{E})_x. \tag{3.16}$$

Let

$$\delta^*\colon \ \mathrm{dom}^*(\delta^*) \to \mathrm{dom}^{*-1}(\delta^{*-1}), \tag{3.17}$$

$$\mathrm{dom}^*(\delta^*) := \{\sigma \in \Omega^*_c(M, E)\colon \vec{n}\sigma = 0\} \tag{3.18}$$

be the formal adjoint of d^* with respect to the inner product 3.15 and with absolute boundary conditions. Define the *Hodge–Laplacian* with absolute boundary conditions

$$\Delta_* := \delta^{*+1} d^* + d^{*-1}\delta^* \colon \operatorname{dom}(\Delta_*) \to \operatorname{dom}(\Delta_*), \tag{3.19}$$

$$\operatorname{dom}(\Delta_*) := \{\omega \in \Omega_c^*(\widetilde{M}, \widetilde{E}) \colon \vec{n}\omega = \vec{n}d^*\omega = 0\} \subseteq \Omega^*(\widetilde{M}, \widetilde{E}). \tag{3.20}$$

Let $\Omega^*_{(2)}(\widetilde{M}, \widetilde{E}) = \Omega^*_{(2)}(\widetilde{M}, \widetilde{E}, \widetilde{g}, \widetilde{h})$ be the L^2-completion of $\Omega_c^*(\widetilde{M}, \widetilde{E})$ with regards to the previously defined inner product. Together with the extension of the isometric Γ-action on $\Omega_c(\widetilde{M}, \widetilde{E})$, $\Omega^*_{(2)}(\widetilde{M}, \widetilde{E})$ becomes a Hilbert $\mathcal{N}(\Gamma)$-module (although not a finitely generated one). Moreover, the restricted operators d^* and Δ_* each admit unbounded closed, Γ-equivariant extensions (denoted by the same symbol), which can therefore be regarded as morphisms between the corresponding Hilbert $\mathcal{N}(\Gamma)$-modules. We obtain a cochain complex of Hilbert $\mathcal{N}(\Gamma)$-modules

$$0 \to \Omega^0_{(2)}(\widetilde{M}, \widetilde{E}) \xrightarrow{d^0} \Omega^1_{(2)}(\widetilde{M}, \widetilde{E}) \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} \Omega^n_{(2)}(\widetilde{M}, \widetilde{E}) \to 0, \tag{3.21}$$

called the *L^2-de Rham complex* induced by the system $\mathcal{D}$.

For each $0 \leq k \leq n$, the (closed extension of the) formal adjoint δ^k (with absolute boundary conditions) is in fact the Hilbert space adjoint of the differential d^k [34, Proposition 3.4.6]. Furthermore, the (closed extension of the) Laplace operator Δ_k is positive and self-adjoint [34, Theorem 3.4.1]. With t ranging over $\mathbb{R}_{>0}$, let $e^{-t\Delta_k} \colon \Omega^k_{(2)}(\widetilde{M}, \widetilde{E}) \to \Omega^k_{(2)}(\widetilde{M}, \widetilde{E})$ be the 1-parameter, monotonically decreasing family of positive *heat operators* associated to Δ_k, defined via the spectral theorem applied to Δ_k. Each $e^{-t\Delta_k}$ is a *bounded* morphism of Hilbert $\mathcal{N}(\Gamma)$-modules that is also of trace class, i.e. satisfies $\operatorname{tr}_\Gamma(e^{-t\Delta_k}) < \infty$. More precisely, each $e^{-t\Delta_k}$ possesses an *integral kernel* $e^{-t\Delta_k}(\,\cdot\,,\cdot\,)$, a smooth section of a certain naturally derived vector bundle over $\widetilde{M} \times \widetilde{M}$, such that for any arbitrary fundamental domain $\mathcal{F} \subseteq \widetilde{M}$ for the Γ-action on $\widetilde{M}$, one has the equality

$$\operatorname{tr}_\Gamma(e^{-t\Delta_k}) = \int_{\mathcal{F}} \operatorname{tr}(e^{-t\Delta_k}(x,x))\mathrm{d}\mu_g(x), \tag{3.22}$$

see [2, Proposition 4.16] for the details. By dominated convergence, we obtain for each $0 \leq k \leq n$ that

$$\dim_{\mathcal{N}(\Gamma)}(\ker(\Delta_k)) = \lim_{t\to\infty} \operatorname{tr}_\Gamma(e^{-t\Delta_k}) \in \mathbb{R}_{\geq 0}\,. \tag{3.23}$$

In fact, the closed subspace $\ker(\Delta_k) \subseteq \Omega^k_{(2)}$ of *L^2-integrable harmonic k-forms* is not only a finite-dimensional Hilbert $\mathcal{N}(\Gamma)$-module, but also consists entirely of smooth forms and is isomorphic to the k-th L^2-cohomology

$$\mathcal{H}^k_{(2)}(M, E, g, h) := H^k(\Omega^*(\widetilde{M}, \widetilde{E}, \widetilde{g}, \widetilde{h})) \tag{3.24}$$

of $\Omega^*(\widetilde{M}, \widetilde{E}, \widetilde{g}, \widetilde{h})$ [34, Propositions 3.4.2, 4.1.33]. We define the k-th *a-L^2-Betti number* as

$$\mathbf{b}_k^{(2)}(M, E) := \dim_{\mathcal{N}(\Gamma)}(\mathcal{H}^k_{(2)}(M, E, g, h)) = \dim_{\mathcal{N}(\Gamma)}(\ker(\Delta_k)). \tag{3.25}$$

Throughout, the prefix "a" stands for *analytic*.

Similarly, the restriction $\Delta_k^\perp := \Delta_k|_{\ker(\Delta_k)^\perp}$ is a self-adjoint morphism of Hilbert $\mathcal{N}(\Gamma)$-modules, so that

$$\operatorname{tr}_\Gamma(e^{-t\Delta_k^\perp}) = \operatorname{tr}_\Gamma(e^{-t\Delta_k}) - \mathbf{b}_k^{(2)}(M, E) \in \mathbb{R}_{\geq 0} \tag{3.26}$$

for each $t > 0$. For $0 \leq k \leq n$ and $s \in \mathbb{C}$, the (truncated) *zeta-function* $\zeta_k(s)$ is defined as the formal expression

$$\zeta_k(s) := \Gamma(s)^{-1} \int_0^1 t^{s-1} \operatorname{tr}_\Gamma(e^{-t\Delta_k^\perp}) \mathrm{d}t. \tag{3.27}$$

Here, $\Gamma(s)^{-1}$ denotes the (entire) inverse *gamma function*, which should not be confused with the *group* Γ. Due to the rational asymptotic behavior of $\operatorname{tr}_\Gamma(e^{-t\Delta_k^\perp})$ near $t = 0$ [21, Lemma 1.3] (see also [34, Theorem 4.3.2]), there exists a constant $C > 0$, such that $\zeta_k(s)$ determines a holomorphic function on the domain $\{s \in \mathbb{C} \colon \Re(s) \gg C\}$ that extends to a meromorphic function on all of $\mathbb{C}$ with $s = 0$ being a regular point.

Definition 3.5. Let $\mathcal{D} = (M, E, g, h, \nabla_{g'} f)$ be a Morse–Smale system as above. For $0 \leq k \leq n$, the k-th *a-Novikov–Shubin* invariant $\alpha_k^{An}(M, E) \in [0, \infty] \cup \{\infty^+\}$ is defined as

$$\alpha_k^{An}(M, E) := \alpha_k \left(\Omega^*_{(2)}(\widetilde{M}, \widetilde{E}, \widetilde{g}, \widetilde{h})\right). \tag{3.28}$$

The pair (M, E) is said to be *of a-determinant class* if the L^2-de Rham complex $\Omega^*_{(2)}(\widetilde{M}, \widetilde{E}, \widetilde{g}, \widetilde{h})$ is of determinant class. If (M, E) is of a-determinant class, we can define the *analytic L^2-torsion* $T^{An}_{(2)}(E \downarrow M, g, h) \in \mathbb{R}_{>0}$ of the system as

$$\log(T^{An}_{(2)}(E \downarrow M, g, h)) := \sum_{k=0}^n \frac{k}{2}(-1)^{k+1} \left(\frac{\mathrm{d}}{\mathrm{d}s}\zeta_k(s)|_{s=0} + \int_1^\infty t^{-1} \operatorname{tr}_\Gamma(e^{-t\Delta_k^\perp}) \mathrm{d}t\right). \tag{3.29}$$

The a-determinant class condition of (M, E) says that for each $0 \leq k \leq n$, the restriction $d^k|_{\operatorname{im}(d^{k-1})^\perp}$ is of determinant class. By [20, Lemma 3.30], this is equivalent to the operator $\Delta_k^\perp$ being of a-determinant class for each $0 \leq k \leq n$, which in turn [20, Lemma 3.139] implies that $\int_1^\infty t^{-1} \operatorname{tr}_\Gamma(e^{-t\Delta_k^\perp}) \mathrm{d}t < \infty$ for each $0 \leq k \leq n$, showing that $T^{An}_{(2)}(E \downarrow M, g, h)$ is well-defined. Up to bounded, Γ-equivariant isomorphisms, the Hilbert $\mathcal{N}(\Gamma)$-cochain complex $\Omega^*_{(2)}(\widetilde{M}, \widetilde{E}, \widetilde{g}, \widetilde{h})$ is independent of the particular choice of g and h. Therefore, neither $\mathbf{b}_k^{(2)}(M, E)$ nor the a-determinant class conditions depend on g

or h. However, in the general case that we concern ourselves with (i.e. when $\partial M \neq \emptyset$), the quantity $T^{An}_{(2)}(E \downarrow M, g, h)$ does depend on both g and h. The precise metric anomalies, to be presentend in the next section, are of fundamental importance for this paper.

3.3. The metric L^2-torsion $T^{Met}_{(2)}(\mathcal{D})$ and the relative L^2-torsion $\mathcal{R}(\mathcal{D})$

We now describe for a general Morse–Smale system $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$ with M compact the construction of the *relative L^2-torsion* $\mathcal{R}(\mathcal{D}) \in \mathbb{R}$, provided that $E \downarrow M$ is determinant class. To begin with, we are going to define new norms on $\Omega^*_c(\widetilde{M}, \widetilde{E}, \widetilde{g}, \widetilde{h})$, the de Rham complex of compactly supported forms. Throughout, we will denote by $\|\cdot\|_0$ the L^2-norm defined in the previous section. Assume first that $\partial\widetilde{M} = \emptyset$. In this case, we define for each $s \in \mathbb{R}_{>0}$ the s-th Sobolev norm

$$\|\omega\|_s := \|(1+\Delta_k)^{s/4}\omega\|_0, \quad \omega \in \Omega^k_c. \tag{3.30}$$

In case that $\partial\widetilde{M} \neq \emptyset$, we define for each integer $p \in \mathbb{N}$ the p-th Sobolev norm (with absolute boundary conditions) inductively as

$$\|\omega\|^2_p := \|\omega\|^2_{p-1} + \|\mathrm{d}^k\omega\|^2_{p-1} + \|\delta^{k-1}\omega\|^2_{p-1} + \|\vec{n}\omega\|^2_{p-1/2} \quad \omega \in \Omega^k_c. \tag{3.31}$$

For fixed $0 \leq k \leq n$ and integer $p \in \mathbb{N}_0$, the L^2-completion $\mathcal{W}^k_p(\widetilde{M}, \widetilde{E}, \widetilde{g}, \widetilde{h})$ is called the *p-th Sobolev space of k-forms*. Just like in the case $p = 0$, the Γ-action on Ω^k_c extends to an isometric Γ-action on W^k_p, turning it into a Hilbert $\mathcal{N}(\Gamma)$-module. Crucially, we obtain *bounded* extensions

$$d^k : \mathcal{W}^k_{p+1}(\widetilde{M}, \widetilde{E}, \widetilde{g}, \widetilde{h}) \to \mathcal{W}^{k+1}_p(\widetilde{M}, \widetilde{E}, \widetilde{g}, \widetilde{h}) \tag{3.32}$$

for each $0 \leq k \leq n-1$, which is why we can define for fixed $l \geq n$ a cochain complex of Hilbert $\mathcal{N}(\Gamma)$-modules

$$\mathcal{W}^*_{l-*}(\widetilde{M}, \widetilde{E}, \widetilde{g}, \widetilde{h}) \colon \mathcal{W}^0_l(\widetilde{M}, \widetilde{E}, \widetilde{g}, \widetilde{h}) \xrightarrow{\mathrm{d}^0} \mathcal{W}^1_{l-1}(\widetilde{M}, \widetilde{E}, \widetilde{g}, \widetilde{h}) \xrightarrow{\mathrm{d}^1} \cdots \xrightarrow{\mathrm{d}^{n-1}} \mathcal{W}^n_{l-n}(\widetilde{M}, \widetilde{E}, \widetilde{g}, \widetilde{h}). \tag{3.33}$$

Now recall the L^2-Morse–Smale complex $C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}'}\widetilde{f}, \widetilde{E}, \widetilde{h})$ and fix an integer $l > 3n/2 + 1$. Then, it follows from the Sobolev inequality that one has $\sigma \in C^1 \cap L^2$ for each $\sigma \in \mathcal{W}^k_l$ and each $0 \leq k \leq n$. Together with our fixed isomorphism $\widetilde{E} \cong \widetilde{M} \times V$, we deduce that for each $p \in \mathrm{Cr}(\widetilde{f})$ with $\mathrm{ind}(p) = k$, the integral $\int_{W^-(p)} \sigma \in V$ over the k-dimesional unstable manfiold $W^-(p)$ is well-defined. In fact, it holds that $\sum_{\mathrm{ind}(p)=k} \|\int_{W^-(p)} \sigma\|^2_{\widetilde{h}_p} < \infty$, e.g. [16, Lemma 3.2]. Therefore, we can define a map between graded Hilbert $\mathcal{N}(\Gamma)$-modules

$$\mathrm{Int}^*\colon \mathcal{W}^*_{l-*}(\widetilde{M},\widetilde{E},\widetilde{g},\widetilde{h}) \to C^*_{(2)}(\widetilde{M},\nabla_{\widetilde{g}'}\widetilde{f},\widetilde{E},\widetilde{h}), \tag{3.34}$$

$$\mathrm{Int}^k(\sigma) := \sum_{\substack{p\in \mathrm{Cr}(f)\\ \mathrm{ind}(p)=k}} [O_p]\otimes\left(\int_{W^-(p)}\sigma\right) \quad \sigma\in\mathcal{W}^k_{l-k}, \tag{3.35}$$

given by integration of Sobolev forms over the unstable manifolds. By a result of Laudenbach [6, Appendix, Proposition 6], Int^* is a cochain map. Let

$$\pi^*:\ker(\partial^*_{MS})\to\ker(\partial^*_{MS})/\mathrm{clos}(\mathrm{im}(\partial^{*-1}_{MS})) =: H^*_{(2)}(\widetilde{M},\nabla_{\widetilde{g}'}\widetilde{f},\widetilde{E},\widetilde{h}) \tag{3.36}$$

be the projection of the kernel of the L^2-Morse–Smale boundary operator onto the corresponding L^2-Morse–Smale homology. By a theorem of Dodziuk [16], extended by Schick [31] to manifolds with boundary and by Shubin [32] to non-unitary bundles, the map

$$\Theta^*:\ker(\Delta_*)\to H^*_{(2)}(\widetilde{M},\nabla_{\widetilde{g}'}\widetilde{f},\widetilde{E},\widetilde{h}), \tag{3.37}$$

defined as the restriction of $\pi^*\circ\mathrm{Int}^*$ onto the closed subspace $\ker(\Delta_*)\subseteq\mathcal{W}^*_{l-*}(\widetilde{M},\widetilde{E},\widetilde{g},\widetilde{h})$ of L^2-harmonic forms is an isomorphism of finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules. In particular,

$$\mathfrak{b}^{(2)}_k(M,E)=\mathbf{b}^{(2)}_k(M,E)\ \ 0\le k\le n, \tag{3.38}$$

i.e. the combinatorial and analytical L^2-Betti numbers of the pair (M,E) agree. From now on, since c-L^2-acyclicity is equivalent to a-L^2-acyclicity, we simply say that the pair (M,E) is *L^2-acyclic* whenever either of the two equivalent conditions hold. The isomorphism Θ^* now also allows us to define the *metric L^2-torsion* $T^{Met}_{(2)}(\mathcal{D})\in\mathbb{R}_{\ge0}$ of the system $\mathcal{D}=(E\downarrow M,g,h,\nabla_{g'}f)$ as

$$\log T^{Met}_{(2)}(\mathcal{D}) := \sum_{k=0}^{\infty}(-1)^k\log\det\nolimits_\Gamma(\Theta^k)=\frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\log\det\nolimits_\Gamma((\Theta^k)^*\Theta^k). \tag{3.39}$$

Assuming that $E\downarrow M$ is of a-determinant class, we define the *Ray–Singer L^2 Torsion* $T^{RS}_{(2)}(\mathcal{D})\in\mathbb{R}_{\ge0}$ as

$$\log T^{RS}_{(2)}(\mathcal{D}) := \log\left(\frac{T^{An}_{(2)}(E\downarrow M,g,h)}{T^{Met}_{(2)}(\mathcal{D})}\right), \tag{3.40}$$

Of course, if $\ker(\Delta_*)=\{0\}$, i.e. if (M,E) is L^2-acyclic, then $T^{Met}_{(2)}(\mathcal{D})=1$, so that $T^{RS}_{(2)}(\mathcal{D})=T^{An}_{(2)}(E\downarrow M,g,h)$. If (M,E) is both of combinatorial and of analytical determinant class, the *relative L^2-torsion* $\mathcal{R}(\mathcal{D}) \in \mathbb{R}$ of the corresponding Morse–Smale system $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$ can be defined as

$$\mathcal{R}(\mathcal{D}) := \log \left(\frac{T^{RS}_{(2)}(\mathcal{D})}{T^{MS}_{(2)}(E \downarrow M, h, \nabla_{g'} f)} \right). \tag{3.41}$$

We will show in Theorem 6.4 that the condition $E \downarrow M$ being of a-determinant class is equivalent to $E \downarrow M$ being of c-determinant class. Therefore, we are justified to say that $E \downarrow M$ is *of determinant class* whenever either determinant class condition (and therefore both) is satisfied.

Remark 3.6. It should be mentioned that the relative torsion $\mathcal{R}(\mathcal{D}) \in \mathbb{R}$ can be defined even if the corresponding bundle $E \downarrow M$ is not of determinant class. In that case, the individual terms $T^{RS}_{(2)}(E \downarrow M, g, h, \nabla_{g'} f)$ and $T^{MS}_{(2)}(E \downarrow M, g, h, \nabla_{g'} f)$ are not real numbers, but non-vanishing vectors in the same orientation class of a particular 1-dimensional real vector space. Therefore, their quotient yields a *positive* real number, which is why $\mathcal{R}(\mathcal{D})$, the logarithm of the quotient as above, is still well-defined. It can be shown that the main Theorem 4.5 still holds in this case. We refer to [7, 11, 35] for a detailed study of L^2-torsion without the determinant class conditions.

4. Statement of the main results

Using the terminology introduced in the previous section, we are going to formulate the main results, Theorems 4.5 and 4.8.

First, however, we also need to establish the notion of a *local quantity*: Given two systems $\mathcal{D}_i = (E_i \downarrow M_i, g_i, h_i, X_i)$, an isometry $\phi : (M_1, g_1) \to (M_2, g_2)$ between the underlying Riemannian manifolds that satisfies $\phi^* X_2 = X_1$ and extends to a flat bundle isometry $\Phi : (E_1, h_1) \to (E_2, h_2)$ is called an *isomorphism* between the systems.

Definition 4.1 (Local Quantity)**.** An assignment of a form $\alpha = \alpha(\mathcal{D}) \in Y$, where either $Y = \Omega^n(M, \mathcal{O}_M)$, or $Y = \Omega^{n-1}(\partial M, \mathcal{O}_{\partial M})$ for any system $\mathcal{D} = (E \downarrow M, g, h, X)$ is called a *local quantity* of $\mathcal{D}$ if it satisfies the following compatibility conditions:

(1) For any open subset $U \subseteq M$, it holds that $\alpha(\mathcal{D}|_U) = \alpha(\mathcal{D})|_U$.

(2) If $\phi : M_1 \to M_2$ is an isomorphism between two systems $\mathcal{D}_i = (E_i \downarrow M_i, g_i, h_i, X_i)$ (for $i = 1, 2$), then $\phi^* \alpha(\mathcal{D}_2) = \alpha(\mathcal{D}_1)$.

Here, as everywhere else, $\mathcal{O}_M \downarrow M$ is the (real) *orientation line bundle* over M. Elements of $\Omega^n(M, \mathcal{O}_M)$ are called *densities*.

For any system $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$ with (f, g') a Morse–Smale pair, we will now construct a local quantity of the derived system $\mathcal{D} = (E|_{M\setminus \mathrm{Cr}(f)} \downarrow M \setminus \mathrm{Cr}(f), g, h, \nabla_{g'} f)$ that constitutes an integral part in the analysis of the anomaly between L^2-Ray Singer and Morse–Smale torsion.

First off, as carefully explained and constructed by Bismut and Zhang in [6, Section 3], the Levi-Civita connection of the Riemannian metric g gives rise to the *Mathai–Quillen Current*

$$\Psi(M, g) \in \Omega^{n-1}(TM \setminus M, \mathcal{O}_{TM}). \tag{4.1}$$

Here, we have identified $M \subseteq TM$ with its zero section inside TM. The second local quantity of relevance is the 1-form $\theta(h) \in \Omega^1(M)$, which measures the local change of the volume form induced the metric h along M and can be constructed as follows: Let ∇ be the flat connection on $E \downarrow M$ and let $\overline{E^*} \downarrow M$ be the flat bundle over M *conjugate dual* to $E \downarrow M$. The induced endomorphism bundle $\mathrm{End}(E, \overline{E^*}) \downarrow M$ carries a flat connection ∇^* naturally induced by ∇. For the metric h, we now observe that $h \in \Gamma(M, \mathrm{End}(E, \overline{E^*}))$, which allows us to define the 1-form

$$\theta(h) \coloneqq \mathrm{tr}(h^{-1} \nabla^* h) \in \Omega^1(M). \tag{4.2}$$

Definition 4.2. A metric h on a flat bundle $E \downarrow M$ is called *unitary* (or parallel) if $\nabla^* h \equiv 0$. h is called *unimodular* if $\theta(h) \equiv 0$.

The canonical metric associated to a flat unitary bundle $E \downarrow M$, i.e. every bundle coming from a unitary representation $\rho\colon \Gamma \to O(V)$), is unitary. Unitary metrics are obviously unimodular; the converse need not hold. Every unimodular bundle $E \downarrow M$, i.e. every flat bundle corresponding to a finite-dimensional unimodular representation $\rho\colon \Gamma \to SL(V)$ admits a unimodular metric h. Although there is in general no canonical choice of a unimodular metric, such metrics can always be chosen with a lot of flexibility, as the next lemma shows:

Lemma 4.3 ([34, Corollary 5.4.18])**.** *Let $E \downarrow M$ be a flat, unimodular bundle over a connected manifold M and $U = \bigsqcup_{i \in I} U_i \subseteq M$ a subset with each U_i open and connected. Let $x_0 \in \mathrm{Int}(M \setminus U)$ and $x_i \in U_i$ for each $i \in I$ be chosen basepoints with curves $c_i \subseteq M$ connecting x_0 to x_i. Further, let $\widetilde{h}_0$ be a Hermitian metric on E_{x_0} and $\widetilde{h}_i$ a Hermitian metric on E_{x_i} satisfying*

$$\det(\widetilde{h}_i \cdot P^*_{c_i}(\widetilde{h}_0)^{-1}) = 1, \tag{4.3}$$

where $P_{c_i} : \mathrm{GL}(E_{x_0}, \overline{E^*_{x_0}}) \to \mathrm{GL}(E_{x_i}, \overline{E^*_{x_i}})$ denotes the parallel transport along the curve c_i. Then, for any unimodular metric $\bigsqcup h_i$ on $E|_U$ extending $\bigsqcup \widetilde{h}_i$, there exists a global unimodular metric h on E further extending $\bigsqcup h_i \sqcup \widetilde{h}_0$.

Now notice that $\nabla_{g'} f$ determines a smooth *embedding* $\nabla_{g'} f : M \setminus \mathrm{Cr}(f) \to TM \setminus M$. Wedging the corresponding pullback $\nabla_{g'} f^* \Psi(M, g) \in \Omega^{n-1}(M \setminus \mathrm{Cr}(f), O_M)$ with $\theta(h) \in \Omega^1(M)$, we obtain a density over $M \setminus \mathrm{Cr}(f)$ and local quantity of $\mathcal{D}$:

$$\theta(h) \wedge \nabla_{g'} f^* \Psi(M, g) \in \Omega^n(M \setminus \mathrm{Cr}(f), O_M). \tag{4.4}$$

This allows us to, at least formally, define the integral

$$\int_M \theta(h) \wedge \nabla_{g'} f^* \Psi(M, g) := \int_{M \setminus \mathrm{Cr}(f)} \theta(h) \wedge \nabla_{g'} f^* \Psi(M, g). \tag{4.5}$$

Note that since $M \setminus \mathrm{Cr}(f)$ is not compact (unless $\mathrm{Cr}(f) = \emptyset$), the integral need *a priori* not converge. That this is indeed the case has been shown in [6], as an immediate consequence of their main result. Moreover, one can verify either from its explicit construction as done in [6, Chapter III] or immediately from [11, Section 4], that $\theta(h) \wedge \nabla_{g'} f^* \Psi(M, g)$ is a local quantity of the system $\mathcal{D} = (E|_{M \setminus \mathrm{Cr}(f)} \downarrow M \setminus \mathrm{Cr}(f), g, h, \nabla_{g'} f)$, as claimed. The theorem that we wish to generalize is the following result by Zhang:

Theorem 4.4 ([35, Theorem 4.2])**.** *Let $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$ be a system with (f, g') a Morse–Smale pair and M* closed. *Then*

$$\mathcal{R}(\mathcal{D}) = -\frac{1}{2} \int_M \theta(h) \wedge \nabla_{g'} f^* \Psi(M, g). \tag{4.6}$$

With aid of the above theorem, we will derive a similar result in case that M is odd-dimensional with non-empty boundary:

Theorem 4.5. *Let $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$ be a type II Morse–Smale system of product form, where M is an odd-dimensional compact manifold and $h|_{\partial M}$ is unimodular. Further, assume that both $E \downarrow M$ and $E|_{\partial M} \downarrow \partial M$ are of determinant class. Then*

$$\mathcal{R}(\mathcal{D}) = -\frac{\log 2}{4} \chi(\partial M) \dim(E) - \frac{1}{2} \int_M \theta(h) \wedge \nabla_{g'} f^* \Psi(TM, g). \tag{4.7}$$

Remark 4.6. Similarly as in the unitary case (cf. [10, Theorem 4.1]), there is also a version of Theorem 4.5 for relative/mixed, instead of absolute boundary conditions as we assume here throughout. The proof presented here carries over to this case with only minor modifications. Although not relevant for this paper, this generalization will prove to be useful when one wants to extend the gluing formula [10, Theorem 4.3] to non-unitary bundles, which could in turn be used for future computational purposes.

Example 4.7. Set $I = [a, b]$, and let $E_{\mathbb{C}} := \mathbb{C} \times I$ be the trivial 1-dimensional complex vector bundle over I. As metrics, we choose g_0 to be the standard Euclidean metric and h_0 the canonical constant Hermitian form, i.e $\langle z, z' \rangle_{h_0(x)} := z\overline{z'}$ for any $x \in I$ and any pair $z, z' \in \mathbb{C}$. Further, we choose as Morse-function a smooth map $f_0 : [a, b] \to \mathbb{R}$ satisfying

- $f_0(x) \coloneqq \frac{1}{2}(x-(b+a)/2)^2$ away from a neighborhood of $\{a,b\}$,
- $f_0(a+t\epsilon) = f_0(b-t\epsilon) = b - t\epsilon$ for all $t \in [0,1]$ and some small $\epsilon > 0$, and so that
- $(b+a)/2$ is the only critical point of f_0.

One now easily verifies that $\mathcal{D}_I \coloneqq (E_{\mathbb{C}} \downarrow I, g_0, h_0, \nabla_{g_0'} f_0)$ is an admissible system and that $E_{\mathbb{C}} \downarrow I$ is of determinant class. In fact, one can easily compute the corresponding analytic and combinatorial torsion elements [34, Example 6.1.7] and obtain

$$\mathcal{R}(\mathcal{D}_I) = -\frac{\log 2}{2} = -\frac{\log 2}{4}\chi(\{a,b\}) - \frac{1}{2}\int_a^b \overbrace{\theta(h_0)}^{=0} \wedge (\nabla_{g_0'} f_0)^* \Psi(TI, g_0). \tag{4.8}$$

The main part of this paper is devoted to the proof of Theorem 4.5. We will adapt the techniques and strategy developed by Burghelea, Friedlander and Kappeler in [10] to our situation of non-unitary bundles, together with employing several known anomaly results that have been shown since. We remark that Theorem 4.5 has also recently been verified in an (as of now) unpublished paper by Guangxiang Su, employing techniques and methods different from the ones that we are using. Theorem 4.5, together with the main results established by Brüning and Ma in [9], Zhang and Ma in [22], and Zhang in [35], are then used to prove the next key result of this paper:

Theorem 4.8. *Let (M,g) be a compact, connected, odd-dimensional Riemannian manifold. Then, there exists a density $B(g) \in \Omega^{n-1}(\partial M, \mathcal{O}_{\partial M})$ with $B(g) \equiv 0$ when g is product-like near ∂M, such that the following holds:*

Let $E \downarrow M$ be a flat, finite-dimensional complex vector bundle, such that

(a) *E is unimodular,*

(b) *the pair (M,E) is L^2-acyclic and of determinant class,*

(c) *the restriction $(\partial M, E|_{\partial M})$ is of determinant class.*

Then, for any choice of unimodular metric h on E, one has

$$\log\left(\frac{T^{An}_{(2)}(E \downarrow M, g, h)}{T^{Top}_{(2)}(M,E)}\right) = \frac{1}{2}\dim_{\mathbb{C}}(E)\int_{\partial M} B(g). \tag{4.9}$$

In particular, for $i = 1, 2$ and any two representations $E_i \downarrow M$ satisfying the above assertions, it follows that

$$\dim_{\mathbb{C}}(E_2) \log\left(\frac{T_{(2)}^{An}(E_1 \downarrow M, g, h_1)}{T_{(2)}^{Top}(M, E_1)}\right) = \dim_{\mathbb{C}}(E_1) \log\left(\frac{T_{(2)}^{An}(E_2 \downarrow M, g, h_2)}{T_{(2)}^{Top}(M, E_2)}\right), \quad (4.10)$$

for any choice of unimodular metric h_i on $E_i \downarrow M$.

Remark 4.9. Observe that the statement is vacuous in the case that M possesses no flat bundle $E \downarrow M$ so that (M, E) is L^2-acyclic. In particular, this is true whenever $\chi(M) \neq 0$, cf. [20, Theorem 1.35].

Proof. Let ρ be a representation satisfying the assumptions from the theorem. By the previous remark, we must have

$$0 = \chi(M) = \frac{1}{2}\chi(\partial M). \quad (4.11)$$

The last equality follows from the fact that any CW-stucture on M which turns ∂M into a subcomplex naturally induces a CW-structure on the *double* $DM := M \cup_{\partial M} M$ with exactly twice the number of cells in each dimension, except for those cells defined by the natural inclusion $\partial M \hookrightarrow DM$. Together with applying Poincaré duality (with $\mathbb{Z}_2$-coefficients) to the closed, odd-dimensional manifold DM and counting the cells, we obtain

$$0 = \chi(DM) = 2\chi(M) - \chi(\partial M), \quad (4.12)$$

from which the right-hand equality of (4.11) immediately follows.

Choose a Morse function f on M of type II, along a Riemannian metric g' on M that is a product near ∂M and so that (f, g') is a Morse–Smale pair. By Lemma 4.3, we may also choose a unimodular metric h' with $h'|_{\partial M} \equiv h|_{\partial M}$ and so that $\mathcal{D} = (E \downarrow M, g', h', f)$ becomes an admissible system (in particular, h' is of product form near ∂M). First, since h' is unimodular and $E \downarrow M$ is det-L^2-acyclic, we obtain from Theorem 3.4 that

$$T_{(2)}^{MS}(E \downarrow M, h', \nabla_{g'} f) = T_{(2)}^{Top}(M, E). \quad (4.13)$$

Furthermore, we can apply (4.11) and Theorem 4.5 to this situation and obtain

$$\log\left(\frac{T_{(2)}^{An}(E \downarrow M, g', h', f)}{T_{(2)}^{MS}(E \downarrow M, h', \nabla_{g'} f)}\right) = \mathcal{R}(\mathcal{D}) = 0. \quad (4.14)$$

Next, choose a type I Morse function $f' : M \to \mathbb{R}$ on M. As $E \downarrow M$ is by assumption L^2-acyclic, we have $T_{(2)}^{An}(E \downarrow M, g, h) = T_{(2)}^{RS}(E \downarrow M, g, h, f')$ and analogously

$T^{An}_{(2)}(E \downarrow M, g', h') = T^{RS}_{(2)}(E \downarrow M, g', h', f')$. Moreover, by the main result of [22], we have the equality of Ray–Singer anomalies

$$\log\left(\frac{T^{An}_{(2)}(E \downarrow M, g, h)}{T^{An}_{(2)}(E \downarrow M, g', h')}\right) = \log\left(\frac{T^{RS}_{(2)}(E \downarrow M, g, h, f')}{T^{RS}_{(2)}(E \downarrow M, g', h', f')}\right) = \log\left(\frac{T^{RS}(E \downarrow M, g, h, f')}{T^{RS}(E \downarrow M, g', h', f')}\right). \tag{4.15}$$

Here, $T^{RS}(E \downarrow M, g', h')$ is the (ordinary) Ray–Singer-metric as originally introduced in [6, Definition 2.2] and first extended to manifolds with boundary in [8]. Further, it is shown in [9, Theorem 3.4] that there exists a density $B(g) \in \Omega^{n-1}(\partial M, \mathcal{O}_{\partial M})$ with $B(g) \equiv 0$ whenever g is also product-like near ∂M, so that

$$\log\left(\frac{T^{RS}(E \downarrow M, g, h, f')}{T^{RS}(E \downarrow M, g', h', f')}\right) = \frac{1}{2}\dim_{\mathbb{C}}(E)\int_{\partial M} B(g). \tag{4.16}$$

The density $B(g)$ is constructed as in [9, p. 1103]. It depends only on the local geometry of $(\partial M, g|_{\partial M})$ inside (M, g).

Using (4.13)–(4.16), we finally obtain

$$\log\left(\frac{T^{An}_{(2)}(E \downarrow M, g, h)}{T^{Top}_{(2)}(M, E)}\right) = \log\left(\frac{T^{An}_{(2)}(E \downarrow M, g, h)}{T^{An}_{(2)}(E \downarrow M, g', h')}\right) + \log\left(\frac{T^{An}_{(2)}(E \downarrow M, g', h')}{T^{MS}_{(2)}(E \downarrow M, h', \nabla_{g'} f)}\right) = \log\left(\frac{T^{RS}(E \downarrow M, g, h, f')}{T^{RS}(E \downarrow M, g', h', f')}\right) = \frac{1}{2}\dim_{\mathbb{C}}(E)\int_{\partial M} B(g), \tag{4.17}$$

as desired. □

5. Product formulas, determinant class and subdivisions

In this section, we study the effect on L^2-torsion and the local quantities after having taking the product of two systems. Moreover, we will make precise the anomaly of relative torsion that occurs when taking a subdivision of a Morse function and appropriate new metrics.

As hinted towards in the introduction, given two Morse–Smale systems $\mathcal{D}_i = (E_i \downarrow M_i, g_i, h_i, \nabla_{g_i'} f_i)$ for $i = 1, 2$, an integral part of our methods will involve considering the product system $\mathcal{D}_1 \times \mathcal{D}_2 = (E_1 \widehat{\otimes} E_2 \downarrow M_1 \times M_2, g_1 \times g_2, h_1 \widehat{\otimes} h_2, \nabla_{g_1' \times g_2'}(f_1 + f_2))$ and derive meaningful information of $\mathcal{D}_1 \times \mathcal{D}_2$ in terms of $\mathcal{D}_1$ and $\mathcal{D}_2$, and vice versa. Throughout, we assume exclusively that M_1 has non-empty boundary and M_2 has empty boundary. In this case, a problem that we have to address is that *a product of two type II Morse–Smale systems need not be a type II Morse–Smale system anymore*.

The problem is due to the fact that the Morse function $f_1 + f_2$ doesn't necessarily fulfil condition (ii) of Definition 3.1 anymore (in particular, it is not necessarily constant on the boundary $\partial(M_1 \times M_2) = \partial M_1 \times M_2$). This can be remedied by deforming $f_1 + f_2$ in a sufficiently small neighborhood of $\partial M_1 \times M_2$ to be of the type II shape as described in Definition 3.2, which can be arranged in such a way that the resulting Morse function, denoted henceforth by $\underline{f_1 + f_2}$, equals $f_1 + f_2$ outside of a small neighborhood of $\partial M_1 \times M_2$, has the same critical points as $f_1 + f_2$, the same gradient trajectories with respect to $\nabla_{g_1' + g_2'}$ and the same *unstable cells*. We denote the resulting *modified product system* by

$$\underline{\mathcal{D}_1 \times \mathcal{D}_2} := (E_1 \mathbin{\widehat{\otimes}} E_2 \downarrow M_1 \times M_2, g_1 \times g_2, h_1 \mathbin{\widehat{\otimes}} h_2, \nabla_{g_1' \times g_2'}(\underline{f_1 + f_2})), \tag{5.1}$$

and observe that $\underline{\mathcal{D}_1 \times \mathcal{D}_2}$ is of product form, respectively weakly admissible whenever both $\mathcal{D}_1$ and $\mathcal{D}_2$ are of product form, respectively weakly admissible. Moreover, under the assumption that both M_1 and M_2 are compact, it follows immediately from the construction of $\underline{f_1 + f_2}$ that the Morse–Smale cochain complexes corresponding to $\underline{\mathcal{D}_1 \times \mathcal{D}_2}$ and $\mathcal{D}_1 \times \mathcal{D}_2$ are the same (as Hilbert $\mathcal{N}(\Gamma)$-cochain complexes). This immediately implies that

$$\log T_{(2)}^{Met}(\mathcal{D}_1 \times \mathcal{D}_2) = \log T_{(2)}^{Met}(\underline{\mathcal{D}_1 \times \mathcal{D}_2}). \tag{5.2}$$

In case that $E \downarrow M$ is of determinant class, we also get

$$\log T_2^{MS}(\mathcal{D}_1 \times \mathcal{D}_2) = \log T_{(2)}^{MS}(\underline{\mathcal{D}_1 \times \mathcal{D}_2}), \tag{5.3}$$

$$\log T_{(2)}^{An}(\mathcal{D}_1 \times \mathcal{D}_2) = \log T_{(2)}^{An}(\underline{\mathcal{D}_1 \times \mathcal{D}_2}). \tag{5.4}$$

Still, to obtain an admissible system from two admissible systems $\mathcal{D}_1$ and $\mathcal{D}_2$, we need to ensure that $h_1 \mathbin{\widehat{\otimes}} h_2$ is unimodular near $\partial M_1 \times M_2$, which can only be guaranteed if we assume additionally that h_2 is (globally) unimodular. For our purposes, this will provide no restriction at all, since we will always form products, where $E_2 \downarrow M_2$ is in fact a unitary bundle and h_2 is an associated unitary (and flat) metric. Summarizing, we have the following:

Lemma 5.1. *For, $i = 1, 2$, let $\mathcal{D}_i = (E_i \downarrow M_i, g_i, h_i, \nabla_{g_i'} f_i)$ be two type II Morse–Smale systems with $\partial M_1 \neq \emptyset$ and $\partial M_2 = \emptyset$. Then, the modified product system $\underline{\mathcal{D}_1 \times \mathcal{D}_2}$ as in* (5.1) *is also a type II Morse–Smale system. Moreover, if both $\mathcal{D}_1$ and $\mathcal{D}_2$ are additionally of product form/weakly admissible, then also $\underline{\mathcal{D}_1 \times \mathcal{D}_2}$ is of product form/weakly admissible. Lastly, if both $\mathcal{D}_1$ and $\mathcal{D}_2$ are admissible, so that h_2 is globally unimodular, then $\underline{\mathcal{D}_1 \times \mathcal{D}_2}$ is also admissible.*

The first product formula that we state is as follows is as follows

Proposition 5.2 (Product Formula 1)**.** *For $i = 1, 2$, let $\mathcal{D}_i = (E_i \downarrow M_i, g_i, h_i, \nabla_{g_i'} f_i)$ be two type II Morse–Smale systems with M_1 compact, $\partial M_1 \neq \emptyset$ and with M_2 closed. Then, the type II Morse–Smale system $\underline{\mathcal{D}_1 \times \mathcal{D}_2}$ is also of determinant class and we get*

(1) $\log T^{An}_{(2)}(\underline{\mathcal{D}_1 \times \mathcal{D}_2}) = \chi(M_1, E_1) \log T^{An}_{(2)}(\mathcal{D}_2) + \log T^{An}_{(2)}(\mathcal{D}_1)\chi(M_2, E_2),$

(2) $\log T^{Met}_{(2)}(\underline{\mathcal{D}_1 \times \mathcal{D}_2}) = \chi(M_1, E_1) \log T^{Met}_{(2)}(\mathcal{D}_2) + \log T^{Met}_{(2)}(\mathcal{D}_1)\chi(M_2, E_2),$

(3) $\log T^{MS}_{(2)}(\underline{\mathcal{D}_1 \times \mathcal{D}_2}) = \chi(M_1, E_1) \log T^{MS}_{(2)}(\mathcal{D}_2) + \log T^{MS}_{(2)}(\mathcal{D}_1)\chi(M_2, E_2),$

(4) $\mathcal{R}(\underline{\mathcal{D}_1 \times \mathcal{D}_2}) = \chi(M_1, E_1)\mathcal{R}(\mathcal{D}_2) + \mathcal{R}(\mathcal{D}_1)\chi(M_2, E_2).$

Proof. Let's first prove (1)–(3). If we replace $\underline{\mathcal{D}_1 \times \mathcal{D}_2}$ by the genuine product system $\mathcal{D}_1 \times \mathcal{D}_2$, the equalities are well-known. Namely, the proofs presented in [10, Proposition 1.21, Proposition 4.2] can be copied line by line, after changing the definition of $\Lambda^{-,q}(M, E)$ to be the C^∞-closure of $d_q^* \left(\Omega^{q+1}(M, \partial M, E)\right)$. Now apply (5.2)–(5.4). (4) is an immediate consequence of (1)–(3). □

In addition, we will need to analyze the behavior under taking products of the local quantities introduced in the previous section. Here, the assumption that the Hermitian forms are unimodular at the boundary becomes essential.

For this, note first that we have a natural embedding $\Omega^*(M_1) \otimes \Omega^*(M_2) \hookrightarrow \Omega^*(M_1 \times M_2)$ (which is dense under the natural C^∞-topology). By passing to local trivializations over coordinate charts, one easily sees that the 1-form $\theta(h_1 \widehat{\otimes} h_2)$ lies in $\Omega^*(M_1) \otimes \Omega^*(M_2)$ and is of the form

$$\theta(h_1 \widehat{\otimes} h_2) = \theta(h_1) \otimes \dim(E_2) + \dim(E_1) \otimes \theta(h_2). \tag{5.5}$$

Furthermore, it has been shown in [11, p. 63-64] (see also [6, Chapter 4] or [5, Theorem 2.7] for additional details) that

$$\begin{aligned} &\nabla_{g_1' \times g_2'}(f_1 + f_2)^* \Psi(T(M_1 \times M_2), g_1 \times g_2)) \\ &\quad = (\nabla_{g_1'} f_1)^* \Psi(TM_1, g_1) \otimes e(TM_2, g_2) + e(TM_1, g_1) \otimes (\nabla_{g_2'} f_2)^* \Psi(TM_2, g_2) \end{aligned} \tag{5.6}$$

on $M_1 \times M_2 \setminus \mathrm{Cr}(f_1 + f_2) = M_1 \times M_2 \setminus \mathrm{Cr}(f_1) \times C(f_2)$. Here, for a Riemannian manifold (M, g), the *Euler form* $e(M, g) \in \Omega^{\dim(M)}(M, \mathcal{O}_M)$ is a density defined using Chern–Weil theory. It has the property that $e(M, g) \equiv 0$ whenever M is odd-dimensional. Moreover, if M is closed, it is a representative of the Euler class of the tangent bundle $TM \downarrow M$. By the Gauss–Chern–Bonnett theorem, it then follows that

$$\int_M e(M, g) = \chi(M), \tag{5.7}$$

if M is closed. We refer [9, Page 1103] for an explicit formula for $e(M, g)$.

Combining (5.5) with (5.6), we get

$$\begin{aligned}\theta(h_1 \mathbin{\widehat{\otimes}} h_2) \wedge \nabla_{g_1' \times g_2'}(f_1 + f_2)^* \Psi(T(M_1 \times M_2), g_1 \times g_2) \\ = \theta(h_1) \wedge (\nabla_{g_1'} f_1)^* \Psi(TM_1, g_1) \otimes \dim(E_2) e(TM_2, g_2) \\ + \dim(E_1) e(TM_1, g_1) \otimes \theta(h_2) \wedge (\nabla_{g_2'} f_2)^* \Psi(TM_2, g_2) \quad (5.8)\end{aligned}$$

on $M_1 \times M_2 \setminus \mathrm{Cr}(f_1 \times f_2)$. Here, we have used that

$$\theta(h_i) \wedge e(TM_i, g_i) \in \Omega^{\dim(M_i)+1}(M_i, \mathcal{O}_{M_i}) = \{0\} \quad \text{for both } i = 1, 2.$$

Lemma 5.3 (Product Formula 2)**.** *For $i = 1, 2$, let $\mathcal{D}_i := (E_i \downarrow M_i, g_i, h_i, \nabla_{g_i'} f_i)$ be two type II Morse–Smale systems of product form, so that both $h_1|_{\partial M}$ and h_2 are unimodular. Then, it holds that*

$$\begin{aligned}\theta(h_1 \mathbin{\widehat{\otimes}} h_2) \wedge \nabla_{g_1' \times g_2'}(\underline{f_1 + f_2})^* \Phi(T(M_1 \times M_2), g_1 \times g_2), \\ = \theta(h_1) \wedge (\nabla_{g_1'} f_1)^* \Psi(TM_1, g_1) \otimes \dim(E_2) \cdot e(TM_2, g_2) \quad (5.9)\end{aligned}$$

on all of $M \setminus \mathrm{Cr}(\underline{f_1 + f_2})$. In particular, if either M_2 is odd-dimensional or h_1 is also unimodular, then

$$\theta(h_1 \mathbin{\widehat{\otimes}} h_2) \wedge \nabla_{g_1' \times g_2'}(\underline{f_1 + f_2})^* \Phi(T(M_1 \times M_2), g_1 \times g_2) = 0. \quad (5.10)$$

Proof. Due to the assumption that $h_1|_{\partial M_1}$ and h_2 both are unimodular, it follows from (5.5) that $h_1|_{\partial M_1} \mathbin{\widehat{\otimes}} h_2$ determines a unimodular metric on the restriction bundle $E|_{\partial(M_1 \times M_2)} = E|_{\partial M_1 \times M_2}$. Since the system $\mathcal{D}_1$ is of product form, this allows us to choose a small neighborhood U of ∂M_1, so that $\theta(h_1) \equiv 0$ on U. Together with Equation (5.5) and $\theta(h_2) \equiv 0$ everywhere on M_2, we deduce that

$$\theta(h_1 \mathbin{\widehat{\otimes}} h_2) \equiv 0 \qquad \text{on } U \times M_2. \quad (5.11)$$

By choosing U smaller, if necessary, we also have by construction $\underline{f_1 + f_2} = f_1 + f_2$ on $(M_1 \setminus U) \times M_2$, and therefore the equality of gradients

$$\nabla_{g_1' \times g_2'}(\underline{f_1 + f_2}) = \nabla_{g_1' \times g_2'}(f_1 + f_2) \qquad \text{on } (M_1 \setminus U) \times M_2. \quad (5.12)$$

The result now follows from (5.11), (5.12) and the product formula (5.8). □

Apart from considering products of systems, we will also have to investigate in the anomaly of the relative torsion that arises when changing the metrics of a given system. In fact, we will only look at anomalies under the assumption that the metrics are left unchanged in a neighborhood of ∂M. The proposition below covers this situation, generalizing [11, Propositions 5.1, 5.2] onto odd-dimensional Manifolds with boundary with product metrics near ∂M.

Proposition 5.4 (Metric anomaly with boundary conditions). *Let* $\mathcal{D}_i = (E \downarrow M, g_i, h_i, \nabla_g f)$ *for* $i = 1, 2$ *be two Morse–Smale Systems with* M *odd-dimensional, such that either*

(1) *near* ∂M, $g_1 \equiv g_2$ *are of product form and* $h_1|_{\partial M} \equiv h_2|_{\partial M}$, *or*

(2) *near* ∂M, g_1 *and* g_2 *are of product form and* $h_1|_{\partial M} \equiv h_2|_{\partial M}$ *is unimodular.*

Then

$$\mathcal{R}(\mathcal{D}_1) - \mathcal{R}(\mathcal{D}_2) = \sum_{p \in \mathrm{Cr}(f)} (-1)^{\mathrm{ind}(p)} \log \Big(\det(h_1(p)^{-1} \circ h_2(p)) \Big). \tag{5.13}$$

Proof. First, observe that

$$\mathcal{R}(\mathcal{D}_1) - \mathcal{R}(\mathcal{D}_2) = \log \left(\frac{T^{An}_{(2)}(\mathcal{D}_1)}{T^{An}_{(2)}(\mathcal{D}_2)} \right) + \log \left(\frac{T^{Met}_{(2)}(\mathcal{D}_2)}{T^{Met}_{(2)}(\mathcal{D}_1)} \right) + \log \left(\frac{T^{MS}_{(2)}(\mathcal{D}_2)}{T^{MS}_{(2)}(\mathcal{D}_1)} \right). \tag{5.14}$$

Furthermore, we have

$$\frac{T^{Met}_{(2)}(\mathcal{D}_2)}{T^{Met}_{(2)}(\mathcal{D}_1)} = \sum_{k=0}^{n} (-1)^k \log \left(\frac{\det_\Gamma(\Theta_2^k)}{\det_\Gamma(\Theta_1^k)} \right), \tag{5.15}$$

where $\Theta_i^* : \mathcal{H}^*(\widetilde{M}, \widetilde{g_i}, \widetilde{E}, \widetilde{h_i}) \to H^*_{(2)}(\widetilde{M}, E, h_i, \nabla_g f)$ are the isomorphisms of finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules as defined in (3.37). We let

$$\mathbb{1}^*_{[h_1,h_2]} : H^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}} \widetilde{f}, \widetilde{E}, \widetilde{h_1}) \to H^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}} \widetilde{f}, \widetilde{E}, \widetilde{h_2}) \tag{5.16}$$

be the isomorphism of Hilbert $\mathcal{N}(\Gamma)$-modules induced by the (not necessarily unitary) identity map $\mathbb{1}^*_{[h_2,h_1]} : C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}} \widetilde{f}, \widetilde{E}, \widetilde{h_2}) \to C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}} \widetilde{f}, \widetilde{E}, \widetilde{h_1})$. Also, we let

$$\tau^* : \mathcal{H}^*(\widetilde{M}, \widetilde{g_2}, \widetilde{E}, \widetilde{h_2}) \to \mathcal{H}^*(\widetilde{M}, \widetilde{g_1}, \widetilde{E}, \widetilde{h_1}) \tag{5.17}$$

be the isomorphism of Hilbert $\mathcal{N}(\Gamma)$-modules making the diagram below commute.

$$\begin{array}{ccc} \mathcal{H}^*(\widetilde{M}, \widetilde{g_1}, \widetilde{E}, \widetilde{h_1}) & \xrightarrow{\Theta_1^*} & H^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}} \widetilde{f}, \widetilde{E}, \widetilde{h_1}) \\ {\scriptstyle \tau^*} \uparrow & & \downarrow {\scriptstyle \mathbb{1}^*_{[h_1,h_2]}} \\ \mathcal{H}^*(\widetilde{M}, \widetilde{g_1}, \widetilde{E}, \widetilde{h_2}) & \xrightarrow{\Theta_2^*} & H^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}} \widetilde{f}, \widetilde{E}, \widetilde{h_2}) \end{array} \tag{5.18}$$

From the multiplicativity of the Fuglede–Kadison determinant [20, Theorem 3.14], it follows that

$$\det_\Gamma(\tau^*) \det_\Gamma(\mathbb{1}^*_{[h_1,h_2]}) = \det_\Gamma(\Theta_1^*)^{-1} \det_\Gamma(\Theta_2^*). \tag{5.19}$$

Therefore, Equation (5.15) decomposes into

$$\log\left(\frac{T^{Met}_{(2)}(\mathcal{D}_2)}{T^{Met}_{(2)}(\mathcal{D}_1)}\right) = \sum_{k=0}^{n}(-1)^k \log\det(\tau^k) + \sum_{k=0}^{n}(-1)^k \log\det(\mathbb{1}^k_{[h_1,h_2]}). \tag{5.20}$$

By Proposition 2.2, we have

$$\sum_{k=0}^{n}(-1)^k \log\det(\mathbb{1}^k_{[h_1,h_2]}) + \log\left(\frac{T^{MS}_{(2)}(\mathcal{D}_2)}{T^{MS}_{(2)}(\mathcal{D}_1)}\right) = \sum_{p\in \mathrm{Cr}(f)}(-1)^{\mathrm{ind}(p)}\log\Big(\det(h_1(p)^{-1}\circ h_2(p))\Big). \tag{5.21}$$

For the remaining term, it is due to the main Theorem of [22] that we have an equality

$$\log\left(\frac{T^{An}_{(2)}(\mathcal{D}_1)}{T^{An}_{(2)}(\mathcal{D}_2)}\right) + \sum_{k=0}^{n}(-1)^k \log\det(\tau^k) = \log\left(\frac{T^{RS}(\mathcal{D}_1)}{T^{RS}(\mathcal{D}_2)}\right). \tag{5.22}$$

Here, $T^{RS}(\mathcal{D}_i)$ denotes the *Ray–Singer Torsion element* as originally defined in [6, Definition 2.2]. It is shown in [9, Theorem 3.4] that, under the conditions that M is odd-dimensional and either one of the two assertions mentioned in the statement of the proposition is satisfied, one has

$$\log\left(\frac{T^{RS}(\mathcal{D}_1)}{T^{RS}(\mathcal{D}_2)}\right) = 0. \tag{5.23}$$

The result direct follows from (5.14) and (5.20)–(5.23). □

Definition 5.5 (Subdivision)**.** Let M be a compact manifold and for $i = 0, 1$, let (f_i, g_i) be a Morse–Smale pair. Then (f_1, g_1) is called a *subdivision* of (f_0, g_0) if all of the following conditions are satisfied

(1) $\mathrm{Cr}_p(f_0) \subseteq \mathrm{Cr}_p(f_1) \subseteq \bigcup_{x\in \mathrm{Cr}(f_0)} W^-_x(f_0)$ for each $0 \le p \le n$,

(2) $W^-_x(f_1) \subseteq W^-_x(f_0)$ for each $x \in \mathrm{Cr}(f_0)$,

(3) $W^-_x(f_0) = \bigcup_{y\in \mathrm{Cr}(f_1)\cap W^-_x(f_0)} W^-_y(f_1)$, and

(4) $g_0 \equiv g_1$ near $\mathrm{Cr}(f_0) \cup \partial M$ and $f_0 \equiv f_1$ near ∂M.

We now describe the effect on the relative torsion under taking subdivisions. For that, let M be a compact manifold, let (f_i, g_i) be a Morse–Smale pair on M for $i = 0, 1$, so that (f_1, g_1) is a subdivision of (f_0, g_0). Let h be Hermitian form on a flat bundle $E \downarrow M$. By definition, there exists for each $y \in \mathrm{Cr}(f_1)$ a unique $x \in \mathrm{Cr}(f_0)$ satisfying $y \in W^-_x(f_0)$.

Let $\widetilde{h}(y) \in \mathrm{GL}(E_y, \overline{E_y^*})$ be the Hermitian metric on E_y obtained by parallel transport of the metric $h(x) \in \mathrm{GL}(E_x, \overline{E_x^*})$ along a curve connecting x and y that is entirely contained within $W_x^-(f_0)$. Note that since $W_x^-(f)$ is simply-connected, the resulting metric doesn't depend on the particular choice of curve. Note also that $\widetilde{h}(y) = h(y)$ whenever h is a unitary metric.

For each $y \in \mathrm{Cr}(f_1)$, define

$$\omega(y) := \log\det(\widetilde{h}(y)^{-1} \circ h(y)) \in \mathbb{R}_{\geq 0}\,. \tag{5.24}$$

Observe that $\omega \equiv 0$ whenever h is a unimodular metric. The proof of the following statement for closed manifolds is laid out in [11, Proposition 5.3] and carries over to general compact manifolds without further modification:

Proposition 5.6. *In the above situation, we have*

$$\mathcal{R}(E \downarrow M, g, h, \nabla_{g_0} f_0) - \mathcal{R}(E \downarrow M, g, h, \nabla_{g_1} f_1) = \sum_{y \in \mathrm{Cr}(f_1)} (-1)^{\mathrm{ind}(y)} \omega(y). \tag{5.25}$$

Corollary 5.7 (Relative Torsion under subdivision)**.** *Let $\mathcal{D}_0 = (E \downarrow M, g_0, h_0, \nabla_{g_0'} f_0)$ be a weakly admissible system with M odd-dimensional and let (f_1, g_1') be a subdivision of (f_0, g_0'). Then, one finds a Riemannian metric g_1 on M and an Hermitian form h_1 with $g_1 \equiv g_0$ and $h_1 \equiv h_0$ near ∂M on E, so that $\mathcal{D}_1 = (E \downarrow M, g_1, h_1, \nabla_{g_1'} f_1)$ is a weakly admissible system, satisfying*

$$\mathcal{R}(\mathcal{D}_0) = \mathcal{R}(\mathcal{D}_1). \tag{5.26}$$

Proof. For each $y \in \mathrm{Cr}(f_1)$, there exists by the definition of a subdivision a unique $x \in \mathrm{Cr}(f_0)$, such that $y \in W_x^-(f_0)$. As above, we let $\widetilde{h}_1(y) \in \mathrm{GL}(E_y, \overline{E_y}^*)$ be the Hermitian metric on the fiber E_y obtained by parallel transport of the Hermitian metric $h_0(x) \in \mathrm{GL}(E_x, \overline{E_x^*})$ along a curve between x and y contained entirely within $W_x^-(f)$. With

$$\omega(y) := \log\det(\widetilde{h}_1(y)^{-1} \circ h_0(y)),$$

we obtain from Proposition 5.6

$$\mathcal{R}(E \downarrow M, g_0, h_0, \nabla_{g_0'} f_0) = \mathcal{R}(E \downarrow M, g_0, h_0, \nabla_{g_1'} f_1) + \sum_{y \in \mathrm{Cr}(f_1)} (-1)^{\mathrm{ind}(y)} \omega(y). \tag{5.27}$$

In order to construct the metric h_1, choose small disjoint open coordinate neighborhoods $U_y \supset V_y \ni y$ around each $y \in \mathrm{Cr}(f_1)$, each also disjoint from a neighborhood of the boundary, such that $\overline{V_y} \subset U_y$. Define the Hermitian form $h_1 \in \mathrm{GL}(E, \overline{E^*})$ to be an extension of the metrics $\bigcup_{y \in \mathrm{Cr}(f_1)} \widetilde{h}_1(y)$ that is parallel on $\bigcup_{y \in \mathrm{Cr}(f_1)} V_y$ and equal to h_0 on $M \setminus \bigcup_{y \in \mathrm{Cr}(f_1) \setminus \mathrm{Cr}(f_0)} U_y$. Lastly, choose a Riemannian metric satisfying $g_1 \equiv g_1'$ near $\mathrm{Cr}(f_1)$ and $g_1 \equiv g_0$ near ∂M (in particular, g_1 is also of product form near ∂M). By construction of the metrics h_1 and g_1, the system $\mathcal{D}_1 = (E \downarrow M, g_1, h_1, \nabla_{g_1'} f_1)$ is weakly admissible. Moreover, an application of Proposition 5.4 gives

$$\mathcal{R}(E \downarrow M, g_0, h_0, \nabla_{g_1'} f_1) = \mathcal{R}(E \downarrow M, g_1, h_1.\nabla_{g_1'} f_1) + \sum_{y \in \mathrm{Cr}(f_1)} (-1)^{\mathrm{ind}(y)+1} \omega(y). \quad (5.28)$$

The result now follows from (5.27) and (5.28). □

The proof of the last result of this section can be found in [10, Proposition 3.7]

Proposition 5.8 (Determinant Class under Glueing)**.** *For $i = 1, 2$, let $(E_i \downarrow M_i)$ be two flat, complex bundles over a compact manifold, satisfying $E_1|_{\partial M_1} \downarrow \partial M_1 = E_2|_{\partial M_2} \downarrow \partial M_2$. Assume that both $E_i \downarrow M_i$ and $E_i|_{\partial M_i} \downarrow \partial M_i$ are of determinant class. Then, the flat bundle $E \downarrow M$ with $E := E_1 \cup_{\partial E_1} E_2$ and $M := M_1 \cup_{\partial M_1} M_2$ is of determinant class as well.*

6. Witten deformation and asymptotic expansions

This section collects the main technical results achieved by Burghelea et al. in [10] that are detrimental for the proof of Theorem 4.5. Since the methods employed by the authors carry over seamlessly from the unitary case to the general case of flat bundles, the proofs won't be included here.

6.1. Witten deformation

Throughout this section, we fix a countable group Γ and a Γ-invariant Morse–Smale system $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$. For any parameter $t \in \mathbb{R}_{\geq 0}$, the *Witten-deformation* d_t of the exterior derivative d on $\Omega^*(M, E)$ is defined as

$$\mathrm{d}_t := e^{-tf}\, \mathrm{d}e^{tf} = \mathrm{d} + t\mathrm{d}f \wedge : \Omega^*(M, E) \to \Omega^{*+1}(M, E). \quad (6.1)$$

Observe that $\mathrm{d}_t^2 = 0$ for any $t \in \mathbb{R}_{\geq 0}$, which is why we can regard the pair $\Omega_t^*(M, E) := (\Omega^*(M, E), \mathrm{d}_t)$ as a cochain complex. In analogy with the case $t = 0$, let $\delta_t : \Omega^*(M, E, g, h) \to \Omega^{*-1}(M, E, g, h)$ be the formal adjoint of d_t with respect to the inner product (3.15) on $\Omega_c^*(M, E)$ induced by g and h and define

$$\Delta_{*,t} := \delta_t^{*+1} d_t^* + d_t^{*-1} \delta_t^* : \mathrm{dom}(\Delta_{*,t}) \to \mathrm{dom}(\Delta_{*,t}), \quad (6.2)$$

$$\mathrm{dom}(\Delta_{*,t}) := \{\sigma \in \Omega_c^*(M, E) \colon \vec{n}\sigma = \vec{n}\mathrm{d}_t^* \sigma = 0\}. \quad (6.3)$$

Observe that for any $t \geq 0$, $\Delta_{*,t}$ is an elliptic differential operator of order 2 that is symmetric (on its domain) with respect to the inner product on $\Omega_t^*(M, E)$ induced by g and h. Moreover, just as in the case $t = 0$, one verifies that all three operators d_t, δ_t and $\Delta_{*,t}$ are closable when regarded as unbounded operators on the L^2-completion

$\Omega^*_{(2)}(M,E)$. The closed, symmetric operator $\Delta_{*,t}\colon \Omega^*_{(2),t}(M,E) \to \Omega^*_{(2),t}(M,E)$ is called the *Witten–Laplacian* (with absolute boundary conditions) associated to the system $\mathcal{D} = (E \downarrow M, g, h, \nabla'_{g'} f)$. We define the L^2-*Witten–de Rham complex* of the system $\mathcal{D}$ as

$$\Omega^*_{(2),t}(M,E) \coloneqq (\Omega^*_{(2)}(M,E), \mathrm{d}_t), \tag{6.4}$$

where we identify d_t with its minimal L^2-closure. All complexes obtained this way have the same isomorphism type. Namely, one easily sees that for each $t > 0$, multiplying a form ω with the function e^{tf} determines an isomorphism of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes

$$e^{tf} : \Omega^*_{(2),t}(M,E) \to \Omega^*_{(2)}(M,E). \tag{6.5}$$

Furthermore, since $\mathcal{D}$ is a Γ-invariant system, it follows that $\Delta_{*,t}$ is the lift of an elliptic operator defined over a bundle on a compact manifold. With this in mind, one verifies as in the case $t = 0$ that $\Delta_{*,t}$ is in fact self-adjoint (with the imposed absolute boundary conditions). In particular, for each $t \geq 0$, we can define the spectral projections

$$P^*(t) \coloneqq \chi_{[0,1)}(\Delta_{*,t})\colon \Omega^*_{(2),t}(M,E) \to \Omega^*_{(2),t}(M,E), \tag{6.6}$$

of $\Delta_{*,t}$ associated wit the half-open interval $[0,1)$, as well as the *small* and *large* subcomplexes

$$\Omega^*_{Sm,t}(M,E) \coloneqq \left(\bigoplus_{k=0}^{n} \mathrm{im}(P^k(t)), \mathrm{d}_t \right), \tag{6.7}$$

$$\Omega^*_{La,t}(M,E) \coloneqq \left(\bigoplus_{k=0}^{n} \mathrm{im}(\mathbb{1}_{\Omega^k_{(2)}(M,E)} - P^k(t)), \mathrm{d}_t \right). \tag{6.8}$$

Because $\Delta_{*,t}$ commutes with its spectral projections, one verifies inductively that $\mathrm{im}(P^*(t)) \subseteq \mathrm{dom}(\Delta^l_{*,t})$ for each $l \in \mathbb{N}_0$. Together with the ellipticity of $\Delta_{*,t}$, we deduce

$$\Omega^*_{Sm,t}(M,E) \subseteq \bigcap_{l=0}^{\infty} \mathcal{W}^*_l(M,E). \tag{6.9}$$

In particular, the complex $\Omega^*_{Sm,t}(M,E)$ consists entirely of smooth forms. Moreover, observe that we have an orthogonal decomposition of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes

$$\Omega^*_{(2),t}(M,E) = \Omega^*_{Sm,t}(M,E) \oplus \Omega^*_{La,t}(M,E). \tag{6.10}$$

Finally, just as in the case $\partial\widetilde{M} = \emptyset$, one verifies:

Proposition 6.1 ([32, Theorem 4.2])**.** *For each $t \geq 0$, the projection $P^*(t)\colon \Omega^*_{(2),t}(M,E) \to \Omega^*_{Sm,t}(M,E)$ onto the small subcomplex is a chain homotopy equivalence of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes (with chain homotopy inverse given by the inclusion).*

Now assume additionally that the Γ-invariant system $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$ is also weakly Γ-admissible. Recall from the axioms laid out in Definition 3.2 that a Γ-invariant system is weakly Γ-admissible if certain local conditions near $\mathrm{Cr}(f)$ are satisfied: We can choose for each $p \in \mathrm{Cr}(f)$ radii $r_p > 0$, coordinate charts $\phi_p : B_{r_p}(0) \xrightarrow{\cong} U_p \subseteq \mathbb{R}^n$ disjoint from ∂M with $B_{r_p}(0) := \{x \in \mathbb{R}^n : \|x\| < r_p\}$ and $\phi_p(0) = p$, along with a flat bundle isomorphism $\Phi_p : B_{r_p}(0) \times \mathbb{C}^m \xrightarrow{\cong} E|_{U_p}$ that fit into the commutative diagram

$$\begin{array}{ccc} B_{r_p}(0) \times \mathbb{C}^m & \xrightarrow[\cong]{\Phi_p} & E|_{U_p} \\ \Big\downarrow pr_1 & & \Big\downarrow \pi_E \\ B_{r_p}(0) & \xrightarrow[\cong]{\phi_p} & U_p, \end{array} \tag{6.11}$$

and such that all of the following conditions hold:

(H_1) The pullback metric $\phi_p^*(g|_{U_p})$ equals the Euclidean metric on $\mathbb{R}^n$.

(H_2) The pullback Hermitian form $\Phi_p^*(h|_{U_p})$ equals the standard inner product on $\mathbb{C}^m$.

(H_3) One has

$$(f \circ \phi_p)(x_1, \ldots, x_n) = f(p) - \frac{1}{2} \sum_{i=1}^{\mathrm{ind}(p)} x_i^2 + \frac{1}{2} \sum_{i=\mathrm{ind}(p)+1}^{n} x_i^2 .$$

(H_4) The above choices are Γ-invariant, i.e. $\gamma.U_p = U_{\gamma.p}$, $r_p = r_{\gamma.p}$, $\gamma \circ \phi_p = \phi_{\gamma.p}$ and $\gamma \circ \Phi_p = \Phi_{\gamma.p}$ for each $p \in \mathrm{Cr}(f)$ and each $\gamma \in \Gamma$.

It is precisely due to this Γ-invariant shape of f and metric bundle $(E, h) \downarrow (M, g)$ near $\mathrm{Cr}(f)$ that Burghelea et al. were able to prove the next theorem. With the aid of properties (H_1)–(H_4), their proof from [10, Section 3.3] can be adapted, word by word, to our situation of non-unitary bundles without any further modification:

Theorem 6.2. *Let $(E \downarrow M, g, h, \nabla_{g'} f)$ be a weakly Γ-admissible system. Then, for each $t \geq 0$, there exists an isometric embedding of Hilbert $\mathcal{N}(\Gamma)$-modules*

$$J^*(t) := \bigoplus_{k=0}^{n} J^k(t) : C^*_{(2)}(M, \nabla_{g'} f, E, h) \to \Omega^*_{(2)}(M, E),$$

Moreover, for large $t \gg 0$, the composition

$$Q(t) := P^*(t) \circ J^*(t) : C^*_{(2)}(M, \nabla_{g'} f, E, h) \to \Omega^*_{Sm,t}(M, E)$$

is an isomorphism of Hilbert $\mathcal{N}(\Gamma)$-modules.

We stress the fact that the map of Hilbert $\mathcal{N}(\Gamma)$-modules $J^*(t)$ from the previous theorem (and therefore also the isomorphism $Q^*(t)$) is in general *not* a map of *cochain complexes*. This is why the maps $Q^*(t)$ alone cannot be used to reach our desired conclusion, namely that the complexes $C^*_{(2)}(M, \nabla_{g'} f, E, h)$ and $\Omega^*_{Sm,t}(M, E, g, h)$ are chain homotopy equivalent. In spite of this, it still follows that for sufficiently large $t \gg 0$, the isomorphism $Q^*(t)$ can be used to define the isometry

$$I^*(t) := Q^*(t)\,(Q^*(t)^*Q^*(t))^{-1/2} : C^*_{(2)}(M, \nabla_{g'} f, E, h) \to \Omega^*_{Sm,t}(M, E, g, h). \quad (6.12)$$

Moreover, since $(E \downarrow M, g, h, \nabla_{g'} f)$ is the lift of an admissible system with deck group Γ, there are also isomorphisms of Hilbert $\mathcal{N}(\Gamma)$-modules for $t > 0$:

$$\begin{aligned} &S^*(t) : C^*_{(2)}(M, \nabla_{g'} f, E, h) \to C^*_{(2)}(M, \nabla_{g'} f, E, h), \\ &\lambda_p \otimes [p] \mapsto \exp\left(\frac{n - 2\,\mathrm{ind}(p)}{4} \log(\pi/t) - tf(p)\right) \cdot \lambda_p \otimes [p], \quad p \in \mathrm{Cr}(f). \end{aligned} \quad (6.13)$$

Here, we have used the fact that f is Γ-invariant, hence in particular satisfies $f(\gamma.x) = f(x)$ for any $x \in M$.

Recall that because of (6.9), we have the inclusion $\Omega^*_{Sm,t}(M, E) \subseteq \bigcap_{l\in\mathbb{N}} \mathcal{W}^*_l(M, E)$. This allows us to define the morphism of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes

$$F^*(t) := \mathrm{Int}^* \circ e^{tf} : \Omega^*_{Sm,t}(M, E, g, h) \to C^*_{(2)}(M, \nabla_{g'} f, E, h), \quad (6.14)$$

as restricting to the subcomplex $\Omega^*_{Sm,t}(M, E)$ the composition of the isomorphism

$$e^{tf} : \Omega^*_{(2),t}(M, E) \to \Omega^*_{(2)}(M, E)$$

from (6.5) with the integration map

$$\mathrm{Int}^* : \mathcal{W}^*_{l-*}(M, E) \to C^*_{(2)}(M, \nabla_{g'} f, E, h),$$

defined as in (3.34). Just as before, the proof of the next theorem, laid out for unitary bundles in [10, Section 3.3], can be adapted to our setting without any modifications:

Theorem 6.3. *Under the previous assumptions, we obtain for large $t \gg 0$, that*

$$S^*(t) \circ F^*(t) \circ I^*(t) = \mathbb{1} + O(t^{-1}). \quad (6.15)$$

Consequently, for large $t \gg 0$, the map $F^*(t)\colon \Omega^*_{Sm,t}(M, E, g, h) \to C^*_{(2)}(M, \nabla_{g'} f, E, h)$ is an isomorphism of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes.

Combining (6.5) with Proposition 6.1 and Theorem 6.3, we arrive at the following very important intermediate result:

Theorem 6.4. *Let $\mathcal{D} = (M, E, g, h, \nabla_{g'} f)$ be a weakly admissible type II-Morse Smale system with M compact, let $\widetilde{M}$ be the universal cover of M and let $\widetilde{\mathcal{D}} = (\widetilde{M}, \widetilde{E}, \widetilde{g}, \widetilde{h}, \nabla_{\widetilde{g}'} \widetilde{f})$*

be the corresponding lift of $\mathcal{D}$. Then, there is a chain homotopy equivalence of Hilbert $\mathcal{N}(\Gamma)$-cochain complexes

$$\begin{array}{ccc} \Omega^*_{(2)}(\widetilde{M}, \widetilde{E}, \widetilde{g}, \widetilde{h}) & \xrightarrow{\ \simeq\ } & C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}'}\widetilde{f}, \widetilde{E}, \widetilde{h}) \\ {\scriptstyle (6.5)}\Big\downarrow {\scriptstyle e^{-t\widetilde{f}}} & & {\scriptstyle F^*(t)}\Big\uparrow {\scriptstyle (6.3)} \\ \Omega^*_{(2),t}(\widetilde{M}, \widetilde{E}, \widetilde{g}, \widetilde{h}) & \xrightarrow[(6.1)]{P^*(t)} & \Omega^*_{Sm,t}(\widetilde{M}, \widetilde{E}, \widetilde{g}, \widetilde{h}), \end{array} \tag{6.16}$$

with $t \gg 0$ chosen sufficiently large. In particular, we obtain:

(1) *For each $0 \leq k \leq n$, it holds that $\alpha_k^{An}(M, E) = \alpha_k^{Top}(M, E)$.*

(2) *(M, E) is of a-determinant class if and only if it is of c-determinant class.*

6.2. **Asymptotic expansions**

Let $\mathcal{D} = (E \downarrow M, g, h, \nabla_g f)$ be a weakly admissible system with $\Gamma := \pi_1(M)$ and let $\widetilde{\mathcal{D}} := (\widetilde{E} \downarrow \widetilde{M}, \widetilde{g}, \widetilde{h}, \nabla_{\widetilde{g}}\widetilde{f})$ be the Γ-invariant lift of $\mathcal{D}$ (throughout this subsection, we assume that $g = g'$). We set $b := f^{-1}(\partial M)$. For $t \geq 0$, let $\Omega^*_{(2),t}(\widetilde{M}, \widetilde{E})$ be the Witten-deformed complex defined in the previous section (with metric induced by $\widetilde{g}$ and $\widetilde{h}$ implicit, in order to simplify notation) with Witten-deformed Laplacian $\Delta_{*,t}[\widetilde{E}] : \Omega^*_{(2),t}(\widetilde{M}, \widetilde{E}) \to \Omega^*_{(2),t}(\widetilde{M}, \widetilde{E})$. Further, we define $\Theta^*(t) : \ker(\Delta_{*,t}[E]) \to H^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \widetilde{E}, \widetilde{h})$ to be the isomorphim of finitely-generated Hilbert $\mathcal{N}(\Gamma)$-modules that is the composition $\Theta^* \cdot e^{t\widetilde{f}}$, where $\Theta^* : \ker(\Delta_{*,0}[E]) \to H^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}}\widetilde{f}, \widetilde{E}, \widetilde{h})$ is the isomorphism from (3.37). Introduce

$$\mathrm{Vol}(\mathcal{D})(t) := \prod_{k=0}^{n} \det_\Gamma(\Theta^k(t))^{(-1)^k}. \tag{6.17}$$

Observe that

$$\mathrm{Vol}(\mathcal{D})(0) = T^{Met}_{(2)}(E \downarrow M, g, h, \nabla_g f). \tag{6.18}$$

Moreover, recall the orthogonal decomposition of subcomplexes

$$\Omega^*_{(2),t}(\widetilde{M}, \widetilde{E}) = \Omega^*_{Sm,t}(\widetilde{M}, \widetilde{E}) \oplus \Omega^*_{La,t}(\widetilde{M}, \widetilde{E}),$$

which implies the following: Provided that $E \downarrow M$ is of determinant class, the torsion elements $T^{An}_{(2)}(E \downarrow M, g, h)(t), T^{Sm}_{(2)}(\mathcal{D})(t)$ and $T^{La}_{(2)}(\mathcal{D})(t)$ of the complexes $\Omega^*_{(2),t}(\widetilde{M}, \widetilde{E})$, $\Omega^*_{Sm,t}(\widetilde{M}, \widetilde{E})$, respectively $\Omega^*_{La,t}(\widetilde{M}, \widetilde{E})$ are all well-defined positive real numbers, so that

$$T^{An}_{(2)}(E \downarrow M, g, h)(0) = T^{An}_{(2)}(M, E, g, h), \tag{6.19}$$

$$T^{An}_{(2)}(E \downarrow M, g, h)(t) = T^{Sm}_{(2)}(\mathcal{D})(t) \cdot T^{La}_{(2)}(\mathcal{D})(t). \tag{6.20}$$

A function $F : \mathbb{R} \to \mathbb{R}$ is said to admit an *asymptotic expansion*, if there exists an integer $N \in \mathbb{N}$ and constants $(a_j)_{j=0}^N$, $(b_j)_{j=0}^N$ such that for $t \to +\infty$

$$F(t) = \sum_{j=0}^{N} \left(a_j + b_j \log(t)\right) t^j + o(1). \tag{6.21}$$

The coefficient a_0 in the expansion is called the *free term* of F and is denoted by $\mathrm{FT}(F)$. In the special case that $E \downarrow M$ is a unitary bundle and h a flat unitary metric, the proof of the next proposition has been carried out in [10, Theorem 3.13]. In fact, the same proof still works without further modification in the more general case that the bundle $E \downarrow M$ is of product from near ∂M, and will therefore be omitted (See also [34, Proposition 6.4.1] for a slightly different proof).

Proposition 6.5 (Asymptotic expansion for the analytic torsion)**.** *There exists a constant $C \in \mathbb{R}$, such that the following holds: For any weakly admissible system $\mathcal{D} = (E \downarrow M, g, h, \nabla_g f)$ of determinant class with M odd-dimensional, the function* $\log T_{(2)}^{An}(E \downarrow M, g, h)(t) - \log \mathrm{Vol}(\mathcal{D})(t)$ *admits the following asymptotic expansion:*

$$\log T_{(2)}^{An}(E \downarrow M, g, h) - \log \mathrm{Vol}(\mathcal{D})(0) + Ct \dim(E)\chi(\partial M). \tag{6.22}$$

Proposition 6.6 (Asymptotic expansion for the small torsion)**.** *For any weakly admissible system $\mathcal{D} = (E \downarrow M, g, h, \nabla_g f)$ of determinant class with $n := \dim(M)$, $\mathrm{Cr}_k(f) := \{p \in \mathrm{Cr}(f) \colon \mathrm{ind}(p) = k\}$ and $m_k := \#\mathrm{Cr}_k(f)$, the function* $\log T_{(2)}^{Sm}(\mathcal{D})(t) - \log \mathrm{Vol}(\mathcal{D})(t)$ *admits the asymptotic expansion*

$$\begin{aligned} &\log T_{MS}^{(2)}(M, E, h, \nabla_g f) \\ &\quad + \dim(E)\left(\sum_{k=0}^{n} (-1)^k m_k \frac{n-2k}{4} \log(\pi/t) + t(-1)^{k+1} \sum_{p \in \mathrm{Cr}_k(f)} f(p)\right) + o(1). \end{aligned} \tag{6.23}$$

Proof. For large $t \gg 0$, there exists by Theorem 6.3 an isomorphism of finitely generated Hilbert $\mathcal{N}(\Gamma)$-cochain complexes

$$F^*(t) : \Omega^*_{Sm,t}(\widetilde{M}, \widetilde{E}, \widetilde{g}, \widetilde{h}) \to C^*_{(2)}(\widetilde{M}, \nabla_{\widetilde{g}} \widetilde{f}, \widetilde{E}, \widetilde{h}). \tag{6.24}$$

From Proposition 2.2, it then follows that

$$\begin{aligned} &\log T_{(2)}^{Sm}(M, E, g, h, f)(t) + \log \mathrm{Vol}(t) \\ &\qquad = \log T_{MS}^{(2)}(M, E, h, \nabla_g f) + \sum_{k=0}^{n} (-1)^k \log \det\nolimits_\Gamma F^k(t). \end{aligned} \tag{6.25}$$

Recall also from Theorem 6.3 the formula $S^k(t) \circ F^k(t) \circ I^k(t) = \mathbb{1}_{C^k_{(2)}} + O(t^{-1})$, where $I^k(t)$ is the isometry from (6.12) and $S^k(t)$ is the scaling isomorphism from (6.13). Consequently, by the multiplicativity of the Fuglede–Kadison determinant in this setting [20, Theorem 3.14], it holds that

$$\log \det_\Gamma F^k(t) = -\log \det_\Gamma S^k(t) + o(1). \tag{6.26}$$

From the explicit formula of $S^k(t)$ (6.13), we obtain

$$\det_\Gamma S^k(t) = \left(\prod_{p \in \mathrm{Cr}_k(f)} (\pi/t)^{\frac{n-2k}{4}} e^{-tf(p)} \right)^{\dim(E)}. \tag{6.27}$$

The result now is an immediate consequence of (6.25)–(6.27). □

Corollary 6.7 (Asymptotic expansion for the large torsion)**.** *Let $\mathcal{D} = (E \downarrow M, g, h, \nabla_g f)$ be a weakly admissible system of determinant class with M odd-dimensional. Then, the following assertions hold*

(1) *The function* $\log T^{La}_{(2)}(\mathcal{D})(t)$ *admits an asymptotic expansion. More precisely, there exists a polynomial* $\Phi(\mathcal{D})(t) : \mathbb{R} \to \mathbb{R}$ *in t and* $\log(t)$*, such that for $t \to \infty$*

$$\log T^{La}_{(2)}(\mathcal{D})(t) = R(\mathcal{D}) + \Phi(\mathcal{D})(t) + o(1). \tag{6.28}$$

Finally, for any arbitrary small neighborhood U of $\mathrm{Cr}(f) \cup \partial M$*, the polynomial* $\Phi(\mathcal{D})$ *depends only on the isomorphism class of the system* $\mathcal{D}^f|_U := (E|_U \downarrow U, g|_U, h|_U, f|_U)$.

(2) *Suppose that* $\mathcal{D}_1 = (E_1 \downarrow M_1, g_1, h_1, \nabla_{g_1} f_1)$ *is another weakly admissible system, such that there exists neighborhoods $U \subseteq M$ of* $\mathrm{Cr}(f) \cup \partial M$ *and $U_1 \subset M_1$ of* $\mathrm{Cr}(f_1) \cup \partial M_1$ *with the property that the derived systems* $\mathcal{D}^f|_U := (E|_U \downarrow U, g|_U, h|_U, f|_U)$ *and* $\mathcal{D}_1^{f_1}|_{U_1} := (E_1|_{U_1} \downarrow U_1, g|_{U_1}, h|_{U_1}, f_1|_{U_1})$ *are isomorphic (in particular* $\#\mathrm{Cr}_k(f) = \#\mathrm{Cr}_k(f_1)$ *for each* $0 \le k \le n$*). Then*

$$R(\mathcal{D}) - R(\mathcal{D}_1) = \mathrm{FT}\left(\log T^{La}_{(2)}(\mathcal{D})\right) - \mathrm{FT}\left(\log T^{La}_{(2)}(\mathcal{D}_1)\right). \tag{6.29}$$

(3) *Under the assumptions of* (2)*, there exists local quantities* $\alpha(\mathcal{D}) \in \Omega^n(M \setminus \mathrm{Cr}(f), \mathcal{O}_M)$ *and* $\alpha(\mathcal{D}_1) \in \Omega^n(M_1 \setminus \mathrm{Cr}(f_1), \mathcal{O}_{M_1})$ *of the derived systems* $\mathcal{D}|_{M\setminus \mathrm{Cr}(f)}$ *and* $\mathcal{D}_1|_{M_1 \setminus \mathrm{Cr}(f_1)}$*, such that one has*

$$\mathrm{FT}\left(\log T^{La}_{(2)}(\mathcal{D})\right) - \mathrm{FT}\left(\log T^{La}_{(2)}(\mathcal{D}_1)\right) = \int_{M \setminus \mathrm{Cr}(f)} \alpha(\mathcal{D}) - \int_{M_1 \setminus \mathrm{Cr}(f_1)} \alpha(\mathcal{D}_1). \tag{6.30}$$

Proof. (1). We have $\log T^{An}_{(2)}(E \downarrow M, g, h)(t) = \log T^{Sm}_{(2)}(\mathcal{D})(t) + \log T^{La}_{(2)}(\mathcal{D})(t)$, hence also in particular

$$\log T^{An}_{(2)}(E \downarrow M, g, h)(t) - \log \mathrm{Vol}(\mathcal{D})(t) \\ - \log T^{Sm}_{(2)}(\mathcal{D})(t) - \log \mathrm{Vol}(\mathcal{D})(t) = \log T^{La}_{(2)}(\mathcal{D})(t).$$

Since the left-hand side of the equation admits an asymptotic expansion, given by the sum of the explicit formulas (6.22) and (6.23), the result follows.

(2). Observe that

$$\mathrm{FT}\big(\log T^{La}_{(2)}(\mathcal{D})\big) = R(\mathcal{D}) + \mathrm{FT}(\Phi(\mathcal{D}))$$

and analogously

$$\mathrm{FT}\big(\log T^{La}_{(2)}(\mathcal{D}_1)\big) = R(\mathcal{D}_1) + \mathrm{FT}(\Phi(\mathcal{D}_1)).$$

Since the systems $\mathcal{D}^f|_U$ and $\mathcal{D}^{f_1}|_{U_1}$ are isomorphic by assumption, assertion (1) implies that $\Phi(\mathcal{D}) \equiv \Phi(\mathcal{D}_1)$ and the result follows.

(3). In case that $\partial M = \emptyset$, this is proven in [12, Theorem B, Section 6.2] for unitary bundles (whose proof is also referred to in [11, Proposition 4.2] for arbitrary flat bundles). The same proof works without any modifications in the case that $\partial M \neq \emptyset$. □

7. Proof of Theorem 4.5

Armed with the results of the previous two sections, we will closely follow the strategy of [10] and use Zhang's result in Theorem 4.4 to prove Theorem 4.5.

Proposition 7.1. *For $i = 1, 2$, let $\mathcal{D}_i = (M_i, E_i, g_i, h_i, \nabla_{g_i} f_i)$ be two weakly admissible systems satisfying the assumptions of Corollary 6.7(2). Moreover, assume that there exists a flat bundle $E_3 \downarrow M_3$ with M_3 compact, satisfying*

(1) $(E_3|_{\partial M_3}) \downarrow \partial M_3 = E_i|_{\partial M_i} \downarrow \partial M_i$, *and*

(2) *the bundle $\overline{E_i} \downarrow N_i$ is of determinant class, where $N_i \coloneqq M_3 \cup_{\partial M_3} M_i$ and $\overline{E_i} \coloneqq E_3 \cup_{E_3|_{\partial M_3}} E_i$.*

Then

$$\mathcal{R}(\mathcal{D}_1) + \frac{1}{2}\int_{M_1} \theta(h_1) \wedge (\nabla_{g_1} f_1)^* \Psi(TM_1, g_1) \\ = \mathcal{R}(\mathcal{D}_2) + \frac{1}{2}\int_{M_2} \theta(h_2) \wedge (\nabla_{g_2} f_2)^* \Psi(TM_2, g_2). \quad (7.1)$$

Proof. Choose a smooth function $f_3 : M_3 \to \mathbb{R}$ on M_3 with $f_3|_{\partial M_3} = f_i|_{\partial M_i}$ for $i = 1, 2$ and such that the function $\overline{f}_i := f_3 \cup_{\partial M_3} f_i : N_i \to \mathbb{R}$ is a Morse function. Furthermore, choose a Riemannian metric g_3 on M_3 with $g_3|_{\partial M_3} = g_i|_{\partial M_i}$ for $i = 1, 2$, such that for the metric $\overline{g}_i = g_3 \cup_{\partial M_3} g_i$ on N_i, the pair $(\overline{f}_i, \overline{g}_i)$ is a Morse–Smale pair (since N_i is closed, there is no distinction between type I and type II). Lastly, choose a Hermitian form h_3 on the flat bundle $E_3 \downarrow M_3$ with $h_3|_{\partial M_3} = h_i|_{\partial M_i}$ for $i = 1, 2$ with $\overline{h}_i := h_3 \cup_{\partial M_3} h_i$, such that the system

$$\overline{\mathcal{D}}_i := (\overline{E_i} \downarrow N_i, \overline{g}_i, \overline{h}_i, \nabla_{\overline{g}_i} \overline{f}_i) \tag{7.2}$$

is weakly admissible. By construction, the pair $\overline{\mathcal{D}}_i$ also satisfies the assumptions of Corollary 6.7 (2). Applying Corollary 6.7 (3), we can find densities α_i on $M_i \setminus \mathrm{Cr}(f_i)$ and $\overline{\alpha_i}$ on $N_i \setminus \mathrm{Cr}(\overline{f_i})$, so that

$$R(\mathcal{D}_1) - R(\mathcal{D}_2) = \int_{M_1 \setminus \mathrm{Cr}(f_1)} \alpha_1 - \int_{M_2 \setminus \mathrm{Cr}(f_2)} \alpha_2, \tag{7.3}$$

$$R(\overline{\mathcal{D}_\infty}) - R(\overline{\mathcal{D}_\epsilon}) = \int_{N_1 \setminus \mathrm{Cr}(\bar{f}_1)} \overline{\alpha_1} - \int_{N_2 \setminus \mathrm{Cr}(\bar{f}_2)} \overline{\alpha_2}. \tag{7.4}$$

Since the densities are local quantities, it follows from the chosen metrics on the respective bundles that $\alpha_i = \overline{\alpha_i}|_{M_i}$ and $\overline{\alpha_1}|_{M_3} = \overline{\alpha_2}|_{M_3}$. Moreover, since $\mathrm{Cr}(\overline{f_i}) \cap M_i = \mathrm{Cr}(f_i)$ by construction, we get from (7.3) and (7.4)

$$R(\mathcal{D}_1) - R(\mathcal{D}_2) = R(\overline{\mathcal{D}}_1) - R(\overline{\mathcal{D}}_2). \tag{7.5}$$

As N_i is closed, we can apply Theorem 4.4 and obtain

$$R(\overline{\mathcal{D}_i}) = -\frac{1}{2} \int_{N_i} \theta(\overline{E_i}, \overline{h_i}) \wedge (\nabla_{\bar{g}_i} \overline{f}_i)^* \Psi(TN_i, \overline{g_i}), \tag{7.6}$$

As mentioned in the introduction, the n-form $\theta(\overline{E_i}, \overline{h_i}) \wedge (\nabla_{\bar{g}_i} \overline{f}_i)^* \Psi(TN_i, \overline{g_i})$ is a local quantity. In particular, it follows both

$$\theta(\overline{E_i}, \overline{h_i}) \wedge (\nabla_{\bar{g}_i} \overline{f_i})^* \Psi(TN_i, \overline{g_i})|_{M_i} = \theta(E_i, h_i) \wedge (\nabla_{g_i} f_i)^* \Psi(TM_i, g_i)$$

and

$$\theta(\overline{E_1}, \overline{h_1}) \wedge (\nabla_{\bar{g}_1} \overline{f}_1)^* \Psi(TN_1, \overline{g_1})|_{M_3} = \theta(\overline{E_2}, \overline{h_2}) \wedge (\nabla_{\bar{g}_2} \overline{f}_2)^* \Psi(TN_2, \overline{g_2})|_{M_3}.$$

Therefore

$$\begin{aligned}
&\int_{N_1} \theta(\overline{E_1}, \overline{h_1}) \wedge (\nabla_{\bar{g}_1} \overline{f}_1)^* \Psi(TN_1, \overline{g_1}) - \int_{N_2} \theta(\overline{E_2}, \overline{h_2}) \wedge (\nabla_{\bar{g}_2} \overline{f}_2)^* \Psi(TN_2, \overline{g_2}) \\
&= \int_{M_1} \theta(E_1, h_1) \wedge (\nabla_{g_1} f_1)^* \Psi(TM_1, g_1) - \int_{M_2} \theta(E_2, h_2) \wedge (\nabla_{g_2} f_2)^* \Psi(TM_2, g_2).
\end{aligned} \tag{7.7}$$

Equation (7.1) now is an immediate consequence of (7.5)–(7.7) and the definition of relative torsion. □

Theorem 7.2. *Assume that $\mathcal{D}_i = (E_i \downarrow M_i, g_i, h_i, \nabla_{g_i'} f_i)$ are two admissible systems with M_i odd-dimensional, $(\partial M_1, g_1|_{\partial M_1}) = (\partial M_2, g_2|_{\partial M_2})$ and $(E_1|_{\partial M_1}, h_1|_{\partial M_1}) = (E_2|_{\partial M_2}, h_2|_{\partial M_2})$. Then, if both $E_i \downarrow M_i$ and $E_i|_{\partial M_i} \downarrow M_i$ are of determinant class, we get*

$$\mathcal{R}(\mathcal{D}_1) + \frac{1}{2}\int_{M_1} \theta(E_1, h_1) \wedge (\nabla_{g_1'} f_1)^*\Psi(TM_1, g_1)$$
$$= \mathcal{R}(\mathcal{D}_2) + \frac{1}{2}\int_{M_2} \theta(E_2, h_2) \wedge (\nabla_{g_2'} f_2)^*\Psi(TM_2, g_2).$$

Proof. We consider different cases:

Case 1: The systems $\mathcal{D}_i$ satisfy the hypotheses of Corollary 6.7(2). Consider the admissible system $\mathcal{D}_{S^2} : (E_{\mathbb{C}}^{S^2} \downarrow S^2, g, h, \nabla_g f)$ with $E_{\mathbb{C}}^{S^2} \downarrow S^2$ the trivial complex line bundle over S^2, (f, g) some Morse–Smale pair on S^2 and h a parallel metric on $E_{\mathbb{C}}^{S^2}$. Since S^2 is simply-connected, the system $\mathcal{D}_{S^2}$ is of determinant class. It follows from Proposition 5.2 that also the modified product systems $\underline{\mathcal{D}_i \times \mathcal{D}_{S^2}}$ are of determinant class, so that

$$\mathcal{R}(\underline{\mathcal{D}_i \times \mathcal{D}_{S^2}}) = 2\mathcal{R}(\mathcal{D}_i), \tag{7.8}$$

where we have used that $\chi(S^2) = 2$, as well as the well-known fact that $\mathcal{R}(\mathcal{D}_{S^2}) = 0$, which follows for example also from Theorem 4.4.

Next, consider the trivial complex line bundle $E_{\mathbb{C}}^{D^3} \downarrow D^3$. Since D^3 is simply-connected, it is of determinant class. Moreover, since $E|_{\partial M_1} \downarrow \partial M_1$ is of determinant class by assumption and ∂M_1 is closed, it follows again from Proposition 5.2 that the product bundle $E|_{\partial M_1} \widehat{\otimes} E_{\mathbb{C}}^{D^3} \downarrow \partial M_1 \times D^3$, as well as its restriction to $\partial(\partial M_1 \times D^3) = \partial M_1 \times \partial D^3$, is of determinant class. Now observe that by construction, the identification $\partial D^3 \cong S^2$ induces an isomorphism of flat bundles $E_1 \widehat{\otimes} E_{\mathbb{C}}^{D^3}|_{\partial M_1 \times \partial D^3} \downarrow \partial M_1 \times \partial D^3 \cong E_i \widehat{\otimes} E_{\mathbb{C}}^{S^2}|_{\partial M_i \times S^2} \downarrow \partial M_i \times S^2$ for $i = 1, 2$. Just as in Proposition 7.1, we can therefore define for $i = 1, 2$

$$N_i := M_i \times S_2 \cup_{\partial M_1 \times S^2} \partial M_1 \times D^3,$$
$$\overline{E_i} := E_i \widehat{\otimes} E_{\mathbb{C}}^{S^2} \cup_{E_i|_{\partial M_i} \hat{\otimes} E_{\mathbb{C}}^{S^2}} E_1 \widehat{\otimes} E_{\mathbb{C}}^{D^3}.$$

By Proposition 5.8, it follows that $\overline{E_i} \downarrow N_i$ is of determinant class. Hence, the modified product systems $\underline{\mathcal{D}_i \times \mathcal{D}_{S^2}}$ satisfy also the assumptions of Proposition 7.1, from which we get

$$\mathcal{R}(\underline{\mathcal{D}_1 \times \mathcal{D}_{S^2}}) + \frac{1}{2}\int_{M_1 \times S^2} \theta(h_1 \widehat{\otimes} h) \wedge \nabla_{g_1 \times g}(\underline{f_1 + f})^*\Psi\left(T(M_1 \times S^2), g_1 \times g\right)$$
$$= \mathcal{R}(\underline{\mathcal{D}_2 \times \mathcal{D}_{S^2}}) + \frac{1}{2}\int_{M_2 \times S^2} \theta(h_2 \widehat{\otimes} h) \wedge (\nabla_{g_2 \times g}(\underline{f_2 + f}))^*\Psi\left(T(M_2 \times S^2), g_2 \times g\right). \tag{7.9}$$

Applying the product formula (5.3), we obtain for $i = 1, 2$

$$\theta(h_i \mathbin{\widehat{\otimes}} h) \wedge \nabla_{g_i \times g}(\underline{f_i + f})^* \Psi \left(T(M_i \times S^2), g_i \times g \right) \\ = \left(\theta(h_i) \wedge (\nabla_{g_i} f_i)^* \Psi(TM_i, g_i) \right) \otimes e(TS^2, g),$$

Since $e(TS^2, g)$ is a representative of the rational Euler class of TS^2, we obtain that $\int_{S^2} e(TS^2, g) = \chi(S^2) = 2$. Together with the previous equation, this implies for $i = 1, 2$, that

$$\int_{M_i \times S^2} \theta(h_i \mathbin{\widehat{\otimes}} h) \wedge \nabla_{g_i \times g}(\underline{f_i + f})^* \Psi \left(T(M_i \times S^2), g_i \times g \right) \\ = 2 \int_{M_i} \theta(h_i) \wedge (\nabla_{g_i} f_i)^* \Psi(TM_i, g_i). \quad (7.10)$$

The result now follows from (7.8)–(7.10).

Case 2: The systems $\mathcal{D}_i$ don't satisfy the hypotheses of Corollary 6.7(2). Since the $\mathcal{D}_i$ are by assumption admissible, we find a neighborhood U_i of ∂M_i, such $\theta(h_i) \equiv 0$ on U_i and $g_i \equiv g_i'$ on $M_i \setminus U_i$, which is why $\theta(h_i) \wedge (\nabla_{g_i'} f_i)^* \Psi(TM_i, g_i) = \theta(h_i) \wedge (\nabla_{g_i} f_i)^* \Psi(TM_i, g_i)$ on *all of* M_i. Moreover, since both g_i' and g_i are of product form near ∂M_i and $h_i|_{\partial M_i}$ is unimodular, it follows from Proposition 5.4 that $\mathcal{R}(\mathcal{D}_i) = \mathcal{R}(E_i \downarrow M_i, g_i', h_i, \nabla_{g_i'} f_i)$. Therefore, we may assume without loss of generality that $g_i \equiv g_i'$ on all of M_i.

Now since the M_i are odd-dimensional with $\partial M_1 = \partial M_2$, we have $\chi(M_1) = \chi(M_2)$. Using this, one proceeds as in [12, Section 6] to show that there exist subdivisions $(\overline{f_i}, \overline{g_i})$ of (f_i, g_i) (with $\overline{g_i} = g_i$ near ∂M_i), neighborhoods U_i of $\mathrm{Cr}(\overline{f_i}) \cup \partial M_i$ and an isometry $\theta : (U_1, \overline{g_1}) \to (U_2, \overline{g_2})$ satisfying $\theta(\mathrm{Cr}(\overline{f_1})) = \mathrm{Cr}(\overline{f_2})$, $\theta(M_1) = M_2$ and $\overline{f_2} \circ \theta = \overline{f_1}$. By Lemma 5.7, one additionally finds a Hermitian form $\overline{h_i}$ on the bundle $E_i \downarrow M_i$ (with $h_i = \overline{h_i}$ near ∂M_i) so that $\overline{\mathcal{D}_i} := (E_i \downarrow M_i, \overline{g_i}, \overline{h_i}, \nabla_{\bar{g}_i} \overline{f_i})$ is an admissible system, satisfying

$$\mathcal{R}(\mathcal{D}_i) = \mathcal{R}(\overline{\mathcal{D}_i}). \quad (7.11)$$

Moreover, since the new systems $\overline{\mathcal{D}_i}$ now also satisfy the assertions of Corollary 6.7 (2), we can apply Case 1 to them to complete the proof. □

Proof of Theorem 4.5. Let $\mathcal{D} = (E \downarrow M, g, h, \nabla_{g'} f)$ be a Morse–Smale system of product form, M odd-dimensional, so that $E|_{\partial M} \downarrow \partial M$ is also of determinant class and that $h|_{\partial M}$ is unimodular. After appropriately pertubing the metric g, it is because of Proposition 5.4 that we may assume without loss of generality that $g \equiv g'$ outside from a neighborhood of ∂M. Similarly, if U is an open neighborhood of $\mathrm{Cr}(f)$ with contractible components and disjoint from a collar neighborhood V of ∂M, and if h' is a parallel Hermitian metric on U that agrees with h on $\mathrm{Cr}(f)$, it is again because of Proposition 5.4 that we may replace h with an arbitrary Hermitian extenstion of the metric $h' \cup h|_V$ over $U \cup V$ onto M without affecting $\mathcal{R}(\mathcal{D})$. Summarizing, we may assume without loss of generality that $\mathcal{D}$ is already an admissible system.

Choose a Morse–Smale pair $(\widehat{f}, \widehat{g})$ on ∂M. Then,

$$\mathcal{D}' := (E|_{\partial M} \downarrow \partial M, g|_{\partial M}, h|_{\partial M}, \nabla_{\hat{g}} \widehat{f})$$

is a Morse–Smale system of determinant class. Since ∂M is closed, we have by Theorem 4.4

$$\mathcal{R}(\mathcal{D}') = -\frac{1}{2} \int_{\partial M} \theta(h|_{\partial M}) \wedge (\nabla_{\hat{g}} \widehat{f})^* \Psi(T\partial M, g|_{\partial M}) = 0, \tag{7.12}$$

where the last equality follows from the assumption that $h|_{\partial M}$ is unimodular, i.e. $\theta(h|_{\partial M}) \equiv 0$.

Now recall the trivial system $\mathcal{D}_0 = (E_{\mathbb{C}} \downarrow I, g_0, h_0, \nabla_{g_0} f_0)$ over the interval $I = [a, b]$ that we have defined in (4.7) and its relative torsion

$$\mathcal{R}(\mathcal{D}_0) = -\frac{\log 2}{2}. \tag{7.13}$$

Since ∂M is closed and $\partial I = \{a, b\}$, we can form the modified product system

$$\underline{\mathcal{D}' \times \mathcal{D}_0} = (E_I \downarrow \partial M \times I, g_I, h_I, \nabla_{\hat{g}_I} \widehat{f}_I), \tag{7.14}$$

with $E_I := E_{\partial M} \widehat{\otimes} E_{\mathbb{C}}$, $g_I := g_{\partial M} \times g_0$, $\widehat{g}_I := \widehat{g} \times g_0$, $h_I := h|_{\partial M} \widehat{\otimes} h_0$ and $\widehat{f}_I$ the sum of the Morse functions $\widehat{f} + f_0$ that is appropriately modified near the boundary $\partial M \times \{a, b\}$, so that $\underline{\mathcal{D}' \times \mathcal{D}_0}$ is a type II Morse–Smale system. By Proposition 5.2, this system is of determinant class as well and satisfies

$$\mathcal{R}(\underline{\mathcal{D}' \times \mathcal{D}_0}) = \mathcal{R}(\mathcal{D}') - \frac{\log 2}{2} \chi(\partial M, E) \overset{(7.12)}{=} -\frac{\log 2}{2} \chi(\partial M) \dim(E). \tag{7.15}$$

Moreover, as $\theta(h_0) \equiv 0$ and $\theta(h|_{\partial M}) = 0$ by assumption, we retrieve from the product formula (5.5) the equality

$$\theta(h_I) = \theta(h|_{\partial M} \widehat{\otimes} h_0) = 0. \tag{7.16}$$

Notice that $\underline{\mathcal{D}' \times \mathcal{D}_0}$ is not necessarily an admissible system. This is due to the fact that neither is g_I trivial nor h_I parallel near $\mathrm{Cr}(\widehat{f}_I)$. However, since $\mathrm{Cr}(\widehat{f}_I)$ is disjoint from $\partial M \times \{a, b\}$, we can pertube the metrics outside of a small neighborhood of ∂M to produce metrics $\widetilde{g_I}$ and $\widetilde{h_I}$, so that $\widetilde{h_I}$ is parallel near $\mathrm{Cr}(f_I)$, and that we have $\widetilde{g_I} \equiv \widehat{g}_I$ outside of a neighborhood of ∂M and near $\mathrm{Cr}(\widehat{f}_I)$. By Lemma 4.3, the pertubation of the Hermitian form h_I can be performed in such way that still, we have

$$\theta(\widetilde{h_I}) \equiv 0, \tag{7.17}$$

$$\widetilde{h_I}(p) = h_I(p), \quad p \in \mathrm{Cr}(\widehat{f}_I). \tag{7.18}$$

For the resulting admissible system $\mathcal{D}_I := (E_I \downarrow \partial M \times I, \widetilde{g_I}, \widetilde{h_I}, \nabla_{\hat{g}_I} \widehat{f}_I)$, we obtain from Proposition 5.4 that

$$\mathcal{R}(\mathcal{D}_I) = \mathcal{R}(\underline{\mathcal{D}' \times \mathcal{D}_0}) = -\frac{\log 2}{2}\chi(\partial M)\dim(E). \tag{7.19}$$

Observe now that by construction, $\mathcal{D}_I$ and the disjoint union $\mathcal{D} \sqcup \mathcal{D} := (E \downarrow M \sqcup E \downarrow M, g \sqcup g, h \sqcup h, \nabla_{g'} f \sqcup \nabla_{g'} f)$ of $\mathcal{D}$ with itself are two admissible systems satisfying the hypotheses of Theorem 7.2. This allows us to finally conclude as follows:

$$\begin{aligned} 2\mathcal{R}(\mathcal{D}) &= \mathcal{R}(\mathcal{D} \sqcup \mathcal{D}) \\ &= \mathcal{R}(\mathcal{D}_I) - \int_M \theta(h) \wedge (\nabla_g f)^* \Psi(TM, g) \\ &\qquad + \frac{1}{2} \int_{\partial M \times I} \theta(\widetilde{h_I}) \wedge (\nabla_{\hat{g}_I} \widehat{f}_I)^* \Psi(T(\partial M \times I), \widetilde{g_I}) \\ &\overset{(7.17)}{=} \mathcal{R}(\mathcal{D}_I) - \int_M \theta(h) \wedge (\nabla_g f)^* \Psi(TM, g) \\ &\overset{(7.19)}{=} -\frac{\log 2}{2}\chi(\partial M)\dim(E) - \int_M \theta(h) \wedge (\nabla_g f)^* \Psi(TM, g). \end{aligned} \tag{7.20}$$

This finishes the proof of Theorem 4.5. □

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