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Touchdown is the Only Finite Time Singularity in a Three-Dimensional MEMS Model

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Abstract

Touchdown is shown to be the only possible finite time singularity that may take place in a free boundary problem modeling a three-dimensional microelectromechanical system. The proof relies on the energy structure of the problem and uses smoothing effects of the semigroup generated in L_1 by the bi-Laplacian with clamped boundary conditions.

La désactivation est la seule singularité en temps fini possible dans un modèle de MEMS tridimensionnel

Résumé

Nous montrons que la désactivation est la seule singularité en temps fini pouvant se produire dans un problème à frontière libre décrivant un microsystème électromécanique tridimensionnel. La démonstration repose sur la structure variationnelle du modèle et utilise les propriétés régularisantes du semi-groupe engendré dans L_1 par le bi-Laplacien avec conditions aux bords encastrées.

1. Introduction

We consider a model for a three-dimensional microelectromechanical system (MEMS) including two components, a rigid ground plate of shape $D \subset \mathbb{R}^2$ and an elastic plate of the same shape (at rest) which is suspended above the rigid one and clamped on its boundary, see Figure 1. Both plates being conducting, holding them at different voltages generates a Coulomb force across the device. This, in turn, induces a deformation of the elastic plate, thereby modifying the geometry of the device and transforming electrostatic energy into mechanical energy. When applying a sufficiently large voltage difference, a well-known phenomenon that might occur is that the two plates come into contact; that is, the elastic plate touches down on the rigid plate. For this feature (usually referred to as *pull-in instability* or *touchdown* [6, 19]) some mathematical models have been developed recently [3, 6, 15, 18, 19]. Since the pioneering works [3, 7, 10, 18], their mathematical analysis has been the subject of numerous papers. We refer to [5, 14] for a more complete account and an extensive list of references.

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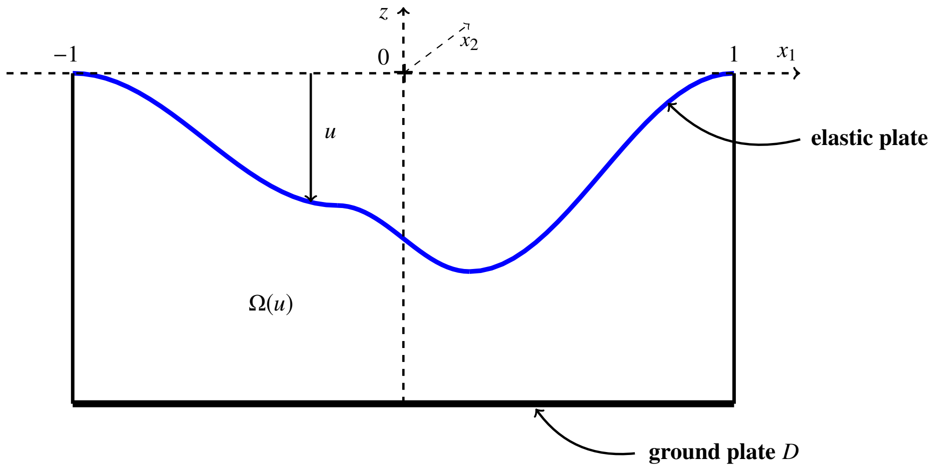

FIGURE 1.1. Cross section of an idealized MEMS device

We focus here on a model describing the evolution of the vertical deformation of the elastic plate from rest and the electrostatic potential between the plates. More precisely, we assume that D is a bounded and convex domain in $\mathbb{R}^2$ with a C^∞-smooth boundary. Then, after an appropriate rescaling and neglecting inertial forces, the ground plate is located at $z = -1$ while the elastic plate's rest position is at $z = 0$, and the evolution of the vertical deformation $u = u(t, x)$ of the elastic plate at time $t > 0$ and position $x \in D$ is given by

$$\partial_t u + \beta\Delta^2 u - \big(\tau + a\|\nabla u\|^2_{L_2(D)}\big)\Delta u = -\lambda\, g(u)\,, \quad x \in D\,, \quad t > 0\,, \tag{1.1a}$$

where

$$g(u(t))(x) := \varepsilon^2|\nabla\psi_{u(t)}(x, u(t,x))|^2 + |\partial_z\psi_{u(t)}(x, u(t,x))|^2\,, \quad x \in D\,, \quad t > 0\,. \tag{1.1b}$$

Throughout the paper, ∇ and Δ denote the gradient and the Laplace operator with respect to $x \in D$, respectively. We supplement (1.1a) with clamped boundary conditions

$$u = \partial_N u = 0\,, \quad x \in \partial D\,, \quad t > 0\,, \tag{1.1c}$$

and initial condition

$$u(0, x) = u^0(x)\,, \quad x \in D\,. \tag{1.1d}$$

As for the electrostatic potential $\psi_{u(t)}(x, z)$, it is defined for $t > 0$ and $(x, z) \in \Omega(u(t))$, where $\Omega(u(t))$ is the three-dimensional cylinder

$$\Omega(u(t)) := \{(x, z) \in D \times (-1, \infty)\ :\ -1 < z < u(t, x)\}$$

enclosed within the rigid ground plate at $z = -1$ and the deflected elastic plate at $z = u(t)$. For each time $t > 0$, the electrostatic potential $\psi_{u(t)}$ solves the rescaled Laplace equation

$$\varepsilon^2 \Delta\psi_{u(t)} + \partial_z^2 \psi_{u(t)} = 0\,, \quad (x,z) \in \Omega(u(t))\,, \quad t > 0\,, \tag{1.2a}$$

supplemented with non-homogeneous Dirichlet boundary conditions

$$\psi_{u(t)}(x,z) = \frac{1+z}{1+u(t,x)}\,, \quad (x,z) \in \partial\Omega(u(t)), \quad t > 0\,. \tag{1.2b}$$

In (1.1)–(1.2), the aspect ratio $\varepsilon > 0$ is the ratio between vertical and horizontal dimensions of the device while $\lambda > 0$ is proportional to the square of the applied voltage difference. The parameters $\beta > 0$, $\tau \geq 0$, and $a \geq 0$ result from the modeling of the mechanical forces and are related to bending and stretching of the elastic plate, respectively. We emphasize that (1.1)–(1.2) is a nonlinear and nonlocal system of partial differential equations featuring a time-varying boundary, which makes its analysis rather involved. Still, its local in time well-posedness can be shown in a suitable functional setting, as we recall below, and the aim of this note is to improve the criterion for global existence derived in [12].

2. Main Result

Expanding upon the above discussion on global existence we recall the following result established in [12, Theorem 1.1].

Theorem 2.1. *Let* $4\xi \in (7/3, 4)$*, and consider an initial value* $u^0 \in W_2^{4\xi}(D)$ *such that* $u^0 > -1$ *in* D *and* $u^0 = \partial_N u^0 = 0$ *on* ∂D.

(i) *There is a unique solution* u *to* (1.1) *on the maximal interval of existence* $[0, T_m)$ *in the sense that*

$$u \in C\big([0,T_m), W_2^{4\xi}(D)\big) \cap C\big((0,T_m), W_2^4(D)\big) \cap C^1\big((0,T_m), L_2(D)\big) \tag{2.1}$$

satisfies (1.1) *together with*

$$u(t,x) > -1\,, \quad (t,x) \in [0,T_m) \times D\,,$$

and $\psi_{u(t)} \in W_2^2\big(\Omega(u(t))\big)$ *solves* (1.2) *in* $\Omega(u(t))$ *for each* $t \in [0,T_m)$.

(ii) *If* $T_m < \infty$*, then*

$$\lim_{t\to T_m} \|u(t)\|_{W_2^{4\xi}(D)} = \infty \qquad or \qquad \lim_{t\to T_m} \min_{x\in\bar{D}} u(t,x) = -1\,. \tag{2.2}$$

It is worth pointing out that, since $\Omega(u(t))$ is only a Lipschitz domain, the W_2^2-regularity of $\psi_{u(t)}$ does not seem to follow from standard elliptic theory. Actually, this property is one of the cornerstones in the proof of Theorem 2.1 and guarantees that the function g in (1.1b) is well-defined (see Proposition 3.1 below).

Further results regarding (1.1)–(1.2) are to be found in [12]. In particular, global existence holds true under additional smallness assumptions on both λ and u^0. Moreover, stationary solutions exist for small values of λ and, when D is a ball in $\mathbb{R}^2$, no stationary solution exists for λ large enough. This last property is actually connected with the touchdown phenomenon already alluded to in the introduction. In the same vein, whether a finite time singularity may occur for the evolution problem for suitable choices of λ and u^0 is yet an open problem, though such a feature is expected on physical grounds.

Coming back to the global existence issue, the criterion (2.2) stated in Theorem 2.1 entails that non-global solutions blow up in finite time in the Sobolev space $W_2^{4\xi}(D)$ or a finite time touchdown of the elastic plate on the ground plate occurs, the occurrence of both simultaneously being not excluded a priori. From a physical point of view, however, only the latter seems possible. For the investigation of the dynamics of MEMS devices it is thus of great importance to rule out mathematically the norm blowup in finite time. In [11] this was done if $D = (-1, 1)$ is one-dimensional, that is, in case the elastic part is a beam or a rectangular plate that is homogeneous in one direction. The situation considered herein, where D is an arbitrary two-dimensional (convex) domain, is more delicate. Indeed, the right-hand side of (1.1) (being given by the square of the gradient trace of the electrostatic potential) has much less regularity properties due to the fact that the moving boundary problem (1.2) for the electrostatic potential is posed in a three-dimensional domain $\Omega(u)$. We shall see, however, that we can overcome this difficulty using the gradient flow structure of the evolution problem along with the regularizing effects of the fourth-order operator in (negative) Besov spaces. More precisely, we shall show the following result.

Theorem 2.2. *Under the assumptions of Theorem 2.1 let u be the unique maximal solution to* (1.1) *on the maximal interval of existence* $[0, T_m)$. *Assume that there are* $T_0 > 0$ *and* $\kappa_0 \in (0, 1)$ *such that*

$$u(t) \geq -1 + \kappa_0 \quad \text{in} \quad D\,, \qquad t \in [0, T_m) \cap [0, T_0]\,. \tag{2.3}$$

Then $T_m \geq T_0$.

Moreover, if, for each $T > 0$, *there is* $\kappa(T) \in (0, 1)$ *such that*

$$u(t) \geq -1 + \kappa(T) \quad \text{in} \quad D\,, \qquad t \in [0, T_m) \cap [0, T]\,,$$

then $T_m = \infty$.

The second statement in Theorem 2.2 obviously follows from the first one applied to an arbitrary $T_0 > 0$. The proof of Theorem 2.2 is given in the next section. As mentioned above and similarly to the case $D = (-1,1)$ considered in [11], it relies on the gradient flow structure of (1.1)–(1.2), where the corresponding energy is given by

$$\mathcal{E}(u) := \mathcal{E}_m(u) - \lambda \mathcal{E}_e(u)$$

with mechanical energy

$$\mathcal{E}_m(u) := \frac{\beta}{2}\|\Delta u\|_{L_2(D)}^2 + \frac{\tau}{2}\|\nabla u\|_{L_2(D)}^2 + \frac{a}{4}\|\nabla u\|_{L_2(D)}^4$$

and electrostatic energy

$$\mathcal{E}_e(u) := \int_{\Omega(u)} \left(\varepsilon^2 |\nabla \psi_u(x,z)|^2 + |\partial_z \psi_u(x,z)|^2\right) \, \mathrm{d}(x,z)\,.$$

We shall see that assuming the lower bound (2.3) on the solution u provides a control on the electrostatic energy. Using the gradient flow structure we thus derive a bound on the (a priori unbounded) mechanical energy and, in turn, on the $W_2^2(D)$-norm of $u(t)$ for $t \in [0,T_m) \cap [0,T_0]$. This yields an $L_1(D)$-bound on the right-hand side of (1.1). We then apply semigroup techniques in negative Besov spaces to obtain a bound on $u(t)$ in the desired Sobolev norm of $W_2^{4\xi}(D)$ for $t \in [0,T_m) \cap [0,T_0]$ which only depends on T_0 and κ_0.

Remark 2.3. It is worth pointing out that the issue whether a norm blowup or touchdown occurs in finite time is still an open problem for the second-order case $\beta = 0$ (and $\tau > 0$), even in the one-dimensional setting $D = (-1,1)$.

3. Proof of Theorem 2.2

Suppose the assumptions of Theorem 2.1 and let u denote the unique maximal solution to (1.1) on the maximal interval of existence $[0,T_m)$. We want to show that, if (2.3) is satisfied, then

$$\|u(t)\|_{W_2^{4\xi}(D)} \le c(T_0,\kappa_0)\,, \quad t \in [0,T_m) \cap [0,T_0]\,,$$

so that Theorem 2.1(ii) in turn implies Theorem 2.2. To this end we first need to derive suitable estimates on the right-hand side $g(u)$ of (1.1) given by the square of the gradient trace of the electrostatic potential ψ_u.

3.1. Estimates on the electrostatic potential

In the following we let $\kappa \in (0,1)$ and set

$$\mathcal{S}(\kappa) := \{v \in W_3^2(D) \,:\, v = 0 \text{ on } \partial D \text{ and } v \ge -1+\kappa \text{ in } D\}\,.$$

We begin with the regularity of the variational solution to (1.2), see [12].

Proposition 3.1. *Given $v \in \mathcal{S}(\kappa)$, there is a unique solution $\psi_v \in W_2^2(\Omega(v))$ to*

$$\varepsilon^2 \Delta\psi + \partial_z^2 \psi = 0\,, \quad (x,z) \in \Omega(v)\,, \tag{3.1a}$$

$$\psi(x,z) = \frac{1+z}{1+v(x)}\,, \quad (x,z) \in \partial\Omega(v)\,, \tag{3.1b}$$

in the cylinder

$$\Omega(v) := \{(x,z) \in D \times (-1,\infty)\ :\ -1 < z < v(x)\}\ .$$

Furthermore, $g(v) \in L_2(D)$.

We recall that the $L_2(D)$-integrability of $g(v)$ is a straightforward consequence of $\psi_v \in W_2^2(\Omega(v))$. Indeed the latter implies that $\big(x \mapsto \nabla\psi_v(x, v(x))\big) \in W_2^{1/2}(D) \hookrightarrow L_4(D)$.

We next provide pointwise estimates on ψ_v.

Lemma 3.2. *Let $v \in \mathcal{S}(\kappa)$. Then, for $(x,z) \in \Omega(v)$,*

$$0 \le \psi_v(x,z) \le \min\left\{1, \frac{1+z}{\kappa}\right\}\ .$$

Proof. Clearly, $(x,z) \mapsto m$ is a solution to (3.1a) for $m = 0, 1$ and $0 \le \psi_v \le 1$ on $\partial\Omega(v)$ since $v = 0$ on ∂D, hence $0 \le \psi_v \le 1$ in $\Omega(v)$ by the comparison principle. Moreover, setting $\Sigma(x,z) := (1+z)/\kappa$ for $(x,z) \in \Omega(v)$, it readily follows that Σ is a supersolution to (3.1) so that $\psi_v \le \Sigma$ in $\Omega(v)$ again by the comparison principle. □

Lemma 3.2 provides uniform estimates on the derivatives of ψ_v on the v-independent part of the boundary of $\Omega(v)$.

Corollary 3.3. *Let $v \in \mathcal{S}(\kappa)$. If $x \in \partial D$ and $z \in (-1,0)$, then $\partial_z\psi_v(x,z) = 1$, while if $x \in D$, then*

$$0 \le \partial_z\psi_v(x,-1) \le \frac{1}{\kappa}\,, \qquad \partial_z\psi_v(x, v(x)) \ge 0\,,$$

and

$$\nabla\psi_v(x,-1) = 0\,, \qquad \nabla\psi_v(x,v(x)) = -\partial_z\psi_v(x,v(x))\nabla v(x)\,.$$

Proof. The first assertion follows from $\psi_v(x,z) = 1+z$, $(x,z) \in \partial D \times (-1,0)$. Next, from (3.1b) and Lemma 3.2 we derive, for $(x,z) \in D \times (-1,0)$,

$$0 \le \frac{\psi_v(x,z) - \psi_v(x,-1)}{1+z} \le \frac{1}{\kappa}\,, \qquad \psi_v(x,v(x)) - \psi_v(x,z) \ge 0\,,$$

hence

$$0 \le \partial_z\psi_v(x,-1) \le \frac{1}{\kappa}\,, \qquad \partial_z\psi_v(x,v(x)) \ge 0\,.$$

The formulas for $\nabla\psi_v$ follow immediately from $\psi_v(x,-1) = 0$ and $\psi_v(x,v(x)) = 1$ for $x \in D$ due to (3.1b). □

Given $v \in \mathcal{S}(\kappa)$ we next introduce the notation

$$\gamma(x) := \partial_z \psi_v(x, v(x)), \qquad \gamma_b(x) := \partial_z \psi_v(x, -1) \tag{3.2}$$

for $x \in D$ and recall the following identity, which is proven in [4, Lemma 5] in the one-dimensional case $D = (-1, 1)$,

Lemma 3.4. *Let $v \in \mathcal{S}(\kappa)$. Then, with the notation* (3.2)*,*

$$\int_D \left(1 + \varepsilon^2 |\nabla v|^2\right) \left(\gamma^2 - 2\gamma\right) \,\mathrm{d}x = \int_D \left(\gamma_b^2 - 2\gamma_b\right) \,\mathrm{d}x \,.$$

Proof. We recall the proof for the sake of completeness and point out that it is somewhat related to the Rellich equality [17, Equation (5.2)]. We multiply the rescaled Laplace equation (3.1a) by $\partial_z \psi_v - 1$ and integrate over $\Omega(v)$. Denoting the outward unit normal vector field to ∂D and the surface measure on ∂D by N and σ, respectively, we deduce from Green's formula that

$$\begin{aligned}
0 &= \int_{\Omega(v)} \left(\varepsilon^2 \Delta \psi_v + \partial_z^2 \psi_v\right) (\partial_z \psi_v - 1) \,\mathrm{d}(x, z) \\
&= \varepsilon^2 \int_{\partial D} \int_{-1}^{0} (\partial_z \psi_v - 1) \nabla \psi_v \cdot N \,\mathrm{d}z \,\mathrm{d}\sigma \\
&\quad - \varepsilon^2 \int_D (\partial_z \psi_v(x, v(x)) - 1) \nabla \psi_v(x, v(x)) \cdot \nabla v(x) \,\mathrm{d}x \\
&\quad - \varepsilon^2 \int_{\Omega(v)} \nabla \psi_v \cdot \partial_z \nabla \psi_v \,\mathrm{d}(x, z) + \int_D \left(\frac{(\partial_z \psi_v(x, v(x)))^2}{2} - \partial_z \psi_v(x, v(x)) \right) \mathrm{d}x \\
&\quad - \int_D \left(\frac{(\partial_z \psi_v(x, -1))^2}{2} - \partial_z \psi_v(x, -1) \right) \mathrm{d}x \,.
\end{aligned}$$

Due to Corollary 3.3 the first integral on the right-hand side vanishes while the others can be simplified to get

$$\begin{aligned}
0 &= -\varepsilon^2 \int_D (\gamma(x) - 1) \nabla \psi(x, v(x)) \cdot \nabla v(x) \,\mathrm{d}x - \frac{\varepsilon^2}{2} \int_D |\nabla \psi_v(x, v(x))|^2 \,\mathrm{d}x \\
&\quad + \frac{\varepsilon^2}{2} \int_D |\nabla \psi_v(x, -1)|^2 \,\mathrm{d}x + \int_D \left(\frac{\gamma^2}{2} - \gamma \right) \mathrm{d}x - \int_D \left(\frac{\gamma_b^2}{2} - \gamma_b \right) \mathrm{d}x \\
&= \varepsilon^2 \int_D (\gamma - 1) \gamma |\nabla v|^2 \,\mathrm{d}x - \frac{\varepsilon^2}{2} \int_D \gamma^2 |\nabla v|^2 \,\mathrm{d}x \\
&\quad + \int_D \left(\frac{\gamma^2}{2} - \gamma \right) \mathrm{d}x - \int_D \left(\frac{\gamma_b^2}{2} - \gamma_b \right) \mathrm{d}x
\end{aligned}$$

$$= \int_D \left(\frac{\gamma^2}{2} - \gamma\right)\left(1 + \varepsilon^2|\nabla v|^2\right)\,\mathrm{d}x - \int_D \left(\frac{\gamma_b^2}{2} - \gamma_b\right)\,\mathrm{d}x\,,$$

which yields the assertion. □

Given $v \in \mathcal{S}(\kappa)$ we recall that

$$g(v)(x) := \varepsilon^2|\nabla\psi_v(x, v(x))|^2 + |\partial_z\psi_v(x, v(x))|^2\,, \quad x \in D\,,$$

with ψ_v still denoting the solution to (3.1). The next result bounds the $L_1(D)$-norm of $g(v)$ in terms of the $H^1(D)$-norm of v.

Corollary 3.5. *For $v \in \mathcal{S}(\kappa)$,*

$$\|g(v)\|_{L_1(D)} \le \left(4 + \frac{2}{\kappa^2}\right)|D| + 4\varepsilon^2\|\nabla v\|^2_{L_2(D)}\,.$$

Proof. Since

$$g(v)(x) = \varepsilon^2|\nabla\psi_v(x, v(x))|^2 + |\partial_z\psi_v(x, v(x))|^2 = \left(1 + \varepsilon^2|\nabla v(x)|^2\right)\gamma(x)^2$$

for $x \in D$ by Corollary 3.3, we deduce from Corollary 3.3 and Lemma 3.4 that

$$\begin{aligned}\|g(v)\|_{L_1(D)} &= 2\int_D \left(1 + \varepsilon^2|\nabla v|^2\right)\gamma\,\mathrm{d}x + \int_D (\gamma_b^2 - 2\gamma_b)\,\mathrm{d}x\\ &\le \frac{1}{2}\int_D \left(1 + \varepsilon^2|\nabla v|^2\right)\gamma^2\,\mathrm{d}x + 2\int_D \left(1 + \varepsilon^2|\nabla v|^2\right)\mathrm{d}x + \frac{|D|}{\kappa^2}\\ &\le \frac{1}{2}\|g(v)\|_{L_1(D)} + 2\varepsilon^2\|\nabla v\|^2_{L_2(D)} + \left(2 + \frac{1}{\kappa^2}\right)|D|\,,\end{aligned}$$

from which the assertion follows. □

We next recall the following identity for the electrostatic energy established in [13, Equation (3.13)] in the one-dimensional case $D = (-1, 1)$. We extend it here to the two-dimensional setting, also providing a simpler proof below.

Lemma 3.6. *For $v \in \mathcal{S}(\kappa)$,*

$$\mathcal{E}_e(v) = |D| - \int_D v\left(1 + \varepsilon^2|\nabla v|^2\right)\gamma\,\mathrm{d}x\,.$$

Proof. We multiply the rescaled Laplace equation (3.1a) by $\psi_v(x,z) - 1 - z$ and integrate over $\Omega(v)$. As in the proof of Lemma 3.4 we use Green's formula to obtain

$$
\begin{aligned}
0 &= \int_{\Omega(v)} \left(\varepsilon^2 \Delta\psi_v + \partial_z^2\psi_v\right)(x,z)\,(\psi_v(x,z) - 1 - z)\ \mathrm{d}(x,z) \\
&= \varepsilon^2 \int_{\partial D}\int_{-1}^{0} (\psi_v(x,z) - 1 - z)\,\nabla\psi_v \cdot N\,\mathrm{d}z\,\mathrm{d}\sigma \\
&\quad - \varepsilon^2 \int_D (\psi_v(x,v(x)) - 1 - v(x))\,\nabla\psi_v(x,v(x)) \cdot \nabla v(x)\,\mathrm{d}x \\
&\quad - \varepsilon^2 \int_{\Omega(v)} |\nabla\psi_v|^2\,\mathrm{d}(x,z) + \int_D (\psi_v(x,v(x)) - 1 - v(x))\,\partial_z\psi_v(x,v(x))\,\mathrm{d}x \\
&\quad - \int_D \psi_v(x,-1)\partial_z\psi_v(x,-1)\,\mathrm{d}x - \int_{\Omega(v)} (\partial_z\psi_v - 1)\,\partial_z\psi_v\,\mathrm{d}(x,z)\,.
\end{aligned}
$$

Employing (1.2b) we see that the first and the fifth term on the right-hand side vanish while the others can be gathered due to Corollary 3.3 as

$$
\begin{aligned}
0 = &-\varepsilon^2 \int_D v|\nabla v|^2\gamma\,\mathrm{d}x - \varepsilon^2 \int_{\Omega(v)} |\nabla\psi_v|^2\,\mathrm{d}(x,z) - \int_D v\gamma\,\mathrm{d}x \\
&- \int_{\Omega(v)} (\partial_z\psi_v)^2\ \mathrm{d}(x,z) + \int_D (\psi_v(x,v(x)) - \psi_v(x,-1))\ \mathrm{d}x\,.
\end{aligned}
$$

The last integral being equal to $|D|$ according to (1.2b), we obtain

$$
\mathcal{E}_e(v) = |D| - \int_D v\left(1 + \varepsilon^2|\nabla v|^2\right)\gamma\,\mathrm{d}x\,,
$$

hence the assertion. □

We are now in a position to derive a lower bound on the total energy.

Corollary 3.7. *For* $v \in \mathcal{S}(\kappa)$*,*

$$
\mathcal{E}(v) \geq \mathcal{E}_m(v) - 3\lambda\varepsilon^2\|\nabla v\|^2_{L_2(D)} - \lambda|D|\left(4 + \frac{1}{2\kappa^2}\right)\,.
$$

Proof. Since $v \geq -1$ in D and, by Corollary 3.3, $\gamma \geq 0$ in D, we infer from Lemma 3.6 that

$$
\begin{aligned}
\mathcal{E}(v) = \mathcal{E}_m(v) - \lambda\mathcal{E}_e(v) &= \mathcal{E}_m(v) - \lambda|D| + \lambda\int_D v\left(1 + \varepsilon^2|\nabla v|^2\right)\gamma\,\mathrm{d}x \\
&\geq \mathcal{E}_m(v) - \lambda|D| - \lambda\int_D \left(1 + \varepsilon^2|\nabla v|^2\right)\gamma\,\mathrm{d}x\,,
\end{aligned}
$$

so that the Cauchy–Schwarz inequality and Corollary 3.5 imply that

$$\begin{aligned}\mathcal{E}(v) &\geq \mathcal{E}_m(v) - \lambda|D| - \lambda\left(\int_D \left(1+\varepsilon^2|\nabla v|^2\right)\,\mathrm{d}x\right)^{1/2} \|g(v)\|_{L_1(D)}^{1/2}\\ &\geq \mathcal{E}_m(v) - \lambda|D|\\ &\quad - \lambda\left(\int_D \left(1+\varepsilon^2|\nabla v|^2\right)\,\mathrm{d}x\right)^{1/2}\left(\frac{2|D|}{\kappa^2} + 4\int_D \left(1+\varepsilon^2|\nabla v|^2\right)\,\mathrm{d}x\right)^{1/2}\\ &\geq \mathcal{E}_m(v) - \lambda|D| - \frac{\sqrt{2|D|}\lambda}{\kappa}\left(\int_D \left(1+\varepsilon^2|\nabla v|^2\right)\,\mathrm{d}x\right)^{1/2}\\ &\quad - 2\lambda\int_D \left(1+\varepsilon^2|\nabla v|^2\right)\,\mathrm{d}x\,.\end{aligned}$$

The assertion follows then from Young's inequality. □

3.2. Estimates on the plate deflection

Under the assumptions of Theorem 2.1 let now u be the unique maximal solution to (1.1) on the maximal interval of existence $[0,T_m)$. We may assume that $7/3 < 4\xi < 3$. Let $\kappa_0 \in (0,1)$ and $T_0 > 0$ be such that (2.3) holds true; that is,

$$u(t,x) \geq -1 + \kappa_0\,, \quad t \in [0,T_m)\cap[0,T_0]\,,\ x \in D\,. \tag{3.3}$$

Throughout this section, c denotes a positive constant which may vary from line to line and depends only on β, τ, a, λ, D, ε, u^0, κ_0, and T_0 (in particular, it does not depend on T_m).

To prove Theorem 2.2 we shall show that

$$\|u(t)\|_{W_2^{4\xi}(D)} \leq c\,, \quad t \in [0,T_m)\cap[0,T_0]\,, \tag{3.4}$$

the assertion then follows from Theorem 2.1 (ii). Note that (3.3) just means that $u(t) \in \mathcal{S}(\kappa_0)$ for $t \in [0,T_m)\cap[0,T_0]$ so that the results of the preceding section apply (with $\kappa = \kappa_0$).

We first provide an $L_2(D)$-bound on u. While the previous computations did not make use of the positivity of the parameter β (i.e. the fourth-order character of (1.1a)), the latter is instrumental in the proof of the next result.

Lemma 3.8. *There is $c > 0$ such that*

$$\|u(t)\|_{L_2(D)} \leq c\,, \quad t \in [0,T_m)\cap[0,T_0]\,.$$

Proof. Let $t \in [0,T_m)$. It readily follows from (1.1) and the lower bounds $u(t) \geq -1$ and $g(u(t)) \geq 0$ in D that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L_2(D)}^2 + 2\mathcal{E}_m(u(t)) = -\lambda\int_D u(t)g(u(t))\,\mathrm{d}x \leq \lambda\|g(u(t))\|_{L_1(D)}\,.$$

Now, Corollary 3.5 along with interpolation and Young's inequality implies, for $t \in [0,T_m) \cap [0,T_0]$,

$$\begin{aligned}
\|g(u(t))\|_{L_1(D)} &\le c\left(1 + \|\nabla u(t)\|_{L_2(D)}^2\right) \\
&\le c\left(1 + \|u(t)\|_{L_2(D)}\,\|\Delta u(t)\|_{L_2(D)}\right) \\
&\le \frac{1}{\lambda}\mathcal{E}_m(u(t)) + c\left(1 + \|u(t)\|_{L_2(D)}^2\right).
\end{aligned}$$

Combining the two inequalities yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L_2(D)}^2 + \mathcal{E}_m(u(t)) \le c\left(1 + \|u(t)\|_{L_2(D)}^2\right), \quad t \in [0,T_m) \cap [0,T_0],$$

from which the assertion follows. □

We next show that the lower bound (3.3) on u implies that the mechanical energy is dominated by the total energy.

Lemma 3.9. *There is $c > 0$ such that*

$$\mathcal{E}(u(t)) \ge \frac{1}{2}\mathcal{E}_m(u(t)) - c, \quad t \in [0,T_m) \cap [0,T_0].$$

Proof. We infer from Corollary 3.7 along with interpolation and Young's inequality that, for some constant $c > 0$,

$$\begin{aligned}
\mathcal{E}(u(t)) &\ge \mathcal{E}_m(u(t)) - c\|u(t)\|_{L_2(D)}\mathcal{E}_m(u(t))^{1/2} - c \\
&\ge \frac{1}{2}\mathcal{E}_m(u(t)) - c\left(1 + \|u(t)\|_{L_2(D)}^2\right)
\end{aligned}$$

for $t \in [0,T_m) \cap [0,T_0]$. Lemma 3.8 yields the claim. □

We next exploit the gradient flow structure of the evolution problem to obtain additional estimates.

Corollary 3.10. *There is $c > 0$ such that*

$$\|u(t)\|_{H^2(D)} + \int_0^t \|\partial_t u(s)\|_{L_2(D)}^2\,\mathrm{d}s \le c, \quad t \in [0,T_m) \cap [0,T_0].$$

Proof. Analogously to [11, Proposition 1.3] (see also [13]) the energy inequality

$$\mathcal{E}(u(t)) + \int_0^t \|\partial_t u(s)\|_{L_2(D)}^2\,\mathrm{d}s \le \mathcal{E}(u^0), \quad t \in [0,T_m),$$

holds; that is, due to Lemma 3.9,

$$\mathcal{E}(u^0) \ge \frac{1}{2}\mathcal{E}_m(u(t)) - c + \int_0^t \|\partial_t u(s)\|_{L_2(D)}^2\,\mathrm{d}s, \quad t \in [0,T_m) \cap [0,T_0].$$

The claim follows then from the fact that $\mathcal{E}(u^0) < \infty$ and the definition of $\mathcal{E}_m$. □

Combining now Corollary 3.5 and Corollary 3.10 we readily obtain an $L_1(D)$-bound on the right-hand side of (1.1).

Corollary 3.11. *There is* $c > 0$ *such that*

$$\|g(u(t))\|_{L_1(D)} \le c\,, \quad t \in [0, T_m) \cap [0, T_0]\,.$$

3.3. Proof of Theorem 2.2

It remains to prove that the $L_1(D)$-bound from Corollary 3.11 implies a bound on u in the Sobolev space $W_2^{4\xi}(D)$, that is, inequality (3.4).

For this purpose we introduce $B^s_{1,1,\mathcal{B}}(D)$ for $s \in \mathbb{R} \setminus \{1, 2\}$, i.e. the Besov space $B^s_{1,1}(D)$ incorporating the boundary conditions appearing in (1.1c) (if meaningful):

$$B^s_{1,1,\mathcal{B}}(D) := \begin{cases} \{w \in B^s_{1,1}(D) \ : \ w = \partial_N w = 0 \text{ on } \partial D\}\,, & s > 2\,, \\ \{w \in B^s_{1,1}(D) \ : \ w = 0 \text{ on } \partial D\}\,, & s \in (1, 2)\,, \\ B^s_{1,1}(D)\,, & s < 1\,. \end{cases}$$

The spaces $W^s_{2,\mathcal{B}}(D)$ are defined analogously with $B^s_{1,1}$ replaced by W^s_2, but for $s > 3/2$, $s \in (1/2, 3/2)$, and $s < 1/2$, respectively.

From now on, we fix $\alpha \in (4\xi - 3, 0)$. Hereafter, the constant c may also depend on ξ and α (but still not on T_m). The dependence upon additional parameters is indicated explicitly.

Lemma 3.12. *The operator A, given by*

$$Av := (-\beta\Delta^2 + \tau\Delta)v\,, \quad v \in B^{4+\alpha}_{1,1,\mathcal{B}}(D)\,,$$

generates an analytic semigroup $\{e^{tA} \ : \ t \ge 0\}$ *on* $B^\alpha_{1,1}(D)$ *and, when restricted to* $W^4_{2,\mathcal{B}}(D)$*, on* $L_2(D)$*. Given* $\theta \in (0, 1)$ *with* $\theta \notin \{(1-\alpha)/4, (2-\alpha)/4\}$*, there are* $c > 0$ *and* $c(\theta) > 0$ *such that, for* $t \in [0, T_0]$*,*

$$\|e^{tA}\|_{\mathcal{L}(W^{4\xi}_{2,\mathcal{B}}(D))} \le c \qquad \text{and} \qquad t^\theta \|e^{tA}\|_{\mathcal{L}(B^\alpha_{1,1}(D), B^{4\theta+\alpha}_{1,1,\mathcal{B}}(D))} \le c(\theta)\,. \tag{3.5}$$

Proof. We shall apply [8, Theorem 2.18] (recalled in Theorem A.1 below) with $\mathcal{A} := -\beta\Delta^2 + \tau\Delta$ (that is, $m = 2$), $\mathcal{B}_1 := \mathrm{tr}$ (i.e. the trace operator on ∂D), $\mathcal{B}_2 := N \cdot \nabla$, and $p = q = 1$. Clearly, the symbol $\mathcal{A}_0(i\zeta) = -\beta|\zeta|^4$ of the principal part $-\beta\Delta^2$ of the operator $\mathcal{A}$ satisfies condition (m), while the boundary operators $\mathcal{B} := (\mathcal{B}_1, \mathcal{B}_2) = (\mathrm{tr}, \partial_N)$ satisfy condition (n) from Theorem A.1. We next check the Lopatinskii–Shapiro condition (o) from Theorem A.1. Given $x \in \partial D$, $\zeta \in \mathbb{R}^2$, $r \ge 0$, and $\vartheta \in [-\pi/2, \pi/2]$ with $\zeta \cdot N(x) = 0$ and $(\zeta, r) \ne (0, 0)$, this condition requires that zero is the only bounded solution on $[0, \infty)$ to

$$\Big(\mathcal{A}_0\big(i\zeta + N(x)\partial_t\big) - re^{i\vartheta}\Big)\, v = 0\,, \qquad \mathcal{B}\big(i\zeta + N(x)\partial_t\big)v(0) = 0\,. \tag{3.6}$$

Now note that the properties $\zeta \cdot N(x) = 0$ and $|N(x)| = 1$ entail that

$$\begin{aligned}\mathcal{A}_0\big(i\zeta + N(x)\partial_t\big)v &= -\beta\left(|N(x)|^2\partial_t^2 + 2i\zeta\cdot N(x)\partial_t + i^2|\zeta|^2\right)^2 v\\ &= -\beta\left(\partial_t^2 - |\zeta|^2\right)^2 v = -\beta\left(\partial_t^4 - 2|\zeta|^2\partial_t^2 + |\zeta|^4\right)v\end{aligned}$$

and

$$\mathcal{B}\big(i\zeta + N(x)\partial_t\big)v(0) = \big(v(0), (i\zeta\cdot N(x) + |N(x)|^2\partial_t)v(0)\big) = \big(v(0), \partial_t v(0)\big),$$

which leads to the explicit formulation of the initial value problem (3.6):

$$\partial_t^4 v(t) - 2|\zeta|^2\partial_t^2 v(t) + \left(|\zeta|^4 + re^{i\vartheta}\right)v(t) = 0, \quad t > 0, \tag{3.7a}$$

$$v(0) = \partial_t v(0) = 0. \tag{3.7b}$$

Introducing

$$M_\pm := \sqrt{|\zeta|^2 \pm \sqrt{r}e^{i(\vartheta+\pi)/2}}, \qquad \operatorname{Re} M_\pm > 0,$$

the solution to (3.7) is

$$\begin{aligned}v(t) = &\left(-\frac{M_- + M_+}{2M_-}k_1 - \frac{M_- - M_+}{2M_-}k_2\right)e^{-M_-t} + k_1e^{-M_+t}\\ &+ \left(-\frac{M_- - M_+}{2M_-}k_1 - \frac{M_- + M_+}{2M_-}k_2\right)e^{M_-t} + k_2e^{M_+t}\end{aligned}$$

for $t \geq 0$ with $k_j \in \mathbb{R}$. Since v must be bounded, $k_1 = k_2 = 0$ and thus $v \equiv 0$ as required. Consequently, assumptions (m), (n), and (o) from [8, Theorem 2.18], see Theorem A.1 below, are satisfied and it follows that the operator A generates an analytic semigroup $\{e^{tA} : t \geq 0\}$ on $B^\alpha_{1,1}(D)$ (recall that $\alpha \in (4\xi - 3, 0) \subset (-2, 1)$). Similarly, [1, Remarks 4.2 (b)] ensures that A restricted to $W^4_{2,\mathcal{B}}(D)$ generates an analytic semigroup $\{e^{tA} : t \geq 0\}$ on $L_2(D)$. Notice then that [9, Proposition 4.13] implies that

$$\big(B^\alpha_{1,1}(D), B^{4+\alpha}_{1,1,\mathcal{B}}(D)\big)_{\theta,1} \doteq B^{4\theta+\alpha}_{1,1,\mathcal{B}}(D), \quad 4\theta \in (0,4)\setminus\{1-\alpha, 2-\alpha\},$$

with $(\cdot,\cdot)_{\theta,1}$ denoting the real interpolation functor. Thus, standard regularizing effects of analytic semigroups [2, II.Lemma 5.1.3] imply (3.5). □

Proof of Theorem 2.2. To finish off the proof of Theorem 2.2 we first recall the continuity of the following embeddings

$$B^{4+\alpha}_{1,1,\mathcal{B}}(D) \hookrightarrow B^s_{1,1,\mathcal{B}}(D) \hookrightarrow B^0_{1,1,\mathcal{B}}(D) \hookrightarrow L_1(D) \hookrightarrow B^\alpha_{1,1}(D), \quad s \in (0, 4+\alpha), \tag{3.8}$$

bearing in mind that $\alpha < 0$. Now, introducing

$$h(t) := -\lambda g(u(t)) + a\|\nabla u(t)\|^2_{L_2(D)}\Delta u(t), \quad t \in [0, T_m),$$

we deduce from (3.8), Corollary 3.10, and Corollary 3.11 that

$$\|h(t)\|_{B^{\alpha}_{1,1}(D)} \le c\,\|h(t)\|_{L_1(D)} \le c\,, \quad t \in [0,T_m) \cap [0,T_0]\,. \tag{3.9}$$

Since $\alpha \in (4\xi - 3, 0)$ we can fix $\theta \in (0,1)$ and $4\xi_1 \in (4\xi, 4) \setminus \{3\}$ such that

$$4\theta + \alpha > 4\xi_1 + 1 > 4\xi + 1$$

and, consequently, (see [1, Section 5] for instance),

$$B^{4\theta+\alpha}_{1,1,\mathcal{B}}(D) \hookrightarrow B^{4\xi_1}_{2,2,\mathcal{B}}(D) \doteq W^{4\xi_1}_{2,\mathcal{B}}(D) \hookrightarrow W^{4\xi}_{2,\mathcal{B}}(D)\,. \tag{3.10}$$

Therefore, from (3.5), (3.9), (3.10), and Duhamel's formula

$$u(t) = e^{tA}u^0 + \int_0^t e^{(t-s)A} h(s)\,\mathrm{d}s\,, \quad t \in [0,T_m)\,,$$

(recall that the linear operator A is defined in Lemma 3.12), it follows that

$$\begin{aligned}
\|u(t)\|_{W^{4\xi}_{2,\mathcal{B}}(D)} &\le \|e^{tA}\|_{\mathcal{L}(W^{4\xi}_{2,\mathcal{B}}(D))}\|u^0\|_{W^{4\xi}_{2,\mathcal{B}}(D)} \\
&\quad + c(\theta)\int_0^t \|e^{(t-s)A}h(s)\|_{B^{4\theta+\alpha}_{1,1,\mathcal{B}}(D)}\,\mathrm{d}s \\
&\le c + c(\theta)\int_0^t \|e^{(t-s)A}\|_{\mathcal{L}(B^{\alpha}_{1,1}(D),B^{4\theta+\alpha}_{1,1,\mathcal{B}}(D))}\|h(s)\|_{B^{\alpha}_{1,1}(D)}\,\mathrm{d}s \\
&\le c(\theta)
\end{aligned}$$

for $t \in [0,T_m) \cap [0,T_0]$. We have thus shown (3.4) and the proof of Theorem 2.2 is complete according to Theorem 2.1. □

Appendix A.

For the sake of completeness, we recall [8, Theorem 2.18], which is at the heart of the proof of Lemma 3.12.

To set the stage, let $\mathcal{O}$ be a bounded open subset of $\mathbb{R}^n$ with C^∞-smooth boundary $\partial\mathcal{O}$ and consider a partial differential operator $\mathcal{A}$ of order $2m \ge 2$ and m boundary differential operators $(\mathcal{B}_k)_{1\le k\le m}$ given by

$$\mathcal{A} := \sum_{|i|\le 2m} a_i \partial_x^i\,, \qquad \mathcal{B}_k := \sum_{|i|\le m_k} b_{k,i}\partial_x^i\,,$$

where $(m_k)_{1\le k\le m} \in \mathbb{N}^m$,

$$a_i \in C^\infty(\overline{\mathcal{O}})\,, \qquad b_{k,i} \in C^\infty(\overline{\mathcal{O}})\,,$$

and where we use the standard notation for multi-indices: $i = (i_j)_{1\le j\le n} \in \mathbb{N}^n$, $|i| = i_1 + \ldots + i_n$, and $\partial_x^i = \partial_{x_1}^{i_1} \ldots \partial_{x_n}^{i_n}$. Let then

$$\mathcal{A}_0(x,\xi) := \sum_{|i|=2m} a_i(x)\xi^i \,, \qquad \mathcal{B}_{k,0}(x,\xi) := \sum_{|i|=m_k} b_{k,i}(x)\xi^i \,, \qquad (x,\xi) \in \overline{\mathcal{O}} \times \mathbb{R}^n \,,$$

denote the symbols of the corresponding principal parts.

Theorem A.1 ([8, Theorem 2.18])**.** *Assume that the following conditions are satisfied:*

(m) *there is* $c \in (0,\infty)$ *such that* $\operatorname{Re} \mathcal{A}_0(x, i\xi) \le -c|\xi|^{2m}$ *for* $(x,\xi) \in \mathcal{O} \times \mathbb{R}^n$*;*

(n) *the operators* $(\mathcal{B}_k)_{1\le k\le m}$ *form a normal system of boundary operators on* $\partial\mathcal{O}$ *in the sense of* [16, Chapitre 2, Section 1.4] *and* $m_k \le 2m-1$ *for* $1 \le k \le m$*;*

(o) *given* $x \in \partial\mathcal{O}$*,* $(\zeta, r) \in \mathbb{R}^n \times (0,\infty) \setminus \{(0,0)\}$*, and* $\theta \in [-\pi/2, \pi/2]$ *such that* $\zeta \cdot N(x) = 0$*, zero is the only bounded solution on* $[0,\infty)$ *to the problem*

$$\begin{aligned} \left[\mathcal{A}_0(x, i\zeta + N(x)\partial_t) - re^{i\theta}\right] v(t) &= 0 \,, \\ \left[\mathcal{B}_{k,0}(x, i\zeta + N(x)\partial_t)\right] v(0) &= 0 \,, \qquad 1 \le k \le m \,, \end{aligned}$$

where $N(x)$ *denotes the outward unit normal vector to* $\mathcal{O}$ *at* $x \in \partial\mathcal{O}$*.*

Let $p \in [1,\infty]$*,* $q \in [1,\infty)$*, and* $\alpha \in \mathbb{R}$ *satisfy*

$$\max_{1\le k\le n} m_k + \frac{1}{p} - 2m < \alpha < \min_{1\le k\le m} m_k + \frac{1}{p} \,.$$

Then the (unbounded) operator $(A, D(A))$ *defined by*

$$\begin{aligned} D(A) &:= \left\{ z \in B_{p,q}^{2m+\alpha}(\mathcal{O}) \;:\; B_k z = 0 \;\text{ on }\; \partial\mathcal{O} \,,\; 1 \le k \le m \right\} \,, \\ Az &:= \mathcal{A}z \,, \quad z \in D(A) \,, \end{aligned}$$

is the infinitesimal generator of an analytic semigroup in $B_{p,q}^{\alpha}(\mathcal{O})$*.*

References

[1] Herbert Amann. Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems. In *Function spaces, differential operators and nonlinear analysis (Friedrichroda, 1992)*, volume 133 of *Teubner-Texte zur Mathematik*, pages 9–126. Teubner, 1993.

[2] Herbert Amann. *Linear and quasilinear parabolic problems. Vol. I. Abstract linear theory*, volume 89 of *Monographs in Mathematics*. Birkhäuser, 1995.

[3] David H. Bernstein, Patrick Guidotti, and John A. Pelesko. Analytical and numerical analysis of electrostatically actuated MEMS devices. In *Proceedings of Modeling and Simulation of Microsystems 2000, San Diego, CA*, pages 489–492. 2000.

[4] Joachim Escher, Philippe Laurençot, and Christoph Walker. Finite time singularity in a free boundary problem modeling MEMS. *C. R. Math. Acad. Sci. Paris*, 351(21-22):807–812, 2013.

[5] Pierpaolo Esposito, Nassif Ghoussoub, and Yujin Guo. *Mathematical analysis of partial differential equations modeling electrostatic MEMS*, volume 20 of *Courant Lecture Notes in Mathematics*. Courant Institute of Mathematical Sciences; American Mathematical Society, 2010.

[6] A. Fargas Marquès, R. Costa Castelló, and A. M. Shkel. Modelling the electrostatic actuation of MEMS: state of the art 2005. Technical Report, Universitat Politècnica de Catalunya, 2005.

[7] G. Flores, G. Mercado, John A. Pelesko, and N. Smyth. Analysis of the dynamics and touchdown in a model of electrostatic MEMS. *SIAM J. Appl. Math.*, 67(2):434–446, 2007.

[8] Davide Guidetti. On elliptic problems in Besov spaces. *Math. Nachr.*, 152:247–275, 1991.

[9] Davide Guidetti. On interpolation with boundary conditions. *Math. Z.*, 207(3):439–460, 1991.

[10] Yujin Guo, Zhenguo Pan, and Michael J. Ward. Touchdown and pull-in voltage behavior of a MEMS device with varying dielectric properties. *SIAM J. Appl. Math.*, 66(1):309–338, 2005.

[11] Philippe Laurençot and Christoph Walker. A free boundary problem modeling electrostatic MEMS: I. Linear bending effects. *Math. Ann.*, 360(1-2):307–349, 2014.

[12] Philippe Laurençot and Christoph Walker. On a three-dimensional free boundary problem modeling electrostatic MEMS. *Interfaces Free Bound.*, 18(3):393–411, 2016.

[13] Philippe Laurençot and Christoph Walker. A variational approach to a stationary free boundary problem modeling MEMS. *ESAIM, Control Optim. Calc. Var.*, 22(2):417–438, 2016.

[14] Philippe Laurençot and Christoph Walker. Some singular equations modeling MEMS. *Bull. Am. Math. Soc.*, 54(3):437–479, 2017.

[15] Philippe Laurençot and Christoph Walker. Heterogeneous dielectric properties in models for microelectromechanical systems. *SIAM J. Appl. Math.*, 78(1):504–530, 2018.

[16] Jacques-Louis Lions and Enrico Magenes. *Problèmes aux limites non homogènes et applications. Vol. 1*, volume 17 of *Travaux et Recherches Mathématiques*. Dunod, 1968.

[17] Jindřich Nečas. *Direct methods in the theory of elliptic equations*. Springer Monographs in Mathematics. Springer, 2012. Translated from the 1967 French original by Gerard Tronel and Alois Kufner, Editorial coordination and preface by Šárka Nečasová and a contribution by Christian G. Simader.

[18] John A. Pelesko. Mathematical modeling of electrostatic MEMS with tailored dielectric properties. *SIAM J. Appl. Math.*, 62(3):888–908, 2001/02.

[19] John A. Pelesko and David H. Bernstein. *Modeling MEMS and NEMS*. Chapman & Hall/CRC, 2003.

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