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Harmonic functions on Manifolds whose large spheres are small.

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Abstract

We study the growth of harmonic functions on complete Riemannian manifolds where the extrinsic diameter of geodesic spheres is sublinear. It is an generalization of a result of A. Kasue. Our estimates also yields a result on the boundedness of the Riesz transform.

Résumé

On étudie la croissance des fonctions harmoniques sur les variétés riemanniennes complètes dont le diamètre des grandes sphères géodésiques croît sous linéairement. Il s'agit d'une généralisation de travaux de A. Kasue. Nous obtenons aussi un résultat de continuité pour la transformée de Riesz

1. Introduction

When (M,g) is a complete Riemannian manifold with non negative Ricci curvature, S-Y. Cheng and S-T. Yau have proven that any harmonic function $h\colon M\to\mathbb{R}$ satisfies the gradient estimate [4]:

$$\sup_{z\in B(x,R)}|dh|(z)\le\frac{C(n)}{R}\sup_{z\in B(x,2R)}|h(z)|.$$

This result implies that such a manifold can not carry non constant harmonic function $h\colon M\to\mathbb{R}$ with sublinear growth:

$$|h(x)|=o\big(d(o,x)\big)\ ,\ d(o,x)\to+\infty\,.$$

A celebrated conjecture of S-T. Yau predicted the finite dimensionality of the space of harmonic functions with polynomial growth on a complete

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Riemannian manifold with non negative Ricci curvature:

$$\mathcal{H}_\nu(M,g)=\left\{h\in\mathcal{C}^2(M)\,,\,\Delta_g h=0,|h(x)|=\mathcal{O}\big(d^\nu(o,x)\big)\right\}.$$

This conjecture has been proven by T. Colding and B. Minicozzi in a much more general setting.

We say that a complete Riemannian manifold (M^n,g) satisfies the *doubling* condition if there is a constant ϑ such that for any $x\in M$ and radius $R>0$:

$$\operatorname{vol}B(x,2R)\le\vartheta\operatorname{vol}B(x,R).$$

If $B\subset M$ is a geodesic ball, we will use the notation $r(B)$ for the radius of B and κB for the ball concentric to B and with radius $\kappa r(B)$. And if f is an integrable function on a subset $\Omega\subset M$, we will note f_Ω its mean over Ω:

$$f_\Omega=\frac{1}{\operatorname{vol}\Omega}\int_\Omega f.$$

We say that a complete Riemannian manifold (M^n,g) satisfies the scaled (L^2) Poincaré inequality if there is a constant μ such that for any ball $B\subset M$ and any function $\varphi\in\mathcal{C}^1(2B)$:

$$\|\varphi-\varphi_B\|^2_{L^2(B)}\le\mu\, r^2(B)\|d\varphi\|^2_{L^2(2B)}\,.$$

Theorem ([5])**.** *If (M,g) is a complete Riemannian manifold that is doubling and that satisfies the scaled Poincaré inequality then for any ν, the space of harmonic function of polynomial growth of order ν has finite dimension:*

$$\dim\mathcal{H}_\nu(M,g)<+\infty.$$

It is well known that a complete Riemannian manifold with non negative Ricci curvature is doubling and satisfies the scaled Poincaré inequality, hence the Yau's conjecture is true.

The proof is quantitative and gives a precise estimation of the dimension of the space of harmonic functions with polynomial growth of order ν. In fact, the condition on the Poincaré inequality can be weakened and the result holds on a doubling manifold (M,g) that satisfies the mean value estimation [6, 11]: for any harmonic function h defined over a geodesic ball $3B$:

$$\sup_{x\in B}|h(x)|\le\frac{C}{\operatorname{vol}2B}\int_{2B}|h|.$$

An example of Riemannian manifold satisfying the above condition are Riemannian surfaces of revolution $(\mathbb{R}^2,g_\gamma)$ where $\gamma\in(0,1]$ and such that on $\mathbb{R}^2 \setminus \{0\} \simeq (0,\infty) \times \mathbb{S}^1$ we have $g_\gamma = (dr)^2 + f_\gamma(r)^2(d\theta)^2$, where for all $r > 1$: $f_\gamma(r) = r^\gamma$ (see [8, Proposition 4.10]). Using the new variable

$$\rho(r) = \exp\left(\int_1^r \frac{ds}{f_\gamma(s)}\right),$$

we see that this metric is conformal to the Euclidean metric $(d\rho)^2+\rho^2(d\theta)^2$ on $\mathbb{R}^2$. In dimension 2, the Laplace equation is conformally invariant hence harmonic functions on $(\mathbb{R}^2, g_\gamma)$ are harmonic functions on the Euclidean space. We know that any harmonic function h on $\mathbb{R}^2$ such that for $h = \mathcal{O}(\rho^\alpha)$ for some $\alpha < 1$ is necessary constant. Hence we see that when $\gamma \in (0,1)$, any harmonic function h on $(\mathbb{R}^2, g_\gamma)$ satisfying for some $\epsilon > 0$:

$$h(x) = \mathcal{O}\left(e^{Cr^{1-\gamma-\epsilon}}\right)$$

is necessary constant. In particular, a harmonic function with polynomial growth is constant.

In [9, 10], A. Kasue has shown that this was a general result for manifold whose Ricci curvature satisfies a quadratic decay lower bound and whose geodesic spheres have sublinear growth (see also [13] for a related results):

Theorem 1.1. *If (M,g) is complete Riemannian manifold with a based point o whose Ricci curvature satisfies a quadratic decay lower bound:*

$$\text{Ricci} \geq -\frac{\kappa^2}{d^2(o,x)} g\ ,$$

and whose geodeosic sphere have sublinear growth:

$$\operatorname{diam} \partial B(o,R) = o(R)\ ,\ R \to +\infty,$$

then any harmonic function with polynomial growth is constant.

Following A. Grigor'yan and L. Saloff-Coste [8], we say that a ball $B(x,r)$ is remote (from a fixed point o) if

$$3r \leq d(o,x).$$

Our first main result is a refinement of A. Kasue's result when the hypothesis of the Ricci curvature is replaced by a scaled Poincaré inequality for remote ball: there is a constant μ such that all remote balls $B = B(x,r)$ satisfy a scaled Poincaré inequality:

$$\forall\, \varphi \in \mathcal{C}^1(2B)\ :\ \|\varphi - \varphi_B\|^2_{L^2(B)} \leq \mu r^2 \|d\varphi\|^2_{L^2(2B)}.$$

Theorem 1.2. *Let (M,g) be a complete Riemannian manifold whose remote balls satisfy the scaled Poincaré inequality and assume that geodesic spheres have sublinear growth:*

$$\operatorname{diam} \partial B(o,R) = o(R) \ , \ \ R \to +\infty.$$

If $h\colon M \to \mathbb{R}$ is a harmonic function such that for $I_R := \int_{B(o,R)} h^2$:

$$\lim_{R\to+\infty} \ \log(I_R)\, \frac{\operatorname{diam} \partial B(o,R/4)}{R} = 0,$$

then h is constant.

For instance, on such a manifold, a harmonic function $h\colon M \to \mathbb{R}$ satisfying:

$$|h(x)| \le C d(o,x)^{\nu} \left(\operatorname{vol} B(o, d(o,x))\right)^{-\frac{1}{2}}$$

is constant.

Moreover, consider (M,g) be a complete Riemannian manifold satisfying the hypothesis of the Theorem 1.2 and assume that for some $\gamma \in (0,1)$, the diameter of geodesic spheres satisfies

$$\operatorname{diam} \partial B(o,R) \le C R^{\gamma} \ .$$

If $h\colon M \to \mathbb{R}$ is a harmonic function satisfying, for some positive constants C and ϵ,

$$|h(x)| \le C e^{C d(o,x)^{1-\gamma-\epsilon}} \operatorname{vol} B(o, d(o,x))^{-\frac{1}{2}} \ ,$$

then h is constant.

Remark 1.3. Our result is a slight improvement of the Theorem 1.1. Indeed if (M,g) is a complete Riemannian manifold with a based point o whose Ricci curvature satisfies a quadratic decay lower bound:

$$\operatorname{Ricci} \ge -\frac{\kappa^2}{d^2(o,x)} g \ .$$

On a remote ball $B \subset M$, the Ricci curvature is bounded from below

$$\operatorname{Ricci} \ge -\frac{\kappa^2}{4r^2(B)} g \ ,$$

hence according to [3, inequality (4.5)], we have the Poincaré inequality:

$$\forall\, \varphi \in \mathcal{C}^1(B) \ : \ \|\varphi - \varphi_B\|^2_{L^2(B)} \le C(n) r^2(B) \|d\varphi\|^2_{L^2(B)}.$$

Moreover a slight variation of the Bishop–Gromov comparison theorem (see for instance [12, Lemma 3.1]) implies that (M, g) has polynomial growth: there is some $N > 0$ such that for all $r > 1$:

$$\operatorname{vol} B(o, r) \leq C r^N.$$

A by product of the proof will imply that on the class of manifold considered by A. Kasue, the doubling condition implies a Cheng–Yau's estimate for for the gradient of harmonic function:

Theorem 1.4. *Let (M^n, g) be a complete Riemannian manifold that is doubling and whose Ricci curvature satisfies a quadratic decay lower bound. Assume that the diameter of geodesic sphere has a sublinear growth*

$$\operatorname{diam} \partial B(o, R) = \sup_{x,y \in \partial B(o,R)} d(x, y) = o(R)\,.$$

Then there is a constant C such that for any geodesic ball $B \subset M$ and any harmonic function $h\colon 3B \to \mathbb{R}$

$$\sup_{x \in B} |dh|^2(x) \leq \frac{C}{\operatorname{vol} 2B} \int_{2B} |dh|^2.$$

This result has consequences for the boundness of the Riesz transform. When (M^n, g) is a complete Riemannian manifold with infinite volume, the Green formula and the spectral theorem yield the equality:

$$\forall\, f \in \mathcal{C}_0^\infty(M)\ ,\ \int_M |df|_g^2 \operatorname{dvol}_g = \langle \Delta f, f \rangle_{L^2} = \int_M \left| \Delta^{\frac{1}{2}} f \right|^2 \operatorname{dvol}_g\ .$$

Hence the Riesz transform

$$R := d\Delta^{-\frac{1}{2}} \colon L^2(M) \to L^2(T^*M)$$

is a bounded operator. It is well known [14] that on a Euclidean space, the Riesz transform has a bounded extension $R\colon L^p(\mathbb{R}^n) \to L^p(T^*\mathbb{R}^n)$ for every $p \in (1, +\infty)$. Also according to D. Bakry, the same is true on manifolds with non-negative Ricci curvature [2]. As it was noticed in [7, Section 5], in the setting of the Theorem 1.4, the analysis of A. Grigor'yan and L. Saloff-Coste [8] implies a scaled L^1-Poincaré inequality: there is a constant C such that any balls $B = B(x, r)$ satisfies:

$$\forall\, \varphi \in \mathcal{C}^1(2B)\ :\ \|\varphi - \varphi_B\|_{L^1(B)} \leq C r^2 \|d\varphi\|_{L^1(2B)}\,.$$

And according to the analysis of P. Auscher and T. Coulhon [1] (see also the explanations in [7, Section 5]), the Theorem 1.4 implies:

Corollary 1.5. *Under the assumption of the Theorem 1.4, the Riesz transform is bounded on L^p for every $p \in (1, +\infty)$.*

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2. Absence of harmonic functions

Recall that when (M, g) is a complete Riemannian manifold and $o \in M$, we say that a geodesic ball $B(x, r)$ is *remote* (from o) if

$$3r \leq d(o, x).$$

We define ρ the radius function by $\rho(t) = \inf_{x \in \partial B(o,t)} \max_{y \in \partial B(o,t)} d(x, y)$, we have

$$\rho(t) \leq \operatorname{diam} \partial B(o, t) \leq 2\rho(t).$$

2.1. An inequality

Lemma 2.1. *Let (M, g) be a complete Riemannian manifold whose all remote balls $B = B(x, r)$ satisfy a scaled Poincaré inequality:*

$$\forall \varphi \in \mathcal{C}^1(2B) \, : \, \|\varphi - \varphi_B\|^2_{L^2(B)} \leq \mu \, r^2(B) \|d\varphi\|^2_{L^2(2B)} \, .$$

Then there are constants $C > 0$ and $\kappa \in (0, 1)$ depending only on μ such that if for some $\varepsilon \in (0, 1/12)$:

$$\forall \, r \in [R, 2R] : \; \rho(r) \leq \varepsilon r \, ,$$

then

$$\int_{B(o,R)} |dh|^2 \leq C \, \kappa^{\frac{1}{\varepsilon}} \int_{B(o,2R)} |dh|^2 \, ,$$

for any harmonic function h on $B(o, 2R)$.

Proof. Let $r \in [R + 4\varepsilon R, 2R - 4\varepsilon R]$, our hypothesis implies that there is some $x \in \partial B(o, r)$ such that

$$B(o, r + \varepsilon R) \setminus B(o, r) \subset B(x, \varepsilon R + \varepsilon r).$$

Let $h\colon B(o,2R)\to\mathbb{R}$ be a harmonic function and $c\in\mathbb{R}$ a real number. We use the Lipschitz function:

$$\chi(x)=\begin{cases}1 & \text{on } B(o,r)\\ \frac{r+\varepsilon R-d(o,x)}{\varepsilon R} & \text{on } B(o,r+\varepsilon R)\setminus B(o,r)\\ 0 & \text{outside } B(o,r+\varepsilon R)\end{cases}$$

Then integrating by parts and using the fact that h is harmonic we get

$$\int_M \chi^2|d(h-c)|^2+2\chi(h-c)\langle d\chi,d(h-c)\rangle=\int_M\langle d((h-c)\chi^2),d(h-c)\rangle=0.$$

So that we have:

$$\begin{aligned}\int_M &|d(\chi(h-c))|^2\\ &=\int_M \chi^2|d(h-c)|^2+2\chi(h-c)\langle d\chi,d(h-c)\rangle+(h-c)^2|d\chi|^2\\ &=\int_{B(o,r+\varepsilon R)}(h-c)^2|d\chi|^2\,,\end{aligned}$$

and hence

$$\begin{aligned}\int_{B(o,r)}|dh|^2&\le\int_{B(o,r+\varepsilon R)}|d(\chi(h-c))|^2=\int_{B(o,r+\varepsilon R)}(h-c)^2|d\chi|^2\\ &\le\frac{1}{\varepsilon^2R^2}\int_{B(o,r+\varepsilon R)\setminus B(o,r)}(h-c)^2\\ &\le\frac{1}{\varepsilon^2R^2}\int_{B(x,\varepsilon R+\varepsilon r)}(h-c)^2.\end{aligned}$$

The hypothesis that $\varepsilon\le 1/12$ implies that the ball $B(x,\varepsilon R+\varepsilon r)$ is remote, hence if we choose

$$c=h_{B(x,\varepsilon(R+r))}=\frac{1}{\operatorname{vol}B(x,,\varepsilon(R+r))}\int_{B(x,,\varepsilon(R+r))}h,$$

then the Poincaré inequality and the fact that $r+R\le 3R$ imply:

$$\int_{B(o,r)}|dh|^2\le 9\mu\int_{B(x,6\varepsilon R)}|dh|^2.$$

But we have:

$$B(x,6\varepsilon R)\subset B(o,r+6\varepsilon R)\setminus B(o,r-6\varepsilon R)\,,$$

hence we get

$$\int_{B(o,r-6\varepsilon R)} |dh|^2 \leq 9\mu \int_{B(o,r+6\varepsilon R)\setminus B(o,r-6\varepsilon R)} |dh|^2.$$

And for all $r \in [R, 2R - 12\varepsilon R]$ we get:

$$\int_{B(o,r)} |dh|^2 \leq \frac{9\mu}{1+9\mu} \int_{B(o,r+12\varepsilon R)} |dh|^2.$$

We iterate this inequality and get

$$\int_{B(o,R)} |dh|^2 \leq \left(\frac{9\mu}{1+9\mu}\right)^N \int_{B(o,2R)} |dh|^2,$$

provide that $N12\varepsilon R \leq R$; hence the result with $C = 1 + \frac{1}{9\mu}$ and

$$\kappa = \left(\frac{9\mu}{1+9\mu}\right)^{\frac{1}{12}}.$$

□

2.2. **Harmonic functions with polynomial growth**

We can now prove the following extension of Kasue's results:

Theorem 2.2. *Let (M,g) be a complete Riemannian manifold whose all remote balls $B = B(x,r)$ satisfy a scaled Poincaré inequality:*

$$\forall\, \varphi \in \mathcal{C}^1(2B) \,:\, \|\varphi - \varphi_B\|^2_{L^2(B)} \leq \mu r^2(B) \|d\varphi\|^2_{L^2(2B)}.$$

Assume that balls anchored at o have polynomial growth:

$$\operatorname{vol} B(o,R) \leq C R^\mu$$

and that geodesic spheres have sublinear diameter growth:

$$\lim_{t\to+\infty} \frac{\rho(t)}{t} = 0.$$

Then any harmonic function on (M,g) with polynomial growth is constant.

Proof. Let $h\colon M \to \mathbb{R}$ be a harmonic function with polynomial growth:

$$h(x) \leq C(1 + d(o,x))^\nu.$$

We will define

$$E_R = \int_{B(o,R)} |dh|^2 \quad \text{and} \quad \epsilon(r) = \sup_{t\geq r} \frac{\rho(t)}{t}.$$

We remark first that using the cut off function ξ defined by

$$\xi(x)=\begin{cases}1 & \text{on } B(o,R)\\ \frac{2R-d(o,x)}{R} & \text{on } B(o,2R)\setminus B(o,R)\\ 0 & \text{outside } B(o,2R).\end{cases}$$

We obtain

$$E_R\leq\int_{B(o,2R)}|d(\xi h)|^2=\int_{B(o,2R)}|h|^2|d\xi|^2\leq CR^{2\nu+\mu-2}. \tag{2.1}$$

If we iterate the inequality obtained in the Lemma 2.1, we get for all R such that $\epsilon(R)\leq 1/12$:

$$E_R\leq C^\ell\kappa^{\sum_{j=0}^{\ell-1}\frac{1}{\epsilon(2^jR)}}E_{2^\ell R}\ .$$

Using the estimation (2.1), we get

$$E_R\leq C(R)e^{\ell\left(\frac{\log\kappa}{\ell}\sum_{j=0}^{\ell-1}\frac{1}{\epsilon(2^jR)}+\log(2)(2\nu+\mu-2)+\log C\right)}. \tag{2.2}$$

But the Cesaro theorem convergence implies that:

$$\lim_{\ell\to+\infty}\frac{1}{\ell}\sum_{j=0}^{\ell-1}\frac{1}{\epsilon\left(2^jR\right)}=+\infty,$$

hence if we let $\ell\to+\infty$ in the inequality (2.2) we get $E_R=0$ and this for all sufficiently large R, hence h is constant. □

2.3. An extension

A slight variation of the arguments yields the following extension, which implies the Theorem 1.2:

Theorem 2.3. *Let (M,g) be a complete Riemannian manifold whose all remote balls $B=B(x,r)$ satisfies a scaled Poincaré inequality:*

$$\forall\,\varphi\in\mathcal{C}^1(2B)\ :\ \|\varphi-\varphi_B\|^2_{L^2(B)}\leq\mu r^2(B)\|d\varphi\|^2_{L^2(2B)}.$$

Assume that the geodesic spheres have sublinear diameter growth:

$$\lim_{t\to+\infty}\frac{\rho(t)}{t}=0\ \text{ and let }\ \epsilon(r)=\sup_{t\geq r}\frac{\rho(t)}{t}\ .$$

Let $h\colon M \to \mathbb{R}$ be a harmonic function such that $I_R = \int_{B(o,R)} h^2$ satisfies

$$\log I(R) = o\left(\int_1^{R/4} \frac{dt}{t\epsilon(t)}\right),$$

then h is constant.

Proof. Indeed, the above argumentation shows that if R is large enough then

$$E_R \leq M(\ell, R) I(2^{\ell+1}R)\, 4^{-\ell} R^{-2},$$

where

$$\begin{aligned}\log(M(\ell,R)) &= \log\left(C^\ell \kappa^{\sum_{j=0}^{\ell-1} \frac{1}{\epsilon(2^j R)}}\right)\\ &= \ell \log C + \log\kappa \left(\sum_{j=0}^{\ell-1} \frac{1}{\epsilon\left(2^j R\right)}\right).\end{aligned}$$

But

$$\sum_{j=0}^{\ell-1} \frac{1}{\epsilon\left(2^j R\right)} \geq \frac{1}{\log 2} \sum_{j=0}^{\ell-1} \int_{2^{j-1}R}^{2^j R} \frac{dt}{t\epsilon\left(t\right)} \geq \frac{1}{\log 2} \int_{R/2}^{2^{\ell-1}R} \frac{dt}{t\epsilon\left(t\right)}.$$

Hence we get the inequality:

$$\log E_R \leq \log I\left(2^{\ell+1}R\right) - \ell\log(4) + \ell \log C + \frac{\log\kappa}{\log 2} \int_{R/2}^{2^{\ell-1}R} \frac{dt}{t\epsilon\left(t\right)} - 2\log R.$$

It is then follows from the above inequality and the Cesaro theorem convergence that h is constant. □

3. Lipschitz regularity of harmonic functions

We are going to prove that a Lipschitz regularity for harmonic function analogous to the Cheng–Yau gradient inequality:

Theorem 3.1. *Let (M^n, g) be a complete Riemannian manifold that satisfy the doubling condition: there is a constant ϑ such that for any $x \in M$ and radius $R > 0$:*

$$\operatorname{vol} B(x, 2R) \leq \vartheta \operatorname{vol} B(x, R)$$

and assume moreover that the Ricci curvature satisfies a quadratic decay lower bound

$$\mathrm{Ricci} \geq -\frac{\kappa^2}{r^2(x)} g\ ,$$

where for a fixed point $o \in M$: $r(x) := d(o,x)$.

Assume that the diameters of geodesic spheres growth slowly

$$\mathrm{diam}\, \partial B(o,R) = \sup_{x,y \in \partial B(o,R)} d(x,y) = o(R).$$

Then there is a constant C such that for any geodesic ball $B \subset M$ and any harmonic function $h \colon 3B \to \mathbb{R}$

$$\sup_{x \in B} |dh|^2(x) \leq \frac{C}{\mathrm{vol}\, 2B} \int_{2B} |dh|^2.$$

Proof. According to [7, Proposition 5.3], we need only to show that there is a constant C such that if $R > 0$ and if $h \colon B(o,2R) \to \mathbb{R}$ is a harmonic function then for any $s \leq \sigma \leq R$:

$$\frac{1}{\mathrm{vol}\, B(o,s)} \int_{B(o,s)} |dh|^2 \leq \frac{C}{\mathrm{vol}\, B(o,\sigma)} \int_{B(o,\sigma)} |dh|^2. \tag{3.1}$$

According to the Remark 1.3, we can apply the Lemma 2.1: for all $\eta > 0$, there is a $R_0 > 0$ such that for all $R \geq R_0$, then

$$\int_{B(o,R)} |dh|^2 \leq \eta \int_{B(o,2R)} |dh|^2.$$

Hence for all $R \geq R_0$:

$$\frac{1}{\mathrm{vol}\, B(o,R)} \int_{B(o,R)} |dh|^2 \leq \eta\, \vartheta\, \frac{1}{\mathrm{vol}\, B(o,2R)} \int_{B(o,2R)} |dh|^2.$$

Choose $\eta = \vartheta^{-1}$, then we get that for all $R_0 \leq s \leq \sigma \leq R$:

$$\frac{1}{\mathrm{vol}\, B(o,s)} \int_{B(o,s)} |dh|^2 \leq \frac{\vartheta}{\mathrm{vol}\, B(o,\sigma)} \int_{B(o,\sigma)} |dh|^2.$$

The Ricci curvature being bounded on $B(o,3R_0)$, the Cheng and Yau gradient estimate yields a constant B such that for all $x \in B(o,R_0)$:

$$|dh|^2(x) \leq \frac{B}{\mathrm{vol}\, B(o,2R_0)} \int_{B(o,2R_0)} |dh|^2.$$

Hence the estimate (3.1) holds with $C = \max\{B\vartheta, \vartheta\}$. □

References

[1] P. Auscher & T. Coulhon – "Riesz transform on manifolds and Poincaré inequalities", *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **4** (2005), no. 3, p. 531–555.

[2] D. Bakry – "Étude des transformations de Riesz dans les variétés riemanniennes à courbure de Ricci minorée", in *Séminaire de Probabilités, XXI*, Lecture Notes in Math., vol. 1247, Springer, Berlin, 1987, p. 137–172.

[3] P. Buser – "A note on the isoperimetric constant", *Ann. Sci. École Norm. Sup. (4)* **15** (1982), no. 2, p. 213–230.

[4] S.-Y. Cheng & S.-T. Yau – "Differential equations on Riemannian manifolds and their geometric applications", *Comm. Pure Appl. Math.* **28** (1975), no. 3, p. 333–354.

[5] T. H. Colding & W. P. Minicozzi, II – "Harmonic functions on manifolds", *Ann. Math. (2)* **146** (1997), no. 3, p. 725–747.

[6] ———, "Liouville theorems for harmonic sections and applications", *Comm. Pure Appl. Math.* **51** (1998), no. 2, p. 113–138.

[7] C. Gilles – "Riesz transform on manifolds with quadratic curvature decay", `https://arxiv.org/abs/1403.6278`, to appear in *Rev. Mat. Iberoam.*, 2014.

[8] A. Grigor'yan & L. Saloff-Coste – "Stability results for Harnack inequalities", *Ann. Inst. Fourier* **55** (2005), no. 3, p. 825–890.

[9] A. Kasue – "Harmonic functions with growth conditions on a manifold of asymptotically nonnegative curvature. I", in *Geometry and analysis on manifolds (Katata/Kyoto, 1987)*, Lecture Notes in Math., vol. 1339, Springer, Berlin, 1988, p. 158–181.

[10] ———, "Harmonic functions of polynomial growth on complete manifolds. II", *J. Math. Soc. Japan* **47** (1995), no. 1, p. 37–65.

[11] P. Li – "Harmonic functions of linear growth on Kähler manifolds with nonnegative Ricci curvature", *Math. Res. Lett.* **2** (1995), no. 1, p. 79–94.

[12] J. LOTT & Z. SHEN – "Manifolds with quadratic curvature decay and slow volume growth", *Ann. Sci. École Norm. Sup. (4)* **33** (2000), no. 2, p. 275–290.

[13] C. SORMANI – "Harmonic functions on manifolds with nonnegative Ricci curvature and linear volume growth", *Pacific J. Math.* **192** (2000), no. 1, p. 183–189.

[14] E. M. STEIN – *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, vol. 30, Princeton University Press, Princeton, N.J., 1970.

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