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Coefficient inequality for transforms of parabolic starlike and uniformly convex functions

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Abstract

The objective of this paper is to obtain sharp upper bound to the second Hankel functional associated with the k^{th} root transform $\left[f(z^k)\right]^{\frac{1}{k}}$ of normalized analytic function $f(z)$ belonging to parabolic starlike and uniformly convex functions, defined on the open unit disc in the complex plane, using Toeplitz determinants.

1. Introduction

Let A denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions. Let the functions F and G be analytic in the unit disc E. Then F is said to be subordinate to G, written $F \prec G$, if there exists an analytic function $w(z)$ in the open unit disc E satisfying $w(0) = 1$ and $|w(z)| < 1, \forall z \in E$ called the Schwarz's function such that

$$F(z) = G(w(z)), \forall z \in E. \tag{1.2}$$

If $F \prec G$ and $G(z)$ is univalent in the open unit disc E, then the subordination is equivalent to $F(0) = G(0)$ and range $F(z) \subseteq$ range G(z). For a univalent function in the class A, it is well known that the n^{th} coefficient is bounded by n. The bounds for the coefficients give information about the geometric properties of these functions. For example, the bound for the second coefficient of normalized univalent function readily yields

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the growth and distortion properties for univalent functions. The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [18] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, (a_1 = 1).$$

This determinant has been considered by many authors in the literature . For example, Noor [17] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in S with bounded boundary. Ehrenborg [8] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [13]. In 1966, Pommerenke [18] investigated the Hankel determinant of areally mean $p-$valent functions, also studied by Noonan and Thomas [16], univalent functions as well as starlike functions. In the recent years, several authors have investigated bounds for the Hankel determinant of functions belonging to various subclasses of univalent and multivalent functions [1, 12, 11]. In particular cases, $q = 2, n = 1, a_1 = 1$ and $q = 2, n = 2, a_1 = 1$, the Hankel determinant simplifies respectively to

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2,$$

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.$$

We refer to $H_2(2)$ as the second Hankel determinant. It is fairly well known that for the univalent functions of the form given in (1.1) the sharp inequality $H_2(1) =| a_3 - a_2^2 |\leq 1$ holds true [7]. For a family $\mathcal{T}$ of functions in S, the more general problem of finding sharp estimates for the functional $| a_3 - \mu a_2^2 |$ ($\mu \in \mathbb{R}$ or $\mu \in \mathbb{C}$) is popularly known as the Fekete-Szegö problem for $\mathcal{T}$. Ali [3] found sharp bounds for the first four coefficients and sharp estimate for the Fekete-Szegö functional $| \gamma_3 - t\gamma_2^2 |$, where t is real for the inverse function of f defined as $f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$ when it belongs to the class of strongly starlike functions of order α $(0 < \alpha \leq 1)$ denoted by $\widetilde{ST}(\alpha)$. R. M. Ali, S. K. Lee, V. Ravichandran and S. Supramaniam [5] obtained sharp bounds for the Fekete-Szegö functional denoted by $| b_{2k+1} - \mu b_{k+1}^2 |$ associated with the k^{th} root transform $\left[f(z^k)\right]^{\frac{1}{k}}$ of the function given in (1.1), belonging to certain subclasses of S. The k^{th} root transform for the function f given in (1.1) is defined as

$$F(z) := \left[f(z^k)\right]^{\frac{1}{k}} = z + \sum_{n=1}^{\infty} t_{kn+1} z^{kn+1} \tag{1.3}$$

Motivated by the results obtained by R. M. Ali, S. K. Lee, V. Ravichandran and S. Supramaniam [5], in the present paper, we obtain sharp upper bound to the functional $\mid t_{k+1}t_{3k+1} - t_{2k+1}^2 \mid$, called the second Hankel determinant for the k^{th} root transform of the function f when it belongs to certain subclasses of S, defined as follows.

Definition 1.1. A function $f(z) \in A$ is said to be parabolic starlike function, if and only if

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < Re\left\{\frac{zf'(z)}{f(z)}\right\}, \forall z \in E \tag{1.4}$$

The class of all parabolic starlike functions is introduced by Ronning [20] and is denoted by S_p. Geometrically, (see [4]) S_p is the class functions f, for which $\left\{\frac{zf'(z)}{f(z)}\right\}$ takes its value in the interior of the parabola in the right half plane symmetric about the real axis with vertex at $(\frac{1}{2}, 0)$.

Definition 1.2. A function $f \in A$ is said to be in UCV, if and only if

$$\left|\frac{zf''(z)}{f'(z)}\right| < Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\}, \forall z \in E. \tag{1.5}$$

Goodman [9] introduced the class UCV of uniformly convex functions consisting of convex functions $f \in A$ with the property that for every circular arc γ contained in the unit disc E with centre also in E, the image arc $f(\gamma)$ is a convex arc. Ma and Minda [15] and Ronning [20] independently developed a one-variable characterization for the functions in the class UCV. From the Definitions 1.1 and 1.2, we have the relation between UCV and S_p is given in terms of an Alexander type Theorem [2] by Ronning (see [4]) as follows.

$$f \in UCV \Leftrightarrow zf' \in S_p. \tag{1.6}$$

Further, Ali [4] obtained sharp bounds on the first four coefficients and Fekete-Szegö inequality for the functions in the class S_p. Ali and Singh [6] showed that the normalized Riemann mapping function $q(z)$ from E onto the domain $D = \{w = u + iv : v^2 < 4u\} = \{w : |w - 1| < 1 + Re(w)\}$, denotes the parabolic region in the right half plane of the complex plane given by

$$q(z) = \left[1 + \frac{4}{\pi^2}\left\{\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right\}^2\right] = \left[1 + \sum_{n=1}^{\infty} B_n z^n\right], \forall z \in E. \quad (1.7)$$

It can be observed that if $f(z) \in S_p$ then

$$\frac{zf'(z)}{f(z)} \prec q(z), \forall z \in E, \quad (1.8)$$

where $q(z)$ is given in (1.7).

Some preliminary lemmas required for proving our results are as follows:

2. Preliminary Results

Let $\mathbb{P}$ denote the class of functions consisting of p, such that

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... = \left[1 + \sum_{n=1}^{\infty} c_n z^n\right], \quad (2.1)$$

which are regular in the open unit disc E and satisfy $\text{Re}\{p(z)\} > 0$ for any $z \in E$. Here $p(z)$ is called the Carathèdory function [7].

Lemma 2.1. *([19, 21]) If $p \in \mathbb{P}$, then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $\left(\frac{1+z}{1-z}\right)$.*

Lemma 2.2. *([10]) The power series for $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ given in (2.1) converges in the open unit disc E to a function in $\mathbb{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, n = 1, 2, 3....$$

and $c_{-k} = \overline{c}_k$, are all non-negative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k P_0(e^{it_k} z)$, $\rho_k > 0$, t_k real and $t_k \neq t_j$, for $k \neq j$, where $P_0(z) = \left(\frac{1+z}{1-z}\right)$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$.

This necessary and sufficient condition found in [10] is due to Carathèodory and Toeplitz. We may assume without restriction that $c_1 > 0$. On using Lemma 2.2, for $n = 2$, we have

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c}_1 & 2 & c_1 \\ \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} = [8 + 2Re\{c_1^2 c_2\} - 2 \mid c_2 \mid^2 - 4|c_1|^2] \geq 0,$$

which is equivalent to

$$2c_2 = \{c_1^2 + x(4 - c_1^2)\}, \text{ for some } x, |x| \leq 1. \tag{2.2}$$

For $n = 3$,

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c}_1 & 2 & c_1 & c_2 \\ \overline{c}_2 & \overline{c}_1 & 2 & c_1 \\ \overline{c}_3 & \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} \geq 0$$

and is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{2.3}$$

From the relations (2.2) and (2.3), after simplifying, we get

$$4c_3 = \{c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z\}$$
$$\text{for some } z, \text{ with } |z| \leq 1. \tag{2.4}$$

To obtain our results, we refer to the classical method initiated by Libera and Zlotkiewicz [14] and used by several authors in the literature.

3. Main Results

Theorem 3.1. *If f given by (1.1) belongs to S_p and F is the k^{th} root transformation of f given by (1.3) then*

$$| t_{k+1}t_{3k+1} - t_{2k+1}^2 | \leq \left[\frac{8}{k\pi^2}\right]^2$$

and the inequality is sharp.

Proof. For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_p$, by virtue of Definition 1.1, we have

$$\left[\frac{zf'(z)}{f(z)}\right] \prec q(z), \quad \forall z \in E. \tag{3.1}$$

By the subordination principle, there exist a Schwarz's function $w(z)$ such that

$$\left[\frac{zf'(z)}{f(z)}\right] \prec [q\{w(z)\}], \forall z \in E. \tag{3.2}$$

Define a function $h(z)$ such that

$$h(z) = \left[\frac{zf'(z)}{f(z)}\right] = 1 + b_1 z + b_2 z^2 + b_3 z^3 + ... = \left[1 + \sum_{n=1}^{\infty} b_n z^n\right]$$
$$\Leftrightarrow [zf'(z)] = [f(z)h(z)]. \tag{3.3}$$

Using the series representations for $f(z)$, $f'(z)$ and $h(z)$ in (3.3), we have

$$z\left\{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}\right\} = \left\{z + \sum_{n=2}^{\infty} a_n z^n\right\}\left\{1 + \sum_{n=1}^{\infty} b_n z^n\right\}. \tag{3.4}$$

Upon simplification, we obtain

$$1 + a_2 z + 2a_3 z^2 + 3a_4 z^3 + ... = 1 + b_1 z + (b_1 a_2 + b_2) z^2 +$$
$$(b_1 a_3 + b_2 a_2 + b_3) z^3 + ... \tag{3.5}$$

Equating the coefficients of like powers of z, z^2 and z^3 respectively on both sides of (3.5), after simplifying, we get

$$a_2 = b_1;\ a_3 = \frac{1}{2}\left(b_2 + b_1^2\right);\ a_4 = \frac{1}{3}\left(b_3 + \frac{3}{2}b_1 b_2 + \frac{b_1^3}{2}\right). \tag{3.6}$$

Since $q(z)$ is univalent in the open unit disc E and $h(z) \prec q(z)$, define a function

$$p(z) = \left[\frac{1 + w(z)}{1 - w(z)}\right] = \left[\frac{1 + q^{-1}\{h(z)\}}{1 - q^{-1}\{h(z)\}}\right] = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ..., \tag{3.7}$$

where $p(z)$ is given in (2.1). Solving $w(z)$ in terms of $p(z)$ in the relation (3.7) and replacing $p(z)$ by its equivalent expression in series, we have

$$w(z) = \left[\frac{p(z) - 1}{p(z) + 1}\right] = \left[\frac{(1 + c_1 z + c_2 z^2 + c_3 z^3 + ...) - 1}{(1 + c_1 z + c_2 z^2 + c_3 z^3 + ...) + 1}\right].$$

Upon simplification, we obtain

$$w(z) = \frac{1}{2}\left\{c_1 z + (c_2 - \frac{c_1^2}{2}) z^2 + (c_3 - c_1 c_2 + \frac{c_1^3}{4}) z^3 + ...\right\}. \tag{3.8}$$

Using the expansion of $\log(1+x) = \left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\right\}$ for $q(z)$ given in (1.7), after simplifying, we get

$$\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2 = \left\{4z + \frac{8}{3}z^2 + \frac{92}{45}z^3 + \frac{176}{105}z^4 + ...\right\}. \quad (3.9)$$

From the relations (1.7) and (3.9), we obtain

$$q(z) = \left\{1 + \frac{16}{\pi^2}z + \frac{32}{3\pi^2}z^2 + \frac{368}{45\pi^2}z^3 + \frac{704}{105\pi^2}z^4 ...\right\}$$
$$= [1 + B_1 z + B_2 z^2 + B_3 z^3 + ...] = \left[1 + \sum_{n=1}^{\infty} B_n z^n\right]. \quad (3.10)$$

Equating the coefficients of like powers of z, z^2 and z^3 respectively, on both sides of (3.10), we get

$$B_1 = \frac{16}{\pi^2};\ B_2 = \frac{32}{3\pi^2};\ B_3 = \frac{368}{45\pi^2}...,$$
$$B_n = \frac{16}{n\pi^2}\sum_{k=0}^{n-1}\frac{1}{(2k+1)}, n = 2, 3, 4... \quad (3.11)$$

From the relations (3.2) and (3.3), we have

$$h(z) = [q\{w(z)\}]. \quad (3.12)$$

In view of (3.12), using (3.8) in (3.10) along with the equivalent expression for $h(z)$ given in (3.3), upon simplification, (3.12) is equivalent to

$$\left[1 + b_1 z + b_2 z^2 + b_3 z^3 + ...\right] = \left[1 + \frac{1}{2}B_1 c_1 z + \left\{\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right\}z^2 + \right.$$
$$\left.\left\{\frac{1}{2}B_1\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) + \frac{1}{2}B_2 c_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{8}B_3 c_1^3\right\}z^3 + ...\right]. \quad (3.13)$$

Equating the coefficients of like powers of z, z^2 and z^3 respectively, on both sides of (3.13), we have

$$b_1 = \frac{1}{2}B_1 c_1;\ b_2 = \left\{\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right\};$$
$$b_3 = \left\{\frac{1}{2}B_1\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) + \frac{1}{2}B_2 c_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{8}B_3 c_1^3\right\}. \quad (3.14)$$

Simplifying the relations (3.11) and (3.14), we get

$$b_1 = \frac{8c_1}{\pi^2};\ b_2 = \frac{8}{\pi^2}\left(c_2 - \frac{c_1^2}{6}\right);\ b_3 = \frac{8}{\pi^2}\left(c_3 - \frac{c_1c_2}{3} + \frac{2}{45}c_1^3\right). \quad (3.15)$$

From the relations (3.6) and (3.15), upon simplification, we obtain

$$a_2 = \frac{8c_1}{\pi^2};\ a_3 = \frac{8}{2\pi^2}\left[c_2 - \left\{\frac{1}{6} - \frac{8}{\pi^2}\right\}c_1^2\right];$$

$$a_4 = \frac{8}{3\pi^2}\left[c_3 - \left\{\frac{1}{3} - \frac{12}{\pi^2}\right\}c_1c_2 + \left\{\frac{2}{45} - \frac{2}{\pi^2} + \frac{32}{\pi^4}\right\}c_1^3\right]. \quad (3.16)$$

For a function f given by (1.1), a computation shows that

$$\left[f(z^k)\right]^{\frac{1}{k}} = \left[z^k + \sum_{n=2}^{\infty} a_n z^{nk}\right]^{\frac{1}{k}}$$

$$= \Big[z + \frac{1}{k}a_2 z^{k+1} + \Big\{\frac{1}{k}a_3 + \frac{(1-k)}{2k^2}a_2^2\Big\}z^{2k+1}$$

$$+\Big\{\frac{1}{k}a_4 + \frac{(1-k)}{k^2}a_2a_3 + \frac{(1-k)(1-2k)}{6k^3}a_2^3\Big\}z^{3k+1} + \cdots\Big]. \quad (3.17)$$

From the equations (1.3) and (3.16) together with (3.17), after simplifying, we get

$$t_{k+1} = \frac{8c_1}{k\pi^2}\ ; \quad t_{2k+1} = \frac{8}{2k\pi^2}\Big[c_2 - \Big\{\frac{1}{6} - \frac{8}{k\pi^2}\Big\}c_1^2\Big]\ ;$$

$$t_{3k+1} = \frac{8}{3k\pi^2}\left[c_3 - \Big\{\frac{1}{3} - \frac{12}{k\pi^2}\Big\}c_1c_2 + \Big\{\frac{2}{45} - \frac{2}{k\pi^2} + \frac{32}{k^2\pi^4}\Big\}c_1^3\right]. \quad (3.18)$$

Substituting the values of t_{k+1}, t_{2k+1} and t_{3k+1} from (3.18) in the second Hankel determinant $\mid t_{k+1}t_{3k+1} - t_{2k+1}^2 \mid$ to the k^{th} transformation for the function $f \in S_p$, upon simplification, we obtain

$$|t_{k+1}t_{3k+1} - t_{2k+1}^2| = \frac{16}{9k^2\pi^4}\left|12c_1c_3 - c_1^2c_2 - 9c_2^2 + \left\{\frac{17}{60} - \frac{192}{k^2\pi^4}\right\}c_1^4\right|, \quad (3.19)$$

which is equivalent to

$$|t_{k+1}t_{3k+1} - t_{2k+1}^2| = \frac{16}{9k^2\pi^4}|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4|, \quad (3.20)$$

$$\text{where } d_1 = 12;\ d_2 = -1;\ d_3 = -9;\ d_4 = \left\{\frac{17}{60} - \frac{192}{k^2\pi^4}\right\}. \quad (3.21)$$

Substituting the values of c_2 and c_3 from (2.2) and (2.4) respectively from Lemma 2.2 on the right-hand side of (3.20), we have

$$\begin{aligned}
&|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \\
&= |d_1c_1 \times \frac{1}{4}\{c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z\}+ \\
&\qquad d_2c_1^2 \times \frac{1}{2}\{c_1^2 + x(4-c_1^2)\} + d_3 \times \frac{1}{4}\{c_1^2 + x(4-c_1^2)\}^2 + d_4c_1^4|. \quad (3.22)
\end{aligned}$$

Using the triangle inequality and the fact that $|z| < 1$, we get

$$\begin{aligned}
4|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| &\leq |(d_1 + 2d_2 + d_3 + 4d_4)c_1^4 + 2d_1c_1(4-c_1^2) \\
&\quad + 2(d_1 + d_2 + d_3)c_1^2(4-c_1^2)|x| - \\
&\quad \left\{(d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3\right\}(4-c_1^2)|x|^2|. \quad (3.23)
\end{aligned}$$

From the relation (3.21), we can now write

$$\begin{aligned}
\{(d_1 + 2d_2 + d_3 + 4d_4) &= \frac{32}{15k^2\pi^4}\left(k^2\pi^4 - 360\right);\ d_1 = 12; \\
&\qquad (d_1 + d_2 + d_3) = 2\}. \quad (3.24)
\end{aligned}$$

$$\left\{(d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3\right\} = 3(c_1 + 2)(c_1 + 6). \quad (3.25)$$

Since $c_1 \in [0, 2]$, using the result $(c_1 + a)(c_1 + b) \geq (c_1 - a)(c_1 - b)$, where $a, b \geq 0$ in the relation (3.25), we get

$$-\left\{(d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3\right\} \leq -3(c_1 - 2)(c_1 - 6). \quad (3.26)$$

Substituting the calculated values from (3.24) and (3.26) on the right-hand side of (3.23), we have

$$\begin{aligned}
4|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| &\leq \left|\frac{32}{15k^2\pi^4}\left(k^2\pi^4 - 360\right)c_1^4 + 24c_1(4-c_1^2)\right. \\
&\quad \left. + 4c_1^2(4-c_1^2)|x| - 3(c_1 - 2)(c_1 - 6)(4-c_1^2)|x|^2\right|. \quad (3.27)
\end{aligned}$$

Choosing $c_1 = c \in [0, 2]$, applying triangle inequality and replacing $|x|$ by μ on the right-hand side of (3.27), we get

$$4|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leq \Big[\frac{32}{15k^2\pi^4}\Big\{360 - k^2\pi^4\Big\}c^4 + 24c(4-c^2) + 4c^2(4-c^2)\mu + 3(c-2)(c-6)(4-c^2)\mu^2\Big] = F(c,\mu),\ \text{for } 0 \leq \mu = |x| \leq 1, \quad (3.28)$$

$$\text{where } F(c,\mu) = \frac{32}{15k^2\pi^4}\Big\{360 - k^2\pi^4\Big\}c^4 + 24c(4-c^2) + 4c^2(4-c^2)\mu + 3(c-2)(c-6)(4-c^2)\mu^2. \quad (3.29)$$

We next maximize the function $F(c, \mu)$ on the closed region $[0, 1] \times [0, 2]$. Differentiating $F(c, \mu)$ in (3.29) partially with respect to μ, we obtain

$$\frac{\partial F}{\partial \mu} = [4c^2 + 6(c-2)(c-6)\mu] \times (4 - c^2). \quad (3.30)$$

For $0 < \mu < 1$ and for fixed c with $0 < c < 2$, from (3.30), we observe that $\frac{\partial F}{\partial \mu} > 0$. Consequently, $F(c, \mu)$ becomes an increasing function of μ and hence it cannot have a maximum value at any point in the interior of the closed region $[0, 1] \times [0, 2]$. Moreover, for fixed $c \in [0, 2]$, we have

$$\max_{0\leq\mu\leq 1} F(c,\mu) = F(c,1) = G(c). \quad (3.31)$$

Therefore, replacing μ by 1 in (3.29), upon simplification, we obtain

$$G(c) = \frac{1}{15k^2\pi^4}\Big\{11520 - 137k^2\pi^4\Big\}c^4 - 8c^2 + 144, \quad (3.32)$$

$$G'(c) = \frac{4}{15k^2\pi^4}\Big\{11520 - 137k^2\pi^4\Big\}c^3 - 16c. \quad (3.33)$$

From (3.33), we observe that $G'(c) \leq 0$, for every $c \in [0, 2]$ and for all values of k. Therefore, $G(c)$ is a monotonically decreasing function of c in the interval $[0, 2]$ and hence its maximum value occurs at $c = 0$ only. From (3.32), we get

$$\max_{0\leq c\leq 2} G(c) = G(0) = 144. \quad (3.34)$$

Simplifying the relations (3.28) and (3.34), we obtain

$$|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \ \leq\ 36. \quad (3.35)$$

From the relations (3.20) and (3.35), after simplifying, we get

$$| t_{k+1}t_{3k+1} - t_{2k+1}^2 | \leq \left[\frac{8}{k\pi^2}\right]^2. \tag{3.36}$$

By setting $c_1 = c = 0$ and selecting $x = -1$ in (2.2) and (2.4), we find that $c_2 = -2$ and $c_3 = 0$. Using these values in (3.35), we observe that equality is attained, which shows that our result it sharp. For these values, we derive that

$$p(z) = \frac{1-z^2}{1+z^2} = 1 - 2z^2 + 2z^4 - ..... \text{ and } w(z) = -z^2. \tag{3.37}$$

Therefore, in this case the extremal function is $\left[\frac{zf'(z)}{f(z)}\right] = \frac{1-z^2}{1+z^2}$. This completes the proof of our Theorem 3.1. □

Theorem 3.2. *If f given by (1.1) belongs to UCV and F is the k^{th} root transformation of f given by (1.3) then*

$$| t_{k+1}t_{3k+1} - t_{2k+1}^2 | \leq \left[\frac{8}{3k\pi^2}\right]^2$$

and the inequality is sharp.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in UCV$, from the Definition 1.2, we have

$$\left[1 + \frac{zf''(z)}{f'(z)}\right] \prec q(z), \quad \forall z \in E.$$

By the subordination principle, there exist a Schwarz's function $w(z)$ such that

$$\left[1 + \frac{zf''(z)}{f'(z)}\right] \prec [q\{w(z)\}], \forall z \in E. \tag{3.38}$$

Define a function $h(z)$ such that

$$h(z) = \left[1 + \frac{zf''(z)}{f'(z)}\right] = \{1 + b_1 z + b_2 z^2 + b_3 z^3 + ...\} = \left[1 + \sum_{n=1}^{\infty} b_n z^n\right]$$
$$\Leftrightarrow [f'(z) + zf''(z)] = [f'(z)h(z)]. \tag{3.39}$$

Replacing $f'(z)$, $f''(z)$ and $h(z)$ by their equivalent expressions in series in the expression (3.39), we have

$$\left[\left\{1+\sum_{n=2}^{\infty} na_n z^{n-1}\right\} + z\left\{\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}\right\}\right] = \left[\left\{1+\sum_{n=2}^{\infty} na_n z^{n-1}\right\}\left\{1+\sum_{n=1}^{\infty} b_n z^n\right\}\right].$$

Upon simplification, we obtain

$$1+2a_2z+6a_3z^2+12a_4z^3+... = 1+b_1z+(2b_1a_2+b_2)z^2 + (3b_1a_3+2b_2a_2+b_3)z^3+.... \quad (3.40)$$

Equating the coefficients of like powers of z, z^2 and z^3 respectively on both sides of (3.40), after simplifying, we get

$$a_2 = \frac{b_1}{2};\ a_3 = \frac{1}{6}(b_2+b_1^2);\ a_4 = \frac{1}{24}(2b_3+3b_1b_2+b_1^3). \quad (3.41)$$

Applying the same procedure as described in Theorem 3.1, we obtain

$$a_2 = \frac{4}{\pi^2}c_1;\ a_3 = \frac{4}{3\pi^2}\left[c_2 - \left\{\frac{1}{6} - \frac{8}{\pi^2}\right\}c_1^2\right]; \quad a_4 = \frac{2}{3\pi^2}\left[c_3 - \left\{\frac{1}{3} - \frac{12}{\pi^2}\right\}c_1c_2 + \left\{\frac{2}{45} - \frac{2}{\pi^2} + \frac{32}{\pi^4}\right\}c_1^3\right]. \quad (3.42)$$

From the equations (1.3) and (3.17) together with (3.42), after simplifying, we get

$$t_{k+1} = \frac{4c_1}{k\pi^2}\ ; \quad t_{2k+1} = \frac{4}{3k\pi^2}\left[c_2 - \left\{\frac{1}{6} - \frac{2(k+3)}{k\pi^2}\right\}c_1^2\right]\ ;$$

$$t_{3k+1} = \frac{2}{3k\pi^2}\left[c_3 + \left\{\frac{-1}{3} + \frac{4(k+2)}{k\pi^2}\right\}c_1c_2 + \left\{\frac{2}{45} - \frac{2(k+2)}{3k\pi^2} + \frac{16(k+1)}{k^2\pi^4}\right\}c_1^3\right]. \quad (3.43)$$

Substituting the values of t_{k+1}, t_{2k+1} and t_{3k+1} from (3.43) in the second Hankel determinant $|t_{k+1}t_{3k+1} - t_{2k+1}^2|$ to the k^{th} transformation for the function $f \in UCV$, upon simplification, we obtain

$$|t_{k+1}t_{3k+1} - t_{2k+1}^2| = \frac{4}{405k^2\pi^8} \times \Big|270\pi^4 c_1c_3 + 30\pi^2\{-\pi^2 + 12\}c_1^2c_2 \\ -180\pi^4c_2^2 + \{7\pi^4 - 60\pi^2 - 720\left(1 + \frac{3}{k^2}\right)\}c_1^4\Big|. \quad (3.44)$$

The above expression is equivalent to

$$|t_{k+1}t_{3k+1} - t_{2k+1}^2| = \frac{4}{405k^2\pi^8}|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4|, \quad (3.45)$$

where $d_1 = 270\pi^4$; $d_2 = 30\pi^2\left\{-\pi^2 + 12\right\}$;

$$d_3 = -180\pi^4;\ d_4 = \left\{7\pi^4 - 60\pi^2 - 720\left(1 + \frac{3}{k^2}\right)\right\}. \quad (3.46)$$

Applying the same procedure as described in Theorem 3.1, we get

$$4|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leq |(d_1 + 2d_2 + d_3 + 4d_4)c_1^4 + 2d_1c_1(4 - c_1^2) \\ + 2(d_1 + d_2 + d_3)c_1^2(4 - c_1^2)|x| - \\ \left\{(d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3\right\}(4 - c_1^2)|x|^2|. \quad (3.47)$$

Using the values of d_1, d_2, d_3 and d_4 from (3.46), upon simplification, we obtain

$$(d_1 + 2d_2 + d_3 + 4d_4) = \left\{58\pi^4 + 480\pi^2 - 2880\left(1 + \frac{3}{k^2}\right)\right\}; \\ d_1 = 270\pi^4;\ (d_1 + d_2 + d_3) = \left\{60\pi^4 + 360\pi^2\right\}. \quad (3.48)$$

$$\left\{(d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3\right\} = 90\pi^4(c_1 + 2)(c_1 + 4). \quad (3.49)$$

Since $c_1 \in [0, 2]$, using the result $(c_1 + a)(c_1 + b) \geq (c_1 - a)(c_1 - b)$, where $a, b \geq 0$ in the relation (3.49), we get

$$-\left\{(d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3\right\} \leq -90\pi^4(c_1 - 2)(c_1 - 4). \quad (3.50)$$

Substituting the calculated values from (3.48) and (3.50) on the right-hand side of (3.47), we obtain

$$4|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4| \leq \Bigg| \left\{58\pi^4+480\pi^2-2880\left(1+\frac{3}{k^2}\right)\right\}c_1^4+ 540\pi^4 c_1(4-c_1^2)+120\pi^2\left(\pi^2+6\right)c_1^2(4-c_1^2)|x| - 90\pi^4(c_1-2)(c_1-4)(4-c_1^2)|x|^2\Bigg|. \quad (3.51)$$

Choosing $c_1 = c \in [0,2]$, applying the triangle inequality and replacing $|x|$ by μ on the right-hand side of the above inequality, we have

$$4|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4| \leq \Bigg[\left\{2880\left(1+\frac{3}{k^2}\right)+480\pi^2-58\pi^4\right\}c^4+ 540\pi^4 c(4-c^2)+120\pi^2\left(\pi^2-6\right)c^2(4-c^2)\mu + 90\pi^4(c-2)(c-4)(4-c^2)\mu^2\Bigg] = F(c,\mu), \text{for } 0\leq\mu=|x|\leq 1, \quad (3.52)$$

$$\text{where } F(c,\mu) = \Bigg[\left\{2880\left(1+\frac{3}{k^2}\right)+480\pi^2-58\pi^4\right\}c^4+540\pi^4c(4-c^2) + 120\pi^2\left(\pi^2-6\right)c^2(4-c^2)\mu+90\pi^4(c-2)(c-4)(4-c^2)\mu^2\Bigg]. \quad (3.53)$$

Applying the same procedure as described in Theorem 3.1, we observe that $\frac{\partial F}{\partial \mu} > 0$, so that $F(c,\mu)$ is an increasing function of μ and hence its maximum value does not occur at any point in the interior of the closed region $[0,1]\times[0,2]$. Further, for fixed $c \in [0,2]$, we have

$$\max_{0\leq\mu\leq1} F(c,\mu) = F(c,1) = G(c). \quad (3.54)$$

Therefore, replacing μ by 1 in (3.53), upon simplification, we obtain

$$G(c) = \left\{2880\left(1+\frac{3}{k^2}\right)-268\pi^4+1200\pi^2\right\}c^4-120\pi^2\left\{24-\pi^2\right\}c^2+2880\pi^4, \quad (3.55)$$

$$G'(c) = 4\left\{2880\left(1+\frac{3}{k^2}\right)-268\pi^4+1200\pi^2\right\}c^3-240\pi^2\left\{24-\pi^2\right\}c. \quad (3.56)$$

From (3.56), for fixed $c \in [0, 2]$ and for every k, we observe that $G'(c) \leq 0$, which shows that $G(c)$ is a monotonically decreasing function of c and hence it attains the maximum value at $c = 0$ only. From (3.55), we get

$$\max_{0 \leq c \leq 2} G(c) = G(0) = 2880\pi^4. \tag{3.57}$$

From (3.52) and (3.57), upon simplification, we obtain

$$|d_1 c_1 c_3 + d_2 c_1^2 c_2 + d_3 c_2^2 + d_4 c_1^4| \leq 720\pi^4. \tag{3.58}$$

Simplifying the relations (3.45) and (3.58), we get

$$| \, t_{k+1} t_{3k+1} - t_{2k+1}^2 \, | \leq \left[\frac{8}{3k\pi^2} \right]^2. \tag{3.59}$$

If we set $c_1 = c = 0$ and take $x = 1$ in (2.2) and (2.4), we find that $c_2 = 2$ and $c_3 = 0$. Using these values in (3.58), we see that equality is attained, which shows that our result it sharp. For these values, we derive that

$$p(z) = \frac{1 + z^2}{1 - z^2} = 1 + 2z^2 + 2z^4 + ... \text{ and } w(z) = z^2. \tag{3.60}$$

Therefore, the extremal function in this case is $\left[1 + \frac{zf''(z)}{f'(z)} \right] = \frac{1+z^2}{1-z^2}$. This completes the proof of our Theorem 3.2. □

Remark 3.3. For the choice of $k = 1$, the result coincides with that of VamsheeKrishna and RamReddy [22].

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