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Annales mathématiques Blaise Pascal **21**, 61-69 (2014)

On 1-cocycles induced by a positive definite function on a locally compact abelian group

Jordan Franks
Alain Valette

Abstract

For φ a normalized positive definite function on a locally compact abelian group G, let π_φ be the unitary representation associated to φ by the GNS construction. We give necessary and sufficient conditions for the vanishing of 1-cohomology $H^1(G, \pi_\varphi)$ and reduced 1-cohomology $\overline{H}^1(G, \pi_\varphi)$. For example, $\overline{H}^1(G, \pi_\varphi) = 0$ if and only if either $\mathrm{Hom}(G, \mathbb{C}) = 0$ or $\mu_\varphi(1_G) = 0$, where 1_G is the trivial character of G and μ_φ is the probability measure on the Pontryagin dual $\hat{G}$ associated to φ by Bochner's Theorem. This streamlines an argument of Guichardet (see Theorem 4 in [7]).

Sur les 1-cocycles induits par une fonction de type positif sur un groupe abélien localement compact

Résumé

Soit φ une fonction de type positif normalisée sur un groupe localement compact abélien G, et π_φ la représentation unitaire de G obtenue par construction GNS. Nous donnons des conditions nécessaires et suffisantes pour l'annulation de la 1-cohomologie $H^1(G, \pi_\varphi)$ et de la 1-cohomologie réduite $\overline{H}^1(G, \pi_\varphi)$. Par exemple, $\overline{H}^1(G, \pi_\varphi) = 0$ si et seulement si ou bien $\mathrm{Hom}(G, \mathbb{C}) = 0$ ou bien $\mu_\varphi(1_G) = 0$, où 1_G est le caractère trivial de G et μ_φ est la mesure de probabilité sur le dual de Pontryagin $\hat{G}$ associée à φ par le théorème de Bochner. Cela simplifie un argument de Guichardet (Théorème 4 de [7]).

1. Introduction

The Gel'fand-Naimark-Segal construction (see [1]) provides a correspondence between positive definite functions φ on a locally compact

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group G and cyclic representations π_φ on Hilbert space. This allows one to construct a dictionary between the functional-analytic and algebro-geometric pictures of φ and π_φ. For example, φ is an extreme point in the cone $\mathcal{P}(G)$ of positive definite functions on G if and only if π_φ is an irreducible representation; or, there exists a constant $a > 0$ such that $\varphi - a$ is again positive definite if and only if π_φ has nonzero fixed vectors (see [4]).

In view of their importance for rigidity questions and Kazhdan's property (T),[1] it seems natural to try to fit 1-cohomology and reduced 1-cohomology of π_φ in that dictionary. Two examples show that the answer will depend heavily on the underlying group G: if φ is the constant function 1, then π_φ is the trivial 1-dimensional representation 1_G of G, and $H^1(G, 1_G) = \mathrm{Hom}(G, \mathbb{C})$; the vanishing of this group depends on the structure of the abelianized group $G/\overline{[G,G]}$ (e.g. if G is compactly generated, then $\mathrm{Hom}(G, \mathbb{C}) = 0$ if and only if $G/\overline{[G,G]}$ is compact). For a second example, assume that G is discrete, and let $\varphi = \delta_1$ be the Dirac measure at the identity; then π_φ is the left regular representation λ_G of G on $\ell^2(G)$, and $H^1(G, \lambda_G) = 0$ if and only if G is non-amenable with vanishing first ℓ^2-Betti number (see [2]).

In this paper we consider the case where G is a locally compact abelian group. This assumption makes the question more tractable and gives us a powerful tool, Bochner's Theorem: φ is the Fourier transform of a positive Borel measure μ_φ on the Pontryagin dual $\hat{G}$ (see [5]). Without relying on the cohomological machinery available in the literature (see [8, 1]), we achieve by completely elementary means the results of this paper, namely, we show that the existence of nontrivial 1-cohomology is determined by two factors: the existence of non-trivial homomorphisms from G to $\mathbb{C}$ and the behavior of μ_φ near the trivial character 1_G. This is all delineated in a precise way in Theorem 1.

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[1] Recall Shalom's results, see Theorems 0.2 and 6.1 in [10]: for a compactly generated group G, the group G has property (T) if and only if $\overline{H}^1(G, \pi) = 0$ for every unitary representation π of G, if and only if $H^1(G, \sigma) = 0$ for every unitary irreducible representation of G.

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2. **Statement of results**

For G a locally compact group and π a unitary representation of G on a Hilbert space $\mathcal{H}_\pi$, recall that the space of *1-cocycles* for π is

$$Z^1(G,\pi) = \{b : G \to \mathcal{H}_\pi : b \text{ continuous}, b(gh) = \pi(g)b(h) + b(g) \text{ for all } g, h \in G\}.$$

The space of *1-coboundaries* for π is:

$$B^1(G,\pi) = \{b \in Z^1(G,\pi) : \exists v \in \mathcal{H}_\pi \text{ such that } b(g) = \pi(g)v - v \text{ for every } g \in G\}.$$

The *1-cohomology* of π is then the quotient

$$H^1(G,\pi) = Z^1(G,\pi)/B^1(G,\pi).$$

Endow $Z^1(G,\pi)$ with the topology of uniform convergence on compact subsets of G. The *reduced 1-cohomology* of π is the quotient of the space of 1-cocycles by the closure of the space of 1-coboundaries, i.e.

$$\overline{H}^1(G,\pi) = Z^1(G,\pi)/\overline{B^1(G,\pi)}.$$

From now on, let G be a locally compact *abelian* group, φ a positive definite function on G. Excluding the zero function, we may without loss of generality take φ to be normalized ($\varphi(e) = 1$). Let μ_φ be the probability measure on the Pontryagin dual $\hat{G}$ provided by Bochner's Theorem, i.e. $\varphi(x) = \int_{\widehat{G}} \xi(x)\, d\mu_\varphi(\xi)$ for $x \in G$. Let $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ be the cyclic representation of G associated to φ through the GNS construction, so that the cyclic vector $\xi_\varphi \in \mathcal{H}_\varphi$ satisfies $\langle \pi_\varphi(x)\xi_\varphi | \xi_\varphi \rangle = \varphi(x)$. Let also ρ_φ be the representation of G on $L^2(\hat{G}, \mu_\varphi)$ given by $(\rho_\varphi(x)f)(\xi) = \xi(x)f(\xi)$ $(\xi \in \hat{G}, f \in L^2(\hat{G}, \mu_\varphi)$.

If λ_G denotes the left regular representation of G, then $(\widehat{\lambda_G(x)h})(\xi) = \int_{\hat{G}} h(x^{-1}g)\overline{\xi(g)}dg = \overline{\xi(x)}\hat{h}(\xi)$ for $h \in L^1(G)$. From Plancherel's Theorem, it follows that the composition of the Fourier transform with conjugation is a unitary equivalence between the regular representation on $L^2(G)$ and the unitary representation defined by $\Big(f(\xi) \mapsto \xi(x)f(\xi)\Big)$ on $L^2(\hat{G})$. This, together with Bochner's Theorem, intuits our introduction of ρ_φ, as well as the following proposition which we prove in Section 3.

Proposition 1. *The representations π_φ and ρ_φ are unitarily equivalent.*

We assume from now on that φ is not the constant function 1, so that μ_φ is not the Dirac measure at the trivial character 1_G of G. This is still equivalent to $\mu_\varphi(1_G) < 1$. Let $\mu_\varphi^\perp$ be the probability measure on $\hat{G}$ defined by $\mu_\varphi = \mu_\varphi(1_G)\delta_{1_G} + (1 - \mu_\varphi(1_G))\mu_\varphi^\perp$.

Let π_φ^0 be the (trivial) subrepresentation of π_φ on the subspace $\mathcal{H}_\varphi^0$ of π_φ-fixed vectors, and $\pi_\varphi^\perp$ be the subrepresentation on the orthogonal complement, so that $\pi_\varphi = \pi_\varphi^0 \oplus \pi_\varphi^\perp$. A simple computation in the ρ_φ-picture shows that $\mathcal{H}_\varphi^0 \neq 0$ if and only if $\mu_\varphi(1_G) > 0$, and in this case $\mathcal{H}_\varphi^0 = \mathbb{C}\delta_{1_G}$. Moreover, the map $L^2(\hat{G} \backslash \{1_G\}, \mu_\varphi^\perp) \to L^2(\hat{G}, \mu_\varphi) : f \mapsto \frac{f}{\sqrt{1-\mu_\varphi(1_G)}}$ is isometric and identifies $\pi_\varphi^\perp$ with the restriction of ρ_φ to $L^2(\hat{G} \backslash \{1_G\}, \mu_\varphi^\perp)$.

Our main result is:

Theorem 1. *Let φ be a nonconstant, normalized positive definite function on a locally compact abelian group G.*

1) *Consider the following statements:*
   i) $H^1(G, \pi_\varphi) = 0$*;*
   ii) *Both of the following properties are satisfied:*
      a) $\mu_\varphi(1_G) = 0$ *or* $Hom(G, \mathbb{C}) = 0$*;*
      b) $1_G \notin supp(\mu_\varphi^\perp)$.

   Then $(ii) \Rightarrow (i)$, and the converse holds if G is σ-compact.

2) *The following are equivalent:*
   i) $\overline{H}^1(G, \pi_\varphi) = 0$*;*
   ii) $\mu_\varphi(1_G) = 0$ *or* $Hom(G, \mathbb{C}) = 0$.

This result will be proved in Section 4. It is essentially equivalent to Theorem 4 in [7], but we emphasize the fact that our proof is direct and based on explicit construction of cocycles and coboundaries. Theorem 1 is sharp in the sense that the implication $(i) \Rightarrow (ii)$ in Part 1 fails in general if G is not assumed σ-compact (see Example 1 below).

3. Proof of Proposition 1

Lemma 1. *The constant function $1 \in L^2(\hat{G}, \mu_\varphi)$ is a cyclic vector for ρ_φ.*

Proof: Consider the associated $L^1(G)$-representation of ρ_φ acting on $L^2(\hat{G},\mu_\varphi^\perp)$ and given by $\rho_\varphi(f) := \int_G f(x)\rho_\varphi(x)\,dx$, $f \in L^1(G)$. Then $(\rho_\varphi(f).1)(\xi) = \int_G f(x)\xi(x)\,dx = \hat{f}(\bar{\xi})$.

Denote by $C_0(\hat{G})$ the space of continuous functions vanishing at infinity on $\hat{G}$, and recall that $\hat{f} \in C_0(\hat{G})$ (the Riemann-Lebesgue Lemma). It is classical that the Fourier transform $L^1(G) \to C_0(\hat{G})$ is a continuous algebra homomorphism with dense image (a consequence of Stone-Weierstrass). Now compose this homomorphism with the continuous inclusion $C_0(\hat{G}) \to L^2(\hat{G},\mu_\varphi) : h \mapsto h.1$. Since continuous functions with compact support are dense in $L^2(\hat{G},\mu_\varphi)$, this inclusion has dense image. Since the map $L^1(G) \to L^2(\hat{G},\mu_\varphi) : f \mapsto \rho_\varphi(f).1$ is the composition of the previous maps, it has dense image, meaning that 1 is a cyclic vector for the $L^1(G)$-representation ρ_φ. Then 1 is also a cyclic vector for the unitary G-representation ρ_φ by a Dirac sequence argument (see [4]). □

Observe that $\langle \rho_\varphi(x).1|1\rangle = \int_{\widehat{G}} \xi(x)\,d\mu_\varphi(\xi) = \varphi(x)$, so Proposition 1 follows from Lemma 1 and the uniqueness statement of the GNS construction. □

4. **Proof of Theorem 1**

Since $\pi_\varphi = \pi_\varphi^0 \oplus \pi_\varphi^\perp$, we have $H^1(G,\pi_\varphi) = H^1(G,\pi_\varphi^0) \oplus H^1(G,\pi_\varphi^\perp)$ and analogously for $\overline{H}^1$. As $B^1(G,\pi_\varphi^0) = 0$, we see that $H^1(G,\pi_\varphi^0) = 0$ if and only if $\overline{H}^1(G,\pi_\varphi^0) = 0$, if and only if either $\mu_\varphi(1_G) = 0$ or $\mathrm{Hom}(G,\mathbb{C}) = 0$: this proves the implications $(i) \Rightarrow (ii)(a)$ in part 1 of Theorem 1, and $(i) \Rightarrow (ii)$ in part 2 of Theorem 1; moreover, it reduces the main result to:

Theorem 2. *Let φ be a nonconstant, normalized positive definite function on a locally compact abelian group G.*

1) *If $1_G \notin supp(\mu_\varphi^\perp)$, then $H^1(G,\pi_\varphi^\perp) = 0$. The converse holds if G is σ-compact.*

2) $\overline{H}^1(G,\pi_\varphi^\perp) = 0$.

Remark: As was pointed out by the referee of a previous version, when G is separable, combining Corollary 1 and Proposition 2 in [6], one gets an explicit description of $Z^1(G,\pi_\varphi^\perp)$: it identifies with the space of measurable functions u on $\hat{G}$ such that, for every $g \in G$, the integral $\int_{\hat{G}} |\xi(g)-1|^2|u(\xi)|^2\, d\mu_\varphi^\perp(\xi)$ is finite, and it tends to zero for g tending to the identity of G (in this picture, $B^1(G,\pi_\varphi^\perp)$ identifies with $L^2(\hat{G},\mu_\varphi^\perp)$). We will, however, not need that result.

Example 1. *In Part 1 of Theorems 1 and 2, the converse implications are false when G is not assumed to be σ-compact. Indeed, let G be an uncountable abelian group with the discrete topology, and take $\varphi = \delta_1$. Then π_φ is the left regular representation λ_G on $\ell^2(G)$, while $\mu_\varphi = \mu_\varphi^\perp$ is the Haar measure on the compact group $\hat{G}$. Since μ_φ has full support, in particular 1_G lies in its support. On the other hand $H^1(G,\lambda_G) = 0$ by Proposition 4.13 in* [3].

To prove the implication "⇒" in part 1 of Theorem 2, we will need:

Lemma 2. *Let F be a closed subset of $\hat{G}$, with $1_G \notin F$.*

a) *There exists a regular Borel probability measure ν_0 on G such that the Fourier transform $\widehat{\nu_0}$ vanishes on F.*

b) *For every $\varepsilon > 0$, there exists a compactly supported regular Borel probability measure ν on G such that $|\hat{\nu}| < \varepsilon$ on F.*

Proof: (a) See Section 1.5.2 in [9].

(b) Let ν_0 be a probability measure on G as in (a). Let δ be a number $0 < \delta < 1$, to be determined later. Let C be a compact subset of G such that $\nu_0(C) > 1-\delta$. Let ν be the probability measure on G defined by $\nu(B) = \frac{\nu_0(B\cap C)}{\nu_0(C)}$, for every Borel subset $B \subseteq G$. By taking δ small enough, the total variation distance $|\nu_0 - \nu|(G)$ between ν_0 and ν can be made arbitrarily small. For any finite signed measure μ on G, we have the classical inequality $|\int_G f(x)\, d\mu(x)| \leq \|f\|_\infty |\mu|(G)$; applied to $\mu = \nu_0 - \nu$ and $f(x) = \xi(x)$ with $\xi \in \hat{G}$, it gives $|\widehat{\nu_0}(\xi) - \hat{\nu}(\xi)| \leq |\nu_0 - \nu|(G)$, so that $\|\widehat{\nu_0} - \widehat{\nu}\|_\infty < \varepsilon$ for δ small enough. □

Proof of "⇒" in part 1 of Theorem 2: We assume that 1_G is not in the support of $\mu_\varphi^\perp$, and prove that $H^1(G,\pi_\varphi^\perp) = 0$. Let $b \in Z^1(G,\pi_\varphi^\perp)$ be a 1-cocycle. Expanding $b(xy) = b(yx)$ using the cocycle relation, we get:

$$(1-\pi_\varphi^\perp(x))b(y) = (1-\pi_\varphi^\perp(y))b(x) \quad (x,y \in G).$$

In the realization of $\pi_\varphi^\perp$ on $L^2(\hat{G}, \mu_\varphi^\perp)$, this gives:

$$(1 - \xi(x))b(y)(\xi) = (1 - \xi(y))b(x)(\xi) \tag{4.1}$$

almost everywhere in ξ (w.r.t. $\mu_\varphi^\perp$). By Lemma 2, we can find a compactly supported probability measure ν on G such that $|1 - \hat{\nu}| \geq \frac{1}{2}$ on $\mathrm{supp}(\mu_\varphi^\perp)$. Define an element $v \in L^2(\hat{G}, \mu_\varphi^\perp)$ by $v := \int_G b(y)\, d\nu(y)$: since b is continuous and ν is compactly supported, the integral exists (in the weak sense) in $L^2(\hat{G}, \mu_\varphi^\perp)$. Integrating (4.1) w.r.t. ν in the variable y, we get:

$$(1 - \xi(x))v(\xi) = (1 - \hat{\nu}(\xi))b(x)(\xi) \tag{4.2}$$

almost everywhere in ξ. Since $|1 - \hat{\nu}| \geq \frac{1}{2}$ on $\mathrm{supp}(\mu_\varphi^\perp)$, the function $w(\xi) := \frac{v(\xi)}{\hat{\nu}(\xi)-1}$ belongs to $L^2(\hat{G}, \mu_\varphi^\perp)$, and by (4.2) its coboundary is exactly b. □

Lemma 3. *Let H be a locally compact group. Let $(\sigma_n)_{n\geq 0}$ be a sequence of unitary representations of H without nonzero fixed vectors, with σ_n acting on a Hilbert space $\mathcal{H}_n$. Assume that, for each $n \geq 0$, there exists a unit vector $\eta_n \in \mathcal{H}_n$ such that the series $\sum_{n=0}^\infty \|\sigma_n(x)\eta_n - \eta_n\|^2$ converges uniformly on compact subsets of H. Set $\sigma = \oplus_{n=0}^\infty \sigma_n$. Then $b(x) := \oplus_{n=0}^\infty(\sigma_n(x)\eta_n - \eta_n)$ defines a nonzero element in $H^1(G, \sigma)$.*

Proof: By assumption $b(x)$ belongs to $\oplus_{n=0}^\infty \mathcal{H}_n$ and the map $H \to \oplus_{n=0}^\infty \mathcal{H}_n$: $x \mapsto b(x)$ is continuous. Let $\eta \in \prod_{n=0}^\infty \mathcal{H}_n$ be defined as $\eta = (\eta_n)_{n\geq 0}$. Since b is the formal coboundary of η, we have $b \in Z^1(H, \sigma)$. To prove that b is not a coboundary, it suffices to show that the associated affine action $\alpha(x)v = \sigma(x)v + b(x)$ on $\oplus_{n=0}^\infty \mathcal{H}_n$ has no fixed point. But $\alpha(x)v = v$ translates into $\sigma_n(x)(v_n + \eta_n) = v_n + \eta_n$ for every $x \in H$ and $n \geq 0$. Since σ_n has no nonzero fixed vector, we have $v_n + \eta_n = 0$ so $\|v_n\| = 1$, which contradicts $\sum_{n=0}^\infty \|v_n\|^2 < +\infty$. □

Proof of "$\Leftarrow$" in part 1 of Theorem 2, assuming G to be σ-compact: We assume that 1_G is in the support of $\mu_\varphi^\perp$, and prove that $H^1(G, \pi_\varphi) \neq 0$. Let $(K_n)_{n\geq 0}$ be an increasing sequence of compact subsets of G, with $G = \cup_{n=0}^\infty K_n$, and $K_0 = \{1\}$. Define a basis $(U_k)_{k\geq 0}$ of open neighborhoods of 1_G in $\hat{G}$ by $U_k = \{\xi \in \hat{G} : \max_{g\in K_k} |\xi(g) - 1| < 2^{-k}\}$ (observe that $U_0 = \hat{G}$). Define a sequence $(k_n)_{n\geq 0}$ inductively by $k_0 = 0$ and $k_{n+1} = \min\{k : k > k_n,\ \mu_\varphi^\perp(U_k) < \mu_\varphi^\perp(U_{k_n})\}$ (since $\mu_\varphi^\perp\{1_G\} = 0$ and 1_G is in the support of $\mu_\varphi^\perp$, this is well-defined). Set then $C_n := U_{k_n} \backslash U_{k_{n+1}}$

for $n \geq 0$. For each $n \geq 0$ let $\mathcal{H}_n$ be the space of functions in $L^2(\hat{G}, \mu_\varphi^\perp)$ which are $\mu_\varphi^\perp$-almost everywhere 0 on $\hat{G} \backslash C_n$. Then $\mathcal{H}_n$ is a closed, ρ_φ-invariant subspace of $L^2(\hat{G}, \mu_\varphi^\perp)$. Denote by σ_n the restriction of ρ_φ to $\mathcal{H}_n$, so that $L^2(\hat{G}, \mu_\varphi^\perp) = \oplus_{n=0}^\infty \mathcal{H}_n$ and $\rho_\varphi = \oplus_{n=0}^\infty \sigma_n$. Let $\eta_n = \frac{\mathbf{1}_{C_n}}{\sqrt{\mu_\varphi^\perp(C_n)}}$ be the normalized characteristic function of C_n. To appeal to Lemma 3, we still have to check that $x \mapsto \sum_{n=0}^\infty \|\sigma_n(x)\eta_n - \eta_n\|^2$ converges uniformly on every compact subset K of G. Clearly we may assume $K = K_\ell$. For $n \geq \ell$ and $x \in K_\ell$ and $\xi \in C_n$, we have $|\xi(x) - 1| < 2^{-k_n}$, hence

$$\begin{aligned} \max_{x \in K_\ell} \sum_{n=\ell}^\infty \|\sigma(x)\eta_n - \eta_n\|^2 &= \max_{x \in K_\ell} \sum_{n=\ell}^\infty \frac{1}{\mu_\varphi^\perp(C_n)} \int_{C_n} |\xi(x) - 1|^2 \, d\mu_\varphi^\perp(\xi) \\ &\leq \sum_{n=\ell}^\infty 4^{-k_n} \leq \sum_{n=0}^\infty 4^{-n} = \frac{4}{3} \end{aligned}$$

and

$$\begin{aligned} \max_{x \in K_\ell} \sum_{n=0}^\infty \|\sigma(x)\eta_n - \eta_n\|^2 &\leq (\max_{x \in K_\ell} \sum_{n=0}^{\ell-1} \|\sigma(x)\eta_n - \eta_n\|^2) + \frac{4}{3} \\ &\leq 4\ell + \frac{4}{3} < +\infty. \end{aligned}$$

So the result follows from Lemma 3. □

Proof of part 2 of Theorem 2: Let $b \in Z^1(G, \pi_\varphi^\perp)$ be a 1-cocycle. We must show that b is a limit of 1-coboundaries (uniformly on compact subsets of G). Since $\mu_\varphi^\perp(1_G) = 0$, by the regularity of $\mu_\varphi^\perp$, we can find a decreasing sequence of relatively compact open neighborhoods $(V_n)_{n\geq 0}$ of 1_G, such that $\mu_\varphi^\perp(V_n) \to 0$ for $n \to \infty$.
Set $\mathcal{H}_n := \{f \in L^2(\hat{G}, \mu_\varphi^\perp) : f = 0 \text{ a.e. on } V_n\}$; then $\mathcal{H}_n$ is a closed, ρ_φ-invariant subspace, and the sequence $(\mathcal{H}_n)_{n\geq 0}$ is increasing with dense union in $L^2(\hat{G}, \mu_\varphi^\perp)$. Let ρ_n denote the restriction of ρ_φ to $\mathcal{H}_n$, and let b_n denote the projection of b onto $\mathcal{H}_n$. Then $b_n \in Z^1(G, \rho_n)$, and $\lim_{n\to\infty} b_n = b$ (uniformly on compact subsets of G). Since 1_G does not belong to the closed subset $\hat{G} \backslash V_n$, by Lemma 2 and by repeating the proof of the forward direction of part 1 of Theorem 2, we obtain that b_n is a coboundary for each $n \geq 0$. □

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