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The new properties of the theta functions


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The new properties of the theta functions

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Abstract

It is shown, that the function

\[ H(x) = \sum_{k=-\infty}^{\infty} e^{-k^2 x} \]

satisfies the relation

\[ H(x) = \sum_{n=0}^{\infty} \frac{(2\pi)^{2n}}{(2n)!} H^{(n)}(x). \]

The theta function is defined by the following equation:

\[ \Theta_0 = \sum_{k=1}^{\infty} e^{-k^2 \pi t}, \quad t > 0. \] (1.1)

For computational purposes, it is convenient to introduce the function

\[ \Theta(x) = \Theta_0\left(\frac{x}{\pi}\right) = \sum_{k=1}^{\infty} e^{-k^2 x}, \quad x > 0. \]

Theorem 1.1. The following formula holds

\[ \sum_{k=1}^{\infty} e^{-k^2 x} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-x-y^2}}{1 - e^{2(-x+i\sqrt{\pi}y)}} dy. \] (1.2)
Proof. For $x > 0$ we have $|e^{2(-x+i\sqrt{x}y)}| < 1$, therefore

$$
\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-x-y^2}}{1-e^{2(-x+i\sqrt{x}y)}} dy = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x-y^2} \sum_{k=0}^{\infty} e^{2(-x+i\sqrt{x}y)k} dy
$$

$$
= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} e^{-(2k+1)x} \int_{-\infty}^{\infty} e^{-y^2+2iy\sqrt{x}k} dy
$$

$$
= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} e^{-(2k+1)x} \left( \int_{-\infty}^{\infty} e^{-y^2} \cos 2y \sqrt{x}k dy + i \int_{-\infty}^{\infty} e^{-y^2} \sin 2y \sqrt{x}k dy \right)
$$

here the first integral is equal $\sqrt{\pi}e^{-k^2x}$; second integral - zero; therefore we have

$$
= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} e^{-(k^2+2k+1)x} \sqrt{\pi} = \sum_{k=1}^{\infty} e^{-k^2x}
$$

By inserting $x = 1$ into (1.2), we obtain

$$
\sum_{k=1}^{\infty} e^{-k^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-1-y^2}}{1-e^{2(-1+iy)}} dy. \quad (1.3)
$$

In the same way we prove the relation

$$
\sum_{k=1}^{\infty} (-1)^{k-1} e^{-k^2x} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-x-y^2}}{1+e^{2(-x+iy\sqrt{x})}} dy. \quad (1.4)
$$

When $x = 1$ we have

$$
\sum_{k=1}^{\infty} (-1)^{k-1} e^{-k^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-1-y^2}}{1+e^{2(-1+iy)}} dy. \quad (1.5)
$$
The new properties of the theta functions

By virtue of (1.2) we easy find the formula

\[ \sum_{k=1}^{\infty} ke^{-k^2}x = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} \left[ 2 - e^{2(-x+iy\sqrt{x})} \right] \left[ 1 - e^{2(-x+iy\sqrt{x})} \right]^2 dy. \] (1.6)

By taking \( x = 1 \), we obtain

\[ \sum_{k=1}^{\infty} ke^{-k^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} \left[ 2 - e^{2(-1+iy)} \right] \left[ 1 - e^{2(-1+iy)} \right]^2 dy. \] (1.7)

Let us consider the following formula

\[ \sum_{k=0}^{\infty} e^{-k^2}x e^{-2axk} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-y^2}}{1 - e^{-2ax+2iy\sqrt{x}}} dy \quad a > 0. \] (1.8)

Let \( h = e^{-x} \) and \( z = e^{-ax} \) be, then

\[ \sum_{k=0}^{\infty} h^{k^2}z^{2k} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-y^2}}{1 - z^2e^{2iy\sqrt{-\ln h}}} dy. \] (1.9)

If we introduce \( h = e^{-x} \) and \( z = e^{-ax-\frac{1}{2}x} \) in (1.8) then we obtain

\[ \sum_{k=0}^{\infty} h^{(k-\frac{1}{2})^2}z^{2k-1} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-y^2}z^{1/4}}{1 - z^2h^{-1}e^{2iy\sqrt{-\ln h}}} dy. \] (1.10)

We define the function \( H(x) = 2\Theta(x) + 1 \).

**Theorem 1.2.** \( H(x) \) can be expanded in a series

\[ H(x) = e^{-x} \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} H^{(n)}(x). \] (1.11)

**Proof.** Let us consider the functions

\[ \Theta(x) = \sum_{k=1}^{\infty} e^{-k^2}x \quad \text{and} \quad v(x) = \sum_{k=1}^{\infty} ke^{-k^2}x. \]

We evaluate the sums

\[ \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} \Theta^{(n)}(x) = \sum_{k=1}^{\infty} e^{-k^2}x \left[ 1 + \frac{k^2(2x)^2}{2!} + \frac{k^4(2x)^4}{4!} + \cdots \right] \]
\[
\sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} v^{(n)}(x) = \sum_{k=1}^{\infty} e^{-k^2 x} \left[ \frac{k2x}{1!} + \frac{k^3 (2x)^3}{3!} + \frac{k^5 (2x)^5}{5!} + \cdots \right].
\]

The above two series are absolutely convergent, therefore we can change the order of the terms. Addition and subtraction of both sides of equations yields

\[
\sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} \Theta^{(n)}(x) + \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} v^{(n)}(x)
\]

\[
= \sum_{k=1}^{\infty} e^{-k^2 x} e^{2kx} = e^x \Theta(x) + e^x
\]

\[
\sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} \Theta^{(n)}(x) + \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} v^{(n)}(x)
\]

\[
= \sum_{k=1}^{\infty} e^{-k^2 x} e^{-2kx} = e^x \Theta(x) - 1.
\]

We add and subtract the equations once again. Thus we get

\[
\sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} \Theta^{(n)}(x) = e^x \Theta(x) + \frac{1}{2} e^x - \frac{1}{2}
\]

\[
\sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} v^{(n)}(x) = \frac{1}{2} e^x + \frac{1}{2}.
\]

Introduction \( H(x) = 2\Theta(x) + 1 \) and \( U(x) = 2v(x) \) yields

\[
\sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} H^{(n)}(x) = e^x H(x) \quad (1.12)
\]

\[
\sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} U^{(n)}(x) = e^x + 1. \quad (1.13)
\]
The new properties of the theta functions

The following relations for \( H(x) \) and \( U(x) \) hold

\[
\sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} H^{(n+1)}(x) = e^x H(x) \quad (1.14)
\]

\[
\sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} U^{(n)}(x) = e^x U(x) + e^x. \quad (1.15)
\]

Proof of these relations is identical as (1.12) and (1.13).

We give now a generalization of (1.12). Let us consider the following extension of \( \Theta(x) \)

\[
\Theta_1(x) = \sum_{k=1}^{\infty} e^{-k^2 x} e^{-ka} \quad x > 0, \ a \in \mathbb{R}.
\]

Then the more general version of (1.12) is equal

\[
\text{cha} H_1(x) - \text{sha} = e^{-x} \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} H_1^{(n)}(x) \quad (1.16)
\]

where \( H_1(x) = 2\Theta_1(x) + 1 \).

The proof of (1.16) is patterned after this of the Theorem 1.2.

By virtue of the relation

\[
\sum_{k=-\infty}^{\infty} e^{-k^2 x} e^{-ka} = \sum_{k=1}^{\infty} e^{-k^2 x} e^{-ka} + \sum_{k=1}^{\infty} e^{-x} e^{ka} + 1
\]

it is easy to verify that, \( \Psi(x) = \sum_{k=-\infty}^{\infty} e^{-k^2 x} e^{-ka} \) satisfies (1.16) with the member \( \text{sha} \) removed.

**Theorem 1.3.** *The formula (1.12) can be written in the form*

\[
H(x) = \sum_{n=0}^{\infty} \frac{(2\pi)^{2n}}{(2n)!} H^{(n)}(x). \quad (1.17)
\]

To prove theorem 0.3 we need the lemma

**Lemma 1.4.**

\[
[x^{-\frac{1}{2}} H(x)]^{(n)} = (-1)^n \frac{(2n)!}{2^{2n}} \sum_{k=0}^{n} \frac{2^{2k}}{(n-k)! (2k)!} x^{-n-k-\frac{1}{2}} \frac{d^k}{d\left(\frac{1}{x}\right)^k} H\left(\frac{1}{x}\right). \quad (1.18)
\]
Proof. Proof of Lemma 1.4 (by induction) The case \( n = 1 \) is obvious. We suppose, that (1.18) is true for any \( n \) and differentiate it.

\[
[x^{-\frac{1}{2}} H(x)]^{(n+1)}
\]

\[
= (-1)^{n+1} \frac{(2n)!}{2^{2n}} \sum_{k=0}^{n} \frac{2^{2k} (n+k+\frac{1}{2}) x^{-n-k-\frac{3}{2}}}{(n-k)! (2k)!} \frac{d^k}{d(\frac{1}{x})^k} H\left(\frac{1}{x}\right)
\]

\[
+ (-1)^{n+1} \frac{(2n)!}{2^{2n}} \sum_{k=0}^{n} \frac{2^{2k} x^{-n-k-\frac{5}{2}}}{(n-k)! (2k)!} \frac{d^{k+1}}{d(\frac{1}{x})^{k+1}} H\left(\frac{1}{x}\right)
\]

\[
= (-1)^{n+1} \frac{(2n)!}{2^{2n}} \left[ \frac{2^{2n} + 2}{2n!} x^{-n-\frac{3}{2}} H\left(\frac{1}{x}\right) \right]
\]

\[
+ \sum_{k=1}^{n} \frac{2^{2k}}{(n-k)!} \frac{(n+k+\frac{1}{2}) x^{-n-k-\frac{3}{2}}}{(2k)!} \frac{d^k}{d(\frac{1}{x})^k} H\left(\frac{1}{x}\right)
\]

\[
+ (-1)^{n+1} \frac{(2n)!}{2^{2n}} \sum_{k=0}^{n} \frac{2^{2k} x^{-n-k-\frac{5}{2}}}{(n+1-k)! (2k)!} \frac{d^k}{d(\frac{1}{x})^k} H\left(\frac{1}{x}\right)
\]

\[
+ (-1)^{n+1} x^{-2n-\frac{5}{2}} \frac{d^{n+1}}{d(\frac{1}{x})^{n+1}} H\left(\frac{1}{x}\right)
\]

and consequently

\[
[x^{-\frac{1}{2}} H(x)]^{(n+1)} = (-1)^{n+1} \frac{(2n + 2)!}{2^{2n+2} (n+1)!} \frac{x^{-n-\frac{3}{2}}}{(n+1)!} H\left(\frac{1}{x}\right)
\]

\[
+ \sum_{k=1}^{n} \frac{2^{2k}}{(n+1-k)!} \frac{x^{-n-k-\frac{3}{2}}}{(2k)!} \frac{d^k}{d(\frac{1}{x})^k} H\left(\frac{1}{x}\right)
\]

\[
+ (-1)^{n+1} x^{-2n-\frac{5}{2}} \frac{d^{n+1}}{d(\frac{1}{x})^{n+1}} H\left(\frac{1}{x}\right)
\]

therefore

\[
[x^{-\frac{1}{2}} H(x)]^{(n+1)} =
\]

\[
(-1)^{n+1} x^{-n-\frac{3}{2}} \frac{(2n + 2)!}{2^{2n+2}} \sum_{k=0}^{n+1} \frac{2^{2k} x^{-k}}{(n+1-k)! (2k)!} \frac{d^k}{d(\frac{1}{x})^k} H\left(\frac{1}{x}\right).
\]

\[
\square
\]
The new properties of the theta functions

Proof. Proof of the Theorem 1.3 The known functional relation

\[ H(\pi^2 x) = \pi^{-\frac{1}{2}} x^{-\frac{1}{2}} H\left(\frac{1}{x}\right) \]

we apply to (1.12) and obtain

\[ H(\pi^2 x) = \pi^{-\frac{1}{2}} e^{-\pi^2 x} \sum_{n=0}^{\infty} (-1)^n \frac{(2\pi x)^{2n} (\pi^2)^n}{(2n)!} \left[ x^{-\frac{1}{2}} H\left(\frac{1}{x}\right) \right]^n \]

We use now (1.18) and have

\[ H(\pi^2 x) = \pi^{-\frac{1}{2}} x^{-\frac{1}{2}} e^{-\pi^2 x} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(2\pi)^{2k} (\pi^2 x)^{n-k}}{(2k)! (n-k)!} \frac{d^k}{d(\frac{1}{x})^k} H\left(\frac{1}{x}\right) \]

and consequently

\[ H\left(\frac{1}{x}\right) = e^{-\pi^2 x} \left( \sum_{n=0}^{\infty} \frac{(\pi^2 x)^n}{n!} H\left(\frac{1}{x}\right) + \sum_{n=1}^{\infty} \frac{(\pi^2 x)^{n-1} (2\pi)^2}{(n-1)!} \frac{d}{d(\frac{1}{x})} H\left(\frac{1}{x}\right) \right. \]
\[ + \sum_{n=2}^{\infty} \frac{(\pi^2 x)^{n-2} (2\pi)^4}{(n-2)!} \frac{d^2}{d(\frac{1}{x})^2} H\left(\frac{1}{x}\right) + \cdots \]

therefore

\[ H\left(\frac{1}{x}\right) = \sum_{n=0}^{\infty} \frac{(2\pi)^{2n}}{(2n)!} \frac{d^n}{d(\frac{1}{x})^n} H\left(\frac{1}{x}\right). \]

If we replace \( \frac{1}{x} \) by \( x \) we obtain (1.17).

The following formula for \( \Theta_1(x) \) holds

\[ \sum_{k=1}^{\infty} e^{-k^2 x} e^{-ka} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \left( \frac{1}{e^a - 1} \right)^{(2n)} \quad a > 0. \quad (1.19) \]

Proof. We transform the right-hand side of (1.19) and have

\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \left( \sum_{k=1}^{\infty} e^{-ka} \right)^{(2n)} = \sum_{k=1}^{\infty} e^{-ka} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (k^2 x)^n \]
\[ = \sum_{k=1}^{\infty} e^{-k^2 x} e^{-ka}. \]
Using the formula (1.19) with $a = 2x$, we find following expression for the theta function [1, 2]

$$
\Theta(x) = e^{-x}[1 + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \frac{1}{(e^{2x} - 1)^{(2n)}}].
$$

(1.20)

References
