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Modeling of the resonance of an acoustic wave in a torus

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Abstract

A pneumatic tyre in rotating motion with a constant angular velocity Ω is assimilated to a torus whose generating circle has a radius R. The contact of the tyre with the ground is schematized as an ellipse with semi-major axis a. When $(\Omega R/C_0) \ll 1$ and $(a/R) \ll 1$ (where C_0 is the velocity of the sound), we show that at the rapid time scale R/C_0, the air motion within a torus periodically excited on its surface generates an acoustic wave h. A study of this acoustic wave is conducted and shows that the mode associated to $p = 0$ leads to resonance. In resonance the acoustic wave h moves quadratically in time and also decreases asymptotically faster when the mean pressure in the domain is low.

Résumé

Un pneumatique animé d'un mouvement de rotation de vitesse angulaire constante Ω est assimilé à un tore dont le cercle générateur est de rayon R. La surface de contact du pneumatique avec la chausseé est schématisée par une ellipse de demi-grand axe a. Lorsque $(\Omega R/C_0) \ll 1$ et $(a/R) \ll 1$ (C_0 étant la vitesse du son), on montre qu'à l'échelle de temps rapide R/C_0, le mouvement de l'air dans le pneumatique excité périodiquement en surface engendre une onde acoustique h. L'étude montre que le mode $p = 0$ est résonant et l'onde h à la résonance évolue de façon quadratique en temps. On montre également que cette variation de l'onde acoustique, à l'infini est d'autant plus rapide que la pression moyenne dans la chambre est basse.

1. Introduction

We study the air motion in a pneumatic tyre which by a rough schematization we model as a torus $\mathbb{T}$. The generating circle has a radius R while

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its cross section has diameter d. In contact with the ground, the surface is deformed and we assume that the deformation is limited by an area $\mathbb{A}^*$ which we model as an ellipse with axes $2a$ and $2b$ with $a > b$. Relative to a frame $(\overrightarrow{X}, \overrightarrow{Y}, \overrightarrow{z})$ moving with the wheel, the motion of air within $\mathbb{T}$ is created by the fact that $\mathbb{A}^*$ is rotating so that it travels and gets recurrently at the same place with period $\frac{2\pi}{\Omega}$, where Ω is the constant angular velocity of the wheel with axis $\overrightarrow{z}$. In a previous note [2], we have shown that the flow in $\mathbb{T}$ is at small Mach Number. If

$$M_{tr} = \frac{\Omega R}{C_0} \tag{1.1}$$

stands for the Mach number associated to the velocity of translation, we have

$$\frac{R}{C_0} \ll \Omega^{-1} \tag{1.2}$$

where C_0 is the sound speed. In [1], a study of the flow in a time scaled to the period Ω^{-1} has shown that only inertial oscillation occurs in $\mathbb{T}$. But a work of Guiraud and Zeytounian in [4] relative to the flow of the air within a chamber at small Mach number, has shown that an incompressible potential flow is, in some situations, superimposed on acoustic waves. According to their model, acoustic waves occur as a consequence of persistence in time, of high frequency oscillations excited within a starting phase of setting the fluid into motion. Basing one's argument on this work, and taking the aforementioned result of [1] and relation (1.2) into account, we examine here the flow in $\mathbb{T}$ at the characteristic time $\frac{R}{C_0}$, using an asymptotic expansion with M_{tr} as a small perturbation parameter. We show that to the leading order we have an incompressible potential flow and that acoustic waves occur to the first order. Discussing these acoustic equations, we prove that the acoustic oscillation associated to $p = 0$ leads to resonance. We study, theoretically and numerically, this resonance phenomenon.

2. Equations

Considering the model in [2], we suppose that the air into $\mathbb{T}$ is a perfect gas with constant entropy and we neglect dissipation. The equations in the moving frame are

$$\begin{cases} \frac{D\rho}{Dt} + \rho div \overrightarrow{u} = 0 \\ \rho \frac{D\overrightarrow{u}}{Dt} + \rho\Omega^2 \overrightarrow{x} + 2\rho\overrightarrow{\Omega} \wedge \overrightarrow{u} + \overrightarrow{\nabla} p = \overrightarrow{0} \end{cases} \tag{2.1}$$

where $\overrightarrow{x}$ represents the radial vector in the plane perpendicular to the rotating axis, p the pressure, while $\rho\Omega^2\overrightarrow{x}$ and $2\rho\overrightarrow{\Omega} \wedge \overrightarrow{u}$ are respectively the inertial and Coriolis forces. Let us introduce the enthalpy h and the speed of sound C. At rest one finds

$$\overrightarrow{u} = \overrightarrow{0}; \quad h = h_0(\overrightarrow{x}); \quad C = C_0, \tag{2.2}$$

with $h_0(\overrightarrow{x}) + \frac{\Omega^2 x^2}{2}$ constant. We examine the flow generated by the deformation at the characteristic time scale $\frac{R}{C_0}$, and the quantities are made non dimensional by setting

$$\overrightarrow{u} = \Omega a \overrightarrow{\widetilde{u}}, \quad h = h_0\left(\overrightarrow{x}\right) + \Omega^2 Ra\widetilde{h}, \quad t = \frac{R}{C_0}\tau, \quad \overrightarrow{\nabla} = R^{-1}\overrightarrow{\widetilde{\nabla}}, \quad \overrightarrow{x} = R\overrightarrow{\widetilde{x}} \tag{2.3}$$

Equations (2.1) become

$$\begin{cases} \overrightarrow{\widetilde{\nabla}}.\overrightarrow{\widetilde{u}} + M_{TR}\left[1 + (1-\gamma)\delta M_{TR}^2\widetilde{h}\right]\left[\frac{\partial\widetilde{h}}{\partial\tau} + M_{TR}\mathbb{D}_\delta\left(\overrightarrow{\widetilde{u}}, \widetilde{h}, \overrightarrow{\widetilde{x}}\right)\right] = 0 \ \text{ in } \mathbb{T} \\ \frac{\partial\overrightarrow{\widetilde{u}}}{\partial\tau} + M_{TR}\left[\overrightarrow{\widetilde{\nabla}}\widetilde{h} + 2\overrightarrow{z}\Lambda\overrightarrow{\widetilde{u}} + \delta(\overrightarrow{\widetilde{u}}.\overrightarrow{\widetilde{\nabla}})\overrightarrow{\widetilde{u}}\right] = \overrightarrow{0} \ \text{ in } \mathbb{T} \end{cases} \tag{2.4}$$

where $\mathbb{D}_\delta$ is definied by

$$\mathbb{D}_\delta\left(\overrightarrow{\widetilde{u}}, \widetilde{h}, \overrightarrow{\widetilde{x}}\right) = \delta(\overrightarrow{\widetilde{u}}.\overrightarrow{\widetilde{\nabla}})\widetilde{h} - \overrightarrow{\widetilde{u}}.\overrightarrow{\widetilde{x}}, \tag{2.5}$$

and $\delta = a/R$ represents a small geometric parameter. For the boundary conditions to equations (2.4), we write the fact that a point tied to $\mathbb{A}^*$ moves while remaining on the tangent plane. We take into account the influence of vertical oscillations of the wheel which result from the dynamics of the wheel and even from the dynamics of the whole vehicle. To this effect, we assume that the axis of the wheel is at a distance from the ground which is a given function of the time. We set f_r^0 the constant value of the equivalent length in a steady state and $\xi(t)$ the variation due to the oscillations. Taking (2.3) into account, we get by linearization, the nondimensional boundary conditions, namely

$$\overrightarrow{\widetilde{u}}.\overrightarrow{N} = \frac{1}{\delta}\left[\theta + \frac{\pi}{2} + \tau M_{TR} - \delta M_{TR}^2 \widetilde{F}'\right]\left[1 + \delta M_{TR}\widetilde{F}''\right] - \frac{\delta}{2M_{TR}}\frac{d\widetilde{\xi}}{d\tau} \quad \text{on } \mathbb{A}^*. \tag{2.6}$$

where $\overrightarrow{N}$ is the unit vector, normal to the surface of the torus and directed towards the exterior of the tyre, the function $\widetilde{F}(-\tau)$ characterizes the profile of the road and $(\widetilde{x}, \theta)$ are the polar coordinates. To determine the term $\frac{d\widetilde{\xi}}{d\tau}$ introduced in the boundary condition, we investigate the dynamics of the whole vehicle. For simplicity, we consider by a rough schematization, a one-dimensional model, assuming that the bodywork is tied to the wheel by a spring which models the suspension of the vehicle. We denote m and M_c the mass of the wheel and the mass of the bodywork respectively, and k the stiffness of the spring. We get

$$\frac{d\widetilde{\xi}}{d\tau} = -\frac{2M_{TR}^3}{\delta}\widetilde{F}'(-\tau) - 2M_{TR}^2\frac{\alpha_2}{\delta}\widetilde{N}^{-1}\int_0^\tau \widetilde{\xi}(\tau')d\tau' \tag{2.7}$$

where (α_2/δ) is a dimensionless parameter $o(1)$, while $\widetilde{N}$ is an operator defined by

$$\widetilde{N}\left(\widetilde{Z}(\tau)\right) = \widetilde{Z}(\tau) + \frac{M_c\omega}{m\Omega}M_{TR}\int_0^\infty \widetilde{Z}(\tau - \tau_1)\sin\left(\frac{\omega}{\Omega}M_{TR}\tau_1\right)d\tau_1, \ \forall\tau \tag{2.8}$$

with $\omega = \sqrt{k/M_c}$. We study the impulsive setting into motion of the fluid from $\tau = 0$, then we set as initial condition

$$(\overrightarrow{\widetilde{u}}; \widetilde{h}) = (\overrightarrow{0}, 0). \tag{2.9}$$

3. Acoustic wave

We study the problem (2.4) - (2.9), using an asymptotic expansion with M_{TR} as a small perturbation parameter, namely

$$(\overrightarrow{\widetilde{u}}, \widetilde{h}) = (\overrightarrow{\widetilde{u}}_0, \widetilde{h}_0) + M_{TR}(\overrightarrow{\widetilde{u}}_1, \widetilde{h}_1) + M_{TR}^2(\overrightarrow{\widetilde{u}}_2, \widetilde{h}_2) + ... \tag{3.1}$$

To the leading order of the expansion (3.1), we get

$$\begin{cases} \overrightarrow{\widetilde{\nabla}}.\overrightarrow{\widetilde{u}}_0 = 0; \quad \frac{\partial \overrightarrow{\widetilde{u}}_0}{\partial \tau} = \overrightarrow{0} \text{ in } \mathbb{T} \\ \overrightarrow{\widetilde{u}}_0.\overrightarrow{N} = \frac{1}{\delta}\left(\theta + \frac{\pi}{2}\right)\chi_{\mathbb{A}^*} \text{ on } \partial\mathbb{T} \end{cases} \tag{3.2}$$

where $\chi_{\mathbb{A}^*}$ is the characteristic function of $\mathbb{A}^*$. Taking the vorticity of the second equation of (3.2), and due to initial condition (2.9), one can show that $\overrightarrow{rot}\,\overrightarrow{\widetilde{u}}_0 = \overrightarrow{0}$ in any time. As $\mathbb{T}$ is connected, we may introduce a potential velocity $\widetilde{\phi}_0$, such that $\overrightarrow{\widetilde{u}}_0 = \overrightarrow{\widetilde{\nabla}}\widetilde{\phi}_0$. This potential velocity is a solution of the system

$$\begin{cases} \widetilde{\Delta}\widetilde{\phi}_0 = 0 \text{ in } \mathbb{T} \\ \overrightarrow{\widetilde{\nabla}}\widetilde{\phi}_0.\overrightarrow{N} = \frac{1}{\delta}\left(\theta + \frac{\pi}{2}\right)\chi_{\mathbb{A}^*} \quad \text{on } \partial\mathbb{T} \end{cases} \tag{3.3}$$

with the compatibility relation

$$\int_{\mathbb{A}^*}\left(\theta + \frac{\pi}{2}\right) ds_* = 0. \tag{3.4}$$

To the first order of the expansion, the problem (2.4) – (2.9) leads to

$$\begin{cases} \overrightarrow{\widetilde{\nabla}}.\overrightarrow{\widetilde{u}}_1 + \frac{\partial \widetilde{h}_0}{\partial \tau} = 0 \;\; ; \;\; \frac{\partial \overrightarrow{\widetilde{u}}_1}{\partial \tau} + \overrightarrow{\widetilde{\nabla}}\widetilde{h}_0 = -2\overrightarrow{z}\Lambda\overrightarrow{\widetilde{u}}_0 - \delta(\overrightarrow{\widetilde{u}}_0.\overrightarrow{\widetilde{\nabla}})\overrightarrow{\widetilde{u}}_0 \quad \text{in } \mathbb{T} \\ \overrightarrow{\widetilde{u}}_1.\overrightarrow{N} = \left[\frac{\tau}{\delta} + (\theta + \frac{\pi}{2})\widetilde{F}''(-\tau) + \frac{B}{\delta}\widetilde{N}^{-1}\int_0^{\tau}\widetilde{\xi}(\tau')d\tau'\right]\chi_{\mathbb{A}^*} \quad \text{on } \partial\mathbb{T} \\ \left(\overrightarrow{\widetilde{u}}_1(0;.);\widetilde{h}_0(0;.)\right) = (\overrightarrow{0};0). \end{cases} \tag{3.5}$$

We observe that equations (3.5) are dimensionless equations of acoustic oscillations generated in the domain $\mathbb{T}$ by an excitation on the deformed area $\mathbb{A}^*$. Let us notice that

$$\int_{\mathbb{T}}\overrightarrow{\widetilde{\nabla}}.\left(\overrightarrow{\widetilde{u}}_0.\overrightarrow{\widetilde{\nabla}}\,\overrightarrow{\widetilde{u}}_0\right)dV$$
$$= \int_{\mathbb{A}^*}(\overrightarrow{\widetilde{u}}_0.\overrightarrow{\widetilde{\nabla}})\overrightarrow{\widetilde{u}}_0.\overrightarrow{N}dS_* + \frac{2}{\delta}\int_{\mathbb{A}^*}(\overrightarrow{z}\wedge\overrightarrow{\widetilde{u}}_0).\overrightarrow{N}dS_*. \tag{3.6}$$

We look for the solution $\widetilde{h}_0$ setting

$$\widetilde{h}_0(\tau,\overrightarrow{\widetilde{x}}) = \widetilde{h}_0^{(1)}(\tau,\overrightarrow{\widetilde{x}}) + \widetilde{h}_0^{(2)}(\overrightarrow{\widetilde{x}}) \tag{3.7}$$

where due to (3.6), $\widetilde{h}_0^{(2)}(\overrightarrow{\widetilde{x}})$ satisfies

$$\begin{cases} \widetilde{\Delta}\widetilde{h}_0^{(2)} = -\overrightarrow{\widetilde{\nabla}}.\left[\left(\overrightarrow{\widetilde{u}}_0.\overrightarrow{\widetilde{\nabla}}\right)\overrightarrow{\widetilde{u}}_0\right] \text{ in } \mathbb{T} \\ \overrightarrow{\widetilde{\nabla}}\widetilde{h}_0^{(2)}.\overrightarrow{N} = \left[-2\left(\overrightarrow{z}\wedge\overrightarrow{\widetilde{u}}_0\right).\overrightarrow{N} - \delta(\overrightarrow{\widetilde{u}}_0.\overrightarrow{\widetilde{\nabla}})\overrightarrow{\widetilde{u}}_0.\overrightarrow{N}\right]\chi_{\mathbb{A}^*} \text{ on } .\partial\mathbb{T} \end{cases} \tag{3.8}$$

Accordingly, to study the resonance of the acoustic problem, we have to study the component which depends on τ, namely $\widetilde{h}_0^{(1)}(\tau, \overrightarrow{\widetilde{x}})$ solution of

$$\begin{cases} \frac{\partial\overrightarrow{\widetilde{u}}_1}{\partial\tau} + \overrightarrow{\widetilde{\nabla}}\widetilde{h}_0^{(1)} = \overrightarrow{0};\ \overrightarrow{\widetilde{\nabla}}.\overrightarrow{\widetilde{u}}_1 + \frac{\partial\widetilde{h}_0^{(1)}}{\partial\tau} = 0 \text{ in } \mathbb{T} \\ \overrightarrow{\widetilde{u}}_1.\overrightarrow{N} = \left[\frac{\tau}{\delta} + (\theta+\frac{\pi}{2})\frac{d^2\widetilde{F}(-\tau)}{d\tau^2} + \frac{B}{\delta}\widetilde{N}^{-1}\int_0^\tau \widetilde{\xi}(\tau')d\tau'\right]\chi_{\mathbb{A}^*} \text{ on } \partial\mathbb{T} \\ \left(\overrightarrow{\widetilde{u}}_1(0;\overrightarrow{\widetilde{x}});\widetilde{h}_0^{(1)}(0;\overrightarrow{\widetilde{x}})\right) = \left(\overrightarrow{0};-\widetilde{h}_0^{(2)}(\overrightarrow{\widetilde{x}})\right). \end{cases} \tag{3.9}$$

4. The study of the acoustic problem

For simpler notation, we leave the $\sim$ and the subscripts on the variables $\overrightarrow{\widetilde{u}}_1$, $\widetilde{h}_0^{(1)}$ and also on $\overrightarrow{\widetilde{x}}$ and we make the following Laplace transform

$$\left(\widehat{h}(p,.);\widehat{Y}(p)\right) = \int_0^\infty \left(h(\tau,.);\widetilde{F}(-\tau)\right)e^{-\tau p}d\tau \tag{4.1}$$

with $\left(h(\tau,.);\widetilde{F}(-\tau)\right) \to \left(-p\widetilde{h}_0^{(2)}(.);0\right)$ when $\tau \to 0$. For $p \in \mathbb{C}$, problem (3.9) leads to

$$\begin{cases} p^2\widehat{h} - \Delta\widehat{h} = -p\widetilde{h}_0^{(2)} \text{ in } \mathbb{T} \\ \overrightarrow{\nabla}\widehat{h}.\overrightarrow{N} = \left[\widehat{X}(p) - p^3\widehat{Y}(p)\left(\theta+\frac{\pi}{2}\right)\right]\chi_{\mathbb{A}^*} \text{ on } \partial\mathbb{T}, \end{cases} \tag{4.2}$$

where

$$\widehat{X}(p) = -\frac{1}{\delta p} + \frac{2B}{\delta^3}\eta\frac{p^2\left(p^2+\frac{m}{m+M_c}\omega_a^2\right)}{p^4+\left(2B\frac{\lambda^2}{a_0^2}+\omega_a^2\right)p^2+2B\frac{\lambda^2}{a_0^2}\frac{m}{M_c+m}\omega_a^2}\widehat{Y}(p) \tag{4.3}$$

with $\omega_a = \frac{R}{C_0}\sqrt{\frac{k(M_c+m)}{mM_c}}$. In order to simplify the problem, we assume that the sinuosities of the road are regular and characterized by a frequency μ, such that

$$\widehat{Y}(p) = \widehat{\psi}(p) + \frac{e^{(-p+i\mu)T_1}}{p-i\mu},\quad T_1 > 0. \tag{4.4}$$

The function $\widehat{\psi}(.)$ is holomorphic in $\mathcal{R}e(p) > 0$. Finally, we look for $\widehat{h}(p,.)$ solution of (4.2) setting

$$\widehat{h}\,(p,.) = \widehat{h}_1\,(p,.) + \widehat{h}_2\,(p,.) + \widehat{h}_3\,(p,.)\,. \tag{4.5}$$

The function $\widehat{h}_1\,(p,.)$ for $p \in \mathbb{C}$, satisfies

$$\begin{cases} \Delta \widehat{h}_1 = 0 \quad \text{in } \mathbb{T} \\ \overrightarrow{\nabla} \widehat{h}_1.\overrightarrow{N} = -p^3 \widehat{Y}(p)\left(\theta + \frac{\pi}{2}\right)\chi_{\mathbb{A}^*} \quad \text{on } \partial\mathbb{T}. \end{cases} \tag{4.6}$$

Due to relation (3.4), this problem has a solution which may be written as

$$\widehat{h}_1\,(p,.) = -p^3 \widehat{Y}(p)\mathfrak{F}\,(.)\,. \tag{4.7}$$

The uniqueness of this solution is realized if we choose $\mathfrak{F}\,(\overrightarrow{x})$ that satisfies

$$\begin{cases} \Delta \mathfrak{F}\,(\overrightarrow{x}) = 0 \quad \text{in } \mathbb{T} \\ \overrightarrow{\nabla} \mathfrak{F}\,(\overrightarrow{x})\,.\overrightarrow{N} = \left(\theta + \frac{\pi}{2}\right)\chi_{\mathbb{A}^*}(\overrightarrow{x}) \;\; \text{on } \partial\mathbb{T} \\ \displaystyle\int_{\mathbb{T}} \mathfrak{F}\,(\overrightarrow{x})\,dV = 0. \end{cases} \tag{4.8}$$

In the decomposition (4.5), the function $\widehat{h}_2\,(p,.)$ is solution of the problem

$$\begin{cases} p^2 \widehat{h}_2 - \Delta \widehat{h}_2 = 0 \text{ in } \mathbb{T} \\ \overrightarrow{\nabla} \widehat{h}_2.\overrightarrow{N} = \widehat{X}(p)\chi_{\mathbb{A}^*} \quad \text{on } \partial\mathbb{T} \end{cases} \tag{4.9}$$

Scrutinizing this equation, we notice that we may look for the solution setting

$$\widehat{h}_2\,(p,.) = \widehat{X}(p)\widehat{H}(p,.) \tag{4.10}$$

where $\widehat{H}(p,.)$ is solution of the problem

$$\begin{cases} p^2 \widehat{H} - \Delta \widehat{H} = 0 \text{ in } \mathbb{T} \\ \overrightarrow{\nabla} \widehat{H}.\overrightarrow{N} = \chi_{\mathbb{A}^*} \quad \text{on } \partial\mathbb{T}. \end{cases} \tag{4.11}$$

This problem has an analytic solution in p with poles at

$$p = 0 \;\text{ if } mes(\mathbb{A}^*) \neq 0 \;\text{ and }\; p = \pm i\omega_n. \tag{4.12}$$

The ω_n are the acoustic vibration eigenfrequencies of $\mathbb{T}$. In (4.5), the function $\widehat{h}_3(p,.)$ for $p \in \mathbb{C}$ satisfies

$$\begin{cases} p^2 \widehat{h}_3 - \Delta \widehat{h}_3 = p^5 \widehat{Y}(p)\mathfrak{F}(\overrightarrow{x}) - p\widetilde{h}_0^{(2)}(\overrightarrow{x}) \text{ in } \mathbb{T} \\ \overrightarrow{\nabla} \widehat{h}_3.\overrightarrow{N} = 0 \;\text{ on } \partial\mathbb{T}. \end{cases} \tag{4.13}$$

The problem (4.13) has a solution, analytic function of p with poles at $p = i\mu$ and $p = \pm i\omega_n$. Finally, the acoustic resonance phenomenon would occur, when one of the poles in (4.12) coming from the eigenmodes will be close to one of the poles of $\widehat{X}(p)$ or $\widehat{Y}(p)$. Observing these poles, we notice that $p = 0$ may lead to resonance.

5. Study of the acoustic wave associated to the pole zero

Using the reverse transform of (4.1), we get the acoustic solution

$$h(\tau, \overrightarrow{x}) = \frac{1}{2i\pi} \int_{\wedge} \widehat{h}(p;.)e^{p\tau} dp \tag{5.1}$$

where $\widehat{h}(p;.)$ is defined in (4.5) and $\wedge$ is a straight line parallel to $\mathcal{R}e(p) > 0$.

Theorem 5.1. *For complex close to $p = 0$, the function $\widehat{H}(p, \overrightarrow{x})$ defined in (4.11) satisfies the relation:*

$$\widehat{H}(p,.) = \frac{mes(\mathbb{A}^*)}{mes(\mathbb{T})} \frac{1}{p^2} + \widehat{H}_0(.) + o(p) \tag{5.2}$$

where $o(p) \to 0$, when $p \to 0$ and $\widehat{H}_0(\overrightarrow{x})$ is given by :

$$\Delta \widehat{H}_0 = \frac{mes(\mathbb{A}^*)}{mes(\mathbb{T})} \quad in \quad \mathbb{T} \quad ; \quad \overrightarrow{N}.\overrightarrow{\nabla}\widehat{H}_{0|_\Sigma} = \chi_{A^*} \quad on \quad \partial\mathbb{T}. \tag{5.3}$$

Proof. Without loosing generality, for complex close to $p = 0$, we can assume the following formal expansion of $\widehat{H}(p,.)$

$$\widehat{H}(p,.) = \frac{\widehat{H}_{-2}(.)}{p^2} + \frac{\widehat{H}_{-1}(.)}{p} + \widehat{H}_0(.) + o(p). \tag{5.4}$$

The substitution of (5.4) in (4.11), shows that $\widehat{H}_{-2}$ and $\widehat{H}_{-1}$ are constants. Taking the compatibility relations into account, we get

$$\widehat{H}_{-2} = \frac{mes(\mathbb{A}^*)}{mes(\mathbb{T})} \quad \text{and} \quad \widehat{H}_{-1} = 0. \tag{5.5}$$

Then, $\widehat{H}_0$ satisfies

$$\Delta \widehat{H}_0 = \widehat{H}_{-2} \quad \text{in} \quad \mathbb{T} \quad ; \quad \overrightarrow{N}.\overrightarrow{\nabla}\widehat{H}_0 = \chi_{\mathbb{A}^*}(.) \quad \text{on} \quad \partial\mathbb{T}. \tag{5.6}$$

□

For complex p close to zero if we use relations (4.7), (4.10), (5.2) and the corresponding solution of (4.13), the residue theorem applied to (5.1) leads to

$$h(\tau, \overrightarrow{x}) = -\frac{mes(\mathbb{A}^*)}{2\delta mes(\mathbb{T})}\tau^2 - \frac{\widehat{H}_0(\overrightarrow{x})}{\delta}. \tag{5.7}$$

In order to study how the acoustic wave h given by (5.7) moves in resonance, we have to look for $\widehat{H}_0$. With this aim, we introduce the local coordinates $(\varrho, \theta, \varphi)$ where θ locates a cross section centered at a point O of the generating circle, while any point M of this cross section is located by the polar coordinates ϱ and φ, namely

$$\begin{aligned} \overrightarrow{SM} &= \overrightarrow{SO} + \overrightarrow{OM} \\ &= (R + \varrho\cos\varphi)(\cos\varphi \overrightarrow{E}_\varrho - \sin\varphi \overrightarrow{E}_\varphi) \\ &\qquad + \varrho \sin\varphi(\sin\varphi \overrightarrow{E}_\varrho - \cos\varphi \overrightarrow{E}_\varphi). \end{aligned} \tag{5.8}$$

In relation (5.8), $\overrightarrow{E}_\varrho$ and $\overrightarrow{E}_\varphi$ are radial and tangential vectors to the cross section respectively, while $\overrightarrow{E}_\theta$ is the vector directly perpendicular to $\left(\overrightarrow{E}_\varrho, \overrightarrow{E}_\varphi\right)$. Setting

$$\varrho = \left(\frac{d}{2}\right) r, \tag{5.9}$$

the Laplacian operator in the non dimensional system of coordinates (r, θ, φ) may be written as;

$$\begin{aligned} \Delta = &\tfrac{\partial^2}{\partial r^2} + \tfrac{1}{r}\tfrac{\partial}{\partial r} + \tfrac{1}{r^2}\tfrac{\partial^2}{\partial \varphi^2} \\ &+ \tfrac{\varepsilon}{1+\varepsilon r\cos\varphi}\left(\cos\varphi\tfrac{\partial}{\partial r} - \tfrac{\sin\varphi}{r}\tfrac{\partial}{\partial\varphi}\right) + \tfrac{\varepsilon^2}{(1+\varepsilon r\cos\varphi)^2}\tfrac{\partial^2}{\partial\theta^2} \end{aligned} \tag{5.10}$$

where

$$\varepsilon = \frac{d}{2R} \tag{5.11}$$

is a non dimensional parameter which characterizes the deformation of the chamber $\mathbb{T}$. According to [3], the mean pressure P_0 in the chamber and $mes(\mathbb{A}^*)$ are related by

$$mes(\mathbb{A}^*) = \frac{(M_c + m)g}{\pi a b P_0}. \tag{5.12}$$

The non dimensional value $mes(\mathbb{T})$ is given by

$$mes(\mathbb{T}) = 2\pi^2. \tag{5.13}$$

To work in two dimension, we set θ as a constant. Then, if we take relations (5.10), (5.12) and (5.13) into account, equation (5.3) leads to

$$\frac{\partial^2\widehat{H}_0}{\partial r^2}+\frac{1}{r}\frac{\partial\widehat{H}_0}{\partial r}+\frac{1}{r^2}\frac{\partial^2\widehat{H}_0}{\partial\varphi^2} + \frac{\varepsilon}{1+\varepsilon r\cos\varphi}\left(\cos\varphi\frac{\partial\widehat{H}_0}{\partial r}-\frac{\sin\varphi}{r}\frac{\partial\widehat{H}_0}{\partial\varphi}\right)=\frac{(M_c+m)g}{2\pi^3abP_0} \quad (5.14)$$

where $0\leq\varphi<2\pi$ and $0<r<1$. At the point O located by $r=0$, equation (5.14) is not valid. So we rewrite equation (5.3) in the rectangular system of coordinates (x,z). For any point M of the cross section, we get

$$\overrightarrow{SM}=\overrightarrow{SO}+\overrightarrow{OM}=(R+x)\overrightarrow{e}_x+z\overrightarrow{e}_z. \quad (5.15)$$

Setting

$$x=R\widetilde{x} \text{ and } \quad z=\left(\frac{d}{2}\right)\widetilde{z}, \quad (5.16)$$

equation (5.3) at the point O leads to

$$\frac{\partial^2\widehat{H}_0}{\partial\widetilde{x}^2}+\frac{\partial^2\widehat{H}_0}{\partial\widetilde{z}^2}+\frac{\varepsilon^2}{\widetilde{x}}\frac{\partial\widehat{H}_0}{\partial\widetilde{x}}=\frac{(M_c+m)g}{2\pi^3abP_0} \quad (5.17)$$

where $1-\varepsilon<\widetilde{x}<1+\varepsilon$ and $0<\widetilde{z}<1$. Using [5], we make a mesh on the cross section of $\mathbb{T}$ taking, for $i,j\in\mathbb{N}$

$$r_i=(i-1)l_r \quad \text{and} \quad \varphi_j=(j-1)l_\varphi \ ; \ 0<r_i<1 \ , \ 0<\varphi_j\leq 2\pi. \quad (5.18)$$

where l_r and l_φ are respectively the radial and the angular steps. If M_{ij} is the point of the mesh with coordinates (r_i,φ_j), we set H_{ij} the value of $\widehat{H}_0$ at M_{ij} namely

$$H_{ij}=\widehat{H}_0(M_{ij}) \quad \text{where} \quad 0\leq r_i<1 \text{ and } 0<\varphi_j\leq 2\pi. \quad (5.19)$$

Using the following finite difference schemes

$$\left\{\begin{array}{lcl}
\frac{\partial\widehat{H}_0(M_{ij})}{\partial\varphi} & = & \frac{H_{ij+1}-H_{ij-1}}{2l_\varphi}+o\left(l_\varphi^2\right)\\
\frac{\partial^2\widehat{H}_0(M_{ij})}{\partial r^2} & = & \frac{H_{i-1j}-2H_{ij}+H_{i+1j}}{l_r^2}+o\left(l_r^2\right)\\
\frac{\partial\widehat{H}_0(M_{ij})}{\partial r} & = & \frac{H_{i+1j}-H_{i-1j}}{2l_r}+o\left(l_r^2\right)\\
\frac{\partial^2\widehat{H}_0(M_{ij})}{\partial\varphi^2} & = & \frac{H_{ij-1}-2H_{ij}+H_{ij+1}}{l_\varphi^2}+o\left(l_\varphi^2\right)
\end{array}\right. \quad (5.20)$$

equation (5.14) at any point M_{ij} of the mesh located by $0 < r_i < 1$ leads to

$$\alpha_{i-1j}H_{i-1j} + b_i H_{ij} + \gamma_{i+1j}H_{i+1j} + \sigma_{ij-1}H_{ij-1} + \xi_{ij+1}H_{ij+1} = \frac{(M_c+m)g}{2\pi^3 abP_0} \quad (5.21)$$

where coefficients α_{ij}, b_i, γ_{ij}, σ_{ij} and ξ_{ij} are determined by l_r, l_φ and ε. Setting $l_\varphi = \frac{\pi}{2k}$ where k is chosen in $\mathbb{N}^*$, we can write at the point $O = M_{11}$ the following finite difference schemes

$$\left\{\begin{array}{lcl} \frac{\partial^2 \widehat{H}_0(M_{11})}{\partial \widetilde{x}^2} & = & \frac{H_{2,2k+1}-2H_{11}+H_{21}}{l_r^2} + o\left(l_r^2\right) \\ \frac{\partial \widehat{H}_0(M_{11})}{\partial \ \widetilde{x}} & = & \frac{H_{21}-H_{2,2k+1}}{2l_r} + o\left(l_r^2\right) \\ \frac{\partial^2 \widehat{H}_0(M_{11})}{\partial \widetilde{z}^2} & = & \frac{H_{2,k+1}-2H_{11}+H_{2,3k+1}}{l_r^2} + o\left(l_r^2\right). \end{array}\right. \quad (5.22)$$

Applying the finite difference schemes (5.22) to equation (5.17), we get at point M_{11} located at $r_1 = 0$

$$-\frac{4}{l_r^2}H_{11} + \frac{2+\varepsilon^2 l_r}{2l_r^2}H_{21} + \frac{1}{l_r^2}H_{2,k+1} + \frac{2-\varepsilon^2 l_r}{2l_r^2}H_{2,2k+1} + \frac{1}{l_r^2}H_{2,3k+1} = \frac{(M_c+m)g}{2\pi^3 abP_0}. \quad (5.23)$$

The boundary condition of the non dimensional problem (5.3) is defined by

$$\frac{\partial \widehat{H}_0}{\partial r}\Big|_{r=1} = \left\{\begin{array}{ll} 1 & \text{if} \quad -\frac{2b}{d} \le \varphi \le \frac{2b}{d} \\ 0 & \text{else.} \end{array}\right. \quad (5.24)$$

Considering the following finite difference scheme

$$\frac{\partial \widehat{H}_0}{\partial r}\Big|_{r=1} = \frac{3H_{ij} - 4H_{i-1j} + H_{i-2j}}{2l_r} + o(l_r^2), \quad (5.25)$$

at any point M_{ij} in the boundary given by $r_i = 1$ and $0 < \varphi_j \le 2\pi$, the dimensionless boundary condition (5.24) leads to

$$3H_{ij} - 4H_{i-1j} + H_{i-2j} = 2l_r\delta_{1j} \quad (5.26)$$

where (δ_{ij}) is the Kronecker symbol, $i = 1 + \frac{1}{l_r}$ and $1 \le j \le 4k$. Using (5.21), (5.23) and (5.26), we make the numerical simulations taking the following values :

$$\begin{array}{l} d = 0.2m, \quad R = 0.3m, \quad a = 0.06m, \quad b = 0.03m, \\ M_c + m = 615Kg, \quad g = 9.8m/s^2, \quad l_r = \frac{1}{3}, \quad l_\varphi = \frac{\pi}{6}. \end{array} \quad (5.27)$$

For the mean pressure P_0, we take respectively :

$$3.0 \times 10^5 \text{ Pa}, \quad 3.5 \times 10^5 \text{ Pa} \text{ and } 4.0 \times 10^5 \text{ Pa}. \tag{5.28}$$

For points M_{11} and M_{33} of the mesh, we obtain the following graphs:

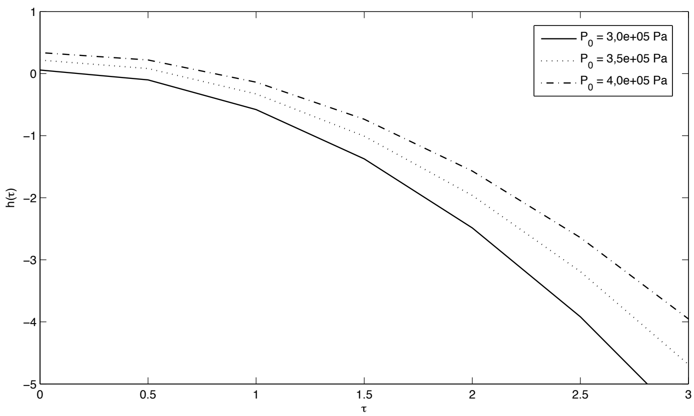

FIGURE 5.1. Evolution of h in M_{11}

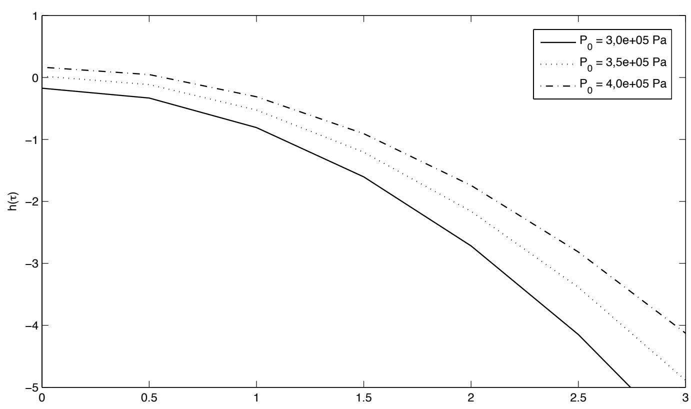

FIGURE 5.2. Evolution of h in M_{33}

For any point of the mesh we get a similar graph. Analyzing the acquired graphs, we note that at resonance, the acoustic wave h moves faster towards infinity when the pressure in the torus $\mathbb{T}$ decreases.

6. Conclusion

Our analysis has shown that at the time scale R/C_0 the motion of the air within a rotating torus periodically excited on its surface, is ruled to the first order of the small mach number M_{TR} by the problem of acoustic waves. We have studied them for the mode corresponding to $p = 0$, and have shown that it may lead to resonance. In resonance, the acoustic oscillation h moves quadratically in time τ, for $\tau \to +\infty$. Furthermore the numerical study of the acoustic wave h for $p = 0$ has shown that under-inflated tires may in resonance lead to waves that decay faster towards infinity.

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