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Examples of polynomial identities distinguishing the Galois objects over finite-dimensional Hopf algebras

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Abstract

We define polynomial H-identities for comodule algebras over a Hopf algebra H and establish general properties for the corresponding T-ideals. In the case H is a Taft algebra or the Hopf algebra $E(n)$, we exhibit a finite set of polynomial H-identities which distinguish the Galois objects over H up to isomorphism.

Exemples d'identités polynomiales distinguant les objets galoisiens d'une algèbre de Hopf de dimension finie

Résumé

Nous définissons le concept d'identité polynomiale pour une algèbre-comodule sur une algèbre de Hopf H. Nous présentons des identités polynomiales explicites distinguant à isomorphisme près les objets galoisiens d'une algèbre de Taft ou de l'algèbre de Hopf $E(n)$.

1. Introduction

By the celebrated Amitsur-Levitzki theorem [4], the standard polynomial of degree $2n$

$$S_{2n} = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(2n)}$$

is a polynomial identity for the algebra $M_n(\mathbb{C})$ of $n \times n$-matrices with complex entries, and $M_n(\mathbb{C})$ has no non-zero polynomial identity of degree less than $2n$. It follows that the identities S_{2n} distinguish the finite-dimensional simple associative algebras over $\mathbb{C}$ up to isomorphism.

When G is an abelian group, Koshlukov and Zaicev [13] established that any finite-dimensional G-graded G-simple associative algebra over

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an algebraically closed field of characteristic zero is determined up to G-graded isomorphism by its G-graded polynomial identities. Aljadeff and Haile [1] extended their result to non-abelian groups. Similar results exist for other classes of algebras.

Let now H be a Hopf algebra over a field k. Consider the class of H-comodule algebras. This class contains the G-graded k-algebras; indeed, such an algebra is nothing but a comodule algebra over the group algebra kG equipped with its standard Hopf algebra structure. Similarly, a comodule algebra over the Hopf algebra of k-valued functions on a finite group G is the same as a G-algebra, i.e., an associative k-algebra equipped with a left G-action by algebra automorphisms.

In this context we may wonder whether the following assertion holds: if H is a Hopf algebra over an algebraically closed field of characteristic zero, then any finite-dimensional simple H-comodule algebra is determined up to H-comodule algebra isomorphism by its polynomial H-identities.

In this note we provide evidence in support of this assertion by means of examples. When H is the n^2-dimensional Taft algebra H_{n^2} or the 2^{n+1}-dimensional Hopf algebra $E(n)$, we exhibit (finitely many) polynomial H-identities that distinguish the H-Galois objects over an algebraically closed field. Denoting the T-ideal of polynomial H-identities for a comodule algebra A by $\mathrm{Id}_H(A)$, we deduce that $\mathrm{Id}_H(A) = \mathrm{Id}_H(A')$ implies that A and A' are isomorphic Galois objects. Since each of our finite sets of identities determines the Galois object A up to isomorphism, it also determines the T-ideal $\mathrm{Id}_H(A)$ completely; in a sense which we shall not make precise, these identities generate the T-ideal.

Before giving the explicit identities, we have to define the concept of a polynomial H-identity for a comodule algebra A over a Hopf algebra H; this is done in full generality in § 2. When A is obtained from H by twisting its product with the help of a two-cocycle, we produce in § 2.3 a universal map detecting all polynomial H-identities for A, i.e., a map whose kernel is exactly the T-ideal $\mathrm{Id}_H(A)$.

In § 3 we deal with the Taft algebra H_{n^2}. After recalling the classification of its Galois objects, we show that the degree $2n$ polynomial

$$(YX - qXY)^n - (1-q)^n X^n Y^n + (1-q)^n c\, E^n X^n$$

is a polynomial H_{n^2}-identity and that it distinguishes the isomorphism classes of the Galois objects over an algebraically closed field. In § 3.3 we extend this to certain monomial Hopf algebras.

We prove a similar result for the Hopf algebra $E(n)$ in § 4, exhibiting a finite set of polynomial $E(n)$-identities which distinguishes the Galois objects over $E(n)$.

2. **Polynomial identities for comodule algebras**

This is a general section in which we define polynomial identities for comodule algebras and state general properties of the corresponding T-ideals.

We fix a ground field k over which all our constructions will be defined. In particular, all linear maps are supposed to be k-linear and unadorned tensor product symbols $\otimes$ mean tensor products over k. Throughout the paper we assume that k is *infinite*.

2.1. **Reminder on comodule algebras**

We suppose the reader familiar with the language of Hopf algebra, as presented for instance in [15, 19]. As is customary, we denote the coproduct of a Hopf algebra by Δ, its counit by ε, and its antipode by S. We also make use of a Heyneman-Sweedler-type notation for the image

$$\Delta(x) = x_1 \otimes x_2$$

of an element x of a Hopf algebra H under its coproduct.

Recall that a (right) *H-comodule algebra* over a Hopf k-algebra H is an associative unital k-algebra A equipped with a right H-comodule structure whose (coassociative, counital) *coaction*

$$\delta : A \to A \otimes H$$

is an algebra map. The subalgebra A^H of *coinvariants* of an H-comodule algebra A is defined by

$$A^H = \{a \in A \,|\, \delta(a) = a \otimes 1\} \,.$$

A *Galois object* over H is an H-comodule algebra A such that $A^H = k\,1_A$ and the map $\beta : A \otimes A \to A \otimes H$ given by $a \otimes a' \mapsto (a \otimes 1)\,\delta(a')$ $(a, a' \in A)$ is a linear isomorphism. For more on Galois objects, see [15, Chap. 8].

Let us now concentrate on a special class of Galois objects, which we call *twisted comodule algebras*. Recall that a *two-cocycle* α on H is a bilinear form $\alpha : H \times H \to k$ satisfying the cocycle condition

$$\alpha(x_1, y_1)\, \alpha(x_2 y_2, z) = \alpha(y_1, z_1)\, \alpha(x, y_2 z_2)$$

for all $x, y, z \in H$. We assume that α is invertible (with respect to the convolution product) and normalized; the latter means that $\alpha(x, 1) = \alpha(1, x) = \varepsilon(x)$ for all $x \in H$.

Let u_H be a copy of the underlying vector space of H. Denote the identity map u from H to u_H by $x \mapsto u_x$ ($x \in H$). We define the algebra ${}^{\alpha}H$ as the vector space u_H equipped with the product given by

$$u_x\, u_y = \alpha(x_1, y_1)\, u_{x_2 y_2}$$

for all x, $y \in H$. This product is associative thanks to the cocycle condition; the two-cocycle α being normalized, u_1 is the unit of ${}^{\alpha}H$.

The algebra ${}^{\alpha}H$ carries an H-comodule algebra structure with coaction $\delta \colon {}^{\alpha}H \to {}^{\alpha}H \otimes H$ given for all $x \in H$ by

$$\delta(u_x) = u_{x_1} \otimes x_2 \, .$$

It is easy to check that the subalgebra of coinvariants of ${}^{\alpha}H$ coincides with $k\, u_1$ and that the map $\beta : {}^{\alpha}H \otimes {}^{\alpha}H \to {}^{\alpha}H \otimes H$ is bijective, turning ${}^{\alpha}H$ into a Galois object over H. Conversely, when H is finite-dimensional, any Galois object over H is isomorphic to a comodule algebra of the form ${}^{\alpha}H$.

2.2. **Polynomial H-identities**

Let us now define the notion of a polynomial H-identity for an H-comodule algebra A. (Polynomial identities for module algebras over a Hopf algebra have been defined e.g. in [5, 8].)

For each $i = 1, 2, \ldots$ consider a copy X_i^H of H; the identity map from H to X_i^H sends an element $x \in H$ to the symbol X_i^x. Each map $x \mapsto X_i^x$ is linear and is determined by its values on a linear basis of H.

Now take the tensor algebra on the direct sum $X_H = \bigoplus_{i \geq 1} X_i^H$:

$$T = T(X_H) = T\left(\bigoplus_{i \geq 1} X_i^H \right) .$$

This algebra is isomorphic to the algebra of *non-commutative* polynomials in the indeterminates $X_i^{x_r}$, where $i = 1, 2, \ldots$ and $\{x_r\}_r$ is a linear basis of H. The algebra T is graded with all generators X_i^x homogeneous of degree 1.

There is a natural H-comodule algebra structure on T whose coaction $\delta : T \to T \otimes H$ is given by

$$\delta(X_i^x) = X_i^{x_1} \otimes x_2 \,.$$

The coaction obviously preserves the grading.

Definition 2.1. An element $P \in T$ is a *polynomial H-identity* for the H-comodule algebra A if $\mu(P) = 0$ for all H-comodule algebra maps $\mu : T \to A$.

Example 2.2. When H is the trivial one-dimensional Hopf algebra k, then a H-comodule algebra A is nothing but an associative unital algebra. In this case a polynomial H-identity for A is a classical polynomial identity, i.e., a non-commutative polynomial $P(X_1, X_2, \ldots)$ such that $P(a_1, a_2, \ldots) = 0$ for all $a_1, a_2, \ldots \in A$.

Example 2.3. When $H = kG$ is a group algebra, a polynomial H-identity is the same as a G-graded polynomial identity, as defined for instance in [6].

Example 2.4. Let H be an arbitrary Hopf algebra and A an H-comodule algebra. Assume that the subalgebra A^H of coinvariants is central in A (such a condition is satisfied e.g. when $A = {}^{\alpha}H$ is a twisted comodule algebra). For $x, y \in H$ consider the following elements of T:

$$P_x = X_1^{x_1} \, X_1^{S(x_2)} \quad \text{and} \quad Q_{x,y} = X_1^{x_1} \, X_1^{y_1} \, X_1^{S(x_2 y_2)} \,.$$

Then the commutators $P_x \, X_2^z - X_2^z \, P_x$ and $Q_{x,y} \, X_2^z - X_2^z \, Q_{x,y}$ are polynomial H-identities for A for all $x, y, z \in H$. Indeed, P_x and $Q_{x,y}$ are coinvariant elements of T by [3, Lemma 2.1]. Thus for any H-comodule algebra map $\mu : T \to A$, the elements $\mu(P_x)$ and $\mu(Q_{x,y})$ are coinvariant, hence central, in A.

Denote the set of all polynomial H-identities for A by $\mathrm{Id}_H(A)$. By definition,

$$I_H(A) = \bigcap_{\mu} \ker \mu \,,$$

where μ runs over all H-comodule algebra maps $T \to A$.

Proposition 2.5. *The set $I_H(A)$ has the following properties:*

(a) it is a graded two-sided ideal of $T = T(X_H)$, i.e.,

$$I_H(A)\, T \subset I_H(A) \supset T\, I_H(A)$$

and

$$I_H(A) = \bigoplus_{r \geq 0} \Big(I_H(A) \bigcap T^r(X_H) \Big) ;$$

(b) it is a right H-coideal of T, i.e.,

$$\delta\big(I_H(A)\big) \subset I_H(A) \otimes H ;$$

(c) any endomorphism f of H-comodule algebras of T preserves $I_H(A)$:

$$f\,\big(I_H(A)\big) \subset I_H(A) .$$

The proof follows the same lines as the proof of [3, Prop. 2.2]. Note that the assumption that k is infinite is needed to establish that the ideal $I_H(A)$ is graded. We summarize property (c) by saying that $I_H(A)$ is a *T-ideal*, a standard concept in the theory of polynomial identities (see [18]).

It is also clear that, if $A \to A'$ is a map of H-comodule algebras, then

$$I_H(A) \subset I_H(A') .$$

In particular, if A and A' are isomorphic H-comodule algebras, then

$$I_H(A) = I_H(A') .$$

In §§ 3–4 we will consider certain finite-dimensional Hopf algebras H such that the equality $I_H(A) = I_H(A')$ for twisted comodule algebras A, A' implies that A and A' are isomorphic.

To this end, we next show how to detect polynomial H-identities for twisted comodule algebras.

2.3. Detecting polynomial identities

Let ${}^{\alpha}H$ be a twisted comodule algebra for some normalized convolution invertible two-cocycle α, as defined in § 2.1. We claim that the polynomial H-identities for ${}^{\alpha}H$ can be detected by a "universal" comodule algebra map

$$\mu_\alpha : T \to S \otimes {}^{\alpha}H ,$$

which we now define.

For each $i = 1, 2, \ldots$, consider a copy t_i^H of H, identifying $x \in H$ linearly with the symbol $t_i^x \in t_i^H$, and define S to be the symmetric algebra on the direct sum $t_H = \bigoplus_{i \geq 1} t_i^H$:

$$S = S(t_H) = S\left(\bigoplus_{i\geq 1} t_i^H\right).$$

The algebra S is isomorphic to the algebra of *commutative* polynomials in the indeterminates $t_i^{x_r}$, where $i = 1, 2, \ldots$ and $\{x_r\}_r$ is a linear basis of H.

The map $\mu_\alpha : T \to S \otimes {}^\alpha H$ is given by

$$\mu_\alpha(X_i^x) = t_i^{x_1} \otimes u_{x_2}\,. \tag{2.1}$$

The algebra $S \otimes {}^\alpha H$ is generated by the symbols $t_i^x u_y$ ($x, y \in H; i \geq 1$) as a k-algebra (we drop the tensor product sign $\otimes$ between the t-symbols and the u-symbols). It is an H-comodule algebra whose $S(t_H)$-linear coaction extends the coaction of ${}^\alpha H$:

$$\delta(t_i^x u_y) = t_i^x u_{y_1} \otimes y_2\,.$$

It is easy to check that $\mu_\alpha : T \to S \otimes {}^\alpha H$ is an H-comodule algebra map. Its *raison d'être* becomes clear in the following statement.

Theorem 2.6. *An element $P \in T$ is a polynomial H-identity for ${}^\alpha H$ if and only if $\mu_\alpha(P) = 0$; equivalently, $I_H({}^\alpha H) = \ker \mu_\alpha$.*

To prove Theorem 2.6 we need the following proposition.

Proposition 2.7. *For every H-comodule algebra map $\mu : T \to {}^\alpha H$, there is a unique algebra map $\chi : S \to k$ such that*

$$\mu = (\chi \otimes \mathrm{id}) \circ \mu_\alpha\,.$$

Proof. By the universal property of the tensor algebra T, the set of H-comodule algebra maps $T \to {}^\alpha H$ is naturally in bijection with the vector space $\mathrm{Hom}^H(X_H, {}^\alpha H)$ of H-colinear maps from X_H to ${}^\alpha H$. Now, since the isomorphism $u : H \to u_H = {}^\alpha H$ is a comodule map, we have a natural identification $\mathrm{Hom}^H(X_H, {}^\alpha H) \cong \mathrm{Hom}^H(X_H, H)$.

On the other hand, by the universal property of the symmetric algebra S, the set of algebra maps $S \to k$ is in bijection with the vector space $\mathrm{Hom}(t_H, k)$ of linear maps from t_H to k.

Recall a basic fact from "colinear algebra": for any right H-comodule M with coaction $\delta : M \to M \otimes H$, the linear map $\mathrm{Hom}^H(M, H) \to \mathrm{Hom}(M, k)$

given by $\mu \mapsto \varepsilon \circ \mu$ is an isomorphism with inverse $\chi \mapsto (\chi \otimes \mathrm{id}) \circ \delta$. As a consequence, any H-comodule map $\mu : M \to H$ is necessarily of the form $\mu = (\chi \otimes \mathrm{id}) \circ \delta$, where $\chi = \varepsilon \circ \mu$.

Combining these observations yields a proof of the proposition. We can be even more precise: the algebra map $\chi : S \to k$ uniquely associated to μ in the statement is determined on the generators t_i^x by $\chi(t_i^x) = \varepsilon(u^{-1}(\mu(X_i^x))$. $\square$

Proof of Theorem 2.6. Let $P \in T$ be in the kernel of μ_α. Since by Proposition 2.7 any H-comodule algebra map $\mu : T \to {}^\alpha H$ is of the form $\mu = (\chi \otimes \mathrm{id}) \circ \mu_\alpha$, it follows that $\mu(P) = 0$. This implies $\ker \mu_\alpha \subset I_H({}^\alpha H)$.

To prove the converse inclusion, start from a polynomial H-identity P and observe that for every algebra map $\chi : S \to k$, the composite map $\mu = (\chi \otimes \mathrm{id}) \circ \mu_\alpha$ from T to ${}^\alpha H$ is a comodule algebra map. By definition of a polynomial H-identity, we thus have $\mu(P) = 0$. Now choose a basis $\{x_r\}_r$ of H and expand $\mu_\alpha(P)$ as $\mu_\alpha(P) = \sum_r \mu_\alpha^{(r)}(P) \otimes u_{x_r}$, where $\mu_\alpha^{(r)}(P)$ belongs to S. Then

$$0 = \mu(P) = \sum_r \chi(\mu_\alpha^{(r)}(P))\, u_{x_r}\, .$$

Since the elements u_{x_r} are linearly independent, we have $\chi(\mu_\alpha^{(r)}(P)) = 0$ for all r. This means that $\mu_\alpha^{(r)}(P) \in S$ vanishes under any evaluation $\chi : S \to k$; in other words, the polynomial $\mu_\alpha^{(r)}(P)$ takes only zero values. The ground field k being infinite, this implies $\mu_\alpha^{(r)}(P) = 0$, and hence $\mu_\alpha(P) = 0$. Therefore, $I_H({}^\alpha H) \subset \ker \mu_\alpha$. $\square$

3. **Taft algebras**

Let n be an integer ≥ 2 and k a field whose characteristic does not divide n. We assume that k contains a primitive n-th root of unity, which we denote by q.

3.1. **Galois objects over a Taft algebra**

The Taft algebra H_{n^2} has the following presentation as a k-algebra:

$$H_{n^2} = k\,\langle\, x, y \,|\, x^n = 1\,,\; yx = qxy\,,\; y^n = 0\,\rangle$$

(see [20]). The set $\{x^i y^j\}_{0\le i,j\le n-1}$ is a basis of the vector space H_{n^2}, which therefore is of dimension n^2.

The algebra H_{n^2} is a Hopf algebra with coproduct Δ, counit ε and antipode S defined by

$$\Delta(x) = x\otimes x\,, \qquad \Delta(y) = 1\otimes y + y\otimes x\,, \tag{3.1}$$

$$\varepsilon(x) = 1\,, \qquad \varepsilon(y) = 0\,, \tag{3.2}$$

$$S(x) = x^{-1} = x^{n-1}\,, \qquad S(y) = -yx^{-1} = -q^{-1}x^{n-1}y\,. \tag{3.3}$$

When $n = 2$, this is the four-dimensional Sweedler algebra.

Given scalars $a, c \in k$ such that $a \neq 0$, we consider the algebra $A_{a,c}$ with the following presentation:

$$A_{a,c} = k\,\langle\, x, y \,|\, x^n = a\,,\ yx = qxy\,,\ y^n = c\,\rangle\,.$$

This is a right H_{n^2}-comodule algebra with coaction given by the same formulas as (3.1).

By [14, Prop. 2.17 and Prop. 2.22] (see also [12]), any Galois object over H_{n^2} is isomorphic to $A_{a,c}$ for some scalars a, c with $a \neq 0$. Moreover, $A_{a,c}$ is isomorphic to $A_{a',c'}$ as a comodule algebra if and only if there is $v \in k^\times = k - \{0\}$ such that $a' = v^n a$ and $c' = c$. It follows that $(a,c) \mapsto A_{a,c}$ induces a bijection between $k^\times/(k^\times)^n \times k$ and the set of isomorphism classes of Galois objects over H_{n^2}. (Note that $k^\times/(k^\times)^n$ is isomorphic to the cohomology group $H^2(G, k^\times)$.)

If k is algebraically closed, then $k^\times = (k^\times)^n$ and any Galois object over H_{n^2} is isomorphic to $A_{1,c}$ for a unique scalar c.

3.2. A polynomial identity distinguishing the Galois objects

Let $A = A_{a,c}$ be a Galois object as defined in § 3.1. Such a comodule algebra is a twisted comodule algebra ${}^{\alpha}H_{n^2}$ for some normalized convolution invertible two-cocycle α. It can be checked that the map $u : H_{n^2} \to A_{a,c}$ is such that $u_1 = 1$, $u_x = x$ and $u_y = y$. This allows us to compute the corresponding universal comodule algebra map $\mu_\alpha : T \to S \otimes A_{a,c}$ on certain elements of T.

For simplicity, we set $E = X_1^1$, $X = X_1^x$, $Y = X_1^y$ for the X-symbols, and $t_1 = t_1^1$, $t_x = t_1^x$, $t_y = t_1^y$ for the t-symbols. In view of (2.1) and (3.1), we have

$$\mu_\alpha(E) = t_1\,, \quad \mu_\alpha(X) = t_x x\,, \quad \mu_\alpha(Y) = t_1 y + t_y x\,. \tag{3.4}$$

(In the previous formulas we consider the commuting t-variables as extended scalars; this allows us to drop the unit u_1 of $A_{a,c}$ and the tensor symbols between the t-variables and the u-variables.)

Proposition 3.1. *The degree $2n$ polynomial*

$$P_c = (YX - qXY)^n - (1-q)^n X^n Y^n + (1-q)^n c\, E^n X^n$$

is a polynomial H_{n^2}-identity for the Galois object $A_{a,c}$.

This polynomial is a generalization of the degree 4 identity

$$(XY + YX)^2 - 4X^2Y^2 + 4c\, E^2X^2$$

obtained for the Sweedler algebra in [3, Cor. 10.4] (in the special case $b = 0$).

Proof. It suffices to check that $\mu_\alpha(P_c) = 0$ using (3.4) and the defining relations of $A_{a,c}$.

Since $yx = qxy$, we have

$$\begin{aligned} \mu_\alpha(YX - qXY) &= (t_1y + t_yx)t_xx - qt_xx(t_1y + t_yx) \\ &= (1-q)t_xt_yx^2 + t_1t_x(yx - qxy) \\ &= (1-q)t_xt_yx^2 . \end{aligned}$$

Therefore, in view of $x^n = a$, we obtain

$$\mu_\alpha\left((YX - qXY)^n\right) = (1-q)^n t_x^n t_y^n x^{2n} = a^2(1-q)^n t_x^n t_y^n .$$

We also have $\mu_\alpha(E^n) = t_1^n$ and $\mu_\alpha(X^n) = t_x^n x^n = at_x^n$.

To compute $\mu_\alpha(Y^n)$, we need the following well-known fact (see [14, Lemma 2.2]): if u and v satisfy the relation $vu = quv$ for some primitive n-root of unity q, then

$$(u+v)^n = u^n + v^n . \tag{3.5}$$

Since $yx = qxy$, we may apply (3.5) to $u = t_yx$ and $v = t_1y$. We thus obtain

$$\mu_\alpha(Y^n) = t_y^n x^n + t_1^n y^n = at_y^n + ct_1^n .$$

Combining the previous equalities, we obtain $\mu_\alpha(P_c) = 0$. □

The following result shows that the polynomial identity of Proposition 3.1 distinguishes the Galois objects of the Taft algebra.

Theorem 3.2. *If k is algebraically closed, then*

$$\mathrm{Id}_{H_{n^2}}(A_{a,c}) = \mathrm{Id}_{H_{n^2}}(A_{a',c'})$$

implies that $A_{a,c}$ and $A_{a',c'}$ are isomorphic comodule algebras.

Proof. By the last remark in § 3.1, we may assume $a = a' = 1$. Consider the elements $P_c \in \mathrm{Id}_{H_{n^2}}(A_{1,c})$ and $P_{c'} \in \mathrm{Id}_{H_{n^2}}(A_{1,c'})$ given by Proposition 3.1. By the equality of T-ideals, both P_c and $P_{c'}$ are polynomial H_{n^2}-identities for $A_{1,c}$. Hence, so is the difference $P_c - P_{c'}$. Therefore, $\mu_\alpha(P_c - P_{c'}) = 0$. Now,

$$\mu_\alpha(P_c - P_{c'}) = \mu_\alpha\left((c-c')(1-q)^n E^n X^n\right) = (c-c')(1-q)^n t_1^n t_x^n \,.$$

Since $(1-q)^n t_1^n t_x^n \neq 0$, we have $c - c' = 0$. This implies $A_{1,c} = A_{1,c'}$. □

The previous proof shows that the theorem holds if we only assume an inclusion $\mathrm{Id}_{H_{n^2}}(A_{a',c'}) \subset \mathrm{Id}_{H_{n^2}}(A_{a,c})$ of T-ideals. Note also that the single polynomial H_{n^2}-identity P_c determines the full T-ideal $\mathrm{Id}_{H_{n^2}}(A_{a,c})$ over an algebraically closed field.

Remark 3.3. If k is *not* algebraically closed, then the equality of T-ideals of Theorem 3.2 implies that $A_{a',c'}$ is a *form* of $A_{a,c}$. Recall that an H_{n^2}-comodule algebra A is a form of $A_{a,c}$ if there is an algebraic extension k' of k such that $k' \otimes_k A$ and $k' \otimes_k A_{a,c}$ are isomorphic comodule algebras over the Hopf algebra $k' \otimes_k H_{n^2}$.

3.3. Monomial Hopf algebras

Taft algebras can be generalized as follows. Let G be a finite group and x a central element of G of order n. We also assume that there exists a homomorphism $\chi : G \to k^\times$ such that $\chi^n = 1$ and $\chi(x) = q$ is the fixed primitive n-root of unity.

To these data we associate a Hopf algebra H as follows: as an algebra, H is the quotient of the free product $kG * k[y]$ by the two-sided ideal generated by the relations

$$y^n = 0 \quad \text{and} \quad yg = \chi(g)\, gy \,. \quad (g \in G)$$

The elements gy^i, where g runs over the elements of G and $i = 0, \ldots, n-1$, form a basis of H, whose dimension is equal to $n|G|$.

The algebra H has a Hopf algebra structure such that kG is a Hopf subalgebra of H and

$$\Delta(y) = 1 \otimes y + y \otimes x\,, \quad \varepsilon(y) = 0\,, \quad S(y) = -yx^{-1}\,.$$

In the literature this Hopf algebra is called a *monomial Hopf algebra of type I* (see [10, Sect. 7], [11]).

When $G = \mathbb{Z}/n$ and x is a generator of G, then H is the Taft algebra H_{n^2}. Note that for an arbitrary finite group G the inclusion $\mathbb{Z}/n\,x \subset G$ induces a natural inclusion $H_{n^2} \subset H$ of Hopf algebras.

Given a two-cocycle $\sigma \in Z^2(G, k^\times)$ of the group G and a scalar $c \in k$, we define $A_{\sigma,c}$ as the algebra generated by the symbols u_y and u_g for all $g \in G$ and the relations

$$u_1 = 1\,, \qquad u_g u_h = \sigma(g,h)\, u_{gh}\,, \tag{3.6}$$

$$u_y^n = c\,, \qquad u_y u_g = \chi(g)\, u_g u_y \tag{3.7}$$

for all $g, h \in G$. The algebra $A_{\sigma,c}$ is an H-comodule algebra with coaction given by

$$\delta(u_y) = 1 \otimes y + u_y \otimes x \quad \text{and} \quad \delta(u_g) = u_g \otimes g\,. \quad (g \in G) \tag{3.8}$$

Bichon proved that any Galois object over H is isomorphic to one of the form $A_{\sigma,c}$. Moreover, $A_{\sigma,c}$ and $A_{\sigma',c'}$ are isomorphic comodule algebras if and only $c = c'$ and the two-cocycles σ and σ' represent the same element of the cohomology group $H^2(G, k^\times)$. In other words, the map $(\sigma, c) \mapsto A_{\sigma,c}$ induces a bijection between $H^2(G, k^\times) \times k$ and the set of isomorphism classes of Galois objects over H (see [9, Th. 2.1]).

Let now introduce the same X-symbols $E = X_1^1$, $X = X_1^x$, $Y = X_1^y$ as in § 3.2. Since H_{n^2} is a Hopf subalgebra of H, we can reproduce the same computation as in the proof of Proposition 3.1. It allows us to conclude that

$$(YX - qXY)^n - (1-q)^n X^n Y^n + (1-q)^n c\, E^n X^n \tag{3.9}$$

is a polynomial H-identity for the Galois object $A_{\sigma,c}$.

Theorem 3.4. *Suppose that k is algebraically closed. If*

$$\mathrm{Id}_H(A_{\sigma,c}) = \mathrm{Id}_H(A_{\sigma',c'})\,,$$

then $A_{\sigma,c}$ and $A_{\sigma',c'}$ are isomorphic comodule algebras.

Proof. Proceeding as in the proof of Theorem 3.2, we deduce $c = c'$ from the identity (3.9). It remains to check that σ and σ' represent the same element of $H^2(G, k^\times)$.

Consider the following diagram:

$$
\begin{array}{ccccccc}
0 \to & \mathrm{Id}_{kG}(k^\sigma G) & \longrightarrow & T(X_{kG}) & \xrightarrow{\mu} & S(t_{kG}) \otimes k^\sigma G \\
& \iota \downarrow & & \iota_T\downarrow & & \iota_S \downarrow \\
0 \to & \mathrm{Id}_H(A_{\sigma,c}) & \longrightarrow & T(X_H) & \xrightarrow{\mu} & S(t_H) \otimes A_{\sigma,c}
\end{array}
$$

Here $k^\sigma G$ is the twisted group algebra generated by the symbols u_g $(g \in G)$ and Relations (3.6); it is the subalgebra of $A_{\sigma,c}$ generated by the elements u_g, where g runs over all elements of G.

The vertical map $\iota_T : T(X_{kG}) \to T(X_H)$ is induced by the inclusion $kG \to H$; it is injective. The map $\iota_S : S(t_{kG}) \otimes k^\sigma G \to S(t_H) \otimes A_{\sigma,c}$ is induced by the previous natural inclusion and the comodule algebra inclusion $k^\sigma G \subset A_{\sigma,c}$; it sends a typical generator $t_i^g u_{g'}$ of $S(t_{kG}) \otimes k^\sigma G$ to the same expression viewed as an element of $S(t_H) \otimes A_{\sigma,c}$. The maps μ are the corresponding universal comodule maps; the horizontal sequences are exact in view of Theorem 2.6. The diagram is obviously commutative. Hence, the restriction ι of ι_T to $\mathrm{Id}_{kG}(k^\sigma G)$ send the latter to $\mathrm{Id}_H(A_{\sigma,c})$ and is injective. Since ι_S is injective, we have

$$\mathrm{Id}_{kG}(k^\sigma G) = T(X_{kG}) \cap \mathrm{Id}_H(A_{\sigma,c})\,.$$

Consequently, the equality of the theorem implies the equality

$$\mathrm{Id}_{kG}(k^\sigma G) = \mathrm{Id}_{kG}(k^{\sigma'} G)$$

of T-ideals of graded identities. We now appeal to [2, Sect. 1], from which it follows that σ and σ' are cohomologous two-cocycles. $\square$

4. The Hopf algebras $E(n)$

We now deal with the Hopf algebras $E(n)$ considered in [7, 10, 16, 17]. When k is an algebraically closed field of characteristic zero, $E(n)$ is up to isomorphism the only 2^{n+1}-dimensional pointed Hopf algebra with coradical $k\mathbb{Z}/2$.

4.1. **Galois objects over** $E(n)$

Fix an integer $n \geq 1$. Assume that the field k is of characteristic $\neq 2$. The algebra $E(n)$ is generated by elements x, $y_1, \ldots, y_n$ subject to the relations

$$x^2 = 1\,, \quad y_i^2 = 0\,, \quad y_i x + x y_i = 0\,, \quad y_i y_j + y_j y_i = 0$$

for all $i, j = 1, \ldots, n$. As a vector space, $E(n)$ is of dimension 2^{n+1}.

The algebra $E(n)$ is a Hopf algebra with coproduct Δ, counit ε and antipode S determined for all $i = 1, \ldots, n$ by

$$\Delta(x) = x \otimes x\,, \qquad \Delta(y_i) = 1 \otimes y_i + y_i \otimes x\,, \tag{4.1}$$

$$\varepsilon(x) = 1\,, \qquad \varepsilon(y_i) = 0\,, \tag{4.2}$$

$$S(x) = x\,, \qquad S(y_i) = -y_i x\,. \tag{4.3}$$

When $n = 1$, the Hopf algebra $E(n)$ coincides with the Sweedler algebra.

The Galois objects over $E(n)$ can be described as follows. Let $a \in k^\times$, $\underline{c} = (c_1, \ldots, c_n) \in k^n$, and $\underline{d} = (d_{i,j})_{i,j=1,\ldots,n}$ be a symmetric matrix with entries in k. To this collection of scalars we associate the algebra $A(a, \underline{c}, \underline{d})$ generated by the symbols u, $u_1, \ldots, u_n$ and the relations

$$u^2 = a\,, \quad u_i^2 = c_i\,, \quad u u_i + u_i u = 0\,, \quad u_i u_j + u_j u_i = d_{i,j} \tag{4.4}$$

for all $i, j = 1, \ldots, n$. It is a comodule algebra with coaction $\delta : A(a, \underline{c}, \underline{d}) \to A(a, \underline{c}, \underline{d}) \otimes E(n)$ given for all $i = 1, \ldots, n$ by

$$\delta(u) = u \otimes x\,, \quad \delta(u_i) = 1 \otimes y_i + u_i \otimes x\,. \tag{4.5}$$

It follows from [17, Sect. 4] completed by [16, Sect. 2] that any Galois object over $E(n)$ is isomorphic to a comodule algebra of the form $A(a, \underline{c}, \underline{d})$. Moreover, $A(a, \underline{c}, \underline{d})$ and $A(a', \underline{c'}, \underline{d'})$ are isomorphic Galois objects if and only if $\underline{c} = \underline{c'}$, $\underline{d} = \underline{d'}$ and $a' = v^2 a$ for some nonzero scalar v.

Consequently, if k is algebraically closed, then $A(a, \underline{c}, \underline{d})$ is isomorphic to $A(1, \underline{c}, \underline{d})$ for any $a \neq 0$, and the Galois objects $A(1, \underline{c}, \underline{d})$ and $A(1, \underline{c'}, \underline{d'})$ are isomorphic if and only if $\underline{c} = \underline{c'}$ and $\underline{d} = \underline{d'}$.

4.2. **Two families of polynomial identities**

Let us now compute the universal comodule algebra map

$$\mu_\alpha : T \to S \otimes A(a, \underline{c}, \underline{d})$$

corresponding to the comodule algebra $A(a, \underline{c}, \underline{d})$.

We set $E = X_1^1$, $X = X_1^x$, $Y_i = X_1^{y_i}$ for the X-symbols, and $t_0 = t_1^1$, $t_x = t_1^x$, $t_i = t_1^{y_i}$ for the corresponding t-symbols. In view of (2.1) and (4.5), we have

$$\mu_\alpha(E) = t_0\,, \quad \mu_\alpha(X) = t_x\, u\,, \quad \mu_\alpha(Y_i) = t_0\, u_i + t_i\, u \tag{4.6}$$

for all $i = 1, \ldots, n$.

Proposition 4.1. *The degree* 4 *polynomials*

$$(XY_i + Y_iX)^2 - 4X^2Y_i^2 + 4c_i\, E^2X^2 \qquad (1 \le i \le n)$$

and

$$2(Y_iY_j + Y_jY_i)X^2 - (XY_i + Y_iX)(XY_j + Y_jX) - 2d_{i,j}\, E^2X^2 \quad (1 \le i \le j \le n)$$

are polynomial $E(n)$-identities for the Galois object $A(a, \underline{c}, \underline{d})$.

Proof. In view of (4.6) and of the defining relations of $A(a, \underline{c}, \underline{d})$, we obtain

$$\mu_\alpha(E^2) = t_0^2\,, \quad \mu_\alpha(X^2) = at_x^2\,, \quad \mu_\alpha(Y_i^2) = at_i^2 + c_it_0^2\,,$$

$$\mu_\alpha(XY_i + Y_iX) = 2at_xt_i\,, \quad \mu_\alpha((Y_iY_j + Y_jY_i) = 2at_it_j + d_{i,j}t_0^2\,.$$

From these equalities, it is easy to check that the above polynomials belong to the kernel of μ_α, hence are polynomial $E(n)$-identities. □

Theorem 4.2. *Suppose that k is algebraically closed. If*

$$\mathrm{Id}_{E(n)}(A(a, \underline{c}, \underline{d})) = \mathrm{Id}_{E(n)}(A(a', \underline{c}', \underline{d}'))\,,$$

then $A(a, \underline{c}, \underline{d})$ and $A(a', \underline{c}', \underline{d}')$ are isomorphic comodule algebras.

Proof. We proceed as in the proof of Theorem 3.2 by using the identities of Proposition 4.1. Note that there is such an identity for each scalar used to parametrize the Galois objects, and each such scalar appears as the coefficient of the monomial E^2X^2; the latter cannot be an identity since its image under the universal comodule map, being equal to $at_0^2t_x^2$, does not vanish. □

We finally note that the set of $n(n+3)/2$ polynomial $E(n)$-identities of Proposition 4.1 determines the T-ideal $\mathrm{Id}_{E(n)}(A(a, \underline{c}, \underline{d}))$.

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