ANNALES MATHÉMATIQUES

BLAISE PASCAL

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Volume 20, n° 1 (2013), p. 37-73.

<http://ambp.cedram.org/item?id=AMBP_2013__20_1_37_0>

Publication éditée par le laboratoire de mathématiques
de l'université Blaise-Pascal, UMR 6620 du CNRS
Clermont-Ferrand — France

Annales mathématiques Blaise Pascal **20**, 37-73 (2013)

Some Remarks on the Boundary Conditions in the Theory of Navier-Stokes Equations

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Abstract

This article addresses some theoretical questions related to the choice of boundary conditions, which are essential for modelling and numerical computing in mathematical fluids mechanics. Unlike the standard choice of the well known non slip boundary conditions, we emphasize three selected sets of slip conditions, and particularly stress on the interaction between the appropriate functional setting and the status of these conditions.

Théorie des Équations de Navier-Stokes : Remarques sur les Conditions aux Limites

Résumé

Cet article traite de quelques questions théoriques relatives au choix des conditions aux limites, essentielles pour la modélisation et la simulation numérique en mécanique des fluides mathématique. Nous marquons la différence avec le choix standard de conditions de non glissement en soulignant trois ensembles de conditions autorisant glissement, et en insistant particulièrement sur l'interaction entre cadre fonctionnel approprié et statut de ces conditions.

Contents

Keywords: Navier-Stokes, Boundary conditions, Weak solutions.
Math. classification: 35Q35, 35A01, 76D05.

1. **Introduction–Motivation**

Let us consider, in a bounded cylindrical domain $Q_T := (0, T) \times \Omega$, the Navier–Stokes equations (NSE) as the fundamental model of classical fluid mechanics

$$\begin{aligned} &\tfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \mathrm{Div}(2\nu\, \mathbb{D}\boldsymbol{u}) - \nabla\pi + \boldsymbol{f} && \text{in } Q_T, \\ &\mathrm{div}\, \boldsymbol{u} = 0 && \text{in } Q_T, \\ &\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0 && \text{in } \Omega, \end{aligned} \tag{1.1}$$

to describe viscous incompressible Newtonian fluids. Due to a constant density, the conservation of mass is expressed by the incompressibility condition $(1.1)_2$. Equation $(1.1)_1$ says the conservation of momentum. The unknowns are the velocity field $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x})$ and the kinematic pressure $\pi = \pi(t, \boldsymbol{x})$ *so-called associated pressure, de facto determined up to an additive constant in terms of the velocity* $\boldsymbol{u}(t, \boldsymbol{x})$. Given data are $\boldsymbol{f}$, the external force density, and $\boldsymbol{u}_0$, the initial velocity (which also prescribes the initial pressure); $\Omega \subset \mathbb{R}^3$ is the geometrical bounded set filled by the fluid, a connected domain with an impermeable smooth boundary Γ, and finally $\nu > 0$ is the kinematic or eddy viscosity. The tensor $\mathbb{D}\boldsymbol{u}$ denotes the deformation tensor, defined by the symmetrized gradient of velocities $(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)/2$.

Appropriate boundary conditions must be introduced to supplement the system of equations (1.1), in a "good agreement" with the particular physical information. They lead to various initial boundary value problems. Essential is the well-posedness of these problems which are of great importance for applications (meteorology, oceanography, environment and engineering) and for the numerical computation of corresponding flows.

The operator P_σ being the Helmholtz-Leray projector, the abstract model for the system $(1.1)_1$ - $(1.1)_2$ reads

$$\frac{\partial \boldsymbol{u}}{\partial t} + P_\sigma[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}] \,=\, \nu P_\sigma \Delta \boldsymbol{u} + P_\sigma \boldsymbol{f}. \tag{1.2}$$

In general (in the case of domains with boundary we consider) the Stokes operator $-P_\sigma\Delta$ is not the Laplacian operator, a known difficulty and a crucial question, especially for numerical simulations and for engineering applications : How to dispose appropriate algorithms with the "good" Stokes operator taking into account the boundary conditions ?

Many known results concern the case of non-slip boundary conditions, but as quoted by Serrin [36], the choice of a so-called non-slip boundary condition is not always suitable since it does not reflect the behavior of the fluid on or near the boundary in the general case, it does not contain the description of the physical boundary layers near the walls.

In order to solve various models for the velocities, we will consider vector fields whose some components on the boundary vanish, the tangential ones or only the normal component. As pointed out by many authors, the geometry and the regularity of Ω play an important role. Then in the hilbertian theory the space

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega);\ \operatorname{div}\boldsymbol{v} = 0 \ \text{ in}\,\Omega,\ \ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \ \text{ on}\,\Gamma\},$$

is the functional space of reference. Useful Helmholtz-type decompositions are required to characterize the considered vector fields, to distinguish and to precise the behavior on the boundary.

The basic spatial boundary condition for the velocity $\boldsymbol{u}$, expresses the impermeability of Γ, and says that the normal component of $\boldsymbol{u}$ is zero

$$u_n := \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma_T, \tag{1.3}$$

where Γ_T denotes the surface $(0,T)\times\Gamma$, and, as usual, $\boldsymbol{n}$ the outward unit vector normal on Γ_T. One can preserve this impermeability condition as a constraint for $\boldsymbol{u}$, then we need two more boundary conditions : different tangential behaviors can be observed along the boundary, related to the velocity or to the vorticity. The tangential components of any vector field $\boldsymbol{v}$ are defined on Γ_T and computed by $\boldsymbol{v_\tau} := \boldsymbol{v}\times\boldsymbol{n}\times\boldsymbol{n} = \boldsymbol{v} - v_n\,\boldsymbol{n}$. Most of the studied Navier–Stokes models follow the Stokes proposal for the velocity

$$\boldsymbol{u_\tau} = \boldsymbol{0} \quad \text{on } \Gamma_T, \tag{1.4}$$

precisely the conditions which express the non slip boundary conditions. A microscopic rugosity of the boundary or the viscosity of the fluid can justify the use of such conditions. The conditions (1.4) together with (1.3) obviously lead to $\boldsymbol{u}|_{\Gamma_T} = \mathbf{0}$, the standard homogeneous Dirichlet–Stokes boundary conditions.

H. Navier [34] have suggested in 1824 another type of complementary boundary conditions, based on a proportionality between the tangential components of the normal dynamic tensor and the velocity

$$2\nu[\mathbb{D}\boldsymbol{u} \cdot \boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \mathbf{0} \quad \text{on } \Gamma_T \tag{1.5}$$

where $\alpha \geq 0$ is the coefficient of friction. It can depend on the rugosity of Γ, it also can depend on the velocity field itself and on the viscosity parameter. The corresponding Navier–Stokes model (1.1), (1.3) with the Navier's slip boundary conditions (1.5) is well-posed and the theory is in recent progress. Navier boundary conditions are often used to simulate flows near rough walls (Amirat-Bresch-Lemoine-Simon [2], Bucur-Feireisl-Necasova [17], Bucur-Feireisl-Necasova-Wolf [18], Jager-Michelic [27] and [28], Bulicek-Malek-Rajagopal [19] (such as in aerodynamics, in weather forecasts and in hemodynamics) as well as perforated walls ([7]). We also mention that such slip boundary conditions are used in the large eddy simulations of turbulent flows.

Taking use of the vorticity vector field $\boldsymbol{\omega} := \nabla \times \boldsymbol{u}$, and using classical identities, one can observe that, in the case of a flat boundary and when $\alpha = 0$, the conditions (1.5), (1.3) may be replaced by

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \qquad \text{and} \qquad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma_T. \tag{1.6}$$

We call them Navier-type boundary conditions. Among other choices of slip boundary conditions, related to the vorticity, we consider the generalized impermeability conditions (see Bellout-Neustupa-Penel [10])

$$\nabla \times \boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{\omega} \cdot \boldsymbol{n} = 0 \qquad \text{and} \qquad \nabla \times \boldsymbol{\omega} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma_T. \tag{1.7}$$

With these two complementary boundary conditions, the Navier–Stokes model (1.1), (1.3), (1.7) looks promising, it presents not less than the same qualitative properties as the standard model with the Dirichlet–Stokes boundary conditions. Both tangential velocity and vorticity traces on the boundary are not constrained. Note that considering the case of the Euler model ($\nu = 0$) there is absolutely no reason to get velocity-solutions for which the tangential components should vanish on the boundary. The question of the inviscid limit is very challenging, of the greatest importance but not our subject here : for the interested reader we mention the famous seminar by Kato [29] and some recents papers [11] [35] [26] [6].

To contrast with the choice of the impermeability boundary condition as a constraint, and coming back to the Navier-Stokes model, we will look on different possibilities to preserve the Dirichlet-Stokes condition $\boldsymbol{u}_{\boldsymbol{\tau}} = \mathbf{0}$ as a constraint on the boundary. A Robin-type condition, including $\partial_{\boldsymbol{n}}\boldsymbol{u}\cdot\boldsymbol{n}$ and $\boldsymbol{u}\cdot\boldsymbol{n}$, is a first example. For a second example, one can introduce a complementary condition on the pressure, following Marusic [32],

$$\pi + \frac{1}{2}|\boldsymbol{u}|^2 = \pi_0 \quad \text{on } \Gamma_T. \tag{1.8}$$

At least on some parts of the boundary, such conditions but in the non homogeneous case $\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n}$ can be considered for physical situations modelling pipes or hydraulic gears using pumps, or blood vessels.

We can also consider the non homogeneous variants of all the previous boundary conditions, respectively $\boldsymbol{u}\cdot\boldsymbol{n} = g_0$ or $\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n}$, $\nabla\times\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n}$, $\nabla\times\boldsymbol{u}\cdot\boldsymbol{n} = g_1$ and $\nabla\times\boldsymbol{u}\cdot\boldsymbol{n} = g_2\ldots$ Conditions of admissibility must be assumed, for instance if $\boldsymbol{u}(t,.) = \boldsymbol{g}(t,.)$ on the boundary, due to the incompressibility we must have $\int_\Gamma \boldsymbol{g}(t,.)\cdot\boldsymbol{n}\,\mathrm{dS} = 0$. We stress that all these boundary conditions are often mixed in realistic models.

We have also the following linear models associated with the Navier–Stokes equations, namely the Stokes equations:

$$\frac{\partial\boldsymbol{u}}{\partial t} - \mathrm{div}(2\nu\,\mathbb{D}\boldsymbol{u}) + \nabla\pi = \boldsymbol{f} \qquad \text{and} \qquad \mathrm{div}\;\boldsymbol{u} = 0 \qquad \text{in}\,\Omega. \tag{1.9}$$

Very few works deal with the mathematical analysis of problems (1.1) and (1.9) with these boundary conditions. In a two-dimensional, simply connected, bounded domain with the boundary condition (1.6), the well-posedness problem has been rigorously established by Yudovich [42]. These two-dimensional results are based on the fact that the vorticity is scalar and satisfies the maximum principle. However, in the three-dimensional case, the standard maximum principle for the vorticity fails, so that the techniques employed in the two-dimensional case cannot be directly extended to this case. Furthermore, the Navier boundary condition causes additional difficulties in developing *a priori* estimates which require to be compatible with the nonlinear convection term.

First results on general Stokes systems, including the case of variable viscosity (see below), were obtained by Solonnikov [38] and [39] in L^p-Sobolev spaces and weighted Hölder spaces in the case of Dirichlet boundary conditions. Bothe and Pruss [16] and Abels [1] obtained unique solvability in L^p-Sobolev spaces in the case of mixed Dirichlet, Neumann and Navier boundary conditions. The case where the viscosity is constant was studied by Shimada [37] for the Stokes problem with Navier boundary conditions. Berselli [14] gives some criteria concerning the vorticity field which imply the global regularity for the Navier-Stokes equations with stress-free boundary conditions (see also Chen-Osborne-Qian [20]).

Employing the Fujita-Kato method, Mitrea and Monniaux [33] prove the existence of a local mild solution of Navier-Stokes equations with the free stress boundary conditions. The basic question of the global well-posedness of those equations is still an open challenging problem. It is not known that starting from a large smooth initial data with respect to some norm, the solution exists for all time and remains regular. This question has an affirmative answer in the case of thin domains (as in meteorology or in oceanography) for the Navier-Stokes equations with Navier slip boundary condition (Hoang [22], Hoang-Sell [23], Iftimie-Raugel [24], Iftimie-Raugel-Sell [25], Iftimie-Sueur [26]). Local existence and uniqueness of the weak solution to the 2D Navier-Stokes system with pressure boundary condition is obtained by Marusic [32].

In contrast, the stationary Stokes equations with boundary conditions (1.6) or (1.8) (here, without the term involving $\boldsymbol{u}$ for the pressure) have been studied by a large number of authors. The first mathematical analysis of the stationary Stokes problem with the conditions (1.3), (1.5) and $\alpha = 0$ was performed in 1973 by Scadilov and Solonnikov [40] which prove the existence of weak solutions and a local regularity result. The existence of weak solutions and regularity for the Navier-Stokes equations with the Navier slip boundary condition has been obtained by Beirao da Veiga [9] for the half-space and by Berselli [15] in the flat boundary case. The situation of bounded domains, eventually multiply-connected with boundary not connected, has been investigated by Begue-Conca-Murat-Pironneau [8] for the linear and nonlinear cases (see also Ebmeyer-Frehse [21] for some mixed boundary conditions in polyhedral domains).

Next, Bernard [12]-[13] and more recently Kozono-Yanagisawa [30] and Amrouche-Seloula [5] and [4] completed this study by developing very useful properties concerning the vector potentials, some Sobolev inequalities for vector fields and a complete L^p-theory to solve Stokes equations with boundary condition (1.6) or (1.8).

The paper consists of three selected parts, with the three considered sets of boundary conditions : Section 2 and Section 3 give some elements of the mathematical theory of the linear Stokes model with the basic impermeability condition and the Navier-type conditions as complementary boundary conditions. We give a variant of the Stokes problem in Section 4. The case of a prescribed pressure on the boundary for a steady Stokes problem will be treated in Section 5. We study in Section 6 the time dependant Stokes problem. In Section 7 we consider the full Navier–Stokes equations with the basic impermeability condition and, as complementary conditions, the generalized impermeability conditions. In all these sections we stress on the nature and the status of the boundary conditions : so the impermeability condition is a scalar constraint into the space

$$\boldsymbol{H} = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\ \operatorname{div} \boldsymbol{v} = 0\ \text{ in}\,\Omega,\ \ \boldsymbol{v}\cdot\boldsymbol{n} = 0\ \text{ on}\,\Gamma\},$$

but not of the same type as the three scalar Dirichlet conditions ("globally" constraining the velocities into very smaller subspaces). In contrast, the generalized impermeability conditions play different roles : we will see that the first complementary condition is a constraint, the second one being "mathematically natural" ... Indeed, as in the case of Navier's conditions, an appropriate weak integral formulation of the considered model can deliver for some complementary boundary conditions a "mathematically natural" status.

Further many boundary value problems result by considering different boundary conditions (nature and status of them can change radically). The mathematically natural status of Neumann-type boundary conditions is well known, more clear again with non homogeneous conditions which directly influence the weak integral formulations. Some classical examples prescribe $\partial_{\boldsymbol{n}}\boldsymbol{u}$ or $2\nu\mathbb{D}\boldsymbol{u}\cdot\boldsymbol{n} - \pi\ \boldsymbol{n}$, or of Robin-type $\partial_{\boldsymbol{n}}\boldsymbol{u}\cdot\boldsymbol{n} + \lambda\boldsymbol{u}\cdot\boldsymbol{n} = 0$ as a complementary condition with $\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0}$, etc. An exhaustive list of choices of convenient boundary conditions would ask for a to vast article. There also exists a nonlinear generalized form of the Navier's conditions

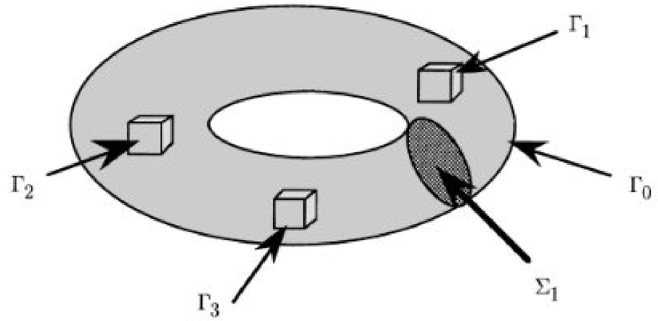

Figure 2.1

(H. Fujita has studied the Stokes model with those nonlinear boundary conditions, a well-posed problem leading to an optimisation model [1993]).

2. Notations and preliminary results

Let Ω be a bounded open set, connected of class $\mathcal{C}^{1,1}$ of $\mathbb{R}^3$ with boundary Γ. Let Γ_i, $0 \leq i \leq I$, denote the connected components of the boundary Γ, Γ_0 being the boundary of the only unbounded connected component of $\mathbb{R}^3 \setminus \Omega$. We do not assume that Ω is simply-connected but we suppose that there exist J connected open surfaces Σ_j, $1 \leq j \leq J$, called 'cuts', contained in Ω, such that each surface Σ_j is an open subset of a smooth manifold. The boundary of each Σ_j is contained in Γ. The intersection $\overline{\Sigma_i} \cap \overline{\Sigma_j}$ is empty for $i \neq j$, and finally the open set $\Omega^\circ = \Omega \setminus \cup_{j=1}^{J} \Sigma_j$ is simply-connected. For $j = 1$ with $I = 3$, see, for example, Figure 2.1.

We denote by $[\cdot]_j$ the jump of a function over Σ_j, *i.e.* the differences of the traces, for $1 \leq j \leq J$ and by $\langle \cdot, \cdot \rangle_{X,X'}$ the duality product between a space X and X'. We shall use bold characters for the vectors or the vector spaces and the non-bold characters for the scalars. The letter C denotes a constant that is not necessarily the same at its various occurrences. Finally, for any function q in $W^{1,p}(\Omega^\circ)$, $\mathbf{grad}\, q$ is the gradient of q in the sense of distributions in $\mathcal{D}'(\Omega^\circ)$. It belongs to $\boldsymbol{L}^p(\Omega^\circ)$ and therefore can be extended to $\boldsymbol{L}^p(\Omega)$. In order to distinguish this extension from the gradient of q in $\mathcal{D}'(\Omega)$, we denote it by $\widetilde{\mathbf{grad}}\, q$.

We are interested in the study of solutions to the Stokes equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \qquad \text{in } Q_T, \tag{2.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } Q_T, \tag{2.2}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \qquad \text{on } \Gamma_T, \tag{2.3}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \text{in } \Omega, \tag{2.4}$$

where $Q_T = \Omega\times]0,T[$ and $\Gamma_T = \Gamma\times]0,T[$. To begin, we study the stationary Stokes problem:

$$\begin{cases} -\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \ \text{ and } \ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} & \text{on } \Gamma, \end{cases} \tag{2.5}$$

(to simplify, we take $\nu = 1$ for the stationary cases). A natural question is what functional space it is resonable to choose the right hand side in order to give a sense to the boundary condition $\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0}$ on Γ. Another interesting question is the uniqueness of the solutions. First, we define the bilinear form

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}$$

on the Sobolev space

$$\boldsymbol{H}^1_{n,\sigma}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega);\ \operatorname{div} \boldsymbol{v} = 0 \text{ in}\, \Omega,\ \ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on}\, \Gamma\}.$$

If Ω is simply connected, we know that for any $\boldsymbol{v} \in \boldsymbol{H}^1_{n,\sigma}(\Omega)$, we have:

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq C\|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \tag{2.6}$$

and then the bilinear form a is coercive and we can applay the Lax-Milgram Lemma to find weak solutions in $\boldsymbol{H}^1(\Omega)$. But, if Ω is multiply connected, the inequality (2.6) is false. Indeed, we introduce the kernel $\boldsymbol{K}^p_n(\Omega)$ for any $1 < p < \infty$:

$$\boldsymbol{K}^p_n(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \operatorname{div} \boldsymbol{v} = 0,\ \mathbf{curl}\, \boldsymbol{v} = \mathbf{0} \text{ in } \Omega \quad \text{and} \quad \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\}.$$

Observe that in this case the kernel $\boldsymbol{K}^p_n(\Omega)$ is not equal to zero and it is of finite dimension and is spanned by the functions $\widetilde{\mathbf{grad}}\, q^n_j$, $j = 1\ldots, J$, where q_j^n is the unique solution in $W^{1,p}(\Omega)$ of the problem

$$\begin{cases} -\Delta q_j^n = 0 \qquad \text{in } \Omega^\circ, \\ \partial_n \, q_j^n = 0 \qquad \text{on } \Gamma, \\ \Big[\, q_j^n \, \Big]_k = \text{constant} \quad \text{ and } \quad [\, \partial_n \, q_j^n \,]_k = 0, \; 1 \leq k \leq J, \\ \Big\langle \partial_n \, q_j^n, \, 1 \Big\rangle_{\Sigma_k} = \delta_{j\,k}, \; 1 \leq k \leq J. \end{cases} \tag{2.7}$$

So, for every function $\boldsymbol{v}$ in $\boldsymbol{W}_n^{1,p}(\Omega)$, we have the following Poincaré's inequality (see [5]):

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \leq C(\|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\, \boldsymbol{v}\|_{\mathrm{L^p(\Omega)}} + \sum_{\mathrm{j=1}}^{\mathrm{J}} |\langle \boldsymbol{v} \cdot \boldsymbol{n}, \, 1 \rangle_{\Sigma_\mathrm{j}}|),$$

where

$$\boldsymbol{W}_n^{1,p}(\Omega) = \{ \boldsymbol{v} \in \, \boldsymbol{W}^{\,1,p}(\Omega); \quad \boldsymbol{v} \cdot \boldsymbol{n} = 0 \ \ \text{on}\, \Gamma \}.$$

Moreover, we must give a sense to the duality brackets $\langle \boldsymbol{f}, \, \boldsymbol{v} \rangle_\Omega$. Unlike the Stokes problem with Dirichlet boundary conditions, the space $\boldsymbol{H}^{-1}(\Omega)$ for $\boldsymbol{f}$ is not adapted to find weak solutions. Let us introduce the space $\boldsymbol{H}_n^p(\mathrm{div}, \Omega)$ for any $1 < p < \infty$ defined by:

$$\boldsymbol{H}_n^p(\mathrm{div}, \Omega) = \{ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \;\; \mathrm{div}\, \boldsymbol{v} \in L^p(\Omega), \;\, \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma \} \, ,$$

for which $\boldsymbol{\mathcal{D}}(\Omega)$ is dense and its dual is characterized by

$$[\boldsymbol{H}_n^p(\mathrm{div}, \Omega)]' = \{ \boldsymbol{F} + \nabla \, \chi, \;\, \boldsymbol{F} \in \boldsymbol{L}^p(\Omega) \text{ and } \chi \in L^p(\Omega) \}.$$

Observe that the test function belongs to the space $\boldsymbol{H}_n^2(\mathrm{div}, \Omega)$ and we can suppose that $\boldsymbol{f}$ belongs to the dual space $[\boldsymbol{H}_n^2(\mathrm{div}, \Omega)]'$ which is a subspace of $\boldsymbol{H}^{-1}(\Omega)$.

We consider the following Stokes problem:

$$(\mathcal{S}_n^0) \begin{cases} -\Delta \, \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \mathrm{div}\, \boldsymbol{u} = 0 \qquad \text{in}\, \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \qquad \text{on}\, \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, \, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J. \end{cases}$$

Now, we are in position to write the following variational formulation

$$\begin{cases} \text{Find } \boldsymbol{u} \in \, \boldsymbol{V}_n^2(\Omega), \text{ such that for any } \boldsymbol{v} \in \, \boldsymbol{V}_n^2(\Omega), \\ \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} = \langle \boldsymbol{f}, \, \boldsymbol{v} \rangle_\Omega, \end{cases} \tag{2.8}$$

where for any $1 < p < \infty$:

$$\boldsymbol{V}_n^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}_n^{1,p}(\Omega);\quad \operatorname{div} \boldsymbol{v} = 0,\quad \langle \boldsymbol{v} \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0\}.$$

Naturally, to solve $(\mathcal{S}_n^0)$, we need the following compatibility conditions on $\boldsymbol{f}$:

$$\forall\, \boldsymbol{v} \in \boldsymbol{K}_n^2(\Omega),\;\; \langle \boldsymbol{f},\, \boldsymbol{v}\rangle_\Omega = 0. \tag{2.9}$$

Next, we can extend (2.8) to any test function in $\boldsymbol{H}^1_{n,\sigma}(\Omega)$. Indeed, we use the following decomposition: for any $\boldsymbol{v} \in \boldsymbol{H}^1_{n,\,\sigma}(\Omega)$,

$$\boldsymbol{v} = \widetilde{\boldsymbol{v}} + \sum_{j=1}^{J} \langle \boldsymbol{v} \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_j}\, \widetilde{\mathbf{grad}}\, q_j^n, \quad \text{with} \quad \widetilde{\boldsymbol{v}} \in \boldsymbol{V}_n^2(\Omega).$$

We observe from (2.9) that problem (2.8) is still valid for any $\boldsymbol{v} \in \boldsymbol{H}^1_{n,\,\sigma}(\Omega)$ and it is equivalent to the Stokes problem $(\mathcal{S}_n^0)$. We can then use the Lax-Milgram Lemma to prove that problem (2.8) has a unique solution $\boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$. The pressure can be found by using a variant of De Rham's Theorem. These results can be summarized by the following theorem (see [4]).

Theorem 2.1. *Let $\boldsymbol{f} \in [\boldsymbol{H}_n^2(\operatorname{div},\,\Omega)]'$ satisfying the compatibility condition* (2.9). *Then, the Stokes problem $(\mathcal{S}_n^0)$ has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ and $\pi \in L^2(\Omega)/\mathbb{R}$ satisfying the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq C \|\boldsymbol{f}\|_{(\boldsymbol{H}_n^2(\operatorname{div},\,\Omega))'}. \tag{2.10}$$

The Stokes problem $(\mathcal{S}_n^0)$ can be also solved by considering other functional spaces for $\boldsymbol{f}$. For example, we can suppose that:

$$\boldsymbol{f} = \mathbf{curl}\, \boldsymbol{\psi} \;\; \text{with} \quad \boldsymbol{\psi} \in \boldsymbol{L}^2(\Omega).$$

The variational formulation (2.8) can be then writen as

$$\begin{cases} \text{Find } \boldsymbol{u} \in \boldsymbol{V}_n^2(\Omega), \text{ such that for any } \boldsymbol{v} \in \boldsymbol{V}_n^2(\Omega), \\ \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} = \int_\Omega \boldsymbol{\psi} \cdot \mathbf{curl}\, \boldsymbol{v}. \end{cases} \tag{2.11}$$

Or equivalently to the problem: Find $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ such that:

$$\begin{cases} -\Delta\, \boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi} \;\text{ and }\; \operatorname{div} \boldsymbol{u} = 0 & \text{in}\, \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad (\mathbf{curl}\, \boldsymbol{u} - \boldsymbol{\psi}) \times \boldsymbol{n} = \mathbf{0} & \text{on}\, \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0,\;\; j = 1, \ldots, J \end{cases} \tag{2.12}$$

which is a Stokes problem with a constant pressure. Observe that if $\boldsymbol{\psi}$ belongs only to $\boldsymbol{L}^2(\Omega)$, then $(\mathbf{curl}\,\boldsymbol{u}-\boldsymbol{\psi})\times\boldsymbol{n}$ belongs to $\boldsymbol{H}^{-1/2}(\Gamma)$, but neither $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}$ nor $\boldsymbol{\psi}\times\boldsymbol{n}$ is defined. However, if $\boldsymbol{\psi}$ belongs to $\boldsymbol{H}_\tau(\mathbf{curl},\,\Omega)$ or to $\boldsymbol{H}(\mathbf{curl},\,\Omega)$, where

$$\boldsymbol{H}(\mathbf{curl},\Omega) = \Big\{\boldsymbol{v}\in\boldsymbol{L}^2(\Omega);\ \mathbf{curl}\,\boldsymbol{v}\in\boldsymbol{L}^2(\Omega)\Big\},$$

$$\boldsymbol{H}_\tau(\mathbf{curl},\Omega) = \Big\{\boldsymbol{v}\in\boldsymbol{L}^2(\Omega);\ \mathbf{curl}\,\boldsymbol{v}\in\boldsymbol{L}^2(\Omega),\quad \boldsymbol{v}\times\boldsymbol{n}=\mathbf{0}\ \text{on}\,\Gamma\Big\},$$

then $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}$ and $\boldsymbol{\psi}\times\boldsymbol{n}$ have a sense in $\boldsymbol{H}^{-1/2}(\Gamma)$.

3. L^p-theory for the Stokes problem

In this section, we establish some regularity results in L^p theory. Before, we note that the assumption on $\boldsymbol{f}$ in Theorem 2.1 can be weakened by considering the space defined for all $1<r,\,p<\infty$ by:

$$\boldsymbol{H}_n^{r,\,p}(\mathrm{div},\,\Omega) = \{\boldsymbol{v}\in\boldsymbol{L}^r(\Omega);\ \ \mathrm{div}\,\boldsymbol{v}\in L^p(\Omega),\ \ \boldsymbol{v}\cdot\boldsymbol{n}=0\ \text{on}\,\Gamma\},$$

which is a Banach space for the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{H}_n^{r,\,p}(\mathrm{div},\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^r(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)},$$

and we will use only in the case

$$r\le p \quad\text{and}\quad \frac{1}{r}\le\frac{1}{p}+\frac{1}{3}. \tag{3.1}$$

We can prove that the space $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{H}_n^{r',\,p'}(\mathrm{div},\,\Omega)$ and its dual space can be characterized as:

$$[\boldsymbol{H}_n^{r',\,p'}(\mathrm{div},\,\Omega)]' = \{\boldsymbol{F}+\nabla\,\chi,\ \ \boldsymbol{F}\in\boldsymbol{L}^r(\Omega),\ \chi\in L^p(\Omega)\}, \tag{3.2}$$

with r' and p' are the conjugate of r and p respectively: $1/p+1/p'=1$. For a future work, the use of this last space will be very useful for the study of the Navier-Stokes equations. We are interested here in the following Stokes problem with non homogeneous boundary conditions:

$$(\mathcal{S}_n)\begin{cases} -\Delta\,\boldsymbol{u}+\nabla\,\pi=\boldsymbol{f} & \text{in }\Omega,\\ \mathrm{div}\,\boldsymbol{u}=\chi & \text{in }\Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n}=g \quad\text{and}\quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{h}\times\boldsymbol{n} & \text{on }\Gamma,\\ \langle\boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}=\alpha_j,\quad j=1,\dots,J, \end{cases}$$

where $\alpha_j \in \mathbb{R}$ are given. We suppose that $\boldsymbol{f}$ belongs to $[\boldsymbol{H}_n^{r',p'}(\mathrm{div},\,\Omega)]'$ and we look to find solutions $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$. We must then suppose that:

$$\chi \in L^p(\Omega),\;\; g \in W^{1-1/p,p}(\Gamma) \;\text{ and }\; \boldsymbol{h} \times \boldsymbol{n} \in \boldsymbol{W}^{-1/p,p}(\Gamma).$$

Observe that the boundary condition on $\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n}$ has a sense. Indeed, we can prove that $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in the space

$$E^{r,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega),\;\; \Delta\,\boldsymbol{v} \in [\boldsymbol{H}_n^{r',p'}(\mathrm{div},\,\Omega)]'\}.$$

As a consequence the mapping $\gamma : \boldsymbol{v} \mapsto \mathbf{curl}\,\boldsymbol{v} \times \boldsymbol{n}$ defined on $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping, still denoted γ from $E^{r,p}(\Omega)$ into $\boldsymbol{W}^{-1/p,p}(\Gamma)$. Moreover, we have the following Green formula: For any $\boldsymbol{v} \in E^{r,p}(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{W}_{n,\sigma}^{1,p'}(\Omega)$:

$$\langle -\Delta\,\boldsymbol{v},\,\boldsymbol{\varphi}\rangle = \int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{\varphi} - \langle \mathbf{curl}\,\boldsymbol{v} \times \boldsymbol{n},\,\boldsymbol{\varphi}\rangle_\Gamma, \tag{3.3}$$

where

$$\boldsymbol{W}_{n,\sigma}^{1,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega),\;\; \mathrm{div}\,\boldsymbol{v} = 0 \;\text{ in}\,\Omega \quad\text{and}\quad \boldsymbol{v}\cdot\boldsymbol{n} = 0 \;\text{ on}\,\Gamma\}.$$

To simplify, we begin by considering the case $\chi = 0$, $g = 0$ and $\alpha_j = 0$. The Stokes problem $(\mathcal{S}_n)$ is then equivalent to the variational formulation:

$$\begin{cases}\text{Find } \boldsymbol{u} \in \boldsymbol{V}_n^p(\Omega), \text{ such that for any } \boldsymbol{\varphi} \in \boldsymbol{W}_{n,\sigma}^{1,p'}(\Omega) : \\ \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{\varphi} = \langle \boldsymbol{f},\,\boldsymbol{\varphi}\rangle_\Omega + \langle \boldsymbol{h} \times \boldsymbol{n},\,\boldsymbol{\varphi}\rangle_\Gamma\end{cases} \tag{3.4}$$

Naturally, to solve problem (3.4), we need the following compatibility condition:

$$\forall \boldsymbol{\varphi} \in \boldsymbol{K}_n^{p'}(\Omega), \quad \langle \boldsymbol{f},\,\boldsymbol{\varphi}\rangle_\Omega + \langle \boldsymbol{h} \times \boldsymbol{n},\,\boldsymbol{\varphi}\rangle_\Gamma = 0. \tag{3.5}$$

Concerning the uniqueness, if we suppose that $\boldsymbol{f} = 0$ and $\boldsymbol{h} \times \boldsymbol{n} = \boldsymbol{0}$, then the pressure is constant. Setting $\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{u}$, we prove that $\boldsymbol{w}$ belongs to the kernel $\boldsymbol{K}_\tau^p(\Omega)$ where

$$\boldsymbol{K}_\tau^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\; \mathrm{div}\,\boldsymbol{v} = 0,\; \mathbf{curl}\,\boldsymbol{v} = \boldsymbol{0} \text{ in } \Omega \text{ and } \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\}.$$

which is of finite dimension and is spanned by the functions ∇q_i^τ, $i = 1\ldots,I$, where q_i^τ is the unique solution in $W^{1,p}(\Omega)$ of the problem

$$\begin{cases}-\Delta q_i^\tau = 0 \qquad \text{in } \Omega, \\ q_i^\tau|_{\Gamma_0} = 0 \quad\text{and}\quad q_i^\tau|_{\Gamma_k} = \text{constant},\;\; 1 \le k \le I, \\ \langle \partial_n\, q_i^\tau,\, 1\rangle_{\Gamma_k} = \delta_{i\,k},\;\; 1 \le k \le I, \quad\text{and}\quad \langle \partial_n\, q_i^\tau,\, 1\rangle_{\Gamma_0} = -1.\end{cases} \tag{3.6}$$

Since $\int_{\Gamma_i} \boldsymbol{w}\cdot\boldsymbol{n} = 0$, for any $1 \le i \le I$, we deduce that $\boldsymbol{w} = \boldsymbol{0}$ in Ω. Then, $\boldsymbol{u}$ belongs to $\boldsymbol{K}^p_n(\Omega)$ and again because $\int_{\Sigma_j} \boldsymbol{u}\cdot\boldsymbol{n} = 0$ for any $1 \le j \le J$, we have $\boldsymbol{u} = \boldsymbol{0}$ in Ω. Finally, to solve problem (3.4), we use the following Inf-Sup condition (see [5]):

$$\inf_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}^{p'}_n(\Omega)\\ \boldsymbol{\varphi}\neq 0}} \sup_{\substack{\boldsymbol{\xi}\in \boldsymbol{V}^{p}_n(\Omega)\\ \boldsymbol{\xi}\neq 0}} \frac{\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\xi}\|_{\boldsymbol{W}^{1,p}(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'}(\Omega)}} > 0. \tag{3.7}$$

Summarize all these results, we obtain the following theorem (see [5]).

Theorem 3.1. (**Weak solutions for** $(\mathcal{S}_{\mathrm{n}})$) *Let* $\boldsymbol{f}$, $\boldsymbol{h}$ *with:*

$$\boldsymbol{f}\in [\boldsymbol{H}^{r',p'}_n(\mathrm{div},\,\Omega)]', \quad \boldsymbol{h}\times\boldsymbol{n}\in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma), \tag{3.8}$$

verifying the compatibility condition (3.5) *and* (3.1)*. Then, the Stokes problem* $(\mathcal{S}_n)$ *has a unique solution* $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbb{R}$ *satisfying the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \le C\big(\|\boldsymbol{f}\|_{(\boldsymbol{H}^{r',p'}_n(\mathrm{div},\Omega))'} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big). \tag{3.9}$$

Now, we give the following Theorem that gives a regularity $\boldsymbol{W}^{2,r}(\Omega)\times W^{1,r}(\Omega)$ for $1 < r < \infty$.

Theorem 3.2. (**Strong solutions for** $(\mathcal{S}_{\mathrm{n}})$) *We suppose that* Ω *is of class* $\mathcal{C}^{2,1}$*. Let* $\boldsymbol{f}\in \boldsymbol{L}^r(\Omega)$, $\boldsymbol{h}\times\boldsymbol{n}\in \boldsymbol{W}^{1-1/r,r}(\Gamma)$ *satisfying the compatibility condition* (3.5)*. Then, the solution* $(\boldsymbol{u},\,\pi)$ *of problem* $(\mathcal{S}_n)$ *given by Theorem 3.1 belongs to* $\boldsymbol{W}^{2,r}(\Omega)\times W^{1,r}(\Omega)$ *and satisfies the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,r}(\Omega)} + \|\pi\|_{W^{1,r}(\Omega)/\mathbb{R}} \le C\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^r(\Omega)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{r},r}(\Gamma)}\big). \tag{3.10}$$

Proof. The pressure $\pi\in W^{1,r}(\Omega)$ is given by:

$$\Delta\pi = \mathrm{div}\,\boldsymbol{f} \ \text{ in}\,\Omega, \qquad (\nabla\pi - \boldsymbol{f})\cdot\boldsymbol{n} = \mathrm{div}_\Gamma(\boldsymbol{h}\times\boldsymbol{n}) \ \text{ on}\,\Gamma.$$

Setting then $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{u}$, we have

$$\boldsymbol{z}\in \boldsymbol{L}^p(\Omega)\subset \boldsymbol{L}^r(\Omega),\ \ \mathrm{div}\,\boldsymbol{z} = 0,\ \ \mathbf{curl}\,\boldsymbol{z}\in \boldsymbol{L}^r(\Omega)\ \text{ and }\ \boldsymbol{z}\cdot\boldsymbol{n}\in \boldsymbol{W}^{1-1/r,r}(\Gamma).$$

Then, $\boldsymbol{z}$ belongs to $\boldsymbol{W}^{1,r}(\Omega)$. Finally, since

$$\boldsymbol{u}\in \boldsymbol{L}^p(\Omega),\ \ \mathrm{div}\,\boldsymbol{u} = 0,\ \ \mathbf{curl}\,\boldsymbol{u}\in \boldsymbol{W}^{1,r}(\Omega),\ \text{ and }\ \boldsymbol{u}\cdot\boldsymbol{n} = 0\ \text{ on}\,\Gamma,$$

we deduce that $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{2,r}(\Omega)$. ☐

Now, we are in position to study the case of non homogeneous conditions.

Theorem 3.3. **i)** *Let* $\boldsymbol{f}$, χ, g, $\boldsymbol{h}$ *and* α_j *with:*

$$\boldsymbol{f} \in [\boldsymbol{H}_n^{r',p'}(\mathrm{div},\,\Omega)]',\ \chi \in L^p(\Omega),\ g \in \boldsymbol{W}^{1-1/p,p}(\Gamma),$$
$$\boldsymbol{h}\times\boldsymbol{n} \in \boldsymbol{W}^{-1/p,p}(\Gamma), \quad (3.11)$$

verifying the compatibility condition (3.5), (3.1) *and*

$$\int_\Omega \chi = \int_\Gamma g. \quad (3.12)$$

Then the Stokes problem $(\mathcal{S}_n)$ *has a unique solution* $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ *satisfying the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \le C\big(\|\boldsymbol{f}\|_{(\boldsymbol{H}_n^{r',p'}(\mathrm{div},\,\Omega))'} + \|\chi\|_{L^p(\Omega)} +$$
$$+ \|g\|_{W^{1-1/p,p}(\Gamma)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big). \quad (3.13)$$

ii) *Moreover, if we suppose that*

$$\chi \in W^{1,r}(\Omega),\ \ g \in W^{2-1/r,r}(\Gamma) \ \text{ and } \ \boldsymbol{h}\times\boldsymbol{n} \in \boldsymbol{W}^{1-1/r,r}(\Gamma), \quad (3.14)$$

then $\boldsymbol{u}$ *belongs to* $\boldsymbol{W}^{2,r}(\Omega)$.
iii) *Moreover, if we suppose that* $\boldsymbol{f} \in \boldsymbol{L}^r(\Omega)$, *then* π *belongs to* $W^{1,r}(\Omega)$.
iv) *Moreover, if we suppose that* Ω *is of class* $\mathcal{C}^{2,1}$ *and*

$$\boldsymbol{f} \in \boldsymbol{L}^r(\Omega),\ \ \chi \in W^{1,r}(\Omega),\ \ g \in W^{2-1/r,r}(\Gamma) \ \text{ and } \ \boldsymbol{h}\times\boldsymbol{n} \in \boldsymbol{W}^{1-1/r,r}(\Gamma), \quad (3.15)$$

then the solution $(\boldsymbol{u},\,\pi)$ *belongs to* $\boldsymbol{W}^{2,r}(\Omega) \times W^{1,r}(\Omega)$.

Proof. It suffices to apply Theorem 3.1 and Theorem 3.2 to solve the problem

$$\begin{cases} -\Delta\,\boldsymbol{w} + \nabla\,\pi = \boldsymbol{f} & \text{in } \Omega,\\ \mathrm{div}\,\boldsymbol{w} = 0 & \text{in } \Omega,\\ \boldsymbol{w}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{w}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{on } \Gamma,\\ \langle \boldsymbol{w}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j} = 0, \quad j = 1,\ldots,J. \end{cases}$$

And then to set:

$$\boldsymbol{u} = \boldsymbol{w} + \nabla\,\theta + \sum_{j=1}^{J}(\alpha_j - \langle\,\nabla\,\theta\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j})\,\widetilde{\mathbf{grad}}\,q_j^n,$$

where θ is the solution of the Neumann problem:

$$\Delta\,\theta = \chi \quad \text{in}\,\Omega \qquad \text{and} \qquad \frac{\partial\,\theta}{\partial\,\boldsymbol{n}} = g \quad \text{on}\,\Gamma. \tag{3.16}$$

$\square$

4. Variant of the Stokes problem $(\mathcal{S}_n)$

Observe that the above relation (3.5) is a necessary condition for the existence of solution for the Stokes problem. Now, our goal is to see what happens precisely, when the data do not satisfy this compatibility condition.

As will appear, the answer strongly depends on the following variant of the Stokes problem $(\mathcal{S}_n)$: Find functions $(\boldsymbol{u},\,\pi,\,\boldsymbol{k})$ such that :

$$(\mathcal{S}'_n) \qquad \begin{cases} -\Delta\,\boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} & \text{in } \Omega,\\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, \ \ \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} - \boldsymbol{k} & \text{on } \Gamma,\\ \langle \boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j} = 0, \quad 1\le j\le J. & \end{cases}$$

We have the following result

Theorem 4.1. *Let* $\boldsymbol{f}\in[\boldsymbol{H}_n^{p'}(\mathrm{div},\,\Omega)]'$ *and* $\boldsymbol{h}\times\boldsymbol{n}\in \boldsymbol{W}^{-1/p,p}(\Gamma)$*. Then the problem* $(\mathcal{S}'_n)$ *has a unique solution* $(\boldsymbol{u},\,\pi,\,\boldsymbol{k})\in \boldsymbol{W}^{1,p}(\Omega)\times(L^p(\Omega)/\mathbb{R})\times \boldsymbol{W}^{1-1/p,p}(\Gamma)$*. Moreover, we have the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p/\mathbb{R}} + \|\boldsymbol{k}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} \le C \quad \Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_n^{p'}(\mathrm{div},\,\Omega)]'} + \\ + \ \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}\Big)$$

Proof. We suppose that $\boldsymbol{f}\in[\boldsymbol{H}_n^{p'}(\mathrm{div},\,\Omega)]'$, $\boldsymbol{h}\times\boldsymbol{n}\in\boldsymbol{W}^{-1/p,p}(\Gamma)$. We note by $(\boldsymbol{y}_j)_j$ an orthonormal basis of $\boldsymbol{K}_n^2(\Omega)$. Observe that the following problem

$$\begin{cases} -\Delta\,\boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} \ \text{ and } \ \operatorname{div}\boldsymbol{u} = 0 & \text{in}\,\Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 & \text{on}\,\Gamma,\\ \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} - \sum_{j=1}^{J}\Big(\langle\boldsymbol{f},\,\boldsymbol{y}_j\rangle\boldsymbol{y}_j + \langle\boldsymbol{h}\times\boldsymbol{n},\,\boldsymbol{y}_j\rangle\boldsymbol{y}_j\Big) & \text{on } \Gamma,\\ \langle\boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j} = 0, \quad 1\le j\le J, & \end{cases}$$

has a unique solution $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ since the compatibility condition (3.5) is verified. We set:

$$\boldsymbol{k} = \sum_{j=1}^{J} \Big(\langle \boldsymbol{f},\, \boldsymbol{y}_j \rangle \boldsymbol{y}_j - \langle \boldsymbol{h} \times \boldsymbol{n},\, \boldsymbol{y}_j \rangle \boldsymbol{y}_j \Big). \tag{4.1}$$

Finally, $(\boldsymbol{u},\,\pi,\,\boldsymbol{k}) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega) \times \boldsymbol{W}^{1-1/p,p}(\Gamma)$ is the solution of $(\mathcal{S}'_n)$. □

5. L^p-Theory for the Stokes problem with pressure boundary condition

In this section we will study the following Stokes problem:

$$(\mathcal{S}_\tau) \begin{cases} -\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = \chi & \text{in}\, \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} \quad \text{and} \quad \pi = \pi_0 & \text{on}\, \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I, \end{cases}$$

where $\boldsymbol{f}$, χ, $\boldsymbol{g}$, and π_0 are given. Due to the non standard boundary conditions:

$$\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n}, \quad \pi = \pi_0 \qquad \text{on } \Gamma,$$

the pressure is decoupled from the system. More precisely, we find that π is a solution of the problem:

$$\Delta\, \pi = \operatorname{div} \boldsymbol{f} + \Delta\, \chi \;\text{ in } \Omega \quad \text{and} \quad \pi = \pi_0 \;\text{ on } \Gamma.$$

Then, π can be found independently of $\boldsymbol{u}$. Observe that if $\operatorname{div} \boldsymbol{f} + \Delta\, \chi = 0$ in Ω and $\pi_0 = 0$ on Γ, the pressure π is zero, unlike the Stokes problem with Dirichlet boundary condition, where the pressure can not be constant.

With π known, we set $\boldsymbol{F} = \boldsymbol{f} - \nabla\, \pi$ and we obtain a system of equations involving only the velocity variable $\boldsymbol{u}$, that is:

$$\begin{cases} -\Delta\, \boldsymbol{u} = \boldsymbol{F} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = \chi & \text{in}\, \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} & \text{on}\, \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I. \end{cases} \tag{5.1}$$

To simplify, we suppose that $\chi = 0$ and $g = 0$. To solve problem (5.1), we need the following Inf-Sup condition (see [5]):

$$\inf_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}_\tau^{p'}(\Omega)\\ \boldsymbol{\varphi}\neq 0}} \sup_{\substack{\boldsymbol{\xi}\in \boldsymbol{V}_\tau^{p}(\Omega)\\ \boldsymbol{\xi}\neq 0}} \frac{\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\xi}\|_{\boldsymbol{W}^{1,p}(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'}(\Omega)}} > 0, \tag{5.2}$$

where

$$\boldsymbol{W}_\tau^{1,p}(\Omega) = \{\boldsymbol{v}\in \boldsymbol{W}^{1,p}(\Omega);\quad \boldsymbol{v}\times\boldsymbol{n} = \mathbf{0}\ \ \text{on}\,\Gamma\}$$

and

$$\boldsymbol{V}_\tau^{p}(\Omega) = \{\boldsymbol{v}\in \boldsymbol{W}_{\tau,\sigma}^{1,p}(\Omega);\ \langle \boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = 0\ \ i = 1,\dots,I\}.$$

We know that for any $\boldsymbol{v}\in \boldsymbol{W}^{1,p}(\Omega)$ with $\operatorname{div}\boldsymbol{v}\in W^{1,p}(\Omega)$ we have the following formula:

$$\operatorname{div}\boldsymbol{v} = \operatorname{div}_\Gamma \boldsymbol{v}_\tau - 2K\boldsymbol{v}\cdot\boldsymbol{n} + \frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}}\cdot\boldsymbol{n}\quad \text{in}\ W^{-1/p,p}(\Gamma) \tag{5.3}$$

where K denotes the mean curvature of Γ, $\boldsymbol{v}_\tau = \boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}$ is the tangential component of $\boldsymbol{v}$ and $\operatorname{div}_\Gamma$ is the surface divergence. So, problem (5.1) is equivalent to: find $\boldsymbol{u}\in \boldsymbol{W}^{1,p}(\Omega)$ such that:

$$\begin{cases} -\Delta\,\boldsymbol{u} = \boldsymbol{F} & \text{in}\,\Omega,\\ \boldsymbol{u}\times\boldsymbol{n} = 0 \quad\text{and}\quad \dfrac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}\cdot\boldsymbol{n} - 2K\boldsymbol{u}\cdot\boldsymbol{n} = 0 & \text{on}\,\Gamma,\\ \langle\boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = 0,\quad 1\le i\le I, \end{cases} \tag{5.4}$$

where the second boundary condition is a Fourier-Robin type boundary condition. More generally, we have the following result:

Theorem 5.1. (**Weak solutions for** $(\mathcal{S}_\tau)$)
i) *Let* $\boldsymbol{f}$, χ, $\boldsymbol{g}$, π_0 *with* $\boldsymbol{f}\in[\boldsymbol{H}_\tau^{p'}(\mathbf{curl},\Omega)]'$, $\chi\in W^{1,p}(\Omega)$, $\boldsymbol{g}\times\boldsymbol{n}\in W^{1-1/p,p}(\Gamma)$, $\pi_0\in W^{1-1/p,p}(\Gamma)$ *and satisfying the compatibility condition:*

$$\forall\boldsymbol{v}\in\boldsymbol{K}_\tau^{p'}(\Omega),\qquad \langle\boldsymbol{f},\,\boldsymbol{v}\rangle_\Omega = \int_\Gamma(\pi_0-\chi)\,\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{s}. \tag{5.5}$$

Then, the Stokes problem $(\mathcal{S}_\tau)$ *has exactly one solution* $\boldsymbol{u}\in\boldsymbol{W}^{1,p}(\Omega)$ *and* $\pi\in W^{1,p}(\Omega)$. *Moreover, there exists a constant* $C>0$ *depending only on* p *and* Ω *such that:*

$$\begin{aligned}\|\,\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \ \le\ & C\Big(\|\,\boldsymbol{f}\,\|_{[\boldsymbol{H}_\tau^{p'}(\mathbf{curl},\Omega)]'} + \|\chi\|_{W^{1,p}(\Omega)} + \\ & + \|\,\boldsymbol{g}\,\|_{W^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{1-1/p,p}(\Gamma)}\Big).\end{aligned}$$

ii) *Moreover, if* Ω *is of classe* $\mathcal{C}^{2,1}$, $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ *and* $\boldsymbol{g} \times \boldsymbol{n} \in \boldsymbol{W}^{2-1/p,p}(\Gamma)$, *then the solution* $\boldsymbol{u}$ *belongs to* $\boldsymbol{W}^{2,p}(\Omega)$.

As it is shown in the previous section, we can also find a variant of the problem $(\mathcal{S}_\tau)$ that we can solve without supposing the compatibility condition (5.5). In this case, we find functions $\boldsymbol{u}$, π and constants c_i for $i = 1, \dots, I$, such that:

$$(\mathcal{S}'_\tau) \quad \begin{cases} -\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \ \text{ and } \ \mathrm{div}\, \boldsymbol{u} = 0 & \text{in}\, \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} & \text{on}\, \Gamma, \\ \pi = \pi_0 \ \text{ on } \Gamma_0 \quad \text{and} \quad \pi = \pi_0 + c_i & \text{on}\, \Gamma_{\mathrm{i}}, \ 1 \leq i \leq I \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \ \ 1 \leq i \leq I. \end{cases}$$

The constants are given by the following theorem.

Theorem 5.2. *Let* $\boldsymbol{f}$, $\boldsymbol{g}$ *and* π_0 *such that:*

$$\boldsymbol{f} \in [\boldsymbol{H}^{p'}_\tau(\mathbf{curl},\, \Omega)]', \quad \boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}(\Gamma), \quad \pi_0 \in W^{1-1/p,p}(\Gamma).$$

Then, the problem $(\mathcal{S}'_\tau)$ *has a unique solution* $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$, $\pi \in W^{1,p}(\Omega)$ *and constants* $c_1, \dots, c_I$ *satisfying the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \\ \leq C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}^{p'}_\tau(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)} + \|\pi_0\|_{W^{1-\frac{1}{p},p}(\Gamma)}\big),$$

and where $c_1, \dots, c_I$ *are given by*

$$c_i = \langle \boldsymbol{f},\, \nabla\, q_i^\tau \rangle_\Omega - \langle \pi_0,\, \nabla\, q_i^\tau \cdot \boldsymbol{n} \rangle_\Gamma. \tag{5.6}$$

In particular, if $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ *and* $\boldsymbol{g} \in \boldsymbol{W}^{2-1/p,p}(\Gamma)$, *then* $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$.

6. Time dependent Stokes problem

In this section, we are interested in the study of the non stationary Stokes problem (2.1)-(2.4). First, we introduce the following space:

$$\boldsymbol{Z}_n(\Omega) = \Big\{ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega); \ \ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \ \text{ on}\, \Gamma, \ \int_\Omega \boldsymbol{v} \cdot \widetilde{\mathbf{grad}}\, q_j^n = 0,\, 1 \leq j \leq J \Big\}.$$

We consider the operator

$$\begin{aligned} \Lambda \ \ : \ \ \boldsymbol{L}^2(\Omega) \bot\, \boldsymbol{K}^2_n(\Omega) &\longrightarrow \boldsymbol{Z}_n(\Omega) \longrightarrow \boldsymbol{L}^2(\Omega) \bot\, \boldsymbol{K}^2_n(\Omega) \\ \boldsymbol{F} &\longmapsto \frac{1}{\nu}\boldsymbol{z} \ \longmapsto \quad \frac{1}{\nu}\boldsymbol{z}, \end{aligned}$$

where $\boldsymbol{z}$ is the solution given by Theorem 2.1 which satisfies:

$$\begin{aligned}\int_\Omega \boldsymbol{z}\cdot\widetilde{\mathbf{grad}}\, q_j^n &= \int_{\Omega^\circ} \boldsymbol{z}\cdot\mathbf{grad}\, q_j^n\\ &= \int_{\Omega^\circ} q_j^n \operatorname{div}\boldsymbol{z} + \sum_{k=1}^{J}\langle \boldsymbol{z}\cdot\boldsymbol{n},\, [q_j^n]\rangle_{\Sigma_k} = 0, \qquad (6.1)\end{aligned}$$

because $[q_j^n]_{\Sigma_k} = \text{cste}$ and $\langle \boldsymbol{z}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_k} = 0$. We set $\boldsymbol{M} = \boldsymbol{L}^2(\Omega)\perp \boldsymbol{K}_n^2(\Omega)$. The operator Λ is clearly linear and continuous from $\boldsymbol{M}$ into $\boldsymbol{Z}_n(\Omega)$. Since Id is compact from $\boldsymbol{Z}_n(\Omega)$ into $\boldsymbol{M}$, the operator Λ is considered as a compact operator from $\boldsymbol{M}$ into itself. This operator is also self-adjoint as

$$\int_\Omega \Lambda\boldsymbol{F}_1\cdot\boldsymbol{F}_2 = \int_\Omega \mathbf{curl}\,\boldsymbol{z}_1\cdot\boldsymbol{z}_2 = \int_\Omega \boldsymbol{F}_1\cdot\Lambda\boldsymbol{F}_2,$$

when $\Lambda\boldsymbol{F}_i = \boldsymbol{z}_i$, $i = 1, 2$. Hence, this operator Λ possesses a hilbertian basis formed by a sequence of eigenfunctions $\boldsymbol{z}_k$:

$$\Lambda\boldsymbol{z}_k = \lambda_k\boldsymbol{z}_k,\ k\geq 1,\ \lambda_k > 1,\ \lambda_k\to\infty,\ k\to\infty$$

$$\boldsymbol{z}_k\in \boldsymbol{V}_n^2(\Omega),\quad \int_\Omega \mathbf{curl}\,\boldsymbol{z}_k\cdot\mathbf{curl}\,\boldsymbol{v} = \lambda_k\int_\Omega \boldsymbol{z}_k\cdot\boldsymbol{v},\quad \forall\boldsymbol{v}\in\boldsymbol{V}_n^2(\Omega). \qquad (6.2)$$

As usual

$$\int_\Omega \boldsymbol{z}_k\cdot\boldsymbol{z}_l = \delta_{kl},\quad \int_\Omega \mathbf{curl}\,\boldsymbol{z}_k\cdot\boldsymbol{z}_l = \lambda_k\delta_{kl}.$$

Remark that (6.2) is also valid for any $\boldsymbol{v}\in\boldsymbol{V}$. Indeed, let $\boldsymbol{v}$ be in $\boldsymbol{V}$ and we set

$$\widetilde{\boldsymbol{v}} = \boldsymbol{v} - \sum_{j=1}^{J}\langle\boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j}\widetilde{\mathbf{grad}}\, q_j^n.$$

Then, we establish

$$\int_\Omega \mathbf{curl}\,\boldsymbol{z}_k\cdot\mathbf{curl}\,\boldsymbol{v} = \int_\Omega \mathbf{curl}\,\boldsymbol{z}_k\cdot\mathbf{curl}\,\widetilde{\boldsymbol{v}} = \lambda_k\int_\Omega \boldsymbol{z}_k\cdot\widetilde{\boldsymbol{v}} = \lambda_k\int_\Omega \boldsymbol{z}_k\cdot\boldsymbol{v},$$

where we have used the fact that (see (6.1))

$$\int_\Omega \boldsymbol{z}_k\cdot\widetilde{\mathbf{grad}}\, q_j^n = 0,\quad \forall j = 1,\ldots,J.$$

Using again Theorem 2.1, we can interpret (6.2) as follows: for each k, there exists $\pi_k \in L^2(\Omega)/\mathbb{R}$ such that:

$$\begin{aligned} -\nu\Delta\, \boldsymbol{z}_k + \nabla\, \pi_k = \lambda_k \boldsymbol{z}_k \qquad & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{z}_k = 0 \qquad & \text{in } \Omega, \\ \boldsymbol{z}_k \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{z}_k \times \boldsymbol{n} = \mathbf{0} \qquad & \text{on } \Gamma, \\ \langle \boldsymbol{z}_k \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0, \ \ \forall 1 \leq j \leq J. & \end{aligned}$$

Now, observe that $\boldsymbol{L}^2(\Omega) = \boldsymbol{M} \oplus \boldsymbol{K}^2_n(\Omega)$. We note by $(\boldsymbol{y}_j)_j$ an orthonormal base of $\boldsymbol{K}^2_n(\Omega)$. We know that each $\boldsymbol{y}_j$ is a linear combination of $\widetilde{\mathbf{grad}}\, q^n_1, \ldots, \widetilde{\mathbf{grad}}\, q^n_J$. Then, the sequence $(\boldsymbol{w}_j)_{j\in\mathbb{N}^*}$ defined by

$$\boldsymbol{w}_j = \begin{cases} \boldsymbol{y}_j & \text{if} \quad 1 \leq j \leq J, \\ \boldsymbol{z}_{j-J} & \text{if} \quad j \geq J+1. \end{cases} \tag{6.3}$$

is a hilbertian base for the space $\boldsymbol{L}^2(\Omega)$ and for any $j \in \mathbb{N}^*$, $\boldsymbol{w}_j \in \boldsymbol{H}$.

We propose the following variational formulation of (2.1)-(2.4):

For $\boldsymbol{f}$ given in $\boldsymbol{L}^2(0,\, T;\, [\boldsymbol{H}^2_n(\mathrm{div};\, \Omega)]')$ and $\boldsymbol{u}_0 \in \boldsymbol{H}$:

find $\boldsymbol{u} \in \boldsymbol{L}^2(0,\, T;\ \boldsymbol{V}) \cap \boldsymbol{L}^\infty(0,\, T;\, \boldsymbol{H})$ such that:

$$\forall \boldsymbol{v} \in \boldsymbol{V}, \qquad \frac{\mathrm{d}}{\mathrm{d}\, t}(\boldsymbol{u}(t),\, \boldsymbol{v}) + \nu \int_\Omega \mathbf{curl}\, \boldsymbol{u}(t) \cdot \mathbf{curl}\, \boldsymbol{v} = \langle \boldsymbol{f}(t),\, \boldsymbol{v}\rangle_\Omega \tag{6.4}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \tag{6.5}$$

with $\langle\cdot,\, \cdot\rangle_\Omega$ as the duality product between $[\boldsymbol{H}^2_n(\mathrm{div};\, \Omega)]'$ and $\boldsymbol{H}^2_n(\mathrm{div};\, \Omega)$.

Remark 6.1. Note that if we consider only the space $\boldsymbol{L}^2(0,\, T;\ \boldsymbol{V})$ as the space for which existence and uniqueness will be proved, the condition (6.5) does not makes sense as we will see later.

We can check that any solution of (6.4)-(6.5) is a solution of (2.1)-(2.4) and conversely.

6.1. Existence of the weak solution

We will now develop the existence theorem of weak solution of the problem (2.1)-(2.4). The proof is based on the construction of an approximate solution by the Galerkin method then a passage to the limit by using a priori estimates. We use the sequence $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m$ and for each m we define an approximate solution $\boldsymbol{u}_m$ of (2.1) as follows:

$$\boldsymbol{u}_m(t) = \sum_{j=1}^{m} g_{jm}(t)\, \boldsymbol{w}_j, \tag{6.6}$$

such that:

$$\int_\Omega \boldsymbol{u}'_m(t)\cdot \boldsymbol{w}_j + \nu \int_\Omega \mathbf{curl}\, \boldsymbol{u}_m(t)\cdot \mathbf{curl}\, \boldsymbol{w}_j = \langle \boldsymbol{f}(t),\, \boldsymbol{w}_j\rangle, \quad 1 \leq j \leq m, \tag{6.7}$$

$$\boldsymbol{u}_m(0) = \boldsymbol{u}_{0m}, \tag{6.8}$$

where, $\boldsymbol{u}_{0m}$ is such that $\boldsymbol{u}_{0m} \longrightarrow \boldsymbol{u}_0$ in $\boldsymbol{H}$ when $m \to \infty$.

We have a linear differential system for the functions g_{im}, $1 \leq i \leq m$, indeed we have:

$$\sum_{i=1}^{m} g'_{im}(t) \int_\Omega \boldsymbol{w}_i \cdot \boldsymbol{w}_j \quad + \quad \sum_{i=1}^{m} \nu\, g_{im}(t) \int_\Omega \mathbf{curl}\, \boldsymbol{w}_i \cdot \mathbf{curl}\, \boldsymbol{w}_j$$
$$= \langle \boldsymbol{f}(t),\, \boldsymbol{w}_j\rangle, \quad 1 \leq j \leq m. \tag{6.9}$$

Since the elements $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m$ are linearly independent, it is well known that the matrix with elements $\int_\Omega \boldsymbol{w}_i \cdot \boldsymbol{w}_j$ $(1 \leq i, j \leq m)$ is non singular, hence by inverting this matrix, we reduce (6.9) to a linear system with constant coefficients:

$$g'_{im}(t) + \sum_{j=1}^{m} \alpha_{ij}\, g_{im}(t) = \sum_{j=1}^{m} \beta_{ij} \langle \boldsymbol{f}(t),\, \boldsymbol{w}_j\rangle, \quad 1 \leq i \leq m, \tag{6.10}$$

where $\alpha_{ij}, \beta_{ij} \in \mathbb{R}$. The condition (6.8) is equivalent to m equations:

$$g_{im}(0) = \text{the i}^{\text{th}} \text{ component of } \boldsymbol{u}_{0m} \tag{6.11}$$

The linear differential system (6.10) together with the initial condition (6.11) admits a unique solution $g_{1m}, \ldots g_{mm}$ on the whole interval $[0, T]$. We will obtain *a priori* estimates independent of m for the functions $\boldsymbol{u}_m$ and $\boldsymbol{u}'_m$ and then pass to the limit.

A priori estimates I:

We multiply the equation (6.7) by $g_{jm}(t)$ and add these equations for $1 \leq j \leq m$. We get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{u}_m(t)|^2 + \nu \int_\Omega |\mathbf{curl}\, \boldsymbol{u}_m(t)|^2 \;=\; \langle \boldsymbol{f}(t),\, \boldsymbol{u}_m(t)\rangle \tag{6.12}$$
$$\leq \; \|\boldsymbol{f}(t)\|_{\mathbb{H}^2_n(\mathrm{div},\,\Omega)'} \|\boldsymbol{u}_m(t)\|_{\boldsymbol{L}^2(\Omega)},$$

where div $\boldsymbol{u}_m = 0$ in Ω. For $m \geq J+1$, we can write (6.6) as

$$\begin{aligned} \boldsymbol{u}_m(t) &= \sum_{j=1}^{J} g_{jm}(t)\,\boldsymbol{y}_j + \sum_{j=J+1}^{m} g_{jm}(t)\,\boldsymbol{z}_{j-J} \\ &= \boldsymbol{a}_m(t) + \boldsymbol{b}_m(t) \end{aligned} \tag{6.13}$$

and hence

$$\|\boldsymbol{u}_m(t)\|_{\boldsymbol{L}^2(\Omega)} \leq \|\boldsymbol{a}_m(t)\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{b}_m(t)\|_{\boldsymbol{L}^2(\Omega)}. \tag{6.14}$$

To estimate the second term in (6.14), observe that $\boldsymbol{b}_m$ satisfies:

$$\operatorname{div} \boldsymbol{b}_m = 0 \text{ in } \Omega, \quad \boldsymbol{b}_m \cdot \boldsymbol{n} = 0 \text{ on } \Gamma \quad \text{and} \quad \langle \boldsymbol{b}_m \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_k} = 0, \;\; 1 \leq k \leq J.$$

According to [3] and [5], we have:

$$\|\boldsymbol{b}_m(t)\|_{\boldsymbol{L}^2(\Omega)} \leq C\|\mathbf{curl}\,\boldsymbol{b}_m\|_{\boldsymbol{L}^2(\Omega)} = C\|\mathbf{curl}\,\boldsymbol{u}_m\|_{\boldsymbol{L}^2(\Omega)}, \tag{6.15}$$

where we have used the fact that $\mathbf{curl}\,\boldsymbol{a}_m = \mathbf{0}$ in Ω.

We shall obtain an estimate for the first term in (6.14). From (6.7), we have for any $1 \leq j \leq J$ and $s > 0$:

$$g'_{jm}(s) = \langle \boldsymbol{f}(s),\, \boldsymbol{w}_j\rangle.$$

Integrating this equality from 0 to t we obtain

$$\begin{aligned} |g_{jm}(t)| &\leq |g_{jm}(0)| + \int_0^t |\langle \boldsymbol{f}(s),\, \boldsymbol{w}_j\rangle|\,\mathrm{d}s \\ &\leq |g_{jm}(0)| + \int_0^t \|\boldsymbol{f}(s)\|_{\boldsymbol{H}^2_n(\mathrm{div};\,\Omega)'}\,\mathrm{d}s. \end{aligned}$$

Moreover,

$$\begin{aligned} |g_{jm}(t)|^2 &\leq 2|g_{jm}(0)|^2 + 2\Big(\int_0^t \|\boldsymbol{f}(s)\|_{\boldsymbol{H}^2_n(\mathrm{div};\,\Omega)'}\,\mathrm{d}s\Big)^2, \\ &\leq 2|g_{jm}(0)|^2 + 2t\int_0^t \|\boldsymbol{f}(s)\|^2_{\boldsymbol{H}^2_n(\mathrm{div};\,\Omega)'}\,\mathrm{d}s. \end{aligned}$$

Then, we have

$$\sum_{j=1}^{J} |g_{jm}(t)|^2 = \|\boldsymbol{a}_m(t)\|^2_{\boldsymbol{L}^2(\Omega)}$$
$$\leq 2\sum_{j=1}^{J} |g_{jm}(t)|^2 + 2t\int_0^T \|\boldsymbol{f}(s)\|^2_{\boldsymbol{H}^2_n(\mathrm{div};\,\Omega)'}\,\mathrm{d}s$$
$$\leq 2\|\boldsymbol{u}_{0m}\|^2_{\boldsymbol{L}^2(\Omega)} + 2t\int_0^T \|\boldsymbol{f}(s)\|^2_{\boldsymbol{H}^2_n(\mathrm{div};\,\Omega)'}. \tag{6.16}$$

Combining (6.15) and (6.16), the right-hand side of (6.12) is majored by:

$$\|\boldsymbol{f}(t)\|_{\boldsymbol{H}^2_n(\mathrm{div};\,\Omega)'}\|\boldsymbol{u}_m(t)\|_{\boldsymbol{L}^2(\Omega)} \leq C\|\boldsymbol{f}(t)\|^2_{\boldsymbol{H}^2_n(\mathrm{div};\,\Omega)'}$$
$$+\frac{\nu}{2}\|\mathbf{curl}\,\boldsymbol{u}_m(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{u}_{0m}\|^2_{\boldsymbol{L}^2(\Omega)} + t\int_0^T \|\boldsymbol{f}(s)\|^2_{\boldsymbol{H}^2_n(\mathrm{div};\,\Omega)'}\,\mathrm{d}s.$$

Therefore

$$\frac{1}{2}\frac{d}{d\,t}\|\boldsymbol{u}_m(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \nu\|\mathbf{curl}\,\boldsymbol{u}_m(t)\|^2_{\boldsymbol{L}^2(\Omega)}$$
$$\leq C\|\boldsymbol{f}(t)\|^2_{\boldsymbol{H}^2_n(\mathrm{div};\,\Omega)'} + \|\boldsymbol{u}_{0m}\|^2_{\boldsymbol{L}^2(\Omega)} + t\int_0^T \|\boldsymbol{f}(s)\|^2_{\boldsymbol{H}^2_n(\mathrm{div};\,\Omega))'}\,\mathrm{d}s \tag{6.17}$$

Integrating (6.17) from 0 to t, we obtain in particular

$$\begin{aligned}\|\boldsymbol{u}_m(t)\|^2_{\boldsymbol{L}^2(\Omega)} &\leq C\Big(\int_0^t \|\boldsymbol{f}(s)\|^2_{\boldsymbol{H}^2_n(\mathrm{div};\,\Omega)'} + T^2\big(\int_0^T \|\boldsymbol{f}(s)\|^2_{\boldsymbol{H}^2_n(\mathrm{div};\,\Omega)'}\big)\Big)\\ &+ 3T\|\boldsymbol{u}_{0m}\|^2_{\boldsymbol{L}^2(\Omega)}.\end{aligned}$$

Hence

$$\sup_{t\in[0,\,T]} \|\boldsymbol{u}_m(t)\|^2_{\boldsymbol{L}^2(\Omega)}$$
$$\leq CT^2\Big(\int_0^T \|\boldsymbol{f}(s)\|^2_{\boldsymbol{H}^2_n(\mathrm{div};\,\Omega)'}\,\mathrm{d}s)\Big) + 3T\|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)} \tag{6.18}$$

The right-hand side of (6.18) is finite and independent of m; therefore,

$$(\boldsymbol{u}_m)_m \text{ is bounded in } \boldsymbol{L}^\infty(0,\,T;\,\boldsymbol{H}) \tag{6.19}$$

In the same way, integrating (6.17) from 0 to T, we obtain:

$$\|\boldsymbol{u}_m(T)\|^2_{\boldsymbol{L}^2(\Omega)} + \nu \int_0^T \|\mathbf{curl}\, \boldsymbol{u}_m(t)\|^2_{\boldsymbol{L}^2(\Omega)} \leq CT^2\Big(\int_0^T \|\boldsymbol{f}(s)\|^2_{\boldsymbol{H}^2_n(\mathrm{div};\,\Omega)'}\,\mathrm{d}s\Big) + 3T\|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)}. \tag{6.20}$$

Therefore,

$$(\boldsymbol{u}_m)_m \text{ is bounded in } \boldsymbol{L}^2(0,\, T;\; \boldsymbol{V}) \tag{6.21}$$

Remark 6.2. In the estimate (6.18) and (6.20), we can replace the factor 3 by $(1+\alpha)$, with $\alpha > 0$ arbitrary.

A priori estimates II:

Let us prove another estimate for the approximate solution $\boldsymbol{u}_m$ constructed by the Galerkin method.
Let $P_m : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega)$ be the projection of $\boldsymbol{H}$ on the space spanned by $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m$. We have:

$$\boldsymbol{u}'_m = -\nu\, P_m A \boldsymbol{u}_m + P_m \boldsymbol{f}, \tag{6.22}$$

where A is the accociated Stokes operator, a linear mapping continuous from $\boldsymbol{V}$ to $\boldsymbol{H}^2_n(\mathrm{div};\,\Omega)'$ such that:

$$\forall \boldsymbol{v} \in \boldsymbol{V}, \quad \langle A\boldsymbol{u}_m,\, \boldsymbol{v}\rangle = \int_\Omega \mathbf{curl}\, \boldsymbol{u}_m(t) \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}. \tag{6.23}$$

The operator P_m is a linear continuous mapping from $\boldsymbol{H}^2_n(\mathrm{div};\,\Omega)$ to $\boldsymbol{H}^2_n(\mathrm{div};\,\Omega)$ and

$$\|P_m\|_{\mathcal{L}(\boldsymbol{H}^2_n(\mathrm{div};\,\Omega):\, \boldsymbol{H}^2_n(\mathrm{div};\,\Omega))} \leq 1.$$

The dual operator $P^*_m = P_m : \boldsymbol{H}^2_n(\mathrm{div};\,\Omega)' \to \boldsymbol{H}^2_n(\mathrm{div};\,\Omega)'$ is also linear continuous and

$$\|P_m\|_{\mathcal{L}(\boldsymbol{H}^2_n(\mathrm{div};\,\Omega)':\, \boldsymbol{H}^2_n(\mathrm{div};\,\Omega)')} \leq 1.$$

From (6.21) and (6.22), we deduce that

$$(\boldsymbol{u}'_m)_m \text{ is bounded in } \boldsymbol{L}^2(0,\, T;\; \boldsymbol{H}^2_n(\mathrm{div};\,\Omega)') \tag{6.24}$$

Passage to the limit: It follows from the bounds (6.19), (6.21) and (6.24) that there exists a subsequence of $(\boldsymbol{u}_m)_m$ still denoted $(\boldsymbol{u}_m)_m$ and a function $\boldsymbol{u} \in \boldsymbol{L}^2(0,\, T;\; \boldsymbol{V}) \cap \boldsymbol{L}^\infty(0,\, T;\; \boldsymbol{H})$, such that

$$\boldsymbol{u}_m \rightharpoonup \boldsymbol{u} \text{ weakly in } \boldsymbol{L}^2(0,\, T;\; \boldsymbol{V}) \text{ and weak}^* \text{ in } \boldsymbol{L}^\infty(0,\, T;\; \boldsymbol{H})$$

Passing to the limit for $m \to \infty$ in (6.12), we get

$$\forall \boldsymbol{v} \in \boldsymbol{V}, \quad \langle \boldsymbol{u}'(t),\; \boldsymbol{v}\rangle_\Omega + \nu \int_\Omega \mathbf{curl}\, \boldsymbol{u}(t) \cdot \mathbf{curl}\, \boldsymbol{v} = \langle \boldsymbol{f}(t),\; \boldsymbol{v}\rangle_\Omega, \tag{6.25}$$

which is exactly (6.4). Finally, it remains to check that $\boldsymbol{u}(0) = \boldsymbol{u}_0$. Since $\boldsymbol{u}_m$ belongs to $\mathcal{C}([0,T];\; \boldsymbol{H}^2_n(\mathrm{div};\, \Omega)')$, we have in particular for $m \to \infty$:

$$\boldsymbol{u}_m(0) = \boldsymbol{u}_{0m} \to \boldsymbol{u}(0) \text{ in } \boldsymbol{H}^2_n(\mathrm{div};\, \Omega)' \text{ weakly.}$$

But $\boldsymbol{u}_{0m} \to \boldsymbol{u}_0$ in $\boldsymbol{H}$ strongly. This implies that $\boldsymbol{u}(0) = \boldsymbol{u}_0$ in the sense of $\boldsymbol{H}^2_n(\mathrm{div};\, \Omega)'$.

Remark 6.3. We wish to make precise on the sense of the condition (6.5). We can write (6.4) as

$$\forall \boldsymbol{v} \in \boldsymbol{V}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{u},\, \boldsymbol{v}\rangle = \langle \boldsymbol{f} - \nu\, A\boldsymbol{u},\, \boldsymbol{v}\rangle \tag{6.26}$$

Since A is linear and continuous from $\boldsymbol{V}$ to $\boldsymbol{H}^2_n(\mathrm{div};\, \Omega)'$ and since $\boldsymbol{u} \in \boldsymbol{L}^2(0,\, T;\, \boldsymbol{V})$, the function $A\boldsymbol{u}$ belong to $\boldsymbol{L}^2(0,\, T;\; \boldsymbol{H}^2_n(\mathrm{div};\, \Omega)')$; hence

$$\boldsymbol{f} - \nu\, A\boldsymbol{u} \in \boldsymbol{L}^2(0,\, T;\; \boldsymbol{H}^2_n(\mathrm{div};\, \Omega)')$$

and (6.26) show that $\boldsymbol{u}' \in \boldsymbol{L}^2(0,\, T;\; \boldsymbol{H}^2_n(\mathrm{div};\, \Omega)')$. So, $\boldsymbol{u}$ is a continuous function from $[0,\, T]$ into $\boldsymbol{H}^2_n(\mathrm{div};\, \Omega)'$, and therefore the condition (6.5) makes sense.

Theorem 6.4. *For given* $\boldsymbol{f} \in \boldsymbol{L}^2(0,\, T;\; \boldsymbol{H}^2_n(\mathrm{div};\, \Omega)')$ *and* $\boldsymbol{u}_0 \in \boldsymbol{H}$*, problem* (2.1)-(2.4) *has a unique solution* $(\boldsymbol{u},\, \pi)$ *where*

$$\boldsymbol{u} \in \boldsymbol{L}^2(0,\, T;\; \boldsymbol{V}) \cap \boldsymbol{L}^\infty(0,\, T;\; \boldsymbol{H}), \quad \boldsymbol{u}' \in \boldsymbol{L}^2(0,\, T;\; \boldsymbol{H}^2_n(\mathrm{div};\, \Omega)')$$

$$\text{and } \pi \in \boldsymbol{L}^2(0,\, T;\; \boldsymbol{L}^2(\Omega))/\mathbb{R}.$$

Moreover, we have the following estimates

$$\begin{aligned} \|\boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2(\Omega)} \;&+\; \nu \int_0^T \|\mathbf{curl}\, \boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2(\Omega)} \leq C_T(1+\alpha)\|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)} \\ &+\; C_T \int_0^T \|\boldsymbol{f}(s)\|^2_{\boldsymbol{H}^2_n(\mathrm{div};\, \Omega)'} \end{aligned} \tag{6.27}$$

If moremover, $\langle \boldsymbol{f}, \widetilde{\mathbf{grad}}\, q_j^T \rangle_{\Sigma_k} = 0$, $\forall\, j = 1, \dots J$, *then*

$$\begin{aligned} \|\boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2(\Omega)} \quad + \quad \nu \int_0^T \|\mathbf{curl}\, \boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2(\Omega)} &\leq \|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)} \\ &+ \frac{C}{\nu} \int_0^T \|\boldsymbol{f}(s)\|^2_{\boldsymbol{H}^2_n(\mathrm{div}\,\Omega)'} \end{aligned} \tag{6.28}$$

Proof. The existence of $\boldsymbol{u}$ is already proved using the Galerkin procedure. The existence of π follows from the De Rham's theorem (see e.g. [41]). To prove that $(\boldsymbol{u}, \pi)$ is unqiue, we can use exactely the same arguments in [41] or in [31]. The estimate (6.27) follows easily from (6.18) by integration from 0 to t and by using Remark 6.2. Suppose now that for any $1 \leq j \leq J$

$$\langle \boldsymbol{f}, \widetilde{\mathbf{grad}}\, q_j^n \rangle_{\Sigma_k} = 0$$

In this case, (6.12) can be written as

$$\frac{1}{2} \frac{\partial}{\partial t} \|\boldsymbol{u}_m(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \nu \|\mathbf{curl}\, \boldsymbol{u}_m(t)\|^2_{\boldsymbol{L}^2(\Omega)} \quad = \quad \langle \boldsymbol{f}(t), \, \boldsymbol{b}_m(t) \rangle$$

and then, the right hand side of this last equality is majorized by :

$$\begin{aligned} \langle \boldsymbol{f}(t), \, \boldsymbol{b}_m(t) \rangle \quad &\leq \quad \|\boldsymbol{f}(t)\|_{\boldsymbol{H}^2_n(\mathrm{div};\,\Omega)'} \|\mathbf{curl}\, \boldsymbol{b}_m(t)\|_{\boldsymbol{L}^2(\Omega)}, \\ &\leq \quad \|\boldsymbol{f}(t)\|_{\mathtt{H}^2_n(\mathrm{div},\,\Omega)'} \|\mathbf{curl}\, \boldsymbol{u}_m(t)\|_{\boldsymbol{L}^2(\Omega)}, \\ &\leq \quad \frac{C}{\nu} \|\boldsymbol{f}(t)\|^2_{\mathtt{H}^2_n(\mathrm{div},\,\Omega)'} + \frac{\nu}{2} \|\mathbf{curl}\, \boldsymbol{u}_m(t)\|^2_{\boldsymbol{L}^2(\Omega)}. \end{aligned}$$

Integrating this inequality from 0 to t, we obtain the estimate (6.28). □

6.2. **Regularity result**

We will now prove a simple regularity result for $\boldsymbol{u}$ and π.

Theorem 6.5. *We suppose that :*

$$\boldsymbol{f} \in \boldsymbol{L}^2(0, T; \, \boldsymbol{L}^2(\Omega)) \quad \text{and} \quad \boldsymbol{u}_0 \in \boldsymbol{V}.$$

Then, the solution $(\boldsymbol{u}, \pi)$ *given in Theorem 6.4 belongs to*

$$\boldsymbol{L}^2(0, T; \, \boldsymbol{H}^2(\Omega)) \times \boldsymbol{L}^2(0, T; \, \boldsymbol{H}^1(\Omega)).$$

Proof. Since $\boldsymbol{u}_0 \in \boldsymbol{V}$, we know that there exists a sequence $\boldsymbol{u}_{0m}$ in the space spanned by $\boldsymbol{w}_1, \dots, \boldsymbol{w}_m$ such that $\boldsymbol{u}_{0m} \longrightarrow \boldsymbol{u}_0$ in $\boldsymbol{V}$ for $m \to 0$. Using (6.23), we can write (6.7) as

$$\langle \boldsymbol{u}'_m(t), \, \boldsymbol{w}_j \rangle_\Omega + \nu \langle A \boldsymbol{u}_m(t), \, \boldsymbol{w}_j \rangle_\Omega = \langle \boldsymbol{f}(t), \, \boldsymbol{w}_j \rangle_\Omega, \quad 1 \leq j \leq m. \tag{6.29}$$

Moreover,

$$\langle A\boldsymbol{w}_j,\, \boldsymbol{v}\rangle_\Omega = \lambda_j \int_\Omega \boldsymbol{w}_j \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} = \int_\Omega \mathbf{curl}\, \boldsymbol{w}_j \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}. \tag{6.30}$$

So, multiplying (6.29) by λ_j, we obtain for $1 \leq j \leq m$:

$$\int_\Omega \mathbf{curl}\, \boldsymbol{u}'_m(t) \cdot \mathbf{curl}\, \boldsymbol{w}_j\, \mathrm{d}\boldsymbol{x} + \nu\langle A\boldsymbol{u}_m(t),\, A\boldsymbol{w}_j\rangle = \langle \boldsymbol{f},\, A\boldsymbol{w}_j\rangle$$

Let us now multiply this last equation by g_{jm} and add, we obtain:

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{curl}\, \boldsymbol{u}_m(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \nu\|A\boldsymbol{u}_m(t)\|^2_{\boldsymbol{L}^2(\Omega)} = \langle \boldsymbol{f}(t),\, A\boldsymbol{u}_m\rangle_\Omega. \tag{6.31}$$

The right-hand side of (6.31) is majorized by:

$$\|\boldsymbol{f}(t)\|_{\boldsymbol{L}^2(\Omega)}\|A\boldsymbol{u}_m(t)\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{1}{2\nu}\|\boldsymbol{f}(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\nu}{2}\|A\boldsymbol{u}_m(t)\|^2_{\boldsymbol{L}^2(\Omega)} \tag{6.32}$$

Substitution this into (6.31) and integration from 0 to t yields

$$\begin{aligned} \|\mathbf{curl}\, \boldsymbol{u}_m(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \nu\int_0^t \|A\boldsymbol{u}_m(s)\|^2_{\boldsymbol{L}^2(\Omega)}\, \mathrm{d}s \quad &\leq \quad \|\mathbf{curl}\, \boldsymbol{u}_{0m}(t)\|^2_{\boldsymbol{L}^2(\Omega)} \\ &+ \quad \frac{1}{\nu}\int_0^t \|\boldsymbol{f}(s)\|^2_{\boldsymbol{L}^2(\Omega)}. \end{aligned}$$

This inequality, in view of the fact that the sequence $\boldsymbol{u}_m$ remains in a bounded set of $L^\infty(0,\, T;\, \boldsymbol{H})$, implies that $\boldsymbol{u}_m \in L^\infty(0,\, T;\, \boldsymbol{V})$ and then $\boldsymbol{u}_m \in L^2(0,\, T;\, \boldsymbol{W})$, with $\boldsymbol{W} = \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)$. $\square$

7. Navier-Stokes model with generalized impermeability b.c.

Let us recall the Navier-Stokes equations in Lamé's form where $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ and $q = \pi + \frac{1}{2}|\boldsymbol{u}|^2$

$$\partial_t \boldsymbol{u} + \boldsymbol{\omega} \times \boldsymbol{u} \;=\; -\nu\, \nabla \times\, \boldsymbol{\omega} - \nabla q + \boldsymbol{f} \qquad \text{in } Q_T, \tag{7.1}$$

$$\operatorname{div} \boldsymbol{u} \;=\; 0 \qquad \text{in } Q_T, \tag{7.2}$$

$$\boldsymbol{u}(.,0) \;=\; \boldsymbol{u}_0 \qquad \text{in } \Omega, \tag{7.3}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} =\; 0 \qquad \text{in } \Gamma_T. \tag{7.4}$$

The chosen complementary boundary conditions concern the vorticity

$$\boldsymbol{\omega} \cdot \boldsymbol{n} = 0 \qquad \nabla \times\, \boldsymbol{\omega} \cdot \boldsymbol{n} = 0 \qquad \text{in } \Gamma_T \tag{7.5}$$

We call them generalized impermeability conditions. The condition (7.4) and $(7.5)_1$ play the role of constraints on the velocity field $\boldsymbol{u}$ to be a weak solution. We introduce the space

$$\boldsymbol{D}^1(\Omega) := \{\boldsymbol{v} \in \boldsymbol{W}^{1,2}_{n,\sigma}(\Omega);\ \nabla \times \boldsymbol{v} \cdot \boldsymbol{n}|_\Gamma = 0\}$$

The operator $\nabla\times$ we use here is precisely the operator $A := \nabla \times |_{\boldsymbol{D}^1(\Omega)}$ whose domain is $\boldsymbol{D}^1(\Omega)$. Remarkable properties are

- The Stokes operator is $S := A^2$, and we have $S = -P_\sigma \Delta = -\Delta P_\sigma$.
- On the boundary Γ, the non-zero tangential behavior of $\boldsymbol{u}$ is described by a potential gradient (a special structure of the divergence free vectors functions from $\boldsymbol{D}^1(\Omega)$ due to a Helmholtz-type decomposition, see Bellout-Neustupa-Penel [10]).
- The surface integral $\int_\Gamma (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{n}\ \mathrm{d}\boldsymbol{s}$ always vanishes for $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{D}^1(\Omega)$.

If Ω is a smooth simply connected domain or if Ω is a convex polyhedron, we know that in $\boldsymbol{D}^1(\Omega)$ the topologies $\|A\cdot\|_{\boldsymbol{L}^2(\Omega)}$ and $\|\cdot\|_{\boldsymbol{H}^1(\Omega}$ are equivalent (Be-Ne-Pe 2004).

The question on the well posedness of the initial boundary value problem (7.1)-(7.5) have been considered in Bellout-Neustupa-Penel [10], in particular one can see that the third condition $(7.5)_2$ is intrinsec to the problem for $\boldsymbol{u}$ weak solution regular enough. So, the condition $(7.5)_2$ is a natural condition for $\boldsymbol{u}$ strong solution.
Let us justify this important fact: For simplicity we consider the linear steady case, assume $\boldsymbol{f}$ given in $\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{u} \in \boldsymbol{D}^1(\Omega) \cap \boldsymbol{H}^2(\Omega)$ solution to

$$\nu \int_\Omega A\boldsymbol{u} \cdot A\boldsymbol{\phi}\ \mathrm{d}\boldsymbol{x} \ = \ \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\phi}\ \mathrm{d}\boldsymbol{x} \quad \text{for all} \quad \boldsymbol{\phi} \in \boldsymbol{D}^1(\Omega),$$

then the solved problem by $\boldsymbol{u}$ is

$$\nu\ A^2\boldsymbol{u} \ = \ P_\sigma \boldsymbol{f} \tag{7.6}$$

and

$$\nabla p \ = (I - P_\sigma)\boldsymbol{f},$$

both equations in $\boldsymbol{L}^2(\Omega)$, where $\boldsymbol{u}$ satisfies the two constraints $\boldsymbol{u}\cdot\boldsymbol{n}|_\Gamma \ = \ 0$ and $A\boldsymbol{u} \cdot \boldsymbol{n}|_\Gamma \ = \ 0$.
Integrating now on Ω the product of (7.6) by $\boldsymbol{\phi} \in \boldsymbol{D}^1(\Omega)$, with $\nabla\psi = \boldsymbol{\phi} - \boldsymbol{\phi}_0$, that decomposition where $\nabla \times \boldsymbol{\phi}_0 = \nabla \times \boldsymbol{\phi} \in \boldsymbol{H}$ being solvable

in $\boldsymbol{H}_0^1(\Omega)$ and $-\Delta\psi = \operatorname{div}\boldsymbol{\phi}_0$ in Ω, $\partial_n\psi|_\Gamma = 0$, so we have

$$
\begin{aligned}
0 &= \int_\Gamma \nabla\times\boldsymbol{u}\cdot\boldsymbol{\phi}\times\boldsymbol{n}\ \ \mathrm{d}\boldsymbol{s}\\
&= \int_\Gamma (\nabla\times\boldsymbol{u}\times\nabla\psi)\cdot\boldsymbol{n}\ \ \mathrm{d}\boldsymbol{s} = \int_\Omega \operatorname{div}(\nabla\times\boldsymbol{u}\times\nabla\psi)\ \mathrm{d}\boldsymbol{x}\\
&= \int_\Omega \nabla\times\ \nabla\times\boldsymbol{u}\cdot\nabla\psi\ \mathrm{d}\boldsymbol{x}\ = \int_\Gamma (\nabla\times\ \nabla\times\boldsymbol{u}\cdot\boldsymbol{n})\ \psi\ \ \mathrm{d}\boldsymbol{s}.
\end{aligned}
$$

Therefore $\nabla\times^2\boldsymbol{u}\cdot\boldsymbol{n}|_\Gamma\ =\ 0$ i.e. the condition $(7.5)_2$.

Remark 7.1. We have observed that the weak formulation of the model delivers the status of a mathematically natural boundary condition to the third condition $(7.5)_2$.

Remark 7.2. The condition $\nabla\times^2\boldsymbol{\omega}\cdot\boldsymbol{n}|_\Gamma\ =\ 0$ also holds, applying the same technic on the vorticity decomposed as $\boldsymbol{\omega}_0 + \nabla\boldsymbol{\chi}$ ..., then with the same series of boundary conditions the induced model for $\boldsymbol{\omega}$ is wellposed

$$
\begin{aligned}
\partial_t\boldsymbol{\omega}\ +\ \nabla\times(\boldsymbol{\omega}\times\boldsymbol{u})\ &=\ -\nu\nabla\times^2\boldsymbol{\omega}+\nabla\times\boldsymbol{f} && \text{in } Q_T, && (7.7)\\
\operatorname{div}\boldsymbol{\omega}\ &=\ 0 && \text{in } Q_T, && (7.8)\\
\boldsymbol{\omega}(.\,,0)\ &=\ \nabla\times\boldsymbol{u}_0 && \text{in } \Omega, && (7.9)\\
\boldsymbol{\omega}\cdot\boldsymbol{n} &= 0 && \text{on } \Gamma_T, && (7.10)\\
\nabla\times\boldsymbol{\omega}\cdot\boldsymbol{n} = 0 \quad \nabla\times^2\boldsymbol{\omega}\cdot\boldsymbol{n} &= 0 && \text{on } \Gamma_T. && (7.11)
\end{aligned}
$$

Note that this situation contrasts with the classical ones when the Navier-Stokes equations are considered with the no slip boundary condition.

Theorem 7.3. *Let $\boldsymbol{u}_0\in\boldsymbol{H}$ and $\boldsymbol{f}\in L^2(0,T;\boldsymbol{L}^2(\Omega))$ be given. There exists $\boldsymbol{u}$ solution to the integral problem such that*
$\boldsymbol{u}\in L^\infty(0,T;\boldsymbol{H})\cap L^2(0,T;\boldsymbol{D}^1(\Omega))$
$\|\boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2(\Omega)} + 2\nu\int_\xi^t \|\nabla\boldsymbol{u}(;\,,s)\|^2_{\boldsymbol{L}^2(\Omega)}\mathrm{ds}\ \le \|\boldsymbol{u}(\xi)\|^2_{\boldsymbol{L}^2(\Omega)} + 2\int_\xi^{\mathrm{t}} \boldsymbol{f}(.,\mathrm{s})\cdot\boldsymbol{u}(.,\mathrm{s})\mathrm{ds}$
$\|\boldsymbol{u}(t)-\boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)}\to 0$ *when* $t\to 0^+$
Moreover if $\boldsymbol{u}_0\in\boldsymbol{D}^1(\Omega)$, then $\int_0^T\|\boldsymbol{u}\|^{2/3}_{\boldsymbol{H}^2(\Omega)}\mathrm{dt}$ is bounded.

The model (7.1)-(7.4) constitutes an alternative to the standard model where only the conditions (7.5) are replaced by the Dirichlet–Stokes conditions $\boldsymbol{u}\times\boldsymbol{n}|_{\Gamma_T} = \boldsymbol{0}$, where in fact only the third condition $(7.5)_2$ differs.

Theorem 7.4. *If* $\boldsymbol{u}_0 \in \boldsymbol{D}^k(\Omega)$ *with* $k = 1$*or* 2*, there exist* $T_k^* > 0$ *and* $\boldsymbol{u}$ *strong solution of the model (7.1)–(7.4) such that* $\boldsymbol{u} \in \mathcal{C}^0(0, T_k^*; \boldsymbol{D}^k(\Omega))$, $A^{k+1}\boldsymbol{u} \in L^2(0, T_k^*; \boldsymbol{H})$ *and* $\partial_t A^{k-1}\boldsymbol{u} \in L^2(0, T_k^*; \boldsymbol{H})$*, where* $\boldsymbol{D}^2(\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{D}^1(\Omega), A^2\boldsymbol{v} \cdot \boldsymbol{n}|_\Gamma = 0\}$.

Remark 7.5. $k = 2$ is the limit case, $A^2\boldsymbol{u}(t) \in \boldsymbol{D}^1(\Omega)$, therefore when $\boldsymbol{f}(t) \in \boldsymbol{D}^1(\Omega)$ the considered Navier–Stokes equation (7.1) is completely solved in $\boldsymbol{D}^1(\Omega)$. Its extension to Γ leads to a natural Dirichlet boundary condition for the pressure $p + \frac{1}{2}|\nabla\phi|^2 = \chi - \partial_t\phi - \nu\ \psi$ where χ, ϕ, ψ are the respective potentials whose gradients describe the respective tangential behaviors of $\boldsymbol{f}, \boldsymbol{u}, A^2\boldsymbol{u}$.

We conclude this section with a non homogeneous simple example, a Stokes model with permeability boundary conditions

$$-\nu\ \Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{7.12}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = g_0 \quad \text{and} \quad \nabla\times\boldsymbol{u}\cdot\boldsymbol{n} = g_1 \qquad \text{on } \Gamma, \tag{7.13}$$

$$\nabla\times^2\boldsymbol{u}\cdot\boldsymbol{n} = g_2 \qquad \text{on } \Gamma. \tag{7.14}$$

We will see that $\boldsymbol{u}$ solution can be constructed in the form $\boldsymbol{u} = \boldsymbol{g} + \boldsymbol{u}_g$.

Lemma 7.6. *Assume that functions* g_0 *and* g_1 *are given in* $H^{-1/2}(\Gamma)$ *and such that* $< g_0, 1 >_\Gamma = < g_1, 1 >_\Gamma = 0$*. There exists a vector function* $\boldsymbol{g} \in \boldsymbol{H}^1(\Omega)$ *such that* $\operatorname{div}\boldsymbol{g} = 0$ *in* Ω*,* $\boldsymbol{g}$ *is harmonic in* Ω *and*

$$\boldsymbol{g}\cdot\boldsymbol{n} = g_0, \qquad \nabla\times\ \boldsymbol{g}\cdot\boldsymbol{n} = g_1 \qquad \text{on } \Gamma. \tag{7.15}$$

Furthermore, $\|\boldsymbol{g}\|_{\boldsymbol{H}^1(\Omega)} \leq c_1(\|g_0\|_{H^{-1/2}(\Gamma)} + \|g_1\|_{H^{-1/2}(\Gamma)})$*, where* $c_1 > 0$ *is independent of* g_0 *and* g_1*.*

Proof. First, we consider two independent Neumann problems whose solutions are unique (up to additive constants) in $W^{1,2}(\Omega)$

$$\Delta\psi_0 = 0 \quad \text{in } \Omega, \qquad \partial_n\psi_0|_\Gamma = g_0, \tag{7.16}$$

$$\Delta\psi_1 = 0 \quad \text{in } \Omega, \qquad \partial_n\psi_1|_\Gamma = g_1. \tag{7.17}$$

Now for any test function $\boldsymbol{\varphi} \in \boldsymbol{D}^1(\Omega)$ decomposed as $\boldsymbol{\varphi}^0 + \nabla\phi$, where only $\nabla\phi$ is uniquely defined, we define a linear functional $\boldsymbol{b}_1$ in $\boldsymbol{H}^{-1/2}(\Gamma)$ by $\langle\boldsymbol{b}_1, \nabla\phi\rangle_\Gamma := \langle g_1, \phi\rangle_\Gamma$. Then we can look for $\boldsymbol{b} \in \boldsymbol{D}^1(\Omega) \cap \boldsymbol{H}^2(\Omega)$ solution to

$$\int_\Omega \nabla\times\ \boldsymbol{b}\cdot\nabla\times\ \boldsymbol{\varphi} \ = \ \int_\Omega \nabla\psi_1\cdot\boldsymbol{\varphi} \ + \ \langle\boldsymbol{b}_1, \nabla\phi\rangle_\Gamma. \tag{7.18}$$

Using the Riesz theorem, there exists a unique solution to the equation (7.18), thus to the corresponding Stokes problem with the boundary conditions $\boldsymbol{b} \cdot \boldsymbol{n} = 0$, $\nabla \times \ \boldsymbol{b} \cdot \boldsymbol{n} = 0$ and $\nabla \times^2 \boldsymbol{b} \cdot \boldsymbol{n} = g_1$. Moreover we have the estimate

$$\|\boldsymbol{b}\|_{\boldsymbol{H}^2(\Omega)} \leq c(\|\nabla \psi_1\|_{\boldsymbol{L}^2(\Omega)} + \|g_1\|_{H^{-1/2}(\Gamma)}). \tag{7.19}$$

According to the regularity of $\boldsymbol{b}$, the third boundary condition is controlled in $\boldsymbol{H}^{-1/2}(\Gamma)$ and such that $\int_\Gamma \nabla \times \ \boldsymbol{b} \cdot (\boldsymbol{n} \times \boldsymbol{\varphi}) \ \mathrm{d}\boldsymbol{s} \ = \ \langle \boldsymbol{b}_1, \nabla\phi \rangle_\Gamma$, thus

$$\begin{aligned} 0 \ &= \ \int_\Gamma \nabla \times \ \boldsymbol{b} \cdot (\boldsymbol{n} \times \nabla \phi) \ \ \mathrm{d}\boldsymbol{s} - \langle \boldsymbol{b}_1, \nabla\phi\rangle_\Gamma \\ &= \ -\int_\Gamma (\nabla\phi \times \nabla \times \ \boldsymbol{b}) \cdot \boldsymbol{n} \ \ \mathrm{d}\boldsymbol{s} - \langle \boldsymbol{b}_1, \nabla\phi\rangle_\Gamma \\ &= \ -\int_\Omega \mathrm{div}(\nabla\phi \times \nabla \times \ \boldsymbol{b}) - \langle \boldsymbol{b}_1, \nabla\phi\rangle_\Gamma \\ &= \ -\int_\Omega \nabla \times^2 \boldsymbol{b} \cdot \nabla\phi - \langle \boldsymbol{b}_1, \nabla\phi\rangle_\Gamma \\ &= \ \int_\Gamma (\nabla \times^2 \boldsymbol{b} \cdot \boldsymbol{n})\phi \ \ \mathrm{d}\boldsymbol{s} - \langle \boldsymbol{b}_1, \nabla\phi\rangle_\Gamma \\ &= \ \langle \nabla \times^2 \boldsymbol{b} \cdot \boldsymbol{n} - g_1, \phi\rangle_\Gamma, \end{aligned}$$

and then $\boldsymbol{b}$ satisfies the third condition. Finally we put $\boldsymbol{g} := \nabla \times \ \boldsymbol{b} + \nabla\psi_0$. The function $\boldsymbol{g}$ is divergence-free because ψ_0 satisfies the equation (7.16). It is harmonic because $\nabla \times^2 \boldsymbol{g} = \nabla \times^3 \ \boldsymbol{b}_1 = 0$. So $\nabla \times^2 \boldsymbol{g} = 0$ also holds in some Ω–interior neighborhood of Γ. The normal component $\boldsymbol{g} \cdot \boldsymbol{n}|_\Gamma = \nabla \times \ \boldsymbol{b} \cdot \boldsymbol{n} + \nabla\psi_0 \cdot \boldsymbol{n} = g_0$. In addition, we have $\nabla \times \boldsymbol{g} \cdot \boldsymbol{n} = \nabla \times^2 \boldsymbol{b} \cdot \boldsymbol{n}\,(= \nabla\psi_1 \cdot \boldsymbol{n}) = g_1$ on Γ. The lemma is established.

We can introduce a second linear functional $\boldsymbol{b}_{2,\nu}$ such that

$$\langle \boldsymbol{b}_{2,\nu}, \boldsymbol{w}\rangle_\Gamma = \ \langle \boldsymbol{b}_{2,\nu}, \nabla\phi\rangle_\Gamma := \ \nu\langle g_2, \phi\rangle^*_\Gamma \tag{7.20}$$

defined for all $\boldsymbol{w} \in \boldsymbol{D}^1(\Omega)$ decomposed as $\boldsymbol{w}^0 + \nabla\phi$. We know $\phi \in H^2(\Omega)$, again $\nabla\phi$ is uniquely defined and $\boldsymbol{w}|_\Gamma = \nabla\phi|_\Gamma$ in $\boldsymbol{H}^{1/2}(\Gamma)$, then $\langle ., . \rangle^*_\Gamma$ denotes the duality between $\boldsymbol{H}^{-3/2}(\Gamma)$ and $\boldsymbol{H}^{3/2}(\Gamma)$ and we assume g_2 given in $H^{-3/2}(\Gamma)$.
We have $\boldsymbol{b}_{2,\nu} \in \boldsymbol{H}^{-1/2}(\Gamma)$. The integral formulation of our problem now reads as follows :

$$\nu \int_\Omega \nabla \times \ \boldsymbol{u}_g \cdot \nabla \times \ \boldsymbol{w} = \langle \boldsymbol{f}_g, \boldsymbol{w}\rangle + \langle \boldsymbol{b}_{2,\nu}, \boldsymbol{w}\rangle_\Gamma \tag{7.21}$$

for all $\boldsymbol{w} \in \boldsymbol{D}^1(\Omega)$, where $\boldsymbol{f}_g = \boldsymbol{f} + \ \nabla \times^2 \boldsymbol{g} = \boldsymbol{f}$ because $\nabla \times^2 \boldsymbol{g} = \boldsymbol{0}$. The function $\boldsymbol{u}_g$ solution lies in $\boldsymbol{D}^1(\Omega)$ and can be also decomposed as $\boldsymbol{u}_g^0 + \nabla\phi_0$.

One can verify that $\boldsymbol{u}_g$ satisfies the relation

$$\nu \ \langle \nabla \times^2 \boldsymbol{u}_g \cdot \boldsymbol{n} - g_2, \phi\rangle^*_\Gamma = 0 \tag{7.22}$$

Remark 7.7. Very important, the third boundary condition (7.14) enters the integral formulation through $b_{2,\nu}$ and, because of this functional $\boldsymbol{b}_{2,\nu}$, is a "mathematically natural" condition.

Remark 7.8. The procedure is the same for the stationary Navier–Stokes model. The main difference consists in the following quadratic form $a_{\nu,g}(\boldsymbol{v},\boldsymbol{v}) = \nu \int_\Omega |\nabla\times\boldsymbol{v}|^2 + \int_\Omega \nabla\times\boldsymbol{v}\times\boldsymbol{g}\cdot\boldsymbol{v}$, the crucial point being the proof of its coercivity in $\boldsymbol{D}^1(\Omega)$. The nonlinear integral formulation reads $a_{\nu,g}(\boldsymbol{u}_g,\boldsymbol{w}) + \int_\Omega[\nabla\times\boldsymbol{g}\times\boldsymbol{u}_g\cdot\boldsymbol{w} + \nabla\times\boldsymbol{u}_g\times\boldsymbol{u}_g\cdot\boldsymbol{w}] = \langle \boldsymbol{f}_g,\boldsymbol{w}\rangle + \langle \boldsymbol{b}_{2,\nu},\boldsymbol{w}\rangle_\Gamma$ where $\boldsymbol{f}_g := \boldsymbol{f} - \nabla\times\boldsymbol{g}\times\boldsymbol{g}$. To prove the existence of $\boldsymbol{u}_g$ solution, a spectral Galerkin method is appropriate.

References

[1] H. Abels – "Nonstationary Stokes system with variable viscosity in bounded and unbounded domains", *Discrete Contin. Dyn. Syst. Ser. S.* **3** (2010), p. 141–157.

[2] Y. Amirat, D. Bresch, J. Lemoine & J. Simon – "Effect of rugosity on a flow governed by stationary Navier-Stokes equations", *Quart. Appl. Math.* **59** (2001), p. 769–785.

[3] C. Amrouche, C. Bernardi, M. Dauge & V. Girault – "Vector potentials in three-dimensional nonsmooth domains", *Math. Meth. Applied Sc.* **21** (1998), p. 823–864.

[4] C. Amrouche & N. Seloula – "On the Stokes equations with the Navier-Type boundary conditions", *Diff. Eq. Appl.* **3-4** (2011), p. 581–607.

[5] ———, "L^p-theory for vector potentials and Sobolev's inequalities for vector fields. application to the Stokes equations with pressure boundary condition", *To appear in Math. Mod. and Meth. in App* **23** (2013), p. 37–92.

[6] C. Bardos, F. Golse & L. Paillard – "The incompressible Euler limit of the Boltzmann equation with accomodation boundary conditions", *Comm. Math. Sci.* **10** (2012), no. 1, p. 159–190.

[7] G. Beavers & D. Joseph – "Boundary conditions at a naturally permeable wall", *J. Fluid Mech.* **30** (1967), no. 1, p. 197–207.

[8] C. Bègue, C. Conca, F. Murat & O. Pironneau – "Les équations de Stokes et de Navier-Stokes avec des conditions aux limites sur la pression", *Nonlinear Partial Differential Equations and their Applications, Collège de France Seminar* **IX** (1989), p. 179–264.

[9] H. Beirao da Veiga – "On the regularity of flows with Ladyzhenskaya shear-dependent viscosity and slip or nonslip boundary conditions", *Comm. Pure Appl. Math.* **58** (2005), no. 4, p. 552–577.

[10] H. Bellout, J. Neustupa & P. Penel – "On the Navier-Stokes equation with boundary conditions based on vorticity", *Math. Nachr.* **269-270** (2004), p. 59–72.

[11] ______, "On a ν-continuous family of strong solutions to the euler or Navier-Stokes equations with the Navier-Type boundary conditions", *Discrete and Continuous Dynamical Systems* **27** (2010), no. 4, p. 1353–1373.

[12] J. M. Bernard – "Non-standard Stokes and Navier-Stokes problem: existence and regularity in stationary case", *Math. Meth. Appl. Sci.* **25** (2002), p. 627–661.

[13] ______, "Time-dependent Stokes and Navier-Stokes problems with boundary conditions involving pressure, existence and regularity", *Nonlinear Anal. Real World Appl.* **4** (2003), no. 5, p. 805–839.

[14] L. C. Berselli – "Some criteria concerning the vorticity and the problem of global regularity for the 3D Navier-Stokes equations", *Ann. Univ. Ferrara Sez. VII Sci. Mat.* **55** (2009), no. 2, p. 209–224.

[15] L. C. Berselli. – "An elementary approach to the 3D Navier-Stokes equations with Navier boundary conditions: existence and uniqueness of various classes of solutions in the flat boundary case", *Discrete Contin. Dyn. Syst. Ser. S* **3** (2010), no. 2, p. 199–219.

[16] D. Bothe & J. Pruss – "L^p-theory for a class of non-newtonian fluids", *SIAM J. Math. Anal.* **39** (2007), no. 2, p. 379–421.

[17] D. Bucur, E. Feireisl & S. Necasova – "Boundary behavior of viscous fluids: influence of wall roughness and friction-driven boundary conditions", *Arch. Ration. Mech. Anal.* **197** (2010), no. 1, p. 117–138.

[18] D. Bucur, E. Feireisl, S. Necasova & J. Wolf – "On the asymptotic limit of the Navier-Stokes system on domains with rough boundaries", *J. Differential Equations* **244** (2008), no. 11, p. 2890–2908.

[19] M. Bulicek, J. Malek & K. R. Rajagopal – "Navier's slip and evolutionary Navier-Stokes-like systems with pressure and shear-rate dependent viscosity", *Indiana Univ. Math. J.* **56** (2007), no. 1, p. 51–85.

[20] G. Q. Chen, D. Osborne & Z. Qian – "The Navier-Stokes equations with the kinematic and vorticity boundary conditions on non-flat boundaries", *Acta Math. Sci. Ser. B Engl. Ed.* **29** (2009), no. 4, p. 919–948.

[21] C. Ebmeyer & J. Frehse – "Steady Navier-Stokes equations with mixed boundary value conditions in three-dimensional Lipschitzian domains", *Math. Ann* **319** (2001), no. 2, p. 349–381.

[22] L. T. Hoang – "Incompressible fluids in thin domains with Navier friction boundary conditions (i)", *J. Math. Fluid Mech.* **12** (2010), no. 3, p. 435–472.

[23] L. T. Hoang & G. R. Sell – "Navier-Stokes equations with Navier boundary conditions for an oceanic model", *J. Dynam. Differential Equations* **22** (2010), no. 3, p. 653–616.

[24] D. Iftimie & G. Raugel – "Some results on the Navier-Stokes equations in thin 3D domains", *J. Diff. Eq.* **169** (2001), p. 281–331.

[25] D. Iftimie, G. Raugel & G. Sell – "Navier-Stokes equations in thin 3D domains with Navier boundary conditions", *Indiana Univ. Math. J.* **56** (2007), no. 3, p. 1083–1156.

[26] D. Iftimie & F. Sueur – "Viscous boundary layers for the Navier-Stokes equations with the Navier slip conditions", *Arch. Ration. Mech. Anal.* **199** (2011), no. 1, p. 145–175.

[27] W. Jager & A. Mikelic – "On the interface boundary condition of Beavers, Joseph, and Saffman", *SIAM J. Appl. Math.* **60** (2000), no. 4, p. 1111–1127.

[28] ———, "On the roughness-induced effective boundary conditions for an incompressible viscous flow", *J. Differential Equations* **170** (2001), no. 1, p. 96–122.

[29] T. Kato – *Remarks on zero viscosity limit for non-stationary Navier-Stokes flows with boundary*, Springer, Seminar on nonlinear PDE (Berkeley), 1984.

[30] H. Kozono & T. Yanagisawa – "L^r-variational inequality for vector fields and the Helmholtz-Weyl decomposition in bounded domains", *Indiana Univ. Math. J.* **58** (2009), no. 4, p. 1853–1920.

[31] J. L. Lions – *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris, 1969.

[32] S. Marusic – "On the Navier-Stokes system with pressure boundary condition", *Ann. Univ. Ferrara Sez. VII Sci. Mat.* **53** (2007), no. 2, p. 319–331.

[33] M. Mitrea & S. Monniaux – "The nonlinear Hodge-Navier-Stokes equations in Lipschitz domains", *Differential Integral Equations* **22** (2009), no. 3-4, p. 339–356.

[34] C. Navier – "Sur les lois de l'équilibre et du mouvement des corps élastiques", *Mem. Acad. R. Sci. Inst.* **6** (1827), no. 369.

[35] J. Neustupa & P. Penel – "Local in time strong solvability of the non-steady Navier-Stokes equations with Navier's boundary conditions and the question of the inviscid limit", *C.R.A.S. Paris* **348** (2010), no. 19-20, p. 1093–1097.

[36] J. Serrin – *Mathematical principles of classical fluid mechanics*, Handbuch der Physik, Springer-Verlag, 1959.

[37] R. Shimada – "On the $L^p - L^q$ maximal regularity for Stokes equations with robin boundary condition in a bounded domain", *Math. Methods Appl. Sci.* **30** (2007), no. 3, p. 257–289.

[38] V. A. Solonnikov – "l^p-estimates for solutions to the initial-boundary-value problem for the generalized Stokes system in a bounded domain", *J. Math. Sci.* **105** (2001), p. 2448–2484.

[39] ______, "Estimates of the solution of model evolution generalized Stokes problem in weighted Hölder spaces", *J. Math. Sci. (N. Y.)* **143** (2007), no. 2, p. 2969–2986.

[40] V. A. Solonnikov & V. E. Scadilov – "A certain boundary value problem for the stationary system of Navier-Stokes equations",

Boundary value problems of mathematical physics, Trudy Mat. Inst. Steklov. **8** (1973), p. 196–210.

[41] R. Temam – *Theory and numerical analysis of the Navier-Stokes equations*, North-Holland, Amsterdam, 1977.

[42] V. Yudovich – "A two-dimensional non-stationary problem on the flow of an ideal compressible fluid through a given region", *Mat. Sb.* **4** (1964), no. 64, p. 562–588.

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