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Michael Pevzner

Abstract

This is an extended version of a lecture given by the author at the summer school "Quasimodular forms and applications" held in Besse in June 2010.

The main purpose of this work is to present Rankin-Cohen brackets through the theory of unitary representations of conformal Lie groups and explain recent results on their analogues for Lie groups of higher rank. Various identities verified by such covariant bi-differential operators will be explained by the associativity of a non-commutative product induced on the set of holomorphic modular forms by a covariant quantization of the associate para-Hermitian symmetric space.

Crochets de Rankin-Cohen et représentations des groupes de Lie conformes

Résumé

Ce texte est une version étendue d'un cours donné par l'auteur lors de l'école d'été *Formes quasimodulaires et applications* qui s'est tenue à Besse en juin 2010.

L'objectif principal de ce travail est de présenter les crochets de Rankin-Cohen dans le cadre de la théorie des représentations unitaires des groupes de Lie conformes et d'expliquer des résultats récents sur leurs analogues pour des groupes de Lie de rang supérieur. Diverses identités que vérifient de tels opérateurs bi-différentiels covariants seront expliquées en terme de l'associativité d'un produit non commutatif induit sur l'ensemble des formes modulaires holomorphes par la quantification covariante de l'espace symétrique para-hermitien associé.

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Contents

1. Introduction

Let $\Pi = \{z = x + iy \,|\, x, \in \mathbb{R},\, y > 0\}$ be the upper half-plane equipped with the Lobachevsky metric. The Lie group $\mathrm{SL}(2,\mathbb{R})$ acts conformally on Π by fractional-linear transformations:

$$z \to \frac{az+b}{cz+d}, \quad z \in \Pi, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}),$$

and consequently it acts on functions on Π.

One says that a function $f \in \mathcal{O}(\Pi)$ is a *holomorphic modular form* of weight $k \in \mathbb{N}$ with respect to an arithmetic subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$ if

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

As far as our technics are algebraic we omit here the extra growth condition on f at infinity that guarantees the Fourier series expansion of f to be of the form $f(z) = \sum_{n\in\mathbb{N}} a_n e^{2\pi i n z}$. For such functions we shall write $f \in M^k(\Gamma)$ or simply $f \in M^k$ if $\Gamma = \mathrm{SL}(2,\mathbb{Z})$.

The set $M(\Gamma) := \bigoplus_{k\in\mathbb{N}} M^k(\Gamma)$ is a graded vector space. One of the purposes of these notes is to discuss the supplementary algebraic structure that this space may be endowed with.

For instance, the product of two modular forms is again a modular form. Furthermore, even though derivations do not preserve the modularity, one may notice that for $f_1 \in M^{k_1}$ and $f_2 \in M^{k_2}$ an appropriate combination of such forms and their first order derivatives has the required covariance. Namely,

$$k_1 f_1 f_2' - k_2 f_1' f_2 \in M^{k_1+k_2+2}.$$

Moreover, this bilinear operator, that we denote $RC_1(f_1, f_2) := k_1 f_1 f_2' - k_2 f_1' f_2$ satisfies the Jacobi identity: $\circlearrowleft RC_1(RC_1(f_1, f_2), f_3) = 0$.

Set $RC_0(f_1, f_2) := f_1 \cdot f_2$, then (M, RC_0) is an associative and commutative algebra whereas (M^{*-2}, RC_1) is a graded Lie algebra. In addition, these structures are compatible in the sense that RC_1 is a derivation of RC_0. Thus M tuns into a Poisson algebra.

This observation represents the top of the iceberg. There exists a whole infinite series of bi-differential operators preserving modular forms and satisfying non trivial algebraic identities. The set of all differential operators preserving holomorphic modular forms was described by R.A. Rankin in 1956, and the set of all bi-differential operators preserving such forms was given by H. Cohen in 1975 [3] and studied by D. Zagier (see e.g. [31]).

Namely, for every $f_1 \in M^{k_1}(\Gamma)$, $f_2 \in M^{k_2}(\Gamma)$ and $n \in \mathbb{N}$ the n-th *Rankin-Cohen bracket* RC_n is defined by:

$$RC_n(f_1, f_2) := \sum_{j=0}^{n} (-1)^j \binom{k_2+n-1}{n-j} \binom{k_1+n-1}{j} f_1^{(n-j)} f_2^{(j)}, \tag{1.1}$$

where

$$f^{(k)} = \left(\frac{\partial}{\partial z}\right)^k f\,.$$

It turns out that for every $f_1 \in M^{k_1}(\Gamma)$, $f_2 \in M^{k_2}(\Gamma)$ and $n \in \mathbb{N}$ $RC_n(f_1, f_2) \in M^{k_1+k_2+2n}(\Gamma)$.

We don't prove this statement now but will show a more general result in the next section. Notice that this fact says that Rankin–Cohen brackets are covariant bi-differential operators and their construction was actually described by Gordon and Gundelfinger already in 1886-87. At that time they called such operators Überschiebungen (transvectancts), or Cayley processes. See [2, 12, 13, 20, 21] etc.

We shall develop a slightly different approach to Rankin-Cohen brackets based on the theory of unitary representations of the Lie group $\mathrm{SL}(2, \mathbb{R})$. This method gives a nice framework for further developments of the initial construction to a whole class of the so-called conformal Lie groups and furnishes a uniform explanation of all the identities satisfied by Rankin-Cohen brackets.

2. **Two series of unitary representations of** $\mathrm{SL}(2,\mathbb{R})$

Consider the Lie group $G = \mathrm{SL}(2,\mathbb{R}) = \{g \in \mathrm{Mat}(2,\mathbb{R}) \mid \det g = 1\}$, and its Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) = \{X \in \mathrm{Mat}(2,\mathbb{R}) \mid \mathrm{tr}(X) = 0\}$. Let

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These elements form a basis of $\mathfrak{g}$ and are subject to the following commutation relations:

$$[h, e^{\pm}] = \pm 2e^{\pm}, \quad [e^+, e^-] = h.$$

Let $\mathrm{SO}(2)$ be the set of orthogonal matrices of determinant 1. It is a maximal compact subgroup of $\mathrm{SL}(2,\mathbb{R})$ and its Lie algebra $\mathfrak{so}(2)$ is generated by the element $e^- - e^+ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Let $U(\mathfrak{g})$ be the universal envelopping algebra of $\mathfrak{g}$, that is $U(\mathfrak{g}) \simeq T(\mathfrak{g})/I$, where I is the ideal of the tensor algebra $T(\mathfrak{g})$ generated by $X \otimes Y - Y \otimes X - [X,Y]$, with $X, Y \in \mathfrak{g}$.

The center of $U(\mathfrak{sl}(2,\mathbb{R}))$ contains, and is actually generated by the so-called Casimir element:

$$c = h^2 + 2(e^+e^- + e^-e^+) = h^2 + 2h + 4e^-e^+.$$

Let (ρ, V) be a representation of $\mathfrak{sl}(2,\mathbb{R})$, that is, V is a vector space and ρ is a Lie algebra homomorphism from $\mathfrak{g}$ to $End(V)$. We also will say that V is a $\mathfrak{g}$-module. For $\lambda \in \mathbb{C}$, the generalized h-eigenspace of V is

$$V_\lambda = \{v \in V \,:\, (h - \lambda Id)^n v = 0 \quad \text{for some}\, n \in \mathbb{N}\}.$$

Notice that for $\lambda \neq \mu$ one has $V_\lambda \cap V_\mu = \{0\}$.

One says that the representation ρ is h-admissible if $V = \sum_{\lambda \in \mathbb{C}} V_\lambda$ and $\dim V_\lambda < \infty$ for every $\lambda \in \mathbb{C}$. If these conditions are satisfied and all V_λ's are genuine eigenspaces, that is, $V_\lambda = \{v \in V \,:\, hv = \lambda v\}$ for every $\lambda \in \mathbb{C}$, then one says that the representation ρ is h-semisimple.

The eigenvalues of h, that is, $\{\lambda \in \mathbb{C} \,:\, V_\lambda \neq \{0\}\}$ are called weights. If the Casimir element c acts by a multiple of identity on V one says that the representation ρ is quasi-simple. It is possible, using the following result, to classify all h-admissible, h-semi-simple, quasi-simple representations of $\mathfrak{sl}(2,\mathbb{R})$.

Proposition 2.1. *Let V be a $\mathfrak{sl}(2,\mathbb{R})$-module and $v_0 \in V$ an $h-$eigenvector: $hv_0 = \lambda v_0$.*

(1) *Then,*
   (a) *The vector $v_k = (e^+)^k v_0$ is either 0 or an h-eigenvector of eigenvalue $\lambda + 2k$ for every $k \in \mathbb{N}$.*
   (b) *The vector $v_{-k} = (e^-)^k v_0$ is either 0 or an h-eigenvector of eigenvalue $\lambda - 2k$ for every $k \in \mathbb{N}$.*

   Thus the $\mathfrak{sl}(2,\mathbb{R})$-module V_0 generated by v_0 is h-semi-simple.

(2) *Suppose that moreover v_0 is an eigenvector of the Casimir element: $cv_0 = \mu v_0$ for some $\mu \in \mathbb{C}$. Then, the set of non zero vectors v_k forms a basis of V_0 and the following relations hold*

$$\begin{aligned} e^+ v_k &= v_{k+1}, \quad k \geq 0, \\ e^- v_k &= v_{k-1}, \quad k \leq 0, \\ e^+ v_k &= \frac{1}{4}(\mu - (\lambda + 2(k+1))^2 + 2(\lambda + 2(k+1)))v_{k+1}, \quad k < 0, \\ e^- v_k &= \frac{1}{4}(\mu - (\lambda + 2(k-1))^2 + 2(\lambda + 2(k-1)))v_{k-1}, \quad k > 0. \end{aligned}$$

   In particular V_0 is h-semi-simple and quasi-simple and all non trivial h-eigenspaces of V_0 are one-dimensional.

We omit the full construction, which is classical and may be found for instance in [15], and concentrate on some particular examples.

(1) Lowest weight modules V_λ with $\lambda \in \mathbb{C}$. Such a module has a basis of h-eigenvectors $\{v_j,\, j \in \mathbb{N}\}$ such that

$$\begin{aligned} hv_j &= (\lambda + 2j)v_j, \quad j \in \mathbb{N}, \\ e^+ v_j &= v_{j+1}, \quad j \in \mathbb{N}, \\ e^- v_j &= -j(\lambda + j - 1)v_{j-1}, \quad j \in \mathbb{N} \setminus \{0\}, \\ e^- v_0 &= 0, \\ cv &= (\lambda^2 - 2\lambda)v, \quad v \in V_\lambda. \end{aligned}$$

   The element v_0 is called in this case the *lowest weight vector* and λ – the *lowest weight of the module V_λ*.

(2) Highest weight modules $\bar{V}_\lambda$ with $\lambda \in \mathbb{C}$. Such a module has a basis of h-eigenvectors $\{\bar{v}_j,\, j \in \mathbb{N}\}$ such that

$$\begin{aligned} h\bar{v}_j &= (\lambda - 2j)\bar{v}_j, \quad j \in \mathbb{N}, \\ e^-\bar{v}_j &= \bar{v}_{j+1}, \quad j \in \mathbb{N}, \\ e^+\bar{v}_j &= j(\lambda - j - 1)\bar{v}_{j-1} \quad j \in \mathbb{N} \setminus \{0\}, \\ e^+\bar{v}_0 &= 0, \\ cv &= (\lambda^2 + 2\lambda)v, \quad v \in \bar{V}_\lambda. \end{aligned}$$

The element v_0 is called in this case the *highest weight vector* and λ - the *highest weight of the module* $\bar{V}_\lambda$.

(3) Modules $W(\mu, \lambda)$ with $\lambda, \mu \in \mathbb{C}$ have a basis of h-eigenvectors $\{v_j \,:\, j \in \mathbb{Z}\}$ such that

$$\begin{aligned} hv_j &= (\lambda + 2j)v_j, \quad j \in \mathbb{N}, \\ e^+v_j &= v_{j+1}, \quad j \in \mathbb{N}, \\ e^-v_j &= \frac{1}{4}(\mu - (\lambda + 2j - 1)^2 + 1)v_{j-1}, \quad j \in \mathbb{N}, \\ cv &= \mu v, \quad v \in W_{\lambda,\mu}. \end{aligned}$$

It turns out that these representations of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ extend into unitary representations of the Lie group $\mathrm{SL}(2, \mathbb{R})$ and give rise to the so-called discrete (holomorphic and anti-holomorphic) series in cases (1) and (2) and principal series representations in case (3).

Indeed, let $\lambda > 1$, then V_λ corresponds to the holomorphic discrete series representation T_λ of $\mathrm{SL}(2, \mathbb{R})$ acting on the weighted Bergman space $H^2_\lambda(\Pi) := \mathcal{O}(\Pi) \cap L^2(\mathrm{Im}\,(z)^{\lambda-2} dz \wedge d\bar{z})$ by

$$\left(T_\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} f\right)(z) = (-cz + a)^{-\lambda} f\left(\frac{dz - b}{-cz + a}\right).$$

Considering the corresponding infinitesimal action dT_λ we get:

$$\begin{aligned} dT_\lambda(e^-)f(z) &= \lambda z f(z) + z^2 f'(z), \\ dT_\lambda(e^+)f(z) &= -f'(z), \\ dT_\lambda(h)f(z) &= -\lambda f(z) - 2zf'(z). \end{aligned}$$

Thereby, letting $v_j := \frac{(\lambda+j-1)!}{(\lambda-1)!} z^{-\lambda-j}$ one gets the identification of V_λ with the set of smooth vectors of T_λ in $H^2_\lambda(\Pi)$ that generate finite dimensional spaces under the action of rotation group SO(2). We refer to the last module as to the underlying $(\mathfrak{g}, K)$-module of T_λ. Being finitely generated this module is actually a lowest weight Harish-Chandra module.

Consider now the tensor product $V_{\lambda_1} \otimes V_{\lambda_2}$ as a $\mathfrak{sl}(2,\mathbb{R})$-module via the diagonal embedding of $\mathfrak{sl}(2,\mathbb{R})$ into $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$. This module is reducible and it decomposes multiplicity free and discretely into a direct sum of lowest weight modules of weight $\lambda_1 + \lambda_2 + 2n$ with $n \in \mathbb{N}$ see [26]. In order to describe all the irreducible components we have to determine for every n an element

$$\sum_j a_j (v_j \otimes \widetilde{v}_{n-j}) \in V_{\lambda_1} \otimes V_{\lambda_2},$$

annihilated by the diagonal action of e^-. Notice that in general an h-eigenvector is of the form $v_j \otimes \widetilde{v}_k$ whose h-eigenvalue is $\lambda_1 + \lambda_2 + 2(j+k)$. So, the eigenspace corresponding to the eigenvalue $\lambda_1+\lambda_2+2n$ is generated by vectors $v_j \otimes \widetilde{v}_{n-j}$ with $j = 0, ..., n$.

In order to determine the coefficients a_j we may either calculate directly the kernel of the diagonal action of e^- (i.e. solve a system of linear equations of size $n \times (n+1)$ of very specific form see [9]) or solve the following recurrence relation :

$$a_{j+1}(j+1)(\lambda_1 + j) + a_j(n-j)(\lambda_2 + n - j - 1) = 0,$$

that gives

$$a_j = (-1)^j \frac{(\lambda_1 + n - 1)_{n-j}}{(n-j)!} \frac{(\lambda_2 + n - 1)_j}{j!},$$

where $(x)_k = \frac{x!}{(x-k)!}$ is the Pochhammer symbol.

On the level of the underlying lowest weight Harish-Chandra modules the diagonal action e^- is given by:

$$\Delta e^- = \left(\lambda_1 z + z^2 \frac{d}{dz}\right) \otimes Id + Id \otimes \left(\lambda_2 w + w^2 \frac{d}{dw}\right).$$

Thus, after identification of two models the lowest weight vector in the underlying Harish-Chandra module is

$$\begin{aligned}
& f_{\lambda_1,\lambda_2,n}(z,w) \\
:= & \sum_{j=1}^{n}(-1)^n \begin{pmatrix} n \\ j \end{pmatrix} \frac{(\lambda_1-n+1)!(\lambda_2-n+1)!}{n!(\lambda_1-1)!(\lambda_2-1)!} z^{-\lambda_1-j} w^{-\lambda_2-n+j} \\
= & \frac{1}{n!}\frac{(\lambda_1-n+1)!(\lambda_2-n+1)!}{(\lambda_1-1)!(\lambda_2-1)!} \cdot \frac{(w^{-1}-z^{-1})^n}{z^{\lambda_1}w^{\lambda_2}}.
\end{aligned}$$

Consequently, one may see the lowest weight vector $f_{\lambda_1,\lambda_2,n}(z,w)$ as the image of the element $v_0 \otimes \widetilde{v}_0$ by the application:

$$\sum_{j=1}^{n}(-1)^j \frac{(\lambda_1+n-1)_{n-j}}{(n-j)!}\frac{(\lambda_2+n-1)_j}{j!}\,(e^+)^j \otimes (e^+)^{n-j}.$$

By identifying the elements of $U(\mathfrak{sl}(2,\mathbb{R}))$ with left invariant differential operators on G and hence with differential operators on Π it becomes possible to interpret the Rankin-Cohen brackets (1.1) as elements of $\mathrm{Hom}_{\mathfrak{sl}(2,\mathbb{R})}(V_{k_1} \otimes V_{k_2}, V_{k_1+k_2+2n})$. That is,

$$T_{k_1+k_2+2n}(g)RC_n(f_1,f_2) = RC_n(T_{k_1}(g)f_1, T_{k_2}(g)f_2),$$

for any $g \in G$ and particularly for any g in a given arithmetic subgroup. Hence RC_n do preserve modularity.

The above computations are standard and may be found in many graduate text books (see for instance [15]) but this remark gives us a way to study and develop the theory of Rankin-Cohen brackets through the theory of intertwining operators for discrete series representations of conformal Lie groups. Furthermore, we will see that the numerous algebraic relations between Rankin-Cohen brackets are governed by a higher order structure related to the covariant quantization of causal symmetric spaces.

Before ending this section we would like to mention very roughly some other facets of Rankin-Cohen brackets.

Any Hermitian vector bundle has a covariant differentiation ∇ compatible with both the metric and the complex structures:

$$\nabla = \nabla^{(1,0)} + \overline{\partial}.$$

For instance, in case of the unit disc, the Rankin-Cohen brackets may be expressed in terms of $\nabla^{(1,0)}$:

$$RC_n(f_1, f_2) = \sum_{j=0}^{n}(-1)^j \begin{pmatrix} n \\ j \end{pmatrix} \frac{1}{(k_1)_j(k_2)_j} \prod_{i=0}^{j-1} D_{k_1+2i}(f_1) \prod_{\ell=0}^{n-j-1} D_{k_2+2\ell}(f_2),$$

where $D_k = \partial - k(1-|z|^2)\bar{z}$ (see [32]).

In a series of fundamental papers about the Hopf algebra symmetries of foliations, A. Connes and H. Moscovici interpreted the RC_1 as a representative of the so-called transverse fundamental class in the Hopf cyclic cohomology of a particular Hopf algebra H_1 (kind of noncommutative Poisson structure). Higher order Rankin-Cohen brackets do also admit such a cohomological interpretation that provides a formal deformation of RC_1 see [5, 6] .

3. Covariant quantization on co-adjoint orbits

The $\mathfrak{sl}(2,\mathbb{R})$–modules $W_{\mu,\lambda}$ correspond to the so-called principal series representations of the Lie group $G = \mathrm{SL}(2,\mathbb{R})$ that we recall the construction (in the unitary case).

Let P^- be the parabolic subgroup of G consisting of the lower triangular matrices

$$P^- = \left\{ \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}, c \in \mathbb{R}, a \in \mathbb{R}^* \right\},$$

and let P^+ be the group of upper triangular matrices

$$P^+ = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, b \in \mathbb{R}, a \in \mathbb{R}^* \right\}.$$

The group G acts on the sphere $S = \left\{ s \in \mathbb{R}^2 : \|s\|^2 = 1 \right\}$ and acts transitively on $\widetilde{S} = S/\sim$, where $s \sim s'$ if and only if $s = \pm s'$, by

$$g.s = \frac{g(s)}{\|g(s)\|}.$$

Clearly, $\mathrm{Stab}(\widetilde{0,1}) = P^-$. So $\widetilde{S} \simeq G/P^-$. Similarly $\widetilde{S} \simeq G/P^+$: $\widetilde{S} = G.(\widetilde{1,0})$. If ds is the usual normalized surface measure on S, then

$$d(g.s) = \|g(s)\|^{-2} ds.$$

For $\mu \in \mathbb{C}$, define the character ω_μ of $P^\pm$ by

$$\omega_\mu(p) = |a|^\mu.$$

Consider the parabolically induced representations $\pi_\mu^\pm = \mathrm{Ind}_{P^\pm}^G \omega_{\mp\mu}$. By definition, they act by left translations on the Hilbert completion of the set of smooth sections of the line bundle $G \times_{\omega_{\mp\mu}} P^\pm$. Hence we may realize these representations, usually referred to as the principal series representations, on $C^\infty(\widetilde{S})$ – the space of smooth functions ϕ on S satisfying

$$\phi(-s) = \phi(s), \quad (s \in S).$$

The formula for π_μ^- is

$$\pi_\mu^-(g)\phi(s) = \phi(g^{-1}.s)\|g^{-1}(s)\|^\mu.$$

Let θ be the Cartan involution of G given by $\theta(g) = {}^t g^{-1}$. Then

$$\pi_\mu^+(g)\phi(s) = \phi(\theta(g^{-1}).s)\|\theta(g^{-1})(s)\|^\mu.$$

Since here

$$\theta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = w\begin{pmatrix} a & b \\ c & d \end{pmatrix} w^{-1}$$

with $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, one has that $\pi_\mu^- \sim \pi_\mu^+$.

Let $(\ ,\ )$ denote the standard inner product on $L^2(S)$:

$$(\phi, \psi) = \int_S \phi(s)\overline{\psi(s)}ds.$$

Then this form is invariant with respect to the pairs

$$(\pi_\mu^-, \pi_{-\bar{\mu}-2}^-), \quad \text{and} \quad (\pi_\mu^+, \pi_{-\bar{\mu}-2}^+).$$

Therefore if $\mu = -1 + i\sigma$, then the representations $\pi_\mu^\pm$ are unitary, the inner product being $(\ ,\ )$. They are irreducible for any $\sigma \neq 0$ [18].

G acts also on $\widetilde{S} \times \widetilde{S}$ by

$$g.(u, v) = (g.u, \theta(g)v). \tag{3.1}$$

This action is not transitive: the orbit

$$(\widetilde{S} \times \widetilde{S})^\# = \{(u, v) : \langle u, v\rangle \neq 0\} = G.(\widetilde{(0,1)}, \widetilde{(0,1)})$$

is only dense in $(\widetilde{S} \times \widetilde{S})^\#$. Moreover $(\widetilde{S} \times \widetilde{S})^\#$ is isomorphic to the one-sheeted hyperboloid G/H, where $H = \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right), a \in \mathbb{R}^*\right\} \simeq \mathrm{SO}(1, 1)$.

The symmetric space G/H is a co-adjoint orbit of G and therefore has a canonical symplectic structure.

The map

$$f \to f(u,v)|\langle u,v\rangle|^{-1+i\sigma}, \quad (\sigma \in \mathbb{R}),$$

is a unitary G-isomorphism between $L^2(G/H)$ and

$$\pi^-_{-1+i\sigma}\hat{\otimes}_2\pi^+_{-1+i\sigma}$$

acting on $L^2(\widetilde{S}\times\widetilde{S})$. The latter space is provided with the usual inner product.

Define the operator A_μ on $C^\infty(\widetilde{S})$ by the formula

$$A_\mu\phi(s) = \int_S |\langle s,t\rangle|^{-\mu-2}\phi(t)dt.$$

This integral is absolutely convergent for $\Re\mu < -1$, and can be analytically extended to the whole complex plane as a meromorphic function. It is easily checked that A_μ is an intertwining operator

$$A_\mu\pi^\pm_\mu(g) = \pi^\mp_{-\mu-2}(g)A_\mu.$$

The operator $A_{-\mu-2}\circ A_\mu$ intertwines $\pi^\pm_\mu$ with itself, and is therefore a scalar $c(\mu)$. It can be computed using K-types:

$$c(\mu) = \pi\frac{\Gamma\left(\frac{\mu+1}{2}\right)\Gamma\left(-\frac{\mu+1}{2}\right)}{\Gamma\left(\frac{-\mu}{2}\right)\Gamma\left(1+\frac{\mu}{2}\right)}.$$

One also shows that : $A^*_\mu = A_{\bar{\mu}}$. So that for $\mu = -1+i\sigma$ we get (by abuse of notation):

$$c(\sigma) = \pi\frac{\Gamma\left(\frac{i\sigma}{2}\right)\Gamma\left(-\frac{i\sigma}{2}\right)}{\Gamma\left(\frac{1-i\sigma}{2}\right)\Gamma\left(\frac{1+i\sigma}{2}\right)},$$

and moreover

$$A_{(-1+i\sigma)}\circ A^*_{(-1+i\sigma)} = c(\sigma)I,$$

so that $\pi^{-\frac{1}{2}}\frac{\Gamma\left(\frac{1+i\sigma}{2}\right)}{\Gamma\left(\frac{i\sigma}{2}\right)}A_{(-1+i\sigma)} = d(\sigma)A_{(-1+i\sigma)}$ is a unitary intertwining operator between $\pi^+_{-1+i\sigma}$ and $\pi^-_{-1-i\sigma}$.

We thus get a $\pi^-_{-1+i\sigma}\hat{\otimes}_2\overline{\pi^-_{-1+i\sigma}}$ invariant map from $L^2(G/H)$ onto $L^2(\widetilde{S}\times\widetilde{S})$ given by

$$\begin{aligned} f &\to d(\sigma)\int_S f(u,w)|\langle u,w\rangle|^{-1+i\sigma}|\langle v,w\rangle|^{-1-i\sigma}dw \\ &=: (\mathcal{K}_\sigma f)(u,v), \qquad s\neq 0. \end{aligned} \tag{3.2}$$

This integral does not converge: it has to be considered as obtained by analytic continuation.

Summarizing we have a G-equivariant embedding of square-integrable functions on the symmetric space G/H into the composition algebra of Hilbert-Schmidt operators by means of the following diagram:

$$L^2(G/H)\hookrightarrow \pi^-_\mu\otimes\pi^+_\mu\hookrightarrow\pi^-_\mu\otimes\overline{\pi^-_\mu}\simeq HS(L^2(\widetilde{S})).$$

The first arrow is of geometric nature and it is given by the fact that the symmetric space G/H is an open dense subset of $G/P^-\cap G/\bar{P}^+$. The last isomorphism is given by

$$L^2(\widetilde{S})\otimes\overline{L^2(\widetilde{S})}\simeq HS(L^2(\widetilde{S})).$$

This embedding gives rise to a covariant symbolic calculus on G/H.

Let $\mathrm{Op}(f)$ on $L^2(\widetilde{S})$ be a Hilbert-Schmidt operator with the kernel $(\mathcal{K}_\sigma f)(u,v)$ defined in (3.2). Then we set:

$$\mathrm{Op}(f\#_\sigma g):=\mathrm{Op}(f)\circ\mathrm{Op}(g).$$

This is an associative product such that:

- $\|f\sharp_\sigma g\|_2\leq\|f\|_2\cdot\|g\|_2$.
- $\mathrm{Op}(L_xf)=\pi^-_{-1+i\sigma}(x)\mathrm{Op}(f)\pi^-_{-1+i\sigma}(x^{-1})$, so
$$L_x(f\#_\sigma g)=(L_xf)\#_\sigma(L_xg),\ \text{for } x\in G.$$

Let us write down a formula for $f\#_\sigma g$; we have:

$$d^{-1}(\sigma)(f\#_\sigma g)(u,v)=\int_S\int_S f(u,x)g(y,v)|[u,y,x,v]|^{-1+i\sigma}d\mu(x,y),$$

where $d\mu(x,y)=|\langle x,y\rangle|^{-2}dxdy$ is a G-invariant measure on $\widetilde{S}\times\widetilde{S}$ for the G-action (3.1). Here

$$[u,y,x,v]=\frac{\langle u,x\rangle\langle y,v\rangle}{\langle u,v\rangle\langle y,x\rangle}.$$

As far as the Laplace-Beltrami operator

$$\Delta_{G/H} = (1-xy)^2 \frac{\partial^2}{\partial x \partial y}$$

of the one-sheeted hyperboloid G/H is G-invariant, G acts on its eigenspaces in $L^2(G/H)$. It is known (see [29]) that the spectrum of $\Delta_{G/H}$ consists of a continuous part $[\frac{1}{4}, \infty[$ (those are generalized eigenvalues) together with the set of all numbers of the form $-n(n+1)$ with $n \in \mathbb{N}$.

The eigenspace corresponding to $-n(n+1)$ decomposes into a direct sum $E_n^+ \oplus E_n^-$ of two irreducible representations of G equivalent to the lowest and the highest weight modules V_n and $\bar{V}_n$. The map

$$J_n(f)(z) = c \int_{\widetilde{S} \times \widetilde{S}} f(x,y) \bar{g}_z^{n+1}(x,y) dm(x,y),$$

where $g_z^{n+1}(x,y) = \left(\frac{\langle x,y \rangle}{\langle x,\bar{z} \rangle \langle y,\bar{z} \rangle} \right)^{n+1}$ and dm the G-invariant measure on G/H is an isometry (with an appropriate choice of the normalizing constant c) from E_n^+ onto the Bergman space $H^2_{2n+2}(\Pi)$ on which we did realize the holomorphic discrete series representations T_{2n+2} of G.

The non-commutative product $\sharp_\sigma$ being defined on the whole space $L^2(G/H)$, it induces a ring structure on the set $\oplus_n E_n^+$. Namely the following theorem due to A. and J. Unterberger (see [30, Theorems 3.6 and 4.2]) holds:

Theorem 3.1. *Let $f_1 \in E_{k_1}^+$ and $f_2 \in E_{k_2}^+$ for some $k_1, k_2 \in \mathbb{N}$. Then $f_1 \sharp_s f_2$ is given by an absolutely convergent series $\sum_{n \geq 0} h_n$, of pairwise orthogonal elements of $L^2(G/H)$. More precisely, for every $n \in \mathbb{N}$ the function $h_n \in E^+_{k_1+k_2+1+n}$ and*

$$J_{k_1+k_2+1+n}(h_n) = c_n(k_1,k_2,\sigma) RC_n(J_{k_1}(f_1), J_{k_2}(f_2)).$$

Moreover, the coefficients $c_n(k_1,k_2,\sigma)$ are given by

$$c_n(k_1,k_2,\sigma) = \frac{\Gamma(k_1+1-i\sigma)\Gamma(k_2+1-i\sigma)}{\Gamma(k_1+k_2+n+2-i\sigma)\Gamma(-i\sigma)} P_n(k_1,k_2,\sigma),$$

where $P_n(k_1,k_2,\sigma)$ are even polynomials in σ.

It is noteworthy that up to a constant the coefficients $c_n(k_1,k_2,\sigma)$, that actually encode the associativity of the $\sharp_\sigma$-product, were conjectured by Cohen, Manin and Zagier in [4].

4. **Toward a generalization of Rankin-Cohen brackets**

Before discussing our method let us mention a series of papers by Eholzer, Ibukayama and Ban [8, 1] where a construction of Rankin-Cohen brackets for Siegel modular forms was developed by means of the Howe θ-correspondence. The advantage of the method that we are going to present is twofold. It gives closed explicit formulas for such bi-differential operators and applies for a large class of reductive Lie groups.

4.1. **Underlying geometric setting**

In the previous section two particular symmetric spaces played an important role: the upper half-plane $\Pi \simeq G/K = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ and the one-sheeted hyperboloid $G/H = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$. The first symmetric space $\Pi \simeq G/K$ is Hermitian of tube type. It implies that $\operatorname{rank} G = \operatorname{rank} K$ and thus guarantees that G has holomorphic discrete series representations (i.e. the underlying Harish-Chandra modules are lowest weight modules).

The second one, G/H, is para-Hermitian, that is, the tangent bundle $T(G/H)$ splits into the sum of two isomorphic G-equivariant sub-bundles (see [16] for more details). This structure generalizes the fact that the one-sheeted hyperboloid in $\mathbb{R}^3$ is generated by two families of straight lines. This splitting induces a G-invariant polarization on $T(G/H)$ that one needs in order to define a covariant quantization $f \mapsto \mathrm{Op}(f)$.

It turns out that there exists a whole class of real semi-simple Lie groups having such two types of co-adjoint orbits : G/K being an Hermitian symmetric space of tube type and G/H a para-Hermitian symmetric space. Their infinitesimal classification is given by the following table:

$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{h}$
$\mathfrak{su}(n,n)$	$\mathfrak{su}(n)+\mathfrak{su}(n)+i\mathbb{R}$	$\mathfrak{sl}(n,\mathbb{C})+\mathbb{R}$
$\mathfrak{so}^*(4n)$	$\mathfrak{su}(2n)+i\mathbb{R}$	$\mathfrak{su}^*(2n)+\mathbb{R}$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{su}(n)+i\mathbb{R}$	$\mathfrak{sl}(n,\mathbb{R})+\mathbb{R}$
$\mathfrak{so}(n,2)$	$\mathfrak{so}(n)+i\mathbb{R}$	$\mathfrak{so}(n-1,1)+\mathbb{R}$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_6+i\mathbb{R}$	$\mathfrak{e}_{6(-26)}+\mathbb{R}$

Figure 4.1. Infinitesimal classification

Such Lie groups may be seen as those of all conformal transformations of Euclidean Jordan algebras and the para-Hermitian symmetric spaces G/H are endowed with a natural G-invariant causal structure.

From now we denote by G one of these connected Lie groups.

4.2. **Holomorphic discrete series**

Holomorphic induction from a maximal compact subgroup leads to a series of unitary representations of G, called holomorphic discrete series representations, that one usually realizes on holomorphic sections of holomorphic vector bundles over G/K.

According to our convenience and easiness of presentation we shall use both bounded and unbounded realizations of the symmetric space G/K. Notice that in our setting G/K is of tube type. Therefore it may be realized as (an unbounded) complex domain $T_\Omega = V + i\Omega$, where V is a real vector space (actually a Euclidean Jordan algebra) and Ω a symmetric cone (consisting of Jordan squares) in V. The group G may be seen as the group of holomorhpic automorphisms $Aut(T_\Omega)$ of the tube domain. For instance, if $G = Sp(n, \mathbb{R})$ then, $V = Sym(n, \mathbb{R})$ and $\Omega = Sym^+(n, \mathbb{R})$ the set of positive definite real symmetric matrices. We also denote by Δ the Jordan determinant on V. In case, when $V = Sym(n, \mathbb{R})$ it coincides with the usual matrix determinant.

Consider the invariant spectral norm $|\cdot|$ on $V_\mathbb{C} := V + iV$. Let

$$\mathcal{D} = \{w \in V \,:\, |w| < 1\},$$

be the corresponding open unit ball. Then the Cayley transform $p: z \mapsto (z - ie)(z + ie)^{-1}$ is a holomorphic isomorphism from the tube T_Ω onto the domain $\mathcal{D}$. Thus, the group of holomorphic automorphisms of $\mathcal{D}$ that one denotes $G(\mathcal{D}) = Aut(\mathcal{D})$ is conjugate to $G : G(\mathcal{D}) = pGp^{-1}$. We shall refer to the domain $\mathcal{D}$ as to the Harish-Chandra bounded realization of the symmetric space G/K and will describe it later.

We start with the simplest case of scalar holomorphic discrete series.

For a real parameter ν consider the weighted Bergman spaces $H^2_\nu(T_\Omega)$ of complex valued holomorphic functions $f \in \mathcal{O}(T_\Omega)$ such that

$$\|f\|^2_\nu = \int_{T_\Omega} |f(z)|^2 \Delta^{\nu - 2\frac{n}{r}}(y)dxdy < \infty,$$

where $z = x + iy \in T_\Omega$. Note that the measure $\Delta^{-2\frac{n}{r}}(y)dxdy$ on T_Ω is invariant under the action of the group G. For $\nu > 1+d(r-1)$ these spaces are non empty Hilbert spaces with reproducing kernels. More precisely, the space $H^2_\nu(T_\Omega)$ has a reproducing kernel K_ν which is given by

$$K_\nu(z,w) = c_\nu \Delta\left(\frac{z-\bar{w}}{2i}\right)^{-\nu}, \tag{4.1}$$

where c_ν is some expression involving Gindikin's conical Γ-functions (see [10] p.261).

The action of G on $H^2_\nu(T_\Omega)$ given for every integer $\nu > 1 + d(r-1)$ by

$$\pi_\nu(g)f(z) = \mathrm{Det}^\nu(D_{g^{-1}}(z))f(g^{-1}.z) \tag{4.2}$$

is called a *scalar holomorphic discrete series representation.*[1]

In the above formula $D_g(z)$ denotes the differential of the conformal transformation $z \to g.z$ of the tube T_Ω.

Now we describe some of the vector valued holomorphic discrete series representations. Let $h(z,w)$ be the so-called *canonical polynomial* (see [10, p.262]). It is the pull back of $K_1(z,w)$ by the Cayley transform.

Then the group $G(\mathcal{D})$ acts on the space $H^2_\nu(\mathcal{D})$ of holomorphic functions f on $\mathcal{D}$ such that

$$\|f\|^2_{\nu,\mathcal{D}} = c'_\nu \int_{\mathcal{D}} |f(z)|^2 h(z,z)^{\nu-2\frac{n}{r}} dxdy < \infty$$

by the similar formula $\pi_\nu(g)f(z) = \mathrm{Det}^\nu(D_{g^{-1}}(z))f(g^{-1}.z)$.

More generally let $\mathfrak{g}$ be the Lie algebra of the automorphisms group $G(\mathcal{D})$ with complexification $\mathfrak{g}_c$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Let $\mathfrak{z}$ be the center of $\mathfrak{k}$. In our case the centralizer of $\mathfrak{z}$ in $\mathfrak{g}$ is equal to $\mathfrak{k}$ and the center of $\mathfrak{k}$ is one-dimensional. There is an element $Z_0 \in \mathfrak{z}$ such that $(\mathrm{ad}Z_0)^2 = -1$ on $\mathfrak{p}$. Fixing i a square root of -1, one has $\mathfrak{p}_c = \mathfrak{p} + i\mathfrak{p} = \mathfrak{p}_+ + \mathfrak{p}_-$ where $\mathrm{ad}Z_0|_{\mathfrak{p}_+} = i$, $\mathrm{ad}Z_0|_{\mathfrak{p}_-} = -i$. Then

$$\mathfrak{g}_c = \mathfrak{p}_+ \oplus \mathfrak{k}_c \oplus \mathfrak{p}_-. \tag{4.3}$$

[1]Notice that in general one shows, by use of analytic continuation, that the reproducing kernel (4.1) is positive-definite for a larger set of spectral parameters, namely for every ν in the so-called Wallach set $W(T_\Omega) = \left\{0, \frac{d}{2}, \dots, (r-1)\frac{d}{2}\right\} \cup](r-1)\frac{d}{2}, \infty[$. However we restrict our considerations only to the subset of $W(T_\Omega)$ consisting of integer $\nu > 1 + d(r-1)$ in order to deal with spaces of holomorphic functions.

and $[\mathfrak{p}_\pm, \mathfrak{p}_\pm] = 0, [\mathfrak{p}_+, \mathfrak{p}_-] = \mathfrak{k}_c$ and $[\mathfrak{k}_c, \mathfrak{p}_\pm] = \mathfrak{p}_\pm$. The vector space $\mathfrak{p}_+$ is isomorphic to V and furthermore it inherits its Jordan algebra structure. Let G_c be a connected, simply connected Lie group with Lie algebra $\mathfrak{g}_c$ and K_c, P_+, P_-, G, K, Z the analytic subgroups corresponding to $\mathfrak{k}_c, \mathfrak{p}_+, \mathfrak{p}_-, \mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{z}$ respectively. Then K_cP_- (and K_cP_+) is a maximal parabolic subgroup of G_c with split component $A = \exp i\mathbb{R}Z_0$. So the group $G = G(\mathcal{D})_o$ is closed in G_c.

Moreover, the exponential mapping is a diffeomorphism of $\mathfrak{p}_-$ onto P_- and of $\mathfrak{p}_+$ onto P_+ [14, Ch.VIII, Lemma 7.8]. Furthermore:

Lemma 4.1. *a. The mapping $(q, k, p) \mapsto qkp$ is a diffeomorphism of $P_+ \times K_c \times P_-$ onto an open dense submanifold of G_c containing G.*
b. The set GK_cP_- is open in $P_+K_cP_-$ and $G \cap K_cP_- = K$.

(see [14, Ch VIII, Lemmæ 7.9 and 7.10]).

Thus G/K is mapped on an open, bounded domain $\mathcal{D}$ in $\mathfrak{p}_+$ This is an alternative description of the Harish-Chandra bounded realization of G/K. The group G acts on $\mathcal{D}$ via holomorphic transformations.

Everywhere in this section we shall denote $\bar{g}$ the complex conjugate of $g \in G_c$ with respect to G. Notice that P_+ is conjugate to P_-.

For $g \in P_+K_cP_-$ we shall write $g = (g)_+ \, (g)_0 \, (g)_-$, where $(g)_\pm \in P_\pm$, $(g)_0 \in K_c$. For $g \in G_c$, $z \in \mathfrak{p}_+$ such that $g.\exp z \in P_+K_cP_-$ we define

$$\exp g(z) = (g.\exp z)_+ \tag{4.4}$$
$$J(g, z) = (g.\exp z)_0. \tag{4.5}$$

$J(g, z) \in K_c$ is called the *canonical automorphic factor* of G_c.

Lemma 4.2. [27, Ch.II, Lemma 5.1.] *The map J satisfies*

(i) $J(g, o) = (g)_0$, *for* $g \in P_+K_cP_-$,

(ii) $J(k, z) = k$ *for* $k \in K_c, z \in \mathfrak{p}_+$.

If for $g_1, g_2 \in G_c$ and $z \in \mathfrak{p}_+$, $g_1(g_2(z))$ and $g_2(z)$ are defined, then $(g_1g_2)(z)$ is also defined and

(iii) $J(g_1g_2, z) = J(g_1, g_2(z)) \, J(g_2, z)$.

For $z, w \in \mathfrak{p}_+$ satisfying $(\exp \bar{w})^{-1}. \exp z \in P_+ K_c P_-$ we define

$$\begin{aligned} K(z,w) &= J((\exp \bar{w})^{-1}, z)^{-1} && (4.6)\\ &= ((\exp \bar{w})^{-1}. \exp z)_0^{-1}. && (4.7)\end{aligned}$$

This expression is always defined for $z, w \in \mathcal{D}$, for then

$$(\exp \bar{w})^{-1}. \exp z \in \overline{(GK_cP_-)}^{-1} GK_cP_- = P_+K_cGK_cP_- = P_+K_cP_-.$$

$K(z,w)$, defined on $\mathcal{D}\times\mathcal{D}$, is called the *canonical kernel* on $\mathcal{D}$ (by Satake). $K(z,w)$ is holomorphic in z, anti-holomorphic in w, with values in K_c. Here are a few properties:

Lemma 4.3. [27, Ch.II, Lemma 5.2] *The map K satisfies:*

(i) $K(z,w) = \overline{K(w,z)}^{-1}$ *if $K(z,w)$ is defined,*

(ii) $K(o,w) = K(z,o) = 1$ *for $z, w \in \mathfrak{p}_+$.*

If $g(z), \bar{g}(w)$ and $K(z,w)$ are defined, then $K(g(z), \bar{g}(w))$ is also defined and one has:

(iii) $K(g(z), \bar{g}(w)) = J(g,z)\, K(z,w)\, \overline{J(\bar{g},w)}^{-1}$.

Lemma 4.4. [27, Ch.II, Lemma 5.3.] *For $g \in G_c$ the Jacobian of the holomorphic mapping*
$z \mapsto g(z)$, when it is defined, is given by

$$\mathrm{Jac}\,(z \mapsto g(z)) = \mathrm{Ad}_{\mathfrak{p}_+}(J(g,z)).$$

For any holomorphic character $\chi : K_c \mapsto \mathbb{C}$ we define:

$$\begin{aligned} j_\chi(g,z) &= \chi(J(g,z)), && (4.8)\\ k_\chi(z,w) &= \chi(K(z,w)). && (4.9)\end{aligned}$$

Since $\chi(\bar{k}) = \overline{\chi(k)}^{-1}$ we have :

$$\begin{aligned} k_\chi(z,w) &= \overline{k_\chi(w,z)}, && (4.10)\\ k_\chi(g(z), \bar{g}(w)) &= j_\chi(g,z) k_\chi(z,w) \overline{j_\chi(\bar{g},w)} && (4.11)\end{aligned}$$

in place of Lemma 4.3 (i) and (iii).

The character $\chi_1(k) = \det \mathrm{Ad}_{\mathfrak{p}_+}(k)$, $(k \in K_c)$ is of particular importance. We call the corresponding j_{χ_1}, k_{χ_1}: j_1 and k_1. Notice that

$$j_1(g,z) = \det(\mathrm{Jac}\,(z \mapsto g(z))). \quad (4.12)$$

Because of (4.12), $|k_1(z,z)|^{-1}d\mu(z)$, where $d\mu(z)$ is the Euclidean measure on $\mathfrak{p}_+$, is a $G-$invariant measure on $\mathcal{D}$. Indeed:

$$\begin{array}{rcll} d\mu(g(z)) & = & |j_1(g,z)|^2 d\mu(z), & \\ k_1(g(z),g(z)) & = & j_1(g,z)\, k_1(z,z)\, \overline{j_1(g,z)}, & \qquad \text{for } g\in G. \end{array}$$

One can actually show that $k_1(z,z)>0$ on $\mathcal{D}$. ([27], Ch.II, Lemma 5.8).

Let τ be an irreducible holomorphic representation of K_c on a finite dimensional complex vector space W with scalar product $\langle\ |\ \rangle$, such that $\tau_{|K}$ is unitary.

Lemma 4.5. *For every $k\in K_c$ one has the identity $\tau^*(k)=\tau(\bar{k})^{-1}$.*

This follows easily by writing $k=k_o\cdot\exp iX$ with $k_o\in K$, $X\in\mathfrak{k}$ and using that $\tau_{|K}$ is unitary.

Call $\pi_\tau=\mathrm{Ind}_K^G\tau$ and set W_τ for the representation space of π_τ. Then W_τ consists of maps $f:G\mapsto W$ satisfying

(i) f measurable,

(ii) $f(gk)=\tau^{-1}(k)f(g)$ for $g\in G,\ k\in K$,

(iii) $\int_{G/K}\|f(g)\|^2 d\dot{g}<\infty$,

where $\|f(g)\|^2=\langle f(g)|f(g)\rangle$ and $d\dot{g}$ is an invariant measure on G/K. Let us identify G/K with $\mathcal{D}$ and $d\dot{g}$ with $d_*z=k_1(z,z)^{-1}d\mu(z)$. Then W_τ can be identified with a space of maps on $\mathcal{D}$, setting

$$\varphi(z)=\tau(J(g,o))f(g), \tag{4.13}$$

if $z=g(o), f\in W_\tau$. Indeed, the right-hand side of (4.13) is clearly right $K-$invariant. The inner product becomes

$$(\varphi|\psi)=\int_{\mathcal{D}}\langle\tau^{-1}(J(g,o))\varphi(z)|\tau^{-1}(J(g,o))\psi(z)\rangle d_*z.$$

Since $\tau^{-1}(J(g,o))^*\tau^{-1}(J(g,o)) = \tau^{-1}(J(g,o)\overline{J(g,o)}^{-1}) = \tau^{-1}(K(z,z))$ by Lemma (4.3), we may also write

$$(\varphi|\psi)=\int_{\mathcal{D}}\langle\tau^{-1}(K(z,z))\varphi(z)|\psi(z)\rangle d_*z. \tag{4.14}$$

The G-action on the new space is given by

$$\pi_\tau(g)\varphi(z)=\tau^{-1}(J(g^{-1},z))\varphi(g^{-1}(z)), \qquad (g\in G,\ z\in\mathcal{D}). \tag{4.15}$$

Now we restrict to the closed sub-space of holomorphic maps and call the resulting Hilbert space $\mathcal{H}_\tau$. The space $\mathcal{H}_\tau$ is $\pi_\tau(G)$-invariant. We assume that $\mathcal{H}_\tau \neq \{0\}$.

The pair $(\pi_\tau, \mathcal{H}_\tau)$ is called a *vector-valued holomorphic discrete series* of G.

In a similar way we can define the anti-holomorphic discrete series. We therefore start with $\bar{\tau}$ instead of τ and take anti-holomorphic maps. Then

$$\pi_{\bar{\tau}}(g)\psi(z) = \bar{\tau}^{-1}(J(g^{-1}, z))\psi(g^{-1}(z)). \tag{4.16}$$

for $\psi \in \mathcal{H}_{\bar{\tau}}$. One easily sees that $\mathcal{H}_{\bar{\tau}} = \bar{\mathcal{H}}_\tau$ and $\pi_{\bar{\tau}} = \bar{\pi}_\tau$ in the usual sense. Notice that when the representation τ is one dimensional we recover scalar holomorphic discrete series representations introduced above.

The Hilbert space $\mathcal{H}_\tau$ is known to have a reproducing (or Bergman) kernel $\mathcal{K}_\tau(z,w)$. Its definition is as follows. Set

$$E_z : \varphi \mapsto \varphi(z) \qquad (\varphi \in \mathcal{H}_\tau)$$

for $z \in \mathcal{D}$. Then $E_z : \mathcal{H}_\tau \mapsto W$ is a continuous linear operator, and $\mathcal{K}_\tau(z,w) = E_z E_w^*$, being a $\mathrm{End}(W)$-valued kernel, holomorphic in z, anti-holomorphic in w. In more detail :

$$\langle \varphi(w) |\, \xi\rangle = \int_{\mathcal{D}} \langle \tau^{-1}(K(z,z))\varphi(z) |\, \mathcal{K}_\tau(z,w)\xi\rangle d_* z \tag{4.17}$$

for any $\varphi \in \mathcal{H}_\tau$, $\xi \in W$ and $w \in \mathcal{D}$.

Since $\mathcal{H}_\tau$ is a $G-$module, one easily gets the following transformation property for $\mathcal{K}_\tau(z,w)$:

$$\mathcal{K}_\tau(g(z), g(w)) = \tau(J(g,z))\mathcal{K}_\tau(z,w)\tau(\overline{J(g,w)})^{-1} \qquad (g \in G, z, w \in \mathcal{D}). \tag{4.18}$$

Now consider $H(z,w) = \mathcal{K}_\tau(z,w) \cdot \tau^{-1}(K(z,w))$.

Clearly $H(g(z), g(w)) = \tau(J(g,z))H(z,w)\tau^{-1}(J(g,z))$ for all $z, w \in \mathcal{D}$. So, setting $z = w = o, g \in K$ we see that $H(o,o)$ is a scalar operator, and hence $H(z,z) = H(o,o)$ is so. But then $H(z,w) = H(o,o)$. So, we get

$$\mathcal{K}_\tau(z,w) = c \cdot \tau(K(z,w)), \tag{4.19}$$

where c is a scalar. The same reasoning yields that π_τ is *irreducible*. Indeed, if $\mathcal{H} \subset \mathcal{H}_\tau$ is a closed invariant subspace, then $\mathcal{H}$ has a reproducing kernel, say $K_{\mathcal{H}}$ and it follows that $K_{\mathcal{H}} = c\mathcal{K}_\tau$, so either $\mathcal{H} = \{0\}$ or $\mathcal{H} = \mathcal{H}_\tau$.

Let us briefly recall the analytic realization of (some of) vector-valued holomorphic discrete series representations of G. We start with the irreducible representations of the maximal compact subgroup K which can be realized on the space of polynomials $\mathcal{P}(V)$ and which are parameterized by the weights $\underline{\mathbf{m}} = (m_1, \ldots, m_r) \in \mathbb{Z}^r$ with $m_1 \geq \cdots \geq m_r \geq 0$ and $m_1 + \cdots + m_r = m = |\underline{\mathbf{m}}|$. These representations do not exhaust all irreducible representations of K but they will produce all necessary components for our further discussion.

Let V' be the dual vector space of $V \simeq \mathfrak{p}_+$. Consider the $m-th$ symmetric tensor power of V'. It is naturally identified with the space $\mathcal{P}^m(V)$ of polynomials of degree m on V. It is well known (see for instance [28]) that under the K-action this space decomposes multiplicity free into a direct sum of irreducible sub-representations :

$$\mathcal{P}^m(V) = \sum_{|\underline{\mathbf{m}}|=m}^{\oplus} \mathcal{P}^{\underline{\mathbf{m}}}(V),$$

where $\mathcal{P}^{\underline{\mathbf{m}}}(V)$ are irreducible representations of K of highest weight $\underline{\mathbf{m}}$. This decomposition is often called the Kostant-Hua-Schmid formula. Denote by $P_{\underline{\mathbf{m}}}$ the orthogonal projection of $\mathcal{P}^m(V)$ onto $\mathcal{P}^{\underline{\mathbf{m}}}(V)$.

Let $h(z, w)$ be as before the canonical polynomial on $V \times V$, then according to [10], for a real ν one has

$$h^{-\nu}(z, w) = \sum_{\underline{\mathbf{m}}} (\nu)_{\underline{\mathbf{m}}} K_{\underline{\mathbf{m}}}(z, w),$$

where $K_{\underline{\mathbf{m}}}(z, w)$ is the reproducing kernel of the space $\mathcal{P}^{\underline{\mathbf{m}}}(V)$, and $(\nu)_{\underline{\mathbf{m}}}$ stands for the generalized Pochhammer symbol:

$$(\nu)_{\underline{\mathbf{m}}} = \prod_{j=1}^{r} \left(\nu - \frac{d}{2}(j-1)\right)_{m_j} = \prod_{j=1}^{r} \prod_{k=1}^{m_j} \left(\nu - \frac{d}{2}(j-1) + k - 1\right).$$

Denote $\mathcal{H}_\nu(\mathcal{P}^{\underline{\mathbf{m}}}(V))$ the Hilbert space of holomorphic functions on $\mathcal{D}$ with values in $\mathcal{P}^{\underline{\mathbf{m}}}(V)$ admitting the reproducing kernel

$$h^{-\nu}(z, w) \otimes^m K^t(z, w).$$

Then, for an integer $\nu > 1 + d(r-1)$ and a given weight $\underline{\mathbf{m}}$ the group G acts on its unitarily and irreducibly by

$$\pi_{\nu, \underline{\mathbf{m}}}(g) f(z) = \mathrm{Det}(Dg^{-1}(z))^{\nu} \left(\otimes^m (dg^{-1})^t\right) \cdot f(g^{-1}.z), \tag{4.20}$$

where $\otimes^m (dg^{-1})^t$ on $\mathcal{P}^{\underline{\mathbf{m}}}(V)$ denotes the induced action of $(dg^{-1})^t$ on V.

4.3. Maximal degenerate series

Let $\mathrm{Det}(g)$ be the determinant of a linear transform $g \in G(\Omega) \subset GL(V)$, where $G(\Omega)$ is the group of all linear transformations preserving the cone Ω. We denote by $\chi(g)$ a particular character of this reductive Lie group given by $\chi(g) := \mathrm{Det}(g)^{\frac{r}{n}}$, where r is the rank and n the dimension of the Jordan algebra V.

This character can be trivially extended to the whole parabolic subgroup $\overline{P} = G(\Omega) \ltimes \overline{N}$ by $\chi(h\bar{n}) := \chi(h)$ for every $h \in G(\Omega)$, $\bar{n} \in \overline{N} \simeq V$.

For every $\mu \in \mathbb{C}$ we define a character χ_μ of $\overline{P}$ by $\chi_\mu(\bar{p}) := |\chi(\bar{p})|^\mu$.

The induced representation $\pi_\mu^- = \mathrm{Ind}_{\overline{P}}^G(\chi_\mu)$ of the group G acts on the space

$$\widetilde{I}_\mu := \{f \in C^\infty(G) \mid f(g\bar{p}) = \chi_\mu(\bar{p})f(g), \forall g \in G, \bar{p} \in \overline{P}\},$$

by left translations. A pre-Hilbert structure on $\widetilde{I}_\mu$ is given by $\|f\|^2 = \int_K |f(k)|^2\, dk$, where K is the maximal compact subgroup of G associated with the Cartan involution σ, and dk is the normalized Haar measure of K.

According to the Gelfand-Naimark decomposition a function $f \in \widetilde{I}_\mu$ is determined by its restriction $f_V(x) = f(n_x)$ on $N \simeq V$. Let I_μ be the subspace of $C^\infty(V)$ of functions f_V with $f \in \widetilde{I}_\mu$. The group G acts on I_μ by:

$$\pi_\mu^-(g)f(x) = |A(g,x)|^\mu f(g^{-1}.x),\ g \in G,\ x \in V, \tag{4.21}$$

where $A(g,x) := \chi_\mu((Dg^{-1})_x)$. These representations are usually called the *maximal degenerate series representations* of G.

One shows that the norm of a function $f(n_x) = f_V(x) \in I_\mu$ is given by:

$$\|f\|^2 = \int_V |f_V(x)|^2 h(x,-x)^{2\Re\mu + \frac{n}{r}}\, dx, \tag{4.22}$$

where $h(z,w)$ is the *canonical polynomial* introduced above. Formula (4.22) implies that for $\Re\mu = -\frac{n}{2r}$ the space I_μ is contained in $L^2(V)$ and the representation π_μ^- extends as a unitary representation on $L^2(V)$.

Analogously the character χ can be extended to the subgroup P and one defines in a similar way the representation $\pi_\mu^+ = \mathrm{Ind}_P^G(\chi_{-\mu})$.

Following the standard procedure we introduce an intertwiner between π_μ^- and $\pi^+_{\mu-\frac{n}{r}}$. Consider the map $\widetilde{A}_\mu$ defined on $\widetilde{I}_\mu$ by

$$f \longrightarrow (\widetilde{A}_\mu f)(g) := \int_N f(gn)dn, \quad \forall g \in G, \tag{4.23}$$

where dn is a left invariant Haar measure on N. One shows that this integral converges for $\Re\mu > \frac{n}{2r_0}$.

Proposition 4.6. *For every $f \in \widetilde{I}_\mu$ the function $\widetilde{A}_\mu f$ belongs to $\widetilde{I}_{-\mu}$ and the map $\widetilde{A}_\mu$ given by (4.23) intertwines the corresponding representations of G:*

$$\widetilde{\pi}^+_{\mu-\frac{n}{r}}(g)(\widetilde{A}_\mu f) = \widetilde{A}_\mu(\widetilde{\pi}_\mu^-(g)f),\ \forall f \in \widetilde{I}_\mu,\ g \in G. \tag{4.24}$$

See [24] for more details.

Similarly to the construction of Section 3 we may define an operator calculus based on intertwining operators A_μ and define a composition formula of two symbols $f_1, f_2 \in L^2(G/H)$.

5. Rankin-Cohen brackets for conformal Lie groups

Adopting the philosophy of the previous section we define generalized Rankin-Cohen brackets as intertwining operators between tensor products of scalar holomorphic discrete series of a given conformal Lie group and their irreducible components. Let us start by introducing all the ingredients of this construction.

One says that a symmetric space G/H has discrete series representations if the set of representations of G on minimal closed invariant subspaces of $L^2(G/H)$ is nonempty. According to a fundamental result of Flensted-Jensen [11] the discrete series for G/H is nonempty and infinite if

$$\operatorname{rank}(G/H) = \operatorname{rank}(K/K \cap H).$$

For any symmetric space G/H given by the classification table 4.1 this condition is fulfilled and on can realize one part of its discrete series as holomorphic discrete series representations of the whole group G.

More precisely assume that π is a scalar holomorphic discrete series representation of G, i.e. it acts on $\mathcal{H}_\pi \subset \mathcal{O}(\mathcal{D}) \cap L^2(\mathcal{D}, dm_\pi)$ where $\mathcal{D}$ is some symmetric domain (the image of the tube T_Ω by the Cayley transform) and where $dm_\pi(w)$ is a measure on $\mathcal{D}$ associated to π. In such a case the Hilbert space $\mathcal{H}_\pi$ has a reproducing kernel $K_\pi(z,w)$.

Assume that the representation π occurs as a multiplicity free closed subspace in the Plancherel formula for $L^2(G/H)$ (actually this is the case in our setting).

Consider $\xi_\pi \in \mathcal{H}_\pi^{-\infty}$ the unique up to scalars H-fixed distribution vector associated to π (see [19, p. 142] for the definition of $\xi_\pi = \phi_\lambda(z)$). It gives rise to a continuous embedding map

$$\mathcal{J}_\pi : \mathcal{H}_\pi \hookrightarrow L^2(G/H) \subset (C_c^\infty)'(G/H)$$

given for any analytic vector $v \in \mathcal{H}_\pi^\infty$ by

$$(\mathcal{J}_\pi v)(x) = \langle v, \pi(x)\xi_\pi\rangle, \qquad x \in G/H, \tag{5.1}$$

where by abusing notations we write $\pi(x)$ instead of $\pi(g)$ with $x = g.H \in G/H$.

For any fixed $w \in \mathcal{D}$ let us define the function $v_w := K_\pi(\cdot, w)$ which is actually a real analytic vector in $\mathcal{H}_\pi$.

Consider now the following function :

$$g_w(x) := (\mathcal{J} v_w)(x), \qquad x \in G/H,\ w \in \mathcal{D}.$$

Because of the reproducing property of the Hilbert space $\mathcal{H}_\pi$ for every $f \in \mathcal{H}_\pi$ one can write

$$f(z) = \int_{\mathcal{D}} K_\pi(z,w) f(w) dm_\pi(w).$$

Furthermore, if such a function f is an analytic vector for the representation $\pi : f \in \mathcal{H}_\pi^\infty$, then

$$(\mathcal{J}_\pi f)(x) = \int_{\mathcal{D}} f(w) g_w(x) dm_\pi(w).$$

Choosing an appropriate normalization in (5.1) one can get the embedding $\mathcal{J}_\pi$ isometric. Therefore the subspace generated by $g_w(x), w \in \mathcal{D}$ is a closed subspace of $L^2(G/H)$ isometric to some holomorphic discrete series representation of G. (see [19, Theorem 5.4] for the precise statement).

The dual map $\mathcal{J}_\pi^* : C_c^\infty(G/H) \mapsto \mathcal{H}_\pi$ is defined by

$$\begin{aligned}\langle \mathcal{J}_\pi^* \phi, f\rangle &= \langle \phi, \mathcal{J}_\pi f\rangle \\ &= \int_{G/H}\int_{\mathcal{D}} \phi(x)\overline{f(w) g_w(x)} dm(w) d\nu(x),\ \forall \phi \in C_c^\infty(G/H),\end{aligned}$$

where $d\nu(x)$ denotes the invariant measure on G/H. Therefore we have,

$$(\mathcal{J}_\pi^*\phi)(w) = \int_{G/H} \phi(x)\overline{g_w(x)}d\nu(x).$$

Similar observations are valid for vector-valued holomorphic discrete series representations as well.

Define the set

$$L^2(G/H)_{\mathrm{hol}} = \bigoplus_{\pi\in\hat{G}'_{\mathrm{hol}}} \mathcal{J}_\pi(\mathcal{H}_\pi)$$

where $\hat{G}'_{\mathrm{hol}}$ denotes the set of equivalence classes of unitary irreducible holomorphic discrete series representations of G with corresponding character τ trivial on $H \cap Z$. Notice that the space $L^2(G/H)_{hol}$ decomposes multiplicity free into irreducible subspaces [22].

According to [19] the H-fixed distribution vector $\xi_k = \xi_{\pi_\nu}$, associated with the scalar holomorphic discrete series representation π_ν (see (4.2)) is given up to a constant by

$$\xi_k(z) = \Delta\left(\frac{\eta(z)-\bar{z}}{2i}\right)^{-\frac{\nu}{2}}, \quad z \in V_0 + i\Omega. \tag{5.2}$$

To get more insight in the product structure of $L^2(G/H)_{\mathrm{hol}}$, we rely on a theorem by T.Kobayashi [17, Theorem 7.4].

We are going to show that $L^2(G/H)_{hol}$ is closed under the product $\#_s$. It is, because of the continuity of the product, sufficient to show the following theorem(see [7]).

Theorem 5.1. *Let $\mathcal{H}_\pi$ and $\mathcal{H}_{\pi'}$ be two irreducible closed subspaces of $L^2(G/H)_{\mathrm{hol}}$. Then*

$$\mathcal{J}_\pi(f)\#_s\mathcal{J}_{\pi'}(g) \in L^2(G/H)_{\mathrm{hol}}.$$

for every $f \in \mathcal{H}_\pi$ and $g \in \mathcal{H}_{\pi'}$.

The proof of this theorem uses the following result by T. Kobayashi [17]:

Theorem 5.2. *Let π and π' be holomorphic discrete series representations of G. Then the representation*

$$\pi\hat{\otimes}_2\pi'$$

decomposes discretely into holomorphic discrete series representations of G with finite multiplicities. Moreover, $\pi\hat{\otimes}_2\pi'$ is K-admissible, i.e. every irreducible representation of K occurs in it with finite multiplicity.

In general we do not have a multiplicity free decomposition.

Now let us show Theorem 5.1. The map $f \otimes g \to \mathcal{J}_\pi(f) \#_s \mathcal{J}_{\pi'}(g)$ clearly gives rise to a $K-$ and $\mathcal{U}(\mathfrak{g})-$equivariant linear map

$$(\mathcal{H}_\pi \hat{\otimes}_2 \mathcal{H}_{\pi'})^K = \mathcal{H}_\pi^K \otimes \mathcal{H}_{\pi'}^K \to L^2(G/H),$$

and thus the result follows for f and g $K-$finite, and then, by continuity of the product, for all f and g.

Example. The decomposition of the tensor product of two holomorphic discrete series for $\mathrm{SL}(2,\mathbb{R})$ was obtained by J. Repka [26] in full generality using the Harish-Chandra modules techniques, and it is given by

$$\pi_n \hat{\otimes}_2 \pi_m = \bigoplus_{k=0}^{\infty} \pi_{m+n+2k}.$$

In the general situation we have to consider also vector-valued holomorphic discrete series representations. Indeed, according to Theorem (5.2) and particularly to the result stated in Theorem 3.3 in [23] the tensor product of two scalar holomorphic discrete series representations decomposes multiplicity free in the direct sum of unitary irreducible vector-valued holomorphic discrete series representations:

$$\mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2} = \sum_{\underline{\mathbf{m}} \geq 0} \mathcal{H}_{\nu_1+\nu_2}(\mathcal{P}^{\underline{\mathbf{m}}}(V')),$$

in the case when $\nu_1 \geq \nu_2 > 1 + d(r-1)$.

In order to understand the previous decomposition we have to identify its different ingredients.

First, we see an element of the tensor product $\mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2}$ as a holomorphic function $F(z,w)$ on $\mathcal{D} \times \mathcal{D}$. Therefore, one can write a Taylor expansion formula:

$$F(z,w) = \sum_{j=0}^{m} (F^{(j)}(z), \otimes^j (w-z)) + (F^{(m+1)}(z,w), \otimes^{m+1}(z-w)),$$

where $F^{(j)}(z)$ are $\mathcal{P}^j(V')$-valued holomorphic functions on $\mathcal{D}$, $F^{(m+1)}(z,w)$ is a $\mathcal{P}^{m+1}(V')$-valued holomorphic function on $\mathcal{D} \times \mathcal{D}$ uniquely determined by the data of $F(z,w)$, and $(\,,\,)$ denotes the standard pairing of corresponding vector spaces.

Second, consider an $End(V)$-valued holomorphic differential form on $\mathcal{D}$ defined for every fixed $w_1, w_2 \in V$ and $z \in \mathcal{D}$ by

$$\Omega(z;,w_1,w_2) = d_z B(z,w_1)B(z,w_1)^{-1} - d_z B(z,w_2)B(z,w_2)^{-1},$$

where $B(z,w)$ is the Bergman operator, and denote by $\omega(z;w_1,w_2)$ it trace $-\frac{r}{2n}\mathrm{tr}\Omega(z;w_1,w_2)$. The former differential form plays a crucial role in the construction of intertwining operators for tensor products.

Namely, for fixed w_1 and w_2 the expression

$$h(z,w_1)^{-\nu_1} h(z,w_2)^{-\nu_2} P_{\underline{\mathbf{m}}} \otimes^{|\underline{\mathbf{m}}|} \omega(z;w_1,w_2)$$

can be seen as an element of the space $\overline{\mathcal{H}_{\nu_1}} \otimes \overline{\mathcal{H}_{\nu_2}}$ dual of $\mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2}$. Let $\langle\,,\,\rangle$ stand for the corresponding pairing. Then the operator $I_{\underline{\mathbf{m}}}$ given by

$$I_{\underline{\mathbf{m}}}(f\otimes g)(z) = \langle h(z,\cdot)^{-\nu_1} h(z,\cdot)^{-\nu_2} P_{\underline{\mathbf{m}}} \otimes^{|\underline{\mathbf{m}}|} \omega(z;\cdot,\cdot), f\otimes g\rangle, \tag{5.3}$$

is a G-equivariant map from $(\pi_{\nu_1}\otimes\pi_{\nu_2}, \mathcal{H}_{\nu_1}\otimes\mathcal{H}_{\nu_2})$ to the space of $\mathcal{P}^{\underline{\mathbf{m}}}(V)$-valued holomorphic functions on $\mathcal{D}$ seen as the representation space of $\pi_{\nu_1+\nu_2,\underline{\mathbf{m}}}$ (see (4.20).

Theorem 4.4 in [23] gives a description of this map. Summarizing and using Theorem (5.1), we get

Proposition 5.3. *Let $\nu_1 \geq \nu_2 > 1 + d(r-1)$ and $f \in \mathcal{H}_{\nu_1}$, $g \in \mathcal{H}_{\nu_2}$. Assume that $(\mathcal{H}_{\nu_1}^{-\infty})^H$ and $(\mathcal{H}_{\nu_2}^{-\infty})^H$ are not reduced to $\{0\}$. Then*

$$\mathcal{J}_{\pi_{\nu_1}}(f) \#_s \mathcal{J}_{\pi_{\nu_2}}(g) = \sum_{\underline{\mathbf{m}}\geq 0} c_{\underline{\mathbf{m}},s} \mathcal{J}_{\pi_{\nu_1+\nu_2}}(RC_{\underline{\mathbf{m}},\nu_1,\nu_2}(f,g)),$$

where $c_{\underline{\mathbf{m}},s}$ are fundamental constants given by the $\#_s$ product of the reproducing kernels of the corresponding Bergman spaces $\mathcal{H}_{\nu_1}$ and $\mathcal{H}_{\nu_2}$ and $RC_{\underline{\mathbf{m}},\nu_1,\nu_2}$ is such a bi-differential operator on $\mathcal{H}_{\nu_1}\otimes\mathcal{H}_{\nu_2}$ that

$$I_{\underline{\mathbf{m}}}(RC_{\underline{\mathbf{m}},\nu_1,\nu_2}(f,g)) = \sum_{|\underline{\mathbf{n}}|+|\underline{\mathbf{n}}'|=m} C_{|\underline{\mathbf{m}}|}^{|\underline{\mathbf{n}}|} \cdot \frac{(-1)^{|\underline{\mathbf{n}}|}}{(\nu_1)_{\underline{\mathbf{n}}}(\nu_2)_{\underline{\mathbf{n}}'}} \cdot P_{\underline{\mathbf{m}}}\left(P_{\underline{\mathbf{n}}}\partial^{|\underline{\mathbf{n}}|} f \otimes P_{\underline{\mathbf{n}}'}\partial^{|\underline{\mathbf{n}}'|} g\right),$$

with $\underline{\mathbf{n}}$ and $\underline{\mathbf{n}}'$ being all possible weights such that $|\underline{\mathbf{n}}| + |\underline{\mathbf{n}}'| = |\underline{\mathbf{m}}|$.

The operators $RC_{\underline{\mathbf{m}},\nu_1,\nu_2}$ are covariant bi-differential operators that automatically preserve modularity with respect to any arithmetic subgroup of a given conformal group. It is natural to call them generalized Rankin-Cohen brackets.

As far as the series $I_{\underline{\mathbf{m}}}(RC_{\underline{\mathbf{m}},\nu_1,\nu_2}(f,g))$ defines an associative product it gives rise to an infinite series of algebraic identities involving all Rankin-Cohen brackets and starting from the commutativity of the pointwise product, Jacobi identity for the Poisson bracket, etc [25]. This reach structure is encoded by the the coefficients $c_{\underline{\mathbf{m}},s}$ which, in general, are not known individually.

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