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# Quasimodular forms and quasimodular polynomials

MIN HO LEE

## Abstract

This paper is based on lectures delivered at the Workshop on quasimodular forms held in June, 2010 in Besse, France, and it provides a survey of some recent work on quasimodular forms.

*Formes quasimodulaires et polynômes quasimodulaires*

## Résumé

Ce texte a pour origine des cours donnés à l'École d'été sur les formes quasimodulaires qui s'est tenue en juin 2010 à Besse, France. Il contient une présentation de travaux récents sur les formes quasimodulaires.

## 1. Introduction

Quasimodular forms were introduced by Kaneko and Zagier in [5] and have been studied actively since then in connection with various topics in number theory and other areas of mathematics (see e.g. [4], [12], [13], [15], [16], [17], [19]). They generalize modular forms, and one of their useful properties is that their derivatives are also quasimodular forms. In particular, the derivatives of modular forms are quasimodular forms. On the other hand, it can be shown that each quasimodular form can be expressed in terms of derivatives of some modular forms. In fact, a quasimodular form can be identified with a finite sequence of modular forms. To see this we can consider two types of actions of  $SL(2, \mathbb{R})$  on the space of polynomials over the ring of holomorphic functions on the Poincaré upper half plane as well as an equivariant automorphism. Given a discrete subgroup  $\Gamma$  of  $SL(2, \mathbb{R})$ , quasimodular polynomials and modular polynomials for  $\Gamma$  are invariant polynomials under such actions restricted to  $\Gamma$ . Thus the equivariance property shows that the above automorphism induces an isomorphism between the space of quasimodular polynomials and that of modular polynomials.

The coefficients of modular polynomials are modular forms of certain weights, so that a modular polynomial can be identified with a certain finite sequence of modular forms. On the other hand, quasimodular polynomials correspond to quasimodular forms. Indeed, given integers  $m$  and  $w$  with  $m \geq 0$ , a quasimodular form  $f$  of weight  $w$  and depth at most  $m$  for  $\Gamma$  corresponds to a set of holomorphic functions  $f_0, f_1, \dots, f_m$  on the Poincaré upper half plane  $\mathcal{H}$  in such a way that

$$\begin{aligned} & \frac{1}{(cz+d)^w} f\left(\frac{az+b}{cz+d}\right) \\ &= f_0(z) + f_1(z) \left(\frac{c}{cz+d}\right) + \cdots + f_m(z) \left(\frac{c}{cz+d}\right)^m \end{aligned} \quad (1.1)$$

for all  $z \in \mathcal{H}$  and  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ , and the corresponding quasimodular polynomial has the functions  $f_k$  as its coefficients.

Quasimodular forms are also related to Jacobi-like forms and automorphic pseudodifferential operators studied by Cohen, Manin and Zagier (cf. [2], [20]). Jacobi-like forms are formal Laurent series which generalize Jacobi forms in some sense, and they correspond to certain sequences of modular forms. The coefficients of a Jacobi-like form are quasimodular forms, and naturally there are projection maps sending a Jacobi-like form to its coefficients.

The organization of this paper is as follows. In Section 2, we describe relations among quasimodular forms, quasimodular polynomials and modular polynomials. In Section 3 we construct vector bundles over the quotient space  $\Gamma \backslash \mathcal{H}$ , whose sections can be identified with quasimodular polynomials and therefore with quasimodular forms. This extends the usual interpretation of modular forms as holomorphic sections of line bundles over  $\Gamma \backslash \mathcal{H}$ . We also introduce Poincaré series for quasimodular forms. Section 4 deals with certain differential operators on quasimodular polynomials that are related to heat operators on Jacobi-like forms studied in [8] as well as operators corresponding to some embeddings and projection maps of modular polynomials. In Section 5 we construct linear maps from quasimodular forms for  $\Gamma$  to some cohomology classes of the group  $\Gamma$ , which are equivariant with respect to appropriate Hecke operator actions.

## 2. Quasimodular forms and polynomials

In this section we discuss correspondences among quasimodular forms, quasimodular polynomials, and modular polynomials for a discrete subgroup of  $\mathrm{SL}(2, \mathbb{R})$ .

Let  $\mathcal{H}$  be the Poincaré upper half plane on which the group  $\mathrm{SL}(2, \mathbb{R})$  acts as usual by linear fractional transformations. Thus we may write

$$\gamma z = \frac{az + b}{cz + d}$$

for all  $z \in \mathcal{H}$  and  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ . For the same  $z$  and  $\gamma$ , we set

$$\mathfrak{J}(\gamma, z) = cz + d, \quad \mathfrak{K}(\gamma, z) = \frac{c}{cz + d}. \quad (2.1)$$

The resulting maps  $\mathfrak{J}, \mathfrak{K} : \mathrm{SL}(2, \mathbb{R}) \times \mathcal{H} \rightarrow \mathbb{C}$  can be shown to satisfy

$$\mathfrak{J}(\gamma\gamma', z) = \mathfrak{J}(\gamma, \gamma'z)\mathfrak{J}(\gamma', z), \quad (2.2)$$

$$\mathfrak{K}(\gamma\gamma', z) = \mathfrak{J}(\gamma', z)^{-2}\mathfrak{K}(\gamma, \gamma'z) + \mathfrak{K}(\gamma', z) \quad (2.3)$$

for all  $\gamma, \gamma' \in \mathrm{SL}(2, \mathbb{R})$  and  $z \in \mathcal{H}$ .

Let  $R$  be the ring of holomorphic functions on  $\mathcal{H}$ . We fix a nonnegative integer  $m$  and denote by  $R_m[X]$  the complex vector space of polynomials in  $X$  over  $R$  of degree at most  $m$ . If a polynomial  $\Phi(z, X) \in R_m[X]$  is of the form

$$\Phi(z, X) = \sum_{r=0}^m \phi_r(z)X^r \quad (2.4)$$

and  $\lambda$  is an integer with  $\lambda > 2m$ , we introduce two additional polynomials

$$(\Xi_\lambda^m \Phi)(z), (\Lambda_\lambda^m \Phi)(z) \in R_m[X]$$

defined by

$$(\Xi_\lambda^m \Phi)(z, X) = \sum_{r=0}^m \phi_r^\Xi(z)X^r, \quad (\Lambda_\lambda^m \Phi)(z, X) = \sum_{r=0}^m \phi_r^\Lambda(z)X^r \quad (2.5)$$

where

$$\phi_r^\Xi = \frac{1}{r!} \sum_{j=0}^{m-r} \frac{1}{j!(\lambda - 2r - j - 1)!} \phi_{m-r-j}^{(j)} \tag{2.6}$$

$$\begin{aligned} \phi_r^\Lambda &= (\lambda + 2r - 2m - 1) \sum_{j=0}^r \frac{(-1)^j}{j!} (m - r + j)! \\ &\quad \times (2r + \lambda - 2m - j - 2)! \phi_{m-r+j}^{(j)}, \end{aligned} \tag{2.7}$$

for each  $r \in \{0, 1, \dots, m\}$ . These formulas determine isomorphisms

$$\Lambda_\lambda^m, \Xi_\lambda^m : R_m[X] \xrightarrow{\sim} R_m[X] \tag{2.8}$$

with

$$(\Lambda_\lambda^m)^{-1} = \Xi_\lambda^m$$

(see [6]).

Given  $\gamma \in \text{SL}(2, \mathbb{R})$ ,  $\lambda \in \mathbb{Z}$ ,  $f \in R$  and  $\Phi(z, X) \in R_m[X]$  as in (2.4), we set

$$\begin{aligned} (f \mid_\lambda \gamma)(z) &= \mathfrak{J}(\gamma, z)^{-\lambda} f(\gamma z), \\ (\Phi \mid_\lambda^X \gamma)(z, X) &= \sum_{r=0}^m (\phi_r \mid_{\lambda+2r} \gamma)(z) X^r, \end{aligned} \tag{2.9}$$

$$(\Phi \parallel_\lambda \gamma)(z, X) = \mathfrak{J}(\gamma, z)^{-\lambda} \Phi(\gamma z, \mathfrak{J}(\gamma, z)^2(X - \mathfrak{K}(\gamma, z))) \tag{2.10}$$

for all  $z \in \mathcal{H}$ . Using (2.2) and (2.3), it can be shown that the operations  $\mid_\lambda$ ,  $\mid_\lambda^X$  and  $\parallel_\lambda$  determine right actions of  $\text{SL}(2, \mathbb{R})$  on  $R$  for the first one and on  $R_m[X]$  for the other two. Furthermore, the two actions on  $R_m[X]$  are compatible in such a way that

$$\begin{aligned} ((\Xi_\lambda^m \Phi) \parallel_\lambda \gamma)(z, X) &= \Xi_\lambda^m (\Phi \mid_{\lambda-2m}^X \gamma)(z, X), \\ ((\Lambda_\lambda^m \Phi) \mid_{\lambda-2m}^X \gamma)(z, X) &= \Lambda_\lambda^m (\Phi \parallel_\lambda \gamma)(z, X) \end{aligned}$$

for all  $\gamma \in \text{SL}(2, \mathbb{R})$ , where  $\Xi_\lambda^m$  and  $\Lambda_\lambda^m$  are the isomorphisms in (2.8) (cf. [6]).

We now fix a discrete subgroup  $\Gamma$  of  $\text{SL}(2, \mathbb{R})$  and consider the restrictions of the  $\text{SL}(2, \mathbb{R})$ -actions described above to  $\Gamma$ .

**Definition 2.1.** (i) An element  $f \in R$  is a *modular form for  $\Gamma$  of weight  $\lambda$*  if it satisfies

$$f \mid_\lambda \gamma = f$$

for all  $\gamma \in \Gamma$ .

(ii) A *modular polynomial* for  $\Gamma$  of weight  $\lambda$  and degree at most  $m$  is an element  $F(z, X) \in R_m[X]$  satisfying

$$F \mid_{\lambda}^X \gamma = F$$

for all  $\gamma \in \Gamma$ .

(iii) An element  $\Phi(z, X) \in R_m[X]$  is a *quasimodular polynomial* for  $\Gamma$  of weight  $\lambda$  and degree at most  $m$  if it satisfies

$$\Phi \parallel_{\lambda} \gamma = \Phi$$

for all  $\gamma \in \Gamma$ .

We denote by  $M_{\lambda}(\Gamma)$  the space of modular forms for  $\Gamma$  of weight  $\lambda$  and by  $MP_{\lambda}^m(\Gamma)$  and  $QP_{\lambda}^m(\Gamma)$  the spaces of modular polynomials and quasimodular polynomials, respectively, for  $\Gamma$  of weight  $\lambda$  and degree at most  $m$ . From (2.8) we see that the maps  $\Xi_{\lambda}^m$  and  $\Lambda_{\lambda}^m$  induce the isomorphisms

$$\Xi_{\lambda}^m : MP_{\lambda-2m}^m(\Gamma) \rightarrow QP_{\lambda}^m(\Gamma), \quad \Lambda_{\lambda}^m : QP_{\lambda}^m(\Gamma) \rightarrow MP_{\lambda-2m}^m(\Gamma) \quad (2.11)$$

for each  $\lambda \in \mathbb{Z}$  with  $\lambda > 2m$ .

**Definition 2.2.** Given an integer  $\lambda$ , an element  $\phi \in R$  is a *quasimodular form* for  $\Gamma$  of weight  $\lambda$  and depth at most  $m$  if there are functions  $\phi_0, \phi_1, \dots, \phi_m \in R$  satisfying

$$(\phi \mid_{\lambda} \gamma)(z) = \sum_{r=0}^m \phi_r(z) \mathfrak{K}(\gamma, z)^r \quad (2.12)$$

for all  $z \in \mathcal{H}$  and  $\gamma \in \Gamma$ , where  $\mathfrak{K}(\gamma, z)$  is as in (2.1). We denote by  $QM_{\lambda}^m(\Gamma)$  the space of quasimodular forms for  $\Gamma$  of weight  $\lambda$  and depth at most  $m$ .

As is well-known, the derivative of a quasimodular form is a quasimodular form. Indeed, if  $\phi \in QM_{\lambda}^m(\Gamma)$  satisfies (2.12), we have

$$(\phi' \mid_{\lambda+2} \gamma)(z) = \sum_{k=0}^{m+1} \psi_k(z) \mathfrak{K}(\gamma, z)^k, \quad (2.13)$$

where

$$\psi_0 = \phi'_0, \quad \psi_{m+1} = (\lambda - m)\phi_m, \quad \psi_k = (\lambda - k + 1)\phi_{k-1} + \phi'_k$$

for  $1 \leq k \leq m$ . It is also known that for  $0 \leq k \leq m$  the function  $\phi_k$  is a quasimodular form belonging to  $QM_{\lambda-2k}^{m-k}(\Gamma)$  (see e.g. [13]). In particular, since quasimodular forms of depth 0 are modular forms,  $\phi_m$  is a modular

form belonging to  $M_{\lambda-2m}(\Gamma)$ . We define the polynomial  $(\mathcal{Q}_\lambda^m \phi)(z, X) \in R_m[X]$  associated to  $\phi$  by

$$(\mathcal{Q}_\lambda^m \phi)(z, X) = \sum_{r=0}^m \phi_r(z) X^r. \tag{2.14}$$

Note that  $\phi$  determines the functions  $\phi_r$  uniquely, and therefore  $\mathcal{Q}_\lambda^m \phi$  is well-defined. Furthermore, it can be shown that the formula (2.14) determines the isomorphism

$$\mathcal{Q}_\lambda^m : QM_\lambda^m(\Gamma) \rightarrow QP_\lambda^m(\Gamma) \tag{2.15}$$

for each  $\lambda \in \mathbb{Z}$  whose inverse is given by

$$(\mathcal{Q}_\lambda^m)^{-1}(\Phi(z, X)) = \Phi(z, 0) \tag{2.16}$$

for  $\Phi(z, X) \in QP_\lambda^m(\Gamma)$  (cf. [1]). For  $0 \leq k \leq m$  we define the projection map

$$\Pi_k^m : R_m[X] \rightarrow R \tag{2.17}$$

by setting

$$\Pi_k^m \Phi = \phi_k$$

for  $\Phi(z, X)$  as in (2.4). Then it induces the maps

$$\Pi_k^m : MP_\lambda^m(\Gamma) \rightarrow M_{\lambda+2k}(\Gamma), \quad \Pi_k^m : QP_\lambda^m(\Gamma) \rightarrow QM_{\lambda-2k}^{m-k}(\Gamma).$$

### 3. Vector bundles and Poincaré series

In this section we construct vector bundles over the quotient space  $\Gamma \backslash \mathcal{H}$  whose sections may be identified with quasimodular polynomials and hence with quasimodular forms for  $\Gamma$ . We also introduce Poincaré series for quasimodular forms.

In order to describe vector bundles constructed in [7], we fix integers  $m$  and  $\lambda$  with  $m \geq 0$  as in Section 2, and for integers  $k, r \in \mathbb{Z}$  with  $0 \leq k \leq r \leq m$  consider a map

$$\Xi_r^{\lambda, k} : \mathrm{SL}(2, \mathbb{R}) \times \mathcal{H} \rightarrow \mathbb{C}$$

defined by

$$\Xi_r^{\lambda, k}(\gamma, z) = \binom{k}{r} \mathfrak{J}(\gamma, z)^{\lambda-2r} \mathfrak{K}(\gamma, z)^{k-r}$$

for  $\gamma \in \mathrm{SL}(2, \mathbb{R})$  and  $z \in \mathcal{H}$ , where  $\mathfrak{J}$  and  $\mathfrak{K}$  are as in (2.1). Then it satisfies

$$\Xi_r^{\lambda, k}(\gamma\gamma', z) = \sum_{\ell=r}^k \Xi_r^{\lambda, \ell}(\gamma, \gamma'z) \Xi_\ell^{\lambda, k}(\gamma', z)$$

for all  $\gamma, \gamma' \in \mathrm{SL}(2, \mathbb{R})$  and  $z \in \mathcal{H}$ . We denote by  $\mathbb{C}_m[X]$  the ring of polynomials in  $X$  over  $\mathbb{C}$  of degree at most  $m$ . Given a polynomial of the form

$$F(X) = \sum_{r=0}^m c_r X^r \in \mathbb{C}_m[X]$$

with  $c_0, \dots, c_m \in \mathbb{C}$  and an integer  $\lambda$ , we set

$$\gamma \odot_\lambda^m (z, F(X)) = \left( \gamma z, \sum_{r=0}^m \sum_{k=r}^m c_k \Xi_r^{\lambda, k}(\gamma, z) X^r \right). \quad (3.1)$$

for all  $\gamma \in \mathrm{SL}(2, \mathbb{R})$  and  $z \in \mathcal{H}$ . Then it can be shown that the formula (3.1) determines a left action of  $\mathrm{SL}(2, \mathbb{R})$  on the Cartesian product  $\mathcal{H} \times \mathbb{C}_m[X]$ .

Let  $\Gamma$  be a discrete subgroup of  $\mathrm{SL}(2, \mathbb{R})$ , and set

$$[\mathcal{V}]_\lambda^m = \Gamma \backslash \mathcal{H} \times \mathbb{C}_m[X], \quad (3.2)$$

where the quotient is taken with respect to the action given by (3.1). If we denote the modular curve associated to  $\Gamma$  by  $U = \Gamma \backslash \mathcal{H}$ , then the natural projection map  $\mathcal{H} \times \mathbb{C}_m[X] \rightarrow \mathcal{H}$  induces a surjective map  $\varpi : [\mathcal{V}]_\lambda^m \rightarrow U$  such that  $\varpi^{-1}(x)$  is isomorphic to  $\mathbb{C}_m[X]$  for each  $x \in U$ . Thus  $[\mathcal{V}]_\lambda^m$  has the structure of a complex vector bundle over  $U$  whose fiber is the  $(m+1)$ -dimensional complex vector space  $\mathbb{C}_m[X]$  of polynomials in  $X$ . We denote by  $\mathbf{\Gamma}_0(U, [\mathcal{V}]_\lambda^m)$  the space of all holomorphic sections of  $[\mathcal{V}]_\lambda^m$  over  $U$ .

**Theorem 3.1.** *The space  $\mathbf{\Gamma}_0(U, [\mathcal{V}]_\lambda^m)$  of holomorphic sections of  $[\mathcal{V}]_\lambda^m$  over  $U = \Gamma \backslash \mathcal{H}$  is canonically isomorphic to the space  $QP_\lambda^m(\Gamma)$  of all quasimodular polynomials for  $\Gamma$  of weight  $\lambda$  and depth at most  $m$ .*

*Proof.* See [7]. □

If  $m = 0$ , then the bundle  $[\mathcal{V}]_\lambda^0$  becomes a line bundle and we obtain the isomorphism

$$\mathbf{\Gamma}_0(U, [\mathcal{V}]_\lambda^m) \cong M_\lambda(\Gamma)$$

for each  $\lambda$ , which provides us the usual identification between modular forms and holomorphic sections of a line bundle.



Given a polynomial  $F(z, X) \in R_m[X]$  of the form

$$F(z, X) = \sum_{r=0}^m f_r(z)X^r$$

with  $f_0, \dots, f_m \in R$ , we set

$$(\Delta_p F)(z, X) = \sum_{r=0}^{m-p} \binom{r+p}{p} f_{r+p}(z)X^r.$$

for each integer  $p$  with  $0 \leq p \leq m$ , so that we obtain the complex linear map

$$\Delta_p : R_m[X] \rightarrow R_{m-p}[X], \tag{3.3}$$

which satisfies

$$\Delta_p(QP_p^m(\Gamma)) \subset QP_{\lambda-2p}^{m-p}(\Gamma).$$

Given  $p \in \{0, 1, \dots, m\}$ , we now define the map

$$\tilde{\Delta}_p : \mathcal{H} \times \mathbb{C}_m[X] \rightarrow \mathcal{H} \times \mathbb{C}_{m-p}[X] \tag{3.4}$$

by

$$\tilde{\Delta}_p(z, f(X)) = (z, \Delta_p f(X))$$

for all  $f(X) \in \mathbb{C}_m[X]$ , where  $\Delta_p : \mathbb{C}_m[X] \rightarrow \mathbb{C}_{m-p}[X]$  is the map obtained from (3.3) by restriction. We consider the vector bundles

$$[\mathcal{V}]_\lambda^m = \Gamma \backslash \mathcal{H} \times \mathbb{C}_m[X], \quad [\mathcal{V}]_{\lambda-2p}^{m-p} = \Gamma \backslash \mathcal{H} \times \mathbb{C}_{m-p}[X], \tag{3.5}$$

where the first bundle is as in (3.2) and the second quotient is with respect to the operation  $\odot_{\lambda-2p}^{m-p}$  in (3.1) of  $\Gamma$  on  $\mathcal{H} \times \mathbb{C}_{m-p}[X]$ . Then the map  $\tilde{\Delta}_p$  in (3.4) induces a morphism

$$[\mathcal{V}]_\lambda^m \rightarrow [\mathcal{V}]_{\lambda-2p}^{m-p} \tag{3.6}$$

of vector bundles in (3.5) over  $U = \Gamma \backslash \mathcal{H}$ . In particular, if  $p = m$  in (3.6), then we obtain the morphism

$$[\mathcal{V}]_\lambda^m \rightarrow [\mathcal{V}]_{\lambda-2m}^0$$

from a vector bundle to a line bundle, where the holomorphic sections of the line bundle  $[\mathcal{V}]_{\lambda-2m}^0$  can be identified with modular forms.

In order to discuss Poincaré series introduced in [6], we now assume that  $x$  is a cusp of  $\Gamma$ , so that there is an element  $\sigma \in \mathrm{SL}(2, \mathbb{R})$  such that

$$\sigma \Gamma_x \sigma^{-1} \cdot \{\pm 1\} = \left\{ \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^n \mid n \in \mathbb{Z} \right\} \tag{3.7}$$

for some positive real number  $h$ . Given integers  $w \geq 3$  and  $u \geq 0$ , we set

$$\mathcal{P}_{w,u}^x(z) = \sum_{\gamma \in \Gamma_x \backslash \Gamma} \mathfrak{J}(\sigma\gamma, z)^{-w} \mathbf{e}^{u/h}(\sigma\gamma z) = (\mathbf{e}^{u/h} |_{2(\xi-m+r)} \sigma\gamma)(z) \quad (3.8)$$

for all  $z \in \mathcal{H}$ , where  $\mathbf{e}^\mu(\cdot) = \exp(2\pi i\mu(\cdot))$  for  $\mu \in \mathbb{C}$ . Then it is well-known that the series in (3.8) converges absolutely and uniformly on any compact subset of  $\mathcal{H}$ , and the resulting function  $\mathcal{P}_{w,u}^x : \mathcal{H} \rightarrow \mathbb{C}$  is a Poincaré series for modular forms belonging to  $M_w(\Gamma)$ .

If  $\alpha, u \in \mathbb{Z}$ , we set

$$\eta_{\alpha,u}(z) = \mathfrak{J}(\sigma, z)^{-\alpha} \mathbf{e}^{u/h}(\sigma z)$$

for all  $z \in \mathcal{H}$ , where  $h$  is as in (3.7). Then, using (2.2), we have

$$\begin{aligned} (\eta_{\alpha,u} |_{\alpha} \gamma)(z) &= \mathfrak{J}(\gamma, z)^{-\alpha} \mathfrak{J}(\sigma, \gamma z)^{-\alpha} \mathbf{e}^{u/h}(\sigma\gamma z) \\ &= \mathfrak{J}(\sigma\gamma, z)^{-\alpha} \mathbf{e}^{u/h}(\sigma\gamma z) \end{aligned}$$

Thus the Poincaré series (3.7) can be written in the form

$$\mathcal{P}_{w,u}^x(z) = \sum_{\gamma \in \Gamma_x \backslash \Gamma} (\eta_{\alpha,u} |_{\alpha} \gamma)(z)$$

Given  $\xi \in \mathbb{Z}$ , we consider the polynomial

$$G_{\xi,u}(z, X) = \sum_{r=0}^m \eta_{2(\xi-m+r),u}(z) X^r \in \mathcal{F}_m[X],$$

and set

$$\widehat{\mathcal{P}}_{2\xi,u}^x(z, X) = \sum_{\gamma \in \Gamma_x \backslash \Gamma} ((\Xi_{2\xi}^m G_{\xi,u}) |_{2\xi-2m} \gamma)(z, X) \quad (3.9)$$

for  $z \in \mathcal{H}$ , where  $x$  is a cusp of  $\Gamma$  as above and  $\Xi_{2\xi}^m$  is as in (2.5).

**Theorem 3.2.** (i) *The series  $\widehat{\mathcal{P}}_{2\xi,u}^x(z, X)$  given by (3.9) is a quasimodular polynomial belonging to  $QP_{2\xi}^m(\Gamma)$ .*

(ii) *The series  $\widehat{\mathcal{P}}_{2\xi,u}^x(z, X)$  can be written in the form*

$$\begin{aligned} \widehat{\mathcal{P}}_{2\xi,u}^x(z, X) &= \sum_{\gamma \in \Gamma_x \backslash \Gamma} \sum_{r=0}^m \sum_{\ell=0}^{m-r} \sum_{j=0}^{\ell} \frac{(-1)^{\ell-j} (2\pi i u)^j \ell!}{h^j j! r! (2\xi - 2m - \ell - 1)!} \\ &\quad \times \binom{2\xi - 2r - \ell - 1}{\ell - j} \frac{\mathfrak{K}(\sigma\gamma, z)^{\ell-j}}{\mathfrak{J}(\sigma\gamma, z)^{2\xi-2r-2\ell+2j}} \mathbf{e}^{u/h}(\sigma\gamma z) X^r. \end{aligned} \quad (3.10)$$

(iii) The function  $\widehat{\mathcal{P}}_{2\xi, u}^{x, 0} \in \mathcal{F}$  given by

$$\widehat{\mathcal{P}}_{2\xi, u}^{x, 0}(z) = \sum_{\gamma \in \Gamma_x \setminus \Gamma} \sum_{\ell=0}^m \sum_{j=0}^{\ell} \frac{(-1)^{\ell-j} (2\pi i u)^{j\ell!}}{h^j j! (2\xi - 2m - \ell - 1)!} \\ \times \binom{2\xi - \ell - 1}{\ell - j} \frac{\mathfrak{K}(\sigma\gamma, z)^{\ell-j}}{\mathfrak{J}(\sigma\gamma, z)^{2\xi - 2r - 2\ell + 2j}} e^{u/h(\sigma\gamma z)}$$

for  $z \in \mathcal{H}$  is a quasimodular form belonging to  $QM_{2\xi}^m(\Gamma)$ .

*Proof.* See [6]. □

#### 4. Differential operators on quasimodular polynomials

We first discuss connections of quasimodular forms with Jacobi-like forms. Let  $R$  be the ring of holomorphic functions on  $\mathcal{H}$  as before, and let  $R[[X]]$  be the complex algebra of formal power series in  $X$  with coefficients in  $R$ . If  $\delta$  is an integer, we set

$$R[[X]]_{\delta} = X^{\delta} R[[X]],$$

so that an element  $\Phi(z, X) \in R[[X]]_{\delta}$  can be written in the form

$$\Phi(z, X) = \sum_{k=0}^{\infty} \phi_k(z) X^{k+\delta}$$

with  $\phi_k \in R$  for each  $k \geq 0$ . Thus, if we allow  $\delta$  to be negative, elements of  $R[[X]]_{\delta}$  may be regarded as formal Laurent series in  $X$ . Given an element  $\Phi(z, X) \in R[[X]]_{\delta}$ , a polynomial  $F(z, X) \in R_m[X]$  and an integer  $\lambda$ , we set

$$(\Phi |_{\lambda}^J \gamma)(z, X) = \mathfrak{J}(\gamma, z)^{-\lambda} e^{-\mathfrak{R}(\gamma, z)X} \Phi(\gamma z, \mathfrak{J}(\gamma, z)^{-2} X) \quad (4.1)$$

for all  $\gamma \in \mathrm{SL}(2, \mathbb{R})$  and  $z \in \mathcal{H}$ . Then it can be shown that

$$\Phi |_{\lambda}^J (\gamma\gamma') = (\Phi |_{\lambda}^J \gamma) |_{\lambda}^J \gamma'$$

for all  $\gamma, \gamma' \in \mathrm{SL}(2, \mathbb{R})$ ; hence the operator  $|_{\lambda}^J$  determines a right action of  $\mathrm{SL}(2, \mathbb{R})$  on  $R[[X]]_{\delta}$ .

We consider the surjective map

$$\Pi_m^{\delta} : R[[X]]_{\delta} \rightarrow R_m[X]$$

with  $\delta \in \mathbb{Z}$  defined by

$$(\Pi_m^\delta \Phi)(z, X) = \sum_{r=0}^m \frac{1}{r!} \phi_{m-r}(z) X^r \quad (4.2)$$

for an element  $\Phi(z, X) \in R[[X]]_\delta$  of the form

$$\Phi(z, X) = \sum_{k=0}^{\infty} \phi_k(z) X^{k+\delta}.$$

This map is  $\mathrm{SL}(2, \mathbb{R})$ -equivariant with respect to the operations in (2.10) and (4.1). More precisely, given  $\Phi(z, X) \in R[[X]]_\delta$  and  $\lambda \in \mathbb{Z}$ , we have

$$\Pi_m^\delta(\Phi |_\lambda^J \gamma) = \Pi_m^\delta(\Phi) \parallel_{\lambda+2m+2\delta} \gamma \quad (4.3)$$

for all  $\gamma \in \mathrm{SL}(2, \mathbb{R})$  (see [1])

Given  $\nu \in \mathbb{Z}$ , we now introduce the formal differential operators

$$\mathcal{D}_\nu : R[[X]] \rightarrow R[[X]], \quad \widehat{\mathcal{D}}_\nu : R_m[X] \rightarrow R_{m+1}[X]$$

defined by

$$\begin{aligned} \mathcal{D}_\nu &= \frac{\partial}{\partial z} - \nu \frac{\partial}{\partial X} - X \frac{\partial^2}{\partial X^2}, \\ \widehat{\mathcal{D}}_\nu &= \frac{\partial}{\partial z} + X \left( \nu - X \frac{\partial}{\partial X} \right). \end{aligned} \quad (4.4)$$

It was noted in [8] that operators of the form  $\mathcal{D}_\nu$  correspond to heat operators on Jacobi forms considered by Eichler and Zagier in [3]. Thus  $\mathcal{D}_\nu$  may be regarded as a heat operator on formal Laurent series, and it satisfies

$$(\mathcal{D}_\nu(\Phi) |_{\lambda+2}^J \gamma)(z, X) = \mathcal{D}_\nu(\Phi |_\lambda^J \gamma)(z, X) + (\lambda - \nu) \mathfrak{K}(\gamma, z) (\Phi |_\lambda^J \gamma)(z, X)$$

for all  $\gamma \in \mathrm{SL}(2, \mathbb{R})$  and  $z \in \mathcal{H}$ . In particular, we obtain

$$(\mathcal{D}_\lambda(\Phi) |_{\lambda+2}^J \gamma)(z, X) = \mathcal{D}_\lambda(\Phi |_\lambda^J \gamma)(z, X). \quad (4.5)$$

**Definition 4.1.** A formal Laurent series  $\Phi(z, X) \in R[[X]]_\delta$  with  $\delta \in \mathbb{Z}$  is a *Jacobi-like form of weight  $\xi$*  for  $\Gamma$  if it satisfies

$$(\Phi |_\xi^J \gamma)(z, X) = \Phi(z, X)$$

for all  $z \in \mathcal{H}$  and  $\gamma \in \Gamma$ . We denote by  $\mathcal{J}_\xi(\Gamma)_\delta$  the space of such Jacobi-like forms.

If  $\phi \in QM_\lambda^m(\Gamma)$ , then from (2.13) we see that  $\phi' \in QM_{\lambda+2}^{m+1}(\Gamma)$ ; hence we obtain the derivative operator

$$\partial : QM_\lambda^m(\Gamma) \rightarrow QM_{\lambda+2}^{m+1}(\Gamma),$$

which can be shown to be compatible with the map  $\widehat{\mathcal{D}}_\xi$  in (4.4) with  $\xi \in \mathbb{Z}$  in such a way that

$$\mathcal{Q}_{\xi+2}^{m+1} \circ \partial = \widehat{\mathcal{D}}_\xi \circ \mathcal{Q}_\xi^m. \quad (4.6)$$

This relation provides us with the complex linear map

$$\widehat{\mathcal{D}}_\xi = \mathcal{Q}_{\xi+2}^{m+1} \circ \partial \circ (\mathcal{Q}_\xi^m)^{-1} : QP_\xi^m(\Gamma) \rightarrow QP_{\xi+2}^{m+1}(\Gamma).$$

On the other hand, using (4.3) and (4.5), we see that

$$\mathcal{D}_\lambda(\mathcal{J}_\lambda(\Gamma)) \subset \mathcal{J}_{\lambda+2}(\Gamma), \quad \Pi_m^\delta(\mathcal{J}_\lambda(\Gamma)_\delta) \subset QP_{\lambda+2m+2\delta}^m(\Gamma);$$

hence we obtain the two additional complex linear maps

$$\Pi_m^{\delta,\lambda} : \mathcal{J}_\lambda(\Gamma)_\delta \rightarrow QP_{\lambda+2m+2\delta}^m(\Gamma), \quad \mathcal{D}_\lambda : \mathcal{J}_\lambda(\Gamma)_\delta \rightarrow \mathcal{J}_{\lambda+2}(\Gamma)_{\delta-1} \quad (4.7)$$

for each  $\lambda \in \mathbb{Z}$ , where  $\Pi_m^{\delta,\lambda}$  is the restriction of  $\Pi_m^\delta$  to  $\mathcal{J}_\lambda(\Gamma)_\delta$ .

**Theorem 4.2.** *Given  $\lambda, \delta \in \mathbb{Z}$ , the diagram*

$$\begin{array}{ccc} \mathcal{J}_\lambda(\Gamma)_\delta & \xrightarrow{\Pi_m^\delta} & QP_{\lambda+2m+2\delta}^m(\Gamma) \\ \mathcal{D}_\lambda \downarrow & & \downarrow \widehat{\mathcal{D}}_{\lambda+2m+2\delta} \\ \mathcal{J}_{\lambda+2}(\Gamma)_{\delta-1} & \xrightarrow{\Pi_{m+1}^{\delta-1}} & QP_{\lambda+2m+2\delta+2}^{m+1}(\Gamma) \end{array} \quad (4.8)$$

*commutes if and only if  $\delta = -m - 1$  or  $\delta = -m - \lambda$ .*

*Proof.* See [9]. □

If  $\Pi_m^{\delta,\lambda}$  is the map in (4.7), we consider the corresponding linear map

$$\widehat{\Pi}_m^{\delta,\lambda} : \mathcal{J}_\lambda(\Gamma)_\delta \rightarrow QM_{\lambda+2m+2\delta}^m(\Gamma)$$

defined by

$$\widehat{\Pi}_m^{\delta,\lambda} = (\mathcal{Q}_{\lambda+2m+2\delta}^m)^{-1} \circ \Pi_m^{\delta,\lambda},$$

where  $\mathcal{Q}_{\lambda+2m+2\delta}^m$  is as in (2.15). Then from (2.16) and (4.2) we see that

$$(\widehat{\Pi}_m^{\delta,\lambda}\Phi)(z) = \phi_m(z)$$

for  $\Phi(z) = \sum_{k=0}^{\infty} \phi_k(z)X^{k+\delta} \in \mathcal{J}_\lambda(\Gamma)_\delta$ , and it follows from Theorem 4.2 that the diagram

$$\begin{array}{ccc} \mathcal{J}_\lambda(\Gamma)_\delta & \xrightarrow{\widehat{\Pi}_m^{\delta,\lambda}} & QM_{\lambda+2m+2\delta}^m(\Gamma) \\ \mathcal{D}_\lambda \downarrow & & \downarrow \partial \\ \mathcal{J}_{\lambda+2}(\Gamma)_{\delta-1} & \xrightarrow{\widehat{\Pi}_m^{\delta-1,\lambda+2}} & QM_{\lambda+2m+2\delta+2}^{m+1}(\Gamma) \end{array}$$

is commutative if  $\delta = -m - 1$  or  $\delta = -m - \lambda$ .

We now denote by  $\mathfrak{S}$  the set of sequences  $\sigma = \{\sigma_i\}_{i=0}^{\infty}$  of complex numbers with  $\sigma_i = 0$  for sufficiently large  $i$ . Given a nonnegative integer  $\nu$  and an element  $\sigma = \{\sigma_i\}_{i=0}^{\infty} \in \mathfrak{S}$ , we define the complex linear maps

$$\mathcal{E}_m^{\sigma,\nu} : R_m[X] \rightarrow R_{m+\nu}[X], \quad \mathcal{P}_m^{\sigma,\nu} : R_{m+\nu}[X] \rightarrow R_m[X] \quad (4.9)$$

by setting

$$(\mathcal{E}_m^{\sigma,\nu}\Phi)(z, X) = \sum_{r=0}^{m+\nu} \sigma_r \phi_r(z)X^r, \quad (4.10)$$

$$(\mathcal{P}_m^{\sigma,\nu}\Psi)(z, X) = \sum_{r=0}^m \sigma_r \psi_r(z)X^r \quad (4.11)$$

for

$$\Phi(z, X) = \sum_{r=0}^m \phi_r(z)X^r \in R_m[X], \quad \Psi(z, X) = \sum_{r=0}^{m+\nu} \psi_r(z)X^r \in R_{m+\nu}[X], \quad (4.12)$$

where  $\phi_r = 0$  for  $m < r \leq m + \nu$ . For example, if  $\sigma_i = 1$  for each  $i \in \{0, 1, \dots, m + \nu\}$ , the map  $\mathcal{E}_m^{\sigma,\nu}$  is a natural embedding, while  $\mathcal{P}_m^{\sigma,\nu}$  is a natural projection. If the operator  $|_\lambda^X$  is as in (2.9), we see easily that

$$\mathcal{E}_m^{\sigma,\nu}(\Phi |_\lambda^X \gamma) = (\mathcal{E}_m^{\sigma,\nu}\Phi) |_\lambda^X \gamma, \quad \mathcal{P}_m^{\sigma,\nu}(\Psi |_\lambda^X \gamma) = (\mathcal{P}_m^{\sigma,\nu}\Psi) |_\lambda^X \gamma$$

for all  $\gamma \in \text{SL}(2, \mathbb{R})$ . We also introduce another pair of complex linear maps

$$\mathcal{A}_{m,\lambda}^{\sigma,\nu} : R_m[X] \rightarrow R_{m+\nu}[X], \quad \mathcal{B}_{m,\lambda}^{\sigma,\nu} : R_{m+\nu}[X] \rightarrow R_m[X] \quad (4.13)$$

with  $\lambda \in \mathbb{Z}$  defined by

$$(\mathcal{A}_{m,\lambda}^{\sigma,\nu}\Phi)(z, X) = \sum_{u=0}^{m+\nu} \sum_{\ell=0}^{m+\nu-u} \sum_{j=0}^{\ell} (-1)^{\ell-j} (\lambda + 2\nu - 2u - 2j - 1) \\ \times \frac{(u + \ell - \nu)! (\lambda + 2\nu - 2u - j - \ell - 2)!}{j! u! (\ell - j)! (\lambda + 2\nu - 2u - j - 1)!} \sigma_{m+\nu-u-j} \phi_{u+\ell-\nu}^{(\ell)} X^u,$$

$$(\mathcal{B}_{m,\lambda}^{\sigma,\nu}\Psi)(z, X) \\ = \sum_{v=0}^m \sum_{\ell=0}^{m-v} \sum_{j=0}^{\ell} (-1)^{\ell-j} (\lambda - 2v - 2j - 2\nu - 1) \\ \times \frac{(\nu + v + \ell)! (\lambda - 2v - j - 2\nu - \ell - 2)!}{j! v! (\ell - j)! (\lambda - 2\nu - 2v - j - 1)!} \sigma_{m-v-j} \psi_{v+\ell+\nu}^{(\ell)} X^v$$

for  $\Phi(z, X)$  and  $\Psi(z, X)$  as in (4.12), respectively, with  $\phi_\ell$  being zero for  $\ell > m$ .

**Theorem 4.3.** (i) Given  $\sigma = \{\sigma_i\}_{i=0}^\infty \in \mathfrak{S}$  and  $\lambda \geq 2m + 2\nu + 2$ , we have

$$\mathcal{A}_{m,\lambda}^{\sigma,\nu} \circ \Xi_\lambda^m = \Xi_{\lambda+2m}^{m+\nu} \circ \mathcal{E}_m^{\sigma,\nu}, \quad \Xi_{\lambda-2\nu}^m \circ \mathcal{P}_m^{\sigma,\nu} = \mathcal{B}_{m,\lambda}^{\sigma,\nu} \circ \Xi_\lambda^{m+\nu},$$

where  $\Xi_\lambda^m$  and  $\Xi_{\lambda-2m}^{m+\nu}$  are as in (2.5).

(ii) The maps  $\mathcal{A}_{m,\lambda}^{\sigma,\nu}$  and  $\mathcal{B}_{m,\lambda}^{\sigma,\nu}$  are  $\mathrm{SL}(2, \mathbb{R})$ -equivariant in such a way that

$$\mathcal{A}_{m,\lambda}^{\sigma,\nu}(\Phi \parallel_\lambda \gamma) = (\mathcal{A}_{m,\lambda}^{\sigma,\nu}\Phi) \parallel_{\lambda+2\nu} \gamma, \quad \mathcal{B}_{m,\lambda}^{\sigma,\nu}(\Psi \parallel_\lambda \gamma) = (\mathcal{B}_{m,\lambda}^{\sigma,\nu}\Psi) \parallel_{\lambda-2\nu} \gamma$$

for all  $\Phi(z, X) \in R_m[X]$ ,  $\Psi(z, X) \in R_{m+\nu}[X]$  and  $\gamma \in \mathrm{SL}(2, \mathbb{R})$ .

*Proof.* See [10]. □

From this theorem it follows that the maps  $\mathcal{E}_m^{\sigma,\nu}$ ,  $\mathcal{P}_m^{\sigma,\nu}$ ,  $\mathcal{A}_{m,\lambda}^{\sigma,\nu}$  and  $\mathcal{B}_{m,\lambda}^{\sigma,\nu}$  in (4.9) and (4.13) induce the commutative diagrams

$$\begin{array}{ccc} MP_{\lambda-2m}^m(\Gamma) & \xrightarrow{\mathcal{E}_m^{\sigma,\nu}} & MP_{\lambda-2m}^{m+\nu}(\Gamma) \\ \Xi_\lambda^m \downarrow & & \downarrow \Xi_{\lambda+2\nu}^{m+\nu} \\ QP_\lambda^m(\Gamma) & \xrightarrow{\mathcal{A}_{m,\lambda}^{\sigma,\nu}} & QP_{\lambda+2\nu}^{m+\nu}(\Gamma), \end{array}$$

$$\begin{array}{ccc}
 MP_{\lambda-2m-2\nu}^{m+\nu}(\Gamma) & \xrightarrow{\mathcal{P}_m^{\sigma,\nu}} & MP_{\lambda-2m-2\nu}^m(\Gamma) \\
 \Xi_\lambda^{m+\nu} \downarrow & & \downarrow \Xi_{\lambda-2\nu}^m \\
 QP_\lambda^{m+\nu}(\Gamma) & \xrightarrow{\mathcal{B}_{m,\lambda}^{\sigma,\nu}} & QP_{\lambda-2\nu}^m(\Gamma)
 \end{array} \quad (4.14)$$

for each  $\lambda \geq 2m + 2\nu + 2$ , where the vertical maps are isomorphisms in (2.11).

The formal derivative operator  $\partial_X : R_m[X] \rightarrow R_{m-1}[X]$  with respect to  $X$  can be shown to satisfy

$$\partial_X(QP_\lambda^m(\Gamma)) \subset QP_{\lambda-2}^{m-1}(\Gamma), \quad \partial_X \circ \mathcal{Q}_\lambda^m = \mathcal{Q}_{\lambda-2}^{m-1} \circ \mathcal{S}$$

for each  $\lambda \in \mathbb{Z}$ . Thus it determines the linear map

$$\partial_X : QP_\lambda^m(\Gamma) \rightarrow QP_{\lambda-2}^{m-1}(\Gamma)$$

of quasimodular polynomials. Given a positive integer  $\nu \leq m$ , by iteration we obtain the maps

$$\partial_X^\nu : QP_\lambda^m(\Gamma) \rightarrow QP_{\lambda-2\nu}^{m-\nu}(\Gamma), \quad \mathcal{S}^\nu : QM_\lambda^m(\Gamma) \rightarrow QM_{\lambda-2\nu}^{m-\nu}(\Gamma) \quad (4.15)$$

satisfying

$$\partial_X^\nu \circ \mathcal{Q}_\lambda^m = \mathcal{Q}_{\lambda-2\nu}^{m-\nu} \circ \mathcal{S}^\nu.$$

On the other hand, if  $\widehat{\mathcal{D}}_\lambda : R_m[X] \rightarrow R_{m+1}[X]$  with  $\lambda \in \mathbb{Z}$  is as in (4.4) and if  $f \in QM_\lambda^m(\Gamma)$  is as in (2.13), from (4.6) we see that

$$\widehat{\mathcal{D}}_\lambda(\mathcal{Q}_\lambda^m f)(z, X) = \sum_{k=0}^{m+1} h_k(z) X^k.$$

Given a positive integer  $\nu$ , by iteration we obtain the maps

$$\partial^\nu : QM_\lambda^m(\Gamma) \rightarrow QM_{\lambda+2\nu}^{m+\nu}(\Gamma), \quad \widehat{\mathcal{D}}_\lambda^\nu : QP_\lambda^m(\Gamma) \rightarrow QP_{\lambda+2\nu}^{m+\nu}(\Gamma)$$

satisfying

$$\widehat{\mathcal{D}}_\lambda^\nu \circ \mathcal{Q}_\lambda^m = \mathcal{Q}_{\lambda+2\nu}^{m+1} \circ \partial^\nu.$$

**Theorem 4.4.** (i) Let  $\varepsilon[m] = \{\varepsilon_i\}_{i=0}^\infty \in \mathfrak{S}$  be the sequence given by

$$\varepsilon_i = \begin{cases} 1 & \text{for } 0 \leq i \leq m; \\ 0 & \text{for } i > m. \end{cases}$$

For each positive integer  $\nu \leq m$  the map in (4.15) can be written as the composite

$$\partial_X^\nu = \mathcal{B}_{m-\nu, \lambda-2\nu+2}^{\varepsilon[m-\nu+1], 1} \circ \cdots \circ \mathcal{B}_{m-1, \lambda}^{\varepsilon[m], 1}$$



of maps in (4.14) for  $\lambda \geq 2m + 2$ .

(ii) Let  $\delta[m] = \{\delta_i\}_{i=0}^\infty \in \mathfrak{S}$  be a sequence given by

$$\delta_i = \begin{cases} (m+1-i)(m+\lambda+i) & \text{for } 0 \leq i \leq m; \\ 0 & \text{for } i \geq m. \end{cases}$$

Then for each positive integer  $\nu \leq m$  we have

$$\widehat{\mathcal{D}}_\lambda^\nu = \mathcal{A}_{\lambda+2m+2\nu-2}^{\delta[m+\nu-1],1} \circ \cdots \circ \mathcal{A}_{\lambda+2m}^{\delta[m],1}.$$

*Proof.* See [10] □

If  $\varepsilon[m]$  is as in Theorem 4.4(i), by (4.11) the map

$$\mathcal{P}_{m-1}^{\varepsilon[m],1} : MP_{\lambda-2m-2}^m(\Gamma) \rightarrow MP_{\lambda-2m-2}^{m-1}(\Gamma)$$

is simply the natural projection map

$$\sum_{r=0}^m \phi_r(z)X^r \mapsto \sum_{r=0}^{m-1} \phi_r(z)X^r,$$

and there is a commutative diagram of the form

$$\begin{array}{ccc} MP_{\lambda-2m}^m(\Gamma) & \xrightarrow{\mathcal{P}_{m-1}^{\varepsilon[m],1}} & MP_{\lambda-2m}^{m-1}(\Gamma) \\ \Xi_\lambda^m \downarrow & & \downarrow \Xi_{\lambda-2}^{m-1} \\ QP_\lambda^m(\Gamma) & \xrightarrow{\partial_X} & QP_{\lambda-2}^{m-1}(\Gamma) \end{array}$$

with  $\partial_X = \mathcal{B}_{m-1,\lambda}^{\varepsilon[m],1}$ . On the other hand, if  $\delta[m]$  is as in Theorem 4.4(ii), by (4.10) the map  $\mathcal{E}_m^{\delta[m],1} : MP_{\lambda-2m}^m(\Gamma) \rightarrow MP_{\lambda-2m}^{m+1}(\Gamma)$  is the embedding

$$\sum_{r=0}^m \phi_r(z)X^r \mapsto \sum_{r=0}^m (m+1-r)(m+\lambda+r)\phi_r(z)X^r,$$

and we obtain the commutative diagram

$$\begin{array}{ccc} MP_{\lambda-2m}^m(\Gamma) & \xrightarrow{\mathcal{E}_m^{\delta[m],1}} & MP_{\lambda-2m}^{m+1}(\Gamma) \\ \Xi_\lambda^m \downarrow & & \downarrow \Xi_{\lambda+2}^{m+1} \\ QP_\lambda^m(\Gamma) & \xrightarrow{\widehat{\mathcal{D}}_\lambda} & QP_{\lambda+2}^{m+1}(\Gamma) \end{array}$$

with  $\widehat{\mathcal{D}}_\lambda = \mathcal{A}_{m,\lambda}^{\delta[m],1}$ .

### 5. Group cohomology

In this section we discuss connections of quasimodular forms with cohomology by constructing a Hecke equivariant map from quasimodular forms to cohomology classes of the corresponding discrete group.

Given a positive integer  $n$ , let  $\{\mathbf{e}_1, \dots, \mathbf{e}_{n+1}\}$  be the standard basis for the complex vector space  $\mathbb{C}^{n+1}$ , whose elements are regarded as column vectors, and set

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^n = \sum_{k=0}^n z_1^{n-k} z_2^k \mathbf{e}_{k+1} \in \mathbb{C}^{n+1}$$

for  $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2$ . Then the  $n$ -th symmetric tensor power

$$\rho_n : \mathrm{GL}(2, \mathbb{C}) \rightarrow \mathrm{GL}(n+1, \mathbb{C})$$

of the standard representation of  $\mathrm{GL}(2, \mathbb{C})$  on  $\mathbb{C}^2$  is given by

$$\rho_n(\gamma) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^n = \left( \gamma \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right)^n$$

for all  $\gamma \in \mathrm{GL}(2, \mathbb{C})$ . We define the vector-valued function  $\mathbf{v}_n : \mathcal{H} \rightarrow \mathbb{C}^{n+1}$  on  $\mathcal{H}$  by

$$\mathbf{v}_n(z) = \begin{pmatrix} z \\ 1 \end{pmatrix}^n = \sum_{k=0}^n z^{n-k} \mathbf{e}_{k+1} = \sum_{k=0}^n z^k \mathbf{e}_{n-k+1}$$

for all  $z \in \mathcal{H}$ . Then for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$  we see that

$$\rho_n(\gamma) \mathbf{v}_n(z) = \begin{pmatrix} az + b \\ cz + d \end{pmatrix}^n = (cz + d)^n \mathbf{v}_n(\gamma z) = \mathfrak{J}(\gamma, z)^n \mathbf{v}_n(\gamma z),$$

where  $\mathfrak{J}(\gamma, z)$  is as in (2.2).

We denote by  $\mathfrak{S}^n(\mathbb{C}^2)$  the complex vector space  $\mathbb{C}^{n+1}$  equipped with the structure of a  $\mathrm{GL}(2, \mathbb{C})$ -module given by

$$(\gamma, v) \mapsto (\det \gamma)^{-n/2} \rho_n(\gamma) v$$

for  $\gamma \in \mathrm{GL}(2, \mathbb{C})$  and  $v \in \mathbb{C}^{n+1}$ . If  $\Gamma$  is a discrete subgroup of  $\mathrm{SL}(2, \mathbb{R}) \subset \mathrm{GL}(2, \mathbb{C})$  as in Section 2, its first cohomology group with coefficients in  $\mathfrak{S}^n(\mathbb{C}^2)$  can be described as follows. The set  $Z^1(\Gamma, \mathfrak{S}^n(\mathbb{C}^2))$  of 1-cocycles consists of all maps  $u : \Gamma \rightarrow \mathbb{C}^{n+1}$  satisfying

$$u(\gamma\gamma') = u(\gamma) + \rho_n(\gamma)u(\gamma')$$

for all  $\gamma, \gamma' \in \Gamma$ . Given an element  $v_0 \in \mathbb{C}^{n+1}$ , the set  $B^1(\Gamma, \mathfrak{S}^n(\mathbb{C}^2))$  of coboundaries consists of the maps  $v : \Gamma \rightarrow \mathbb{C}^{n+1}$  such that

$$v(\gamma) = (\rho_n(\gamma) - 1)v_0$$

for all  $\gamma \in \Gamma$ , where 1 is the identity map on  $\mathbb{C}^{n+1}$ . Then the first cohomology group of  $\Gamma$  with coefficients in  $\mathfrak{S}^n(\mathbb{C}^2)$  is given by

$$H^1(\Gamma, \mathfrak{S}^n(\mathbb{C}^2)) = \frac{Z^1(\Gamma, \mathfrak{S}^n(\mathbb{C}^2))}{B^1(\Gamma, \mathfrak{S}^n(\mathbb{C}^2))}. \quad (5.1)$$

We fix a nonnegative integer  $m$  and a point  $z_0 \in \mathcal{H}$  and consider a quasimodular form  $\phi \in QM_{2\nu}^m(\Gamma)$  satisfying (2.12) with  $\lambda = 2\nu$  for some integer  $\nu > m$ . If  $k$  and  $r$  are integers with  $0 \leq r \leq k \leq m$ , we set

$$\begin{aligned} \widehat{\omega}_{m,\nu}^{k,r}(\phi)(\gamma) \\ = \sum_{\ell=0}^{2(k+\nu-m-1)} \int_{z_0}^{\gamma z_0} \phi_{m-k+r}^{(r)}(z) z^\ell \mathbf{e}_{2(k+\nu-m)-\ell-1} dz \in \mathbb{C}^{2(k+\nu-m)-1} \end{aligned}$$

for all  $\gamma \in \Gamma$ , where  $\{\mathbf{e}_1, \dots, \mathbf{e}_{2(k+\nu-m)-1}\}$  denotes the standard basis for  $\mathbb{C}^{2(k+\nu-m)-1}$ . Note that the integral is independent of the choice of the path  $z_0 \rightarrow \gamma z_0$  because the functions  $\phi_k$  are holomorphic. We now define the map

$$\mathcal{L}_{m,\nu}^k(\phi) : \Gamma \rightarrow \mathbb{C}^{2(k+\nu-m)-1} \quad (5.2)$$

associated to  $\phi$  by

$$\mathcal{L}_{m,\nu}^k(\phi)(\gamma) = \sum_{r=0}^k \frac{(-1)^r}{r!} (2(k+\nu-m-1)-r)!(m-k+r)! \widehat{\omega}_{m,\nu}^{k,r}(\gamma)$$

for all  $\gamma \in \Gamma$ .

**Theorem 5.1.** *The map  $\mathcal{L}_{m,\nu}^k(\phi)$  in (5.2) is a cocycle belonging to*

$$Z^1(\Gamma, \mathfrak{S}^{2(k+\nu-m)}(\mathbb{C}^2)),$$

which induces a complex linear map

$$\mathcal{L}_{m,\nu}^k : QM_{2\nu}^m(\Gamma) \rightarrow H^1(\Gamma, \mathfrak{S}^{2(k+\nu-m-1)}(\mathbb{C}^2)) \quad (5.3)$$

sending a quasimodular form  $\phi(z, X) \in QM_{2\nu}^m(\Gamma)$  to the cohomology class of  $\mathcal{L}_{m,\nu}^k(\phi)$  in  $H^1(\Gamma, \mathfrak{S}^{2(k+\nu-m-1)}(\mathbb{C}^2))$ .

*Proof.* See [11]. □

From Theorem 5.1 we obtain the complex linear map

$$\bigoplus_{k=0}^m \mathcal{L}_{m,\nu}^k : QM_{2\nu}^m(\Gamma) \rightarrow \bigoplus_{k=0}^m H^1(\Gamma, \mathfrak{S}^{2(k+\nu-m-1)}(\mathbb{C}^2)) \quad (5.4)$$

for each  $\nu > m$ .

In order to discuss Hecke operators we now extend the formulas for  $\mathfrak{J}$  and  $\mathfrak{K}$  in (2.1) from  $SL(2, \mathbb{R})$  to the group  $GL^+(2, \mathbb{R})$  of  $2 \times 2$  real matrices of positive determinant. We also extend the operations  $|_\lambda$  and  $\|_\lambda$  of  $SL(2, \mathbb{R})$  in (2.9) and (2.10) to those of  $GL^+(2, \mathbb{R})$  by setting

$$\begin{aligned} (f |_\lambda \alpha)(z) &= (\det \alpha)^{\lambda/2} \mathfrak{J}(\alpha, z)^{-\lambda} f(\alpha z) \\ (F \|_\lambda \alpha)(z, X) &= \det(\alpha)^{\lambda/2} \mathfrak{J}(\alpha, z)^{-\lambda} \\ &\quad \times F(\alpha z, \det(\alpha)^{-1} \mathfrak{J}(\alpha, z)^2 (X - \mathfrak{K}(\alpha, z))) \end{aligned} \quad (5.5)$$

for all  $z \in \mathcal{H}$ ,  $\alpha \in GL^+(2, \mathbb{R})$ ,  $f \in R$  and  $F(z, X) \in R_m[X]$ .

Let  $\Gamma$  be a discrete subgroup of  $SL(2, \mathbb{R})$  as in Section 2, and let  $\tilde{\Gamma}$  be its commensurator, that is, the set of elements  $g \in GL^+(2, \mathbb{R})$  such that  $g\Gamma g^{-1} \cap \Gamma$  has finite index in both  $\Gamma$  and  $g\Gamma g^{-1}$ . Given  $\alpha \in \tilde{\Gamma}$ , the double coset  $\Gamma\alpha\Gamma$  has a decomposition of the form

$$\Gamma\alpha\Gamma = \coprod_{i=1}^s \Gamma\alpha_i \quad (5.6)$$

for some  $\alpha_i \in GL^+(2, \mathbb{R})$  with  $i = 1, \dots, s$ . For the same  $\alpha$  and an integer  $k$  the corresponding Hecke operator

$$T_k(\alpha) : M_k(\Gamma) \rightarrow M_k(\Gamma)$$

on modular forms is given by

$$T_k(\alpha)f = \sum_{i=1}^s (f |_k \alpha_i),$$

for  $f \in M_k(\Gamma)$  (see e.g. [14]). Such operators can be used to introduce the Hecke operator

$$T_\lambda^M(\alpha) : MP_\lambda^m(\Gamma) \rightarrow MP_\lambda^m(\Gamma)$$

on modular polynomials by

$$(T_\lambda^M(\alpha)F)(z, X) = \sum_{r=0}^m (T_{\lambda+2r}(\alpha)\Pi_r^m F)(z)X^r \quad (5.7)$$

for  $F(z, X) \in MP_\lambda^m(\Gamma)$ , where the projection maps  $\Pi_r^m$  are as in (2.17). On the other hand, for quasimodular polynomials the Hecke operator

$$T_\lambda^P(\alpha) : QP_\lambda^m(\Gamma) \rightarrow QP_\lambda^m(\Gamma)$$

is given by

$$(T_\lambda^P(\alpha)F)(z, X) = \sum_{i=1}^s (F \parallel_\lambda \alpha_i)(z, X)$$

for  $F(z, X) \in QP_\lambda^m(\Gamma)$ , where  $\parallel_\lambda$  is as in (5.5) (see [1]). Using this and the isomorphism in (2.15), we can also introduce the corresponding Hecke operator

$$T_\lambda^Q(\alpha) : QM_\lambda^m(\Gamma) \rightarrow QM_\lambda^m(\Gamma)$$

on quasimodular forms by setting

$$T_\lambda^Q(\alpha)f = \sum_{i=1}^s (\mathcal{Q}_\lambda^m)^{-1}((\mathcal{Q}_\lambda^m f) \parallel_\lambda \alpha) = ((\mathcal{Q}_\lambda^m)^{-1} \circ T_\lambda^P(\alpha) \circ \mathcal{Q}_\lambda^m)f$$

for  $f \in QM_\lambda^m(\Gamma)$ , so that

$$T_\lambda^P(\alpha) \circ \mathcal{Q}_\lambda^m = \mathcal{Q}_\lambda^m \circ T_\lambda^Q(\alpha). \tag{5.8}$$

Then it is known that the linear isomorphisms  $\Xi_\lambda^m$  and  $\Lambda_\lambda^m$  in (2.11) are Hecke equivariant in the sense that

$$T_\lambda^P(\alpha) \circ \Xi_\lambda^m = \Xi_\lambda^m \circ T_{\lambda-2m}^M(\alpha), \quad T_{\lambda-2m}^M(\alpha) \circ \Lambda_\lambda^m = \Lambda_\lambda^m \circ T_\lambda^P(\alpha) \tag{5.9}$$

for each  $\alpha \in \tilde{\Gamma}$ . From (5.7), (5.8) and (5.9) we see that the diagram

$$\begin{array}{ccccccc} QM_\lambda^m(\Gamma) & \xrightarrow{\mathcal{Q}_\lambda^m} & QP_\lambda^m(\Gamma) & \xrightarrow{\Lambda_\lambda^m} & MP_{\lambda-2m}^m(\Gamma) & \xrightarrow{\Pi_r^m} & M_{\lambda-2m+2r}^m(\Gamma) \\ T_\lambda^Q(\alpha) \downarrow & & T_\lambda^P(\alpha) \downarrow & & T_{\lambda-2m}^M(\alpha) \downarrow & & T_{\lambda-2m+2r}(\alpha) \downarrow \\ QM_\lambda^m(\Gamma) & \xrightarrow{\mathcal{Q}_\lambda^m} & QP_\lambda^m(\Gamma) & \xrightarrow{\Lambda_\lambda^m} & MP_{\lambda-2m}^m(\Gamma) & \xrightarrow{\Pi_r^m} & M_{\lambda-2m+2r}^m(\Gamma) \end{array}$$

is commutative for each  $\alpha \in \tilde{\Gamma}$  and  $r \in \{0, 1, \dots, m\}$ .

We now describe Hecke operators on cohomology. Given an element  $\alpha \in \tilde{\Gamma} \subset \text{GL}^+(2, \mathbb{R})$  such that the corresponding Hecke double coset has a decomposition as in (5.6), the corresponding Hecke operator on the cohomology group  $H^1(\Gamma, \mathfrak{S}^n(\mathbb{C}^2))$  in (5.1) can be determined as follows. If  $\gamma \in \Gamma$ , since  $\Gamma\alpha\Gamma = \Gamma\alpha\Gamma$ , the decomposition in (5.6) can be written as

$$\Gamma\alpha\Gamma = \coprod_{i=1}^s \Gamma\alpha_i\gamma;$$

hence for  $1 \leq i \leq s$ , we see that

$$\alpha_i \gamma = \xi_i(\gamma) \alpha_{i(\gamma)}$$

for some element  $\xi_i(\gamma) \in \Gamma$ . We note that the set  $\{\alpha_{1(\gamma)}, \dots, \alpha_{s(\gamma)}\}$  is a permutation of  $\{\alpha_1, \dots, \alpha_s\}$ . Then the Hecke operator  $T_n^H(\alpha)$  on the cohomology space  $H^1(\Gamma, \mathfrak{S}^n(\mathbb{C}^2))$  is given by

$$(T_n^H(\alpha)(\phi))(\gamma) = \sum_{i=1}^s (\det \alpha_i)^{n/2} \rho_n(\alpha_i) \phi(\xi_i(\gamma))$$

for each 1-cocycle  $\phi$  and  $\gamma \in \Gamma$  (see [18]).

**Theorem 5.2.** *The linear map  $\mathcal{L}_{m,\nu}^k$  with  $k, \nu \in \mathbb{Z}$  and  $0 \leq k \leq m < \nu$  in (5.3) satisfies*

$$\mathcal{L}_{m,\nu}^k \circ T_{2\nu}^Q = T_{2(k+\nu-m-1)}^H(\alpha) \circ \mathcal{L}_{m,\nu}^k$$

for each  $\alpha \in \tilde{\Gamma}$ .

*Proof.* See [11]. □

If we define the Hecke operator  $\bigoplus_{k=0}^m T_{2(k+\nu-m-1)}^H(\alpha)$  on

$$\bigoplus_{k=0}^m H^1(\Gamma, \mathfrak{S}^{2(k+\nu-m-1)}(\mathbb{C}^2))$$

component-wise, then we see that it is compatible with the Hecke operator  $T_{2\nu}^Q(\alpha)$  on  $QM_{2\nu}^m(\Gamma)$  under the map  $\bigoplus_{k=0}^m \mathcal{L}_{m,\nu}^k$  in (5.4). Thus we obtain the commutative diagram

$$\begin{array}{ccc} QM_{2\nu}^m(\Gamma) & \xrightarrow{\bigoplus_{k=0}^m \mathcal{L}_{m,\nu}^k} & \bigoplus_{k=0}^m H^1(\Gamma, \mathfrak{S}^{2(k+\nu-m-1)}(\mathbb{C}^2)) \\ T_{2\nu}^Q(\alpha) \downarrow & & \downarrow \bigoplus_{k=0}^m T_{2(k+\nu-m-1)}^H(\alpha) \\ QM_{2\nu}^m(\Gamma) & \xrightarrow{\bigoplus_{k=0}^m \mathcal{L}_{m,\nu}^k} & \bigoplus_{k=0}^m H^1(\Gamma, \mathfrak{S}^{2(k+\nu-m-1)}(\mathbb{C}^2)) \end{array}$$

for each  $\alpha \in \tilde{\Gamma}$  and  $\nu > m$ .

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