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Hatem Mejjaoli

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Hatem Mejjaoli

Abstract

We introduce and study the linear symmetric systems associated with the modified Cherednik operators. We prove the well-posedness results for the Cauchy problem for these systems. Eventually we describe the finite propagation speed property of it.

Les systèmes symétriques linéaires associés aux opérateurs de Cherednik modifiés et applications

Résumé

Nous présentons et étudions les systèmes symétriques linéaires associés aux opérateurs de Cherednik modifiés. Nous prouvons que le problème de Cauchy pour ces systèmes sont bien posé. Finalement nous en décrivons le principe de vitesse finie.

Dedicated to Khalifa Trimèche for his 66th birthday

1. Introduction

Let $\mathfrak{a}$ be a real Euclidean vector space of dimension d and let R be a root system in $\mathfrak{a}$. A multiplicity function is a complex-valued function k on R which is invariant with respect to the Weyl group of R. In the mid 1990s, Ivan Cherednik associated with a triplet $(\mathfrak{a}, R, k)$ a commutative family of first order differential-reflection operators, nowadays known as Cherednik

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operators or trigonometric Dunkl operators. The original motivation for the study of these operators came from the theory of invariant differential operators: if the triplet $(\mathfrak{a}, R, k)$ arises from the structure theory of a Riemannian symmetric space of the non-compact type G/K, then it is possible to explicitly construct all radial components of the $W-$invariant differential operators on G/K using the Cherednik operators. The joint spectral theory of Cherednik operators is therefore naturally related to the harmonic analysis on Reimannian symmetric spaces (and to the more general theory of hypergeometric functions in several variables of Heckman and Opdam). But it is also related with the representation theory of the graded Hecke algebra of Lusztig. There are many references on the subject. Our starting point will be the following references (cf. [3, 12, 13, 15]).

In this paper, we are interested in studying to the modified Cherednik-linear symmetric system

$$(S)\begin{cases}\partial_t u(t,x) - \sum_{j=1}^{d} A_j T_j u(t,x) - A_0 u(t,x) = f(t,x), \quad (t,x) \in I \times \mathbf{R}^d \\ u_{\,|t=0} = v,\end{cases}$$

where T_j, $j = 1,\dots,d$, are the modified Cherednik operators, I be an interval of $\mathbf{R}$, $(A_p)_{0\leq p\leq d}$ a family of functions from $I \times \mathbf{R}^d$ into the space of $m\times m$ matrices with real coefficients $a_{p,i,j}(t,x)$ which are W-invariant with respect to x, symmetric (i.e. $a_{p,i,j}(t,x) = a_{p,j,i}(t,x)$) and whose all derivatives in $x \in \mathbf{R}^d$ are bounded and continuous functions of (t,x), the initial data belongs to generalized Sobolev spaces $[H^s_k(\mathbf{R}^d)]^m$ and f is a continuous function on an interval I with value in $[H^s_k(\mathbf{R}^d)]^m$. In the classical case, the Cauchy problem for symmetric hyperbolic systems of first order, it has been introduced and studied by Friedrichs (cf. [6]). The Cauchy problem will be solved with the aid of energy integral inequalities, developed for this purpose by Friedrichs. Such energy inequalities have been employed by Weber [18], Zaremba [19] to derive various uniqueness theorems, and by Courant-Friedrichs-Lewy [5], Friedrichs [6] to derive existence theorems. In all these treatments the energy inequality is used to show that the solution, at some later time, depends boundedly on the initial values in an appropriate norm. To derive an existence theorem however one needs, in addition to the a priori energy estimates, auxiliary constructions. Thus motivated by these methods we will prove by energy methods and

Friedrichs approach local well-posedness and principle of finite speed of propagation for the system (S).

Let us first summarize our well-posedness results and finite speed of propagation (Theorem 4.3 and Theorem 5.2).

Well-posedness for DLS. For all given $f \in [C(I, H_k^s(\mathbf{R}^d))]^m$ and $v \in [H_k^s(\mathbf{R}^d)]^m$, there exists a unique solution u of the system (S) in the space

$$[C(I, H_k^s(\mathbf{R}^d))]^m \bigcap [C^1(I, H_k^{s-1}(\mathbf{R}^d))]^m.$$

In the classical case, a similar result can be found in [2], where the authors used another method based on the symbolic calculations for the pseudo-differential operators that we can not adapt for the system (S) at the moment. Our method use some ideas inspired by the works [2, 6, 7, 8, 9, 10, 11, 5, 14].

Finite speed of propagation. Let (S) as above. We assume that f belongs to $[C(I, H_k^1(\mathbf{R}^d))]^m$ and $v \in [H_k^1(\mathbf{R}^d)]^m$.

- There exists a positive constant C_0 such that, for any positive real R satisfying

$$\begin{cases} f(t,x) & \equiv 0 \quad \text{for} \quad \|x\| < R - C_0 t \\ v(x) & \equiv 0 \quad \text{for} \quad \|x\| < R, \end{cases}$$

  the unique solution u of the system (S) verifies

$$u(t,x) \equiv 0 \quad \text{for} \quad \|x\| < R - C_0 t.$$

- If the given f and v are such that

$$\begin{cases} f(t,x) & \equiv 0 \quad \text{for} \quad \|x\| > R + C_0 t \\ v(x) & \equiv 0 \quad \text{for} \quad \|x\| > R, \end{cases}$$

  then the unique solution u of the system (S) satisfies

$$u(t,x) \equiv 0 \quad \text{for} \quad \|x\| > R + C_0 t.$$

In the classical case, similar results can be found in [2, 11, 16].

A standard example of the modified Cherednik linear symmetric system is the generalized wave equations with variable coefficients defined by:

$$\partial_t^2 u - \mathrm{div}_k[A.\nabla_{k,x} u] + Q(t, x, \partial_t u, T_x u), \ t \in \mathbf{R}, \ x \in \mathbf{R}^d,$$

where

$$\nabla_{k,x} u := (T_1 u, \dots, T_d u), \; \mathrm{div}_k (v_1, \dots, v_d) := \sum_{i=1}^{d} T_i v_i,$$

A is a real symmetric matrix which verifies some hypotheses (see subsection 5.1) and $Q(t, x, \partial_t u, T_x u)$ is differential-difference operator of degree 1 such that these coefficients are C^∞, and all derivatives are bounded. From the previous results we deduce the well-posedness of the generalized wave equations (Theorem 5.1):

Well-posedness for GDW. For all $s \in \mathbf{N}$, $u_0 \in H_k^{s+1}(\mathbf{R}^d)$, $u_1 \in H_k^s(\mathbf{R}^d)$ and $f \in C(\mathbf{R}, H_k^s(\mathbf{R}^d))$, there exists a unique

$$u \in C^1(\mathbf{R}, H_k^s(\mathbf{R}^d)) \cap C(\mathbf{R}, H_k^{s+1}(\mathbf{R}^d))$$

such that

$$\begin{cases} \partial_t^2 u - \mathrm{div}_k[A.\nabla_{k,x} u] + Q(t, x, \partial_t u, T_x u) = f \\ u_{|t=0} = u_0 \\ \partial_t u_{|t=0} = u_1. \end{cases}$$

The paper is organized as follows. In Section 2 we recall the main results about the harmonic analysis associated with the modified Cherednik operators. In Section 3 we introduce the generalized Sobolev spaces associated with modified Cherednik operators and we study these properties. Section 4 is devoted to study the generalized Cauchy problem of the modified Cherednik linear symmetric systems. In the last sections we give many applications. More precisely, we prove the well-posedness of the generalized wave equations associated with the modified Cherednik operators. Next, we prove the principle of finite speed of propagation of the linear Cherednik symetric systems.

Throughout this paper by C we always represent a positive constant not necessarily the same in each occurrence.

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2. **Preliminaries**

This section gives an introduction to the theory of modified Cherednik operators, generalized Fourier transform, and generalized convolution operator. Main references are [1, 4].

2.1. **The eigenfunctions of the modified Cherednik operators**

The basic ingredient in the theory of modified Cherednik operators is finite reflection groups, acting on $\mathbf{R}^d$ with the standard Euclidean scalar product $\langle .,. \rangle$ and $\|x\| = \sqrt{\langle x, x\rangle}$. On $\mathbf{C}^d$, $\|.\|$ denotes also the standard Hermitian norm, while $\langle z, w\rangle = \sum_{j=1}^{d} z_j \overline{w}_j$.

Let $(e_j)_{j=1,\dots,d}$ be the Euclidean bases of $\mathbb{R}^d$, let $e_j^\vee = 2e_j$ be the coroot associated to e_j and let

$$r_{e_j}(x) = x - \langle e_j^\vee, x\rangle e_j \tag{2.1}$$

be the reflection in the hyperplane $H_{e_j} \subset \mathbb{R}^d$ orthogonal to e_j. The reflections $r_{e_j}, j = 1, \dots, d$, generate a finite group $W \subset O(d)$, called the reflection group associated with $(e_j)_{j=1,\dots,d}$.

Moreover, let A_k denotes the weight function

$$\forall\, x \in \mathbb{R}^d \,, A_k(x) = 2^\gamma \prod_{j=1}^{d} |\sinh(\langle e_j \,, x\rangle)|^{2k_j} \cosh^{2l_j}(\langle e_j \,, x\rangle), \tag{2.2}$$

with $k_j \geq l_j \geq 0$ and $k_j \neq 0$, and $\gamma := \sum_{j=0}^{d} (k_j + l_j)$.

In the following we denote by

- $C(\mathbf{R}^d)$ the space of continuous functions on $\mathbf{R}^d$.
- $C_0(\mathbf{R}^d)$ the space of continuous functions on $\mathbf{R}^d$ vanishing at infinity.
- $C^p(\mathbf{R}^d)$ the space of functions of class C^p on $\mathbf{R}^d$.
- $C_b^p(\mathbf{R}^d)$ the space of bounded functions of class C^p.
- $\mathcal{E}(\mathbf{R}^d)$ the space of C^∞-functions on $\mathbf{R}^d$.
- $\mathcal{S}(\mathbf{R}^d)$ the Schwartz space of rapidly decreasing functions on $\mathbf{R}^d$.

- $D(\mathbf{R}^d)$ the space of C^∞-functions on $\mathbf{R}^d$ which are of compact support.
- $\mathcal{S}'(\mathbf{R}^d)$ the space of temperate distributions on $\mathbf{R}^d$.

The modified Cherednik operators $T_j,\ j = 1, \dots, d$, on $\mathbf{R}^d$ are given by

$$T_j f(x) := \frac{\partial}{\partial x_j} f(x) + \Big[k_j \coth(\langle e_j, x\rangle) + l_j \tanh(\langle e_j, x\rangle)\Big]\Big(f(x) - f(r_{e_j}(x))\Big). \tag{2.3}$$

Some properties of the T_j, $j = 1, \dots, d$, are given in the following: for all f and g in $C^1(\mathbf{R}^d)$ with at least one of them is W-invariant, we have

$$T_j(fg) = (T_j f)g + f(T_j g), \quad j = 1, \dots, d. \tag{2.4}$$

For f of class C^1 on $\mathbb{R}^d$ with compact support and g of class C^1 on $\mathbb{R}^d$, we have for $j = 1, 2, \dots, d$:

$$\int_{\mathbb{R}^d} T_j f(x) g(x) A_k(x) dx = -\int_{\mathbb{R}^d} f(x) T_j g(x) A_k(x) dx. \tag{2.5}$$

The modified Cherednik operators form a commutative system of differential-difference operators. The modified Heckman-Opdam Laplacian $\triangle_k$ is defined by

$$\begin{aligned} \triangle_k f(x) &:= \sum_{j=1}^d T_j^2 f(x) \qquad (2.6)\\ &= \triangle f(x) + \sum_{j=1}^d \Big[2k_j \coth(\langle e_j, x\rangle) + 2l_j \tanh(\langle e_j, x\rangle)\Big]\langle \nabla f(x), e_j\rangle \\ &\quad - \sum_{j=1}^d \Big[\frac{k_j}{(\sinh(\langle e_j, x\rangle))^2} - \frac{l_j}{(\cosh(\langle e_j, x\rangle))^2}\Big]\Big(f(x) - f(r_{e_j}(x))\Big), \end{aligned}$$

where $\triangle$ and ∇ are respectively the Laplacian and the gradient on $\mathbb{R}^d$.

The modified Heckman-Opdam Laplacian on W-invariant functions in denoted by $\triangle_k^W$ and has the expression

$$\triangle_k^W f(x) = \triangle f(x) + \sum_{j=1}^d \Big[2k_j \coth(\langle e_j, x\rangle) + 2l_j \tanh(\langle e_j, x\rangle)\Big]\langle \nabla f(x), e_j\rangle.$$

Example 2.1. For $d = 1$, and $k \geq l \geq 0$ and $k \neq 0$, the modified Heckman-Opdam Laplacian $\triangle_k^W$ is the Jacobi operator defined for even functions f of class C^2 on $\mathbb{R}$ by

$$\triangle_k^W f(x) = \frac{d^2}{dx^2} f(x) + \Big[2k \coth(x) + 2l \tanh(x)\Big] \frac{d}{dx} f(x).$$

We denote by G_λ the eigenfunction of the operators T_j, $j = 1, 2, \ldots, d$. It is the unique analytic function on $\mathbb{R}^d$ which satisfies the differential-difference system

$$\begin{cases} T_j u(x) & = \lambda_j u(x), \quad j = 1, 2, \ldots, d, x \in \mathbb{R}^d \\ u(0) & = 1. \end{cases}$$

It is called the modified Opdam-Cherednik kernel.
We consider the function F_λ defined by

$$\forall\, x \in \mathbb{R}^d\,, \quad F_\lambda(x) = \frac{1}{|W|} \sum_{w \in W} G_\lambda(wx).$$

This function is the unique analytic W-invariant function on $\mathbb{R}^d$, which satisfies the differential equations

$$\begin{cases} p(T)u(x) & = p(\lambda)u(x), \quad x \in \mathbb{R}^d, \lambda \in \mathbb{R}^d \\ u(0) & = 1, \end{cases}$$

for all W-invariant polynomial p on $\mathbb{R}^d$ and $p(T) = p(T_1, \ldots, T_d)$. In particular for all $\lambda \in \mathbb{R}^d$ we have

$$\triangle_k^W F_\lambda(x) = \|\lambda\|^2 F_\lambda(x).$$

The function F_λ is called the modified Heckman-Opdam kernel.

The functions G_λ and F_λ possess the following properties

i) For all $x \in \mathbb{R}^d$, the functions G_λ and F_λ are entire on $\mathbb{C}^d$.

ii) There exists a positive constant M_0 such that

$$\forall\, x \in \mathbb{R}^d\,, \forall\, \lambda \in \mathbb{R}^d,\ |G_{i\lambda}(x)| \leq M_0.$$

iii) Let p and q be polynomials of degree m and n. Then there exists a positive constant M' such that for all $\lambda \in \mathbb{C}^d \backslash \{0\}$ and for all

$x \in \mathbb{R}^d$, we have

$$|p(\frac{\partial}{\partial\lambda})q(\frac{\partial}{\partial x})G_\lambda(x)| \leq M' \prod_{j=1}^{d}(1+|x_j|)^n \frac{(1+|\lambda_j|+\gamma)^m}{|\lambda_j|} e^{\max_{w\in W}(\mathrm{Re}\langle \mathrm{w}\lambda, \mathrm{x}\rangle)}. \tag{2.7}$$

Example 2.2. When $d=1$ and $W=\mathbb{Z}_2$, the modified Opdam-Cherednik kernel $G_\lambda(x)$ is given for all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$ by

$$G_\lambda(x) = \begin{cases} \varphi_\mu^{(k-\frac{1}{2},l-\frac{1}{2})}(x) + \frac{1}{\lambda}\frac{d}{dx}\varphi_\mu^{(k-\frac{1}{2},l-\frac{1}{2})}(x), & \text{if} \quad \lambda \in \mathbf{C}\backslash\{0\}, \\ 1, & \text{if} \quad \lambda = 0, \end{cases}$$

with $\lambda^2 = \mu^2 + (k+l)^2$ and $\varphi_\mu^{(k-\frac{1}{2},l-\frac{1}{2})}$ the Jacobi function given by

$$\varphi_\mu^{(k-\frac{1}{2},l-\frac{1}{2})}(x) = {}_2F_1(\frac{k+l+i\mu}{2}, \frac{k+l-i\mu}{2}; k+\frac{1}{2}; -(\sinh x)^2), \tag{2.8}$$

where ${}_2F_1$ is the Gauss hypergeometric function.

In this case the modified Heckman-Opdam kernel $F_\lambda(x)$ is given for all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$ by

$$F_\lambda(x) = \varphi_\lambda^{(k-\frac{1}{2},l-\frac{1}{2})}(x).$$

2.2. **The generalized Fourier transform**

We denote by

$\mathcal{S}_2(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$ such that for all $\ell, n \in \mathbb{N}$, we have

$$\sup_{|\mu|\leq n,\, x\in\mathbb{R}^d} (1+\|x\|)^\ell \prod_{j=1}^{d}(\cosh(x_j))^{2(k_j+l_j)}|D^\mu f(x)| < \infty,$$

where

$$D^\mu = \frac{\partial^{|\mu|}}{\partial^{\mu_1}x_1 \ldots \partial^{\mu_d}x_d}\ , \mu = (\mu_1, \ldots, \mu_d) \in \mathbb{N}^d.$$

$\mathrm{PW}(\mathbb{C}^d)$ the space of entire functions on $\mathbb{C}^d$, which are rapidly decreasing and of exponential type.

$\mathcal{S}_2'(\mathbb{R}^d)$ the topological dual of $\mathcal{S}_2(\mathbb{R}^d)$.

$L^p_{A_k}(\mathbb{R}^d)$, $1 \leq p \leq \infty$, the space of measurable functions f on $\mathbb{R}^d$ satisfying

$$\begin{aligned}\|f\|_{L^p_{A_k}(\mathbb{R}^d)} &= \Big(\int_{\mathbb{R}^d} |f(x)|^p A_k(x)dx\Big)^{1/p} < \infty, \quad \text{if } 1 \leq p < \infty \\ \|f\|_{L^\infty_{A_k}(\mathbb{R}^d)} &= \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} |f(x)| < \infty.\end{aligned}$$

The generalized Fourier transform of a function f in $D(\mathbf{R}^d)$ is given by

$$\mathcal{F}_k(f)(\lambda) = \int_{\mathbf{R}^d} f(x)G_{-i\lambda}(x)A_k(x)dx, \quad \text{for all } \lambda \in \mathbf{R}^d. \tag{2.9}$$

Proposition 2.3. *The transform $\mathcal{F}_k$ is a topological isomorphism from*

(i) $D(\mathbb{R}^d)$ *onto* $\mathrm{PW}(\mathbb{C}^d)$.

(ii) $\mathcal{S}_2(\mathbb{R}^d)$ *onto* $\mathcal{S}(\mathbb{R}^d)$.

The inverse transform is given by

$$\forall\, x \in \mathbb{R}^d,\ (\mathcal{F}_k)^{-1}(h)(x) = \int_{\mathbb{R}^d} h(\lambda)G_{i\lambda}(x)d\nu_k(\lambda), \tag{2.10}$$

where $d\nu_k(\lambda) := C_k(\lambda)d\lambda$ is the spectral measure (cf. [1, 4]*).*

Remark 2.4. The function C_k is a positive, continuous on $\mathbf{R}^d$ and satisfies the estimate

$$\forall\, \lambda \in \mathbf{R}^d, \quad |C_k(\lambda)| \leq const.(1 + \|\lambda\|)^b,$$

for some $b > 0$.

Proposition 2.5.

(i) *Plancherel formula: For all f, g in $D(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) we have*

$$\int_{\mathbb{R}^d} f(x)\overline{g(x)}A_k(x)dx = \int_{\mathbb{R}^d} \mathcal{F}_k(f)(\lambda)\overline{\mathcal{F}_k(g)(\lambda)}d\nu_k(\lambda). \tag{2.11}$$

(ii) *Plancherel theorem: The generalized Fourier transform $\mathcal{F}_k$ extends uniquely to an isometric isomorphism of $L^2_{A_k}(\mathbf{R}^d)$ into $L^2_{\nu_k}(\mathbf{R}^d)$, where $L^2_{\nu_k}(\mathbf{R}^d)$ denotes the space of measurable functions f on $\mathbf{R}^d$ satisfying*

$$\|f\|_{L^2_{\nu_k}(\mathbf{R}^d)} = \Big(\int_{\mathbf{R}^d} |f(x)|^2\, d\nu_k(x)\Big)^{1/2} < \infty.$$

2.3. Generalized convolution operator

Definition 2.6. Let y be in $\mathbf{R}^d$. The generalized translation operator $f \mapsto \tau_y f$ is defined on $\mathcal{S}_2(\mathbf{R}^d)$ by

$$\mathcal{F}_k(\tau_y f)(x) = G_{-iy}(x)\mathcal{F}_k(f)(x), \quad \text{for all } x \in \mathbf{R}^d. \tag{2.12}$$

Using the generalized translation operator, we define the generalized convolution product of functions as follows.

Definition 2.7. The generalized convolution product of f and g in $\mathcal{S}_2(\mathbf{R}^d)$ is the function $f *_k g$ defined by

$$f *_k g(x) = \int_{\mathbf{R}^d} \tau_x f(-y) g(y) A_k(y) dy, \quad \text{for all } x \in \mathbf{R}^d. \tag{2.13}$$

For the remainder of this subsection we collect some results proved in [1].

Proposition 2.8. *Let f be in $L^1_{A_k}(\mathbb{R}^d)$ and g in $L^2_{A_k}(\mathbb{R}^d)$. Then*

i) *The function $f *_k g$ defined almost everywhere on $\mathbb{R}^d$ by*

$$f *_k g(y) = \int_{\mathbb{R}^d} \tau_y g(-x) f(x) A_k(x) dx,$$

belongs to $L^2_{A_k}(\mathbb{R}^d)$ and we have

$$\|f *_k g\|_{L^2_{A_k}(\mathbf{R}^d)} \leq C \|f\|_{L^1_{A_k}(\mathbf{R}^d)} \|g\|_{L^2_{A_k}(\mathbf{R}^d)}.$$

ii) *We have*

$$\mathcal{F}_k(f *_k g) = \mathcal{F}_k(f).\mathcal{F}_k(g).$$

Proposition 2.9. *Let φ be a positive function in $D(\mathbb{R}^d)$ such that supp $\varphi \subset B(0,1)$ and $\|\varphi\|_{L^1_{A_k}(\mathbf{R}^d)} = 1$. For $\varepsilon > 0$, we consider the function φ_ε given by*

$$\forall\, x \in \mathbb{R}^d, \quad \varphi_\varepsilon(x) = \frac{A_k(\frac{x}{\varepsilon})}{\varepsilon^d A_k(x)}\, \varphi(\frac{x}{\varepsilon}).$$

Then for all f in $L^2_{A_k}(\mathbb{R}^d)$ we have

$$\lim_{\varepsilon \to 0} \|f *_k \varphi_\varepsilon - f\|_{L^2_{A_k}(\mathbf{R}^d)} = 0. \tag{2.14}$$

Definition 2.10. The generalized Fourier transform of a distribution τ in $\mathcal{S}'_2(\mathbf{R}^d)$ is defined by

$$\langle \mathcal{F}_k(\tau), \phi \rangle = \langle \tau, \mathcal{F}_k^{-1}(\phi) \rangle, \quad \text{for all } \phi \in \mathcal{S}(\mathbf{R}^d). \tag{2.15}$$

Proposition 2.11. *The generalized Fourier transform $\mathcal{F}_k$ is a topological isomorphism from ${\mathcal{S}'}_2(\mathbf{R}^d)$ onto $\mathcal{S}'(\mathbf{R}^d)$.*

3. The generalized Sobolev spaces

Let τ be in $\mathcal{S}_2'(\mathbb{R}^d)$. We define the distributions $T_j\tau$, $j = 1, \dots, d$, by

$$\langle T_j\tau, \psi\rangle \;=\; -\langle \tau, T_j\psi\rangle, \text{ for all } \psi \in \mathcal{S}_2(\mathbb{R}^d). \tag{3.1}$$

Thus we deduce

$$\langle \triangle_k\tau, \psi\rangle \;=\; \langle \tau, \triangle_k\psi\rangle, \text{ for all } \psi \in \mathcal{S}_2(\mathbb{R}^d). \tag{3.2}$$

These distributions satisfies the following properties

$$\mathcal{F}_D(T_j\tau) \;=\; iy_j\mathcal{F}_D(\tau), \quad j = 1, \dots, d. \tag{3.3}$$

$$\mathcal{F}_D(\triangle_k\tau) \;=\; -\|y\|^2\mathcal{F}_D(\tau). \tag{3.4}$$

Definition 3.1. We define the generalized Sobolev space $H_k^s(\mathbf{R}^d)$ as

$$\Big\{u \in \mathcal{S}_2'(\mathbf{R}^d) : \quad \int_{\mathbf{R}^d}(1 + \|\lambda\|^2)^s|\mathcal{F}_k(u)(\lambda)|^2 d\nu_k(\lambda) < \infty\Big\}.$$

We provide this space with the scalar product

$$\langle u, v\rangle_{H_k^s(\mathbf{R}^d)} := \int_{\mathbf{R}^d}(1 + \|\xi\|^2)^s\mathcal{F}_k(u)(\xi)\overline{\mathcal{F}_k(v)(\xi)}d\nu_k(\xi), \tag{3.5}$$

and the norm

$$\|u\|^2_{H_k^s(\mathbf{R}^d)} := \langle u, u\rangle_{H_k^s(\mathbf{R}^d)}. \tag{3.6}$$

Proposition 3.2.

(i) *The space $H_k^s(\mathbf{R}^d)$ provided with the scalar product $\langle ., .\rangle_{H_k^s(\mathbf{R}^d)}$ is a Hilbert space.*

(ii) *Let s_1, s_2 in $\mathbf{R}$ such that $s_1 \geq s_2$ then*

$$H_k^{s_1}(\mathbf{R}^d) \hookrightarrow H_k^{s_2}(\mathbf{R}^d).$$

(iii) *Let $s \in \mathbf{R}$. Then $D(\mathbf{R}^d)$ is dense in $H_k^s(\mathbf{R}^d)$.*

(iv) *The dual of $H_k^s(\mathbf{R}^d)$ can be identified with $H_k^{-s}(\mathbf{R}^d)$. The relation of the identification is given by*

$$\langle u, v\rangle_k = \int_{\mathbf{R}^d}\mathcal{F}_k(u)(\xi)\mathcal{F}_k(v)(\xi)d\nu_k(\xi),$$

with $u \in H_k^s(\mathbf{R}^d)$ *and* $v \in H_k^{-s}(\mathbf{R}^d)$.

Proof. (i) It is clear that $L^2(\mathbf{R}^d, (1+\|\xi\|^2)^{2s} d\nu_k(\xi))$ is complete and since $\mathcal{F}_k$ is an isomorphism from $\mathcal{S}_2'(\mathbb{R}^d)$ onto $\mathcal{S}'(\mathbf{R}^d)$, $H_k^s(\mathbf{R}^d)$ is then a Hilbert space. The result (ii) is immediately from definition of the generalized Sobolev space. As in [17], we can obtain (iii) and (iv). □

Proposition 3.3. *Let* s_1, s, s_2 *be three real numbers :* $s_1 < s < s_2$*. Then, for all* $\varepsilon > 0$, *there exists a nonnegative constant* C_ε *such that for all* u *in* $H_k^s(\mathbf{R}^d)$

$$\|u\|_{H_k^s(\mathbf{R}^d)} \leq C_\varepsilon \|u\|_{H_k^{s_1}(\mathbf{R}^d)} + \varepsilon \|u\|_{H_k^{s_2}(\mathbf{R}^d)}. \tag{3.7}$$

Proof. We consider $s = (1-t)s_1 + ts_2$, (with $t \in (0,1)$). Moreover it is easy to see

$$\|u\|_{H_k^s(\mathbf{R}^d)} \leq \|u\|^{1-t}_{H_k^{s_1}(\mathbf{R}^d)} \|u\|^t_{H_k^{s_2}(\mathbf{R}^d)}. \tag{3.8}$$

Thus

$$\begin{aligned} \|u\|_{H_k^s(\mathbf{R}^d)} &\leq (\varepsilon^{-\frac{t}{1-t}} \|u\|_{H_k^{s_1}(\mathbf{R}^d)})^{1-t} (\varepsilon \|u\|_{H_k^{s_2}(\mathbf{R}^d)})^t \\ &\leq \varepsilon^{-\frac{t}{1-t}} \|u\|_{H_k^{s_1}(\mathbf{R}^d)} + \varepsilon \|u\|_{H_k^{s_2}(\mathbf{R}^d)}. \end{aligned}$$

Hence the proof is completed for $C_\varepsilon = \varepsilon^{-\frac{t}{1-t}}$. □

A characterization of $H_k^s(\mathbf{R}^d)$, for $s = m$, a positive integer, is given below.

Proposition 3.4.

(i) *For* $m \in \mathbf{N}$ *the space* $H_k^m(\mathbf{R}^d)$ *coincides with the space* E_m *given by*

$$E_m = \Big\{u \in L^2_{A_k}(\mathbf{R}^d) : T^\alpha u \in L^2_{A_k}(\mathbf{R}^d),\ |\alpha| \leq m\Big\}, \tag{3.9}$$

where $T^\alpha = T_1^{\alpha_1} \otimes \cdots \otimes T_d^{\alpha_d}$, $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{N}^d$ *and* $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

(ii) *The norm* $\|.\|_{m,k}$ *is equivalent to the norm*

$$\|u\|^2_{m,k} = \sum_{|\nu| \leq m} \|T^\nu u\|^2_{L^2_{A_k}(\mathbf{R}^d)}. \tag{3.10}$$

For prove this proposition we need the following lemma.

Lemma 3.5. *Let $m \in \mathbf{N}\backslash\{0\}$. For all $\alpha \in \mathbf{N}^d$ with $0 < |\alpha| \leq m$, there exists $C > 0$ such that*

$$\forall \xi \in \mathbf{R}^d,\ \prod_{j=1}^{d} |\xi_j|^{2\alpha_j} \leq (1 + \|\xi\|^2)^m \leq C(1 + \sum_{0<|\alpha|\leq m} \prod_{j=1}^{d} |\xi_j|^{2\alpha_j}). \quad (3.11)$$

Proof of Proposition 3.4. Let u be in $H_k^m(\mathbf{R}^d)$, hence $u \in L^2_{A_k}(\mathbf{R}^d)$. Using Lemma 3.5 we deduce that for all $\alpha \in \mathbf{N}^d$ with $|\alpha| \leq m$, there exists a positive constant C such that

$$\sum_{0<|\alpha|\leq m} \int_{\mathbf{R}^d} (\prod_{j=1}^{d} |\xi_j|^{2\alpha_j}) |\mathcal{F}_k(u)(\xi)|^2 d\nu_k(\xi)$$
$$\leq C \int_{\mathbf{R}^d} (1 + \|\xi\|^2)^m |\mathcal{F}_k(u)(\xi)|^2 d\nu_k(\xi). \quad (3.12)$$

But from (3.3) we have for all $\xi \in \mathbf{R}^d$:

$$\mathcal{F}_k(T^\alpha u)(\xi) = i^{|\alpha|} \xi_1^{\alpha_1} \ldots \xi_d^{\alpha_d} \mathcal{F}_k(u)(\xi).$$

Thus from (3.12) we deduce that

$$\sum_{0<|\alpha|\leq m} \int_{\mathbf{R}^d} |T^\alpha u(x)|^2 A_k(x) \leq C \|u\|^2_{H_k^m(\mathbf{R}^d)}.$$

Then u belongs to E_m, and

$$\|u\|^2_{m,k} \leq C \|u\|^2_{H_k^m(\mathbf{R}^d)}.$$

Reciprocally let u be in E_m. From Lemma 3.5 we deduce that for all $\alpha \in \mathbf{N}^d$ with $|\alpha| \leq m$, there exists a positive constant C' such that

$$\int_{\mathbf{R}^d} (1 + \|\xi\|^2)^m |\mathcal{F}_k(u)(\xi)|^2 d\nu_k(\xi)$$
$$\leq C' \Big[\|u\|^2_{2,k} + \sum_{0<|\alpha|\leq m} \int_{\mathbf{R}^d} |T^\alpha u(x)|^2 A_k(x) \Big].$$

Thus u belongs to $H_k^m(\mathbf{R}^d)$ and

$$\|u\|^2_{H_k^m(\mathbf{R}^d)} \leq C' \|u\|^2_{m,k}.$$

□

Proposition 3.6.

(i) *For $s \in \mathbf{R}$ and $\mu \in \mathbf{N}^d$, the Dunkl operator T^μ is continuous from $H_k^s(\mathbf{R}^d)$ into $H_k^{s-|\mu|}(\mathbf{R}^d)$.*

(ii) *Let $p \in \mathbf{N}$. An element u is in $H_k^s(\mathbf{R}^d)$ if and only if for all $\mu \in \mathbf{N}^d$, with $|\mu| \leq p$, $T^\mu u$ belongs to $H_k^{s-p}(\mathbf{R}^d)$, and we have*

$$\|u\|_{H_k^s(\mathbf{R}^d)} \sim \sum_{|\mu|\leq p} \|T^\mu u\|_{H_k^{s-p}(\mathbf{R}^d)}.$$

Proof. (i) Let u be in $H_k^s(\mathbf{R}^d)$. From (3.3) we have

$$\int_{\mathbf{R}^d} (1+\|\xi\|^2)^{s-|\mu|} |\mathcal{F}_k(T^\mu u)(\xi)|^2 d\nu_k(\xi) = \int_{\mathbf{R}^d} (1+\|\xi\|^2)^{s-|\mu|} \|\xi^\mu\|^2 |\mathcal{F}_k(u)(\xi)|^2 d\nu_k(\xi).$$

Thus

$$\int_{\mathbf{R}^d} (1+\|\xi\|^2)^{s-|\mu|} |\mathcal{F}_k(T^\mu u)(\xi)|^2 d\nu_k(\xi) \leq \int_{\mathbf{R}^d} (1+\|\xi\|^2)^{s} |\mathcal{F}_k(u)(\xi)|^2 d\nu_k(\xi) < +\infty.$$

Then $T^\mu u$ belongs to $H_k^{s-|\mu|}(\mathbf{R}^d)$, and

$$\|T^\mu u\|_{H_k^{s-|\mu|}(\mathbf{R}^d)} \leq \|u\|_{H_k^s(\mathbf{R}^d)}.$$

(ii) We consider $p \in \mathbf{N}$ and $u \in H_k^s(\mathbf{R}^d)$. From (i) and Proposition 3.2 (ii) for all $\mu \in \mathbf{N}^d$, with $|\mu| \leq p$, we have

$$T^\mu u \in H_k^{s-|\mu|}(\mathbf{R}^d) \subset H_k^{s-p}(\mathbf{R}^d).$$

Then there exists a positive constant C such that

$$\|T^\mu u\|_{H_k^{s-p}(\mathbf{R}^d)} \leq C\|u\|_{H_k^s(\mathbf{R}^d)}.$$

This implies that

$$\sum_{|\mu|\leq p} \|T^\mu u\|_{H_k^{s-p}(\mathbf{R}^d)} \leq C'\|u\|_{H_k^s(\mathbf{R}^d)},$$

where C' is a positive constant.

Reciprocally let $p \in \mathbf{N}$ as for all $\mu \in \mathbf{N}^d$ with $|\mu| \leq p$, $T^\mu u$ belongs to $H_k^{s-p}(\mathbf{R}^d)$, then

$$\begin{aligned}\|u\|^2_{H_k^s(\mathbf{R}^d)} &= \int_{\mathbf{R}^d} (1+\|\xi\|^2)^s |\mathcal{F}_k(u)(\xi)|^2 d\nu_k(\xi) \\ &= \int_{\mathbf{R}^d} (1+\|\xi\|^2)^{s-p}(1+\|\xi\|^2)^p |\mathcal{F}_k(u)(\xi)|^2 d\nu_k(\xi).\end{aligned}$$

But from Lemma 3.5, there exists a positive constant C such that

$$(1+\|\xi\|^2)^p \leq C \sum_{|\mu|\leq p} \|\xi^\mu\|^2.$$

Hence from (3.3) we deduce that

$$\|u\|^2_{H_k^s(\mathbf{R}^d)} \leq C \sum_{|\mu|\leq p} \|T^\mu u\|^2_{H_k^{s-p}(\mathbf{R}^d)}.$$

This implies that u is in $H_k^s(\mathbf{R}^d)$. □

Proposition 3.7. *Let $p \in \mathbf{N}$ and $s \in \mathbf{R}$ such that $s > \frac{b+d+p}{2}$, then*

$$H_k^s(\mathbf{R}^d) \hookrightarrow C^p(\mathbf{R}^d),$$

with b the positive constant given in Remark 2.4.

Proof. Let u be in $H_k^s(\mathbf{R}^d)$ with $s \in \mathbf{R}$ such that $s > \frac{b+d}{2}$.
We have

$$\int_{\mathbf{R}^d} |\mathcal{F}_k(u)(\lambda)| d\nu_k(\lambda) = \int_{\mathbf{R}^d} (1+\|\lambda\|^2)^{-\frac{s}{2}}(1+\|\lambda\|^2)^{\frac{s}{2}} |\mathcal{F}_k(u)(\lambda)| d\nu_k(\lambda).$$

Using Hölder inequality we obtain

$$\int_{\mathbf{R}^d} |\mathcal{F}_k(u)(\lambda)| d\nu_k(\lambda) \leq \Big(\int_{\mathbf{R}^d} (1+\|\lambda\|^2)^{-s} d\nu_k(\lambda)\Big)^{\frac{1}{2}} \|u\|_{H_k^s(\mathbf{R}^d)}.$$

Thus from Remark 2.4, we deduce that there exists a positive constant C such that

$$\|\mathcal{F}_k(u)\|_{L^1_{\nu_k}(\mathbf{R}^d)} \leq C\|u\|_{H_k^s(\mathbf{R}^d)}. \tag{3.13}$$

Then

$$\mathcal{F}_k(u) \in L^1_{\nu_k}(\mathbf{R}^d).$$

Thus from (2.10) we have

$$u(x) = \int_{\mathbf{R}^d} \mathcal{F}_k(u)(\lambda) G_{i\lambda}(x) d\nu_k(\lambda), \quad a.e.\, x \in \mathbf{R}^d.$$

We identify u with the second member, then we deduce that u belongs to $C(\mathbf{R}^d)$ and using (3.13) we show that the injection of $H_k^s(\mathbf{R}^d)$ into $C(\mathbf{R}^d)$ is continuous.

Now let u be in $H_k^s(\mathbf{R}^d)$ with $s \in \mathbf{R}$ such that $s > \frac{b+d+p}{2}$ with p belongs to $\mathbf{N}\backslash\{0\}$. From (2.7), for all $x, \lambda \in \mathbf{R}^d$, and $\nu \in \mathbf{N}^d$ such that $|\nu| \leq p$, we have

$$|D_x^\nu G_{i\lambda}(x)| \leq C\|\lambda\|^p.$$

Using the same method as for $p = 0$, and the derivation theorem under the integral sign we deduce that

$$\forall\, x \in \mathbf{R}^d,\ D^\nu u(x) = \int_{\mathbf{R}^d} \mathcal{F}_k(u)(\lambda) D_x^\nu G_{i\lambda}(x) d\nu_k(\lambda).$$

Thus $D^n u$ belongs to $C(\mathbf{R}^d)$, for all $n \in \mathbf{N}$ such that $|\nu| \leq p$. Then we show that u is in $C^p(\mathbf{R}^d)$ and the injection of $H_k^s(\mathbf{R}^d)$ into $C^p(\mathbf{R}^d)$ is continuous. $\square$

4. **Cherednik linear symmetric systems**

Notation 4.1. For any interval I of $\mathbf{R}$ we define the mixed space-time spaces $C(I, H_k^s(\mathbf{R}^d))$, for $s \in \mathbf{R}$, as the spaces of functions from I into $H_k^s(\mathbf{R}^d)$ such that the map

$$t \mapsto \|u(t, .)\|_{H_k^s(\mathbf{R}^d)}$$

is continuous.

In this section, I designates the interval $[0, T)$, $T > 0$ and

$$u = (u_1, \dots, u_m),\ u_p \in C(I, H_k^s(\mathbf{R}^d)),$$

a vector with m components elements of $C(I, H_k^s(\mathbf{R}^d))$. Let $(A_p)_{0\leq p\leq d}$ a family of functions from $I \times \mathbf{R}^d$ into the space of $m \times m$ matrices with real coefficients $a_{p,i,j}(t, x)$ which are W-invariant with respect to x and whose all derivatives in $x \in \mathbf{R}^d$ are bounded and continuous functions of (t, x).

For a given $f \in [C(I, H_k^s(\mathbf{R}^d))]^m$ and $v \in [H_k^s(\mathbf{R}^d)]^m$, we find u in $[C(I, H_k^s(\mathbf{R}^d))]^m$ satisfying the following system (S)

$$\begin{cases} \partial_t u(t, x) - \displaystyle\sum_{j=1}^d (A_j T_j u)(t, x) - (A_0 u)(t, x) = f(t, x),\ (t, x) \in I \times \mathbf{R}^d \\ u|_{t=0} = v. \end{cases}$$

We shall first define the notion of symmetric systems.

Definition 4.2. The system (S) is symmetric, if and only if, for any $p \in \{1, \ldots, d\}$ and any $(t, x) \in I \times \mathbf{R}^d$ the matrices $A_p(t, x)$ are symmetric i.e. $a_{p,i,j}(t, x) = a_{p,j,i}(t, x)$.

In this section, we shall assume $s \in \mathbf{N}$ and denote by $\|u(t)\|_{s,k}$ the norm defined by

$$\|u(t)\|_{s,k}^2 = \sum_{\substack{1 \le p \le m \\ 0 \le |\mu| \le s}} \|T_x^\mu u_p(t)\|_{L^2_{A_k}(\mathbf{R}^d)}^2.$$

The aim of this section is to prove the following theorem.

Theorem 4.3. *Let (S) be a symmetric system. Assume that f is contained in $[C(I, H_k^s(\mathbf{R}^d))]^m$ and v in $[H_k^s(\mathbf{R}^d)]^m$, then there exists a unique solution u of (S) in*

$$[C(I, H_k^s(\mathbf{R}^d))]^m \bigcap [C^1(I, H_k^{s-1}(\mathbf{R}^d))]^m.$$

The proof of this theorem will be made in several steps:

A: We prove a priori estimates for the regular solutions of the system (S).

B: We apply the Friedrichs method.

C: We pass to the limit for regular solutions and we obtain the existence in all cases by the regularization of the Cauchy data.

D: We prove the uniqueness using the existence result of the adjoint system.

A: Energy estimates. The symmetric hypothesis is crucial for the energy estimates which are only valid for regular solutions. More precisely we have

Lemma 4.4. *(Energy Estimate in $[H_k^s(\mathbf{R}^d)]^m$). For any positive integer s, there exists a positive constant λ_s such that, for any function u in $[C^1(I, H_k^s(\mathbf{R}^d))]^m \bigcap [C(I, H_k^{s+1}(\mathbf{R}^d))]^m$, we have*

$$\|u(t)\|_{s,k} \le e^{\lambda_s t} \|u(0)\|_{s,k} + \int_0^t e^{\lambda_s (t-t')} \|f(t')\|_{s,k} dt', \qquad (4.1)$$

for all $t \in I$*,with*

$$f = \partial_t u - \sum_{p=1}^{d} A_p T_p u - A_0 u.$$

To prove Lemma 4.4, we need the following Lemma.

Lemma 4.5. *Let* g *a* C^1*-function on* $[0,T)$*,* a *and* b *two positive continuous functions. We assume*

$$\frac{d}{dt} g^2(t) \leq 2a(t)g^2(t) + 2b(t)|g(t)|. \tag{4.2}$$

Then, for $t \in [0,T)$*, we have*

$$|g(t)| \leq |g(0)| \exp \int_0^t a(s)ds + \int_0^t b(s) \exp\Big(\int_s^t a(\tau)d\tau\Big) ds.$$

Proof. To prove this lemma, let us set for $\varepsilon > 0$, $g_\varepsilon(t) = \Big(g^2(t) + \varepsilon\Big)^{\frac{1}{2}}$; the function g_ε is C^1, and we have $|g(t)| \leq g_\varepsilon(t)$. Thanks to the inequality (4.2), we have

$$\frac{d}{dt}(g^2)(t) \leq 2a(t)g_\varepsilon^2(t) + 2b(t)g_\varepsilon(t).$$

As $\frac{d}{dt}(g^2)(t) = \frac{d}{dt}(g_\varepsilon^2)(t)$. Then

$$\frac{d}{dt}(g_\varepsilon^2)(t) = 2g_\varepsilon(t)\frac{dg_\varepsilon}{dt}(t) \leq 2a(t)g_\varepsilon^2(t) + 2b(t)g_\varepsilon(t).$$

Since for all $t \in [0,T)$ $g_\varepsilon(t) > 0$, we deduce then

$$\frac{dg_\varepsilon}{dt}(t) \leq a(t)g_\varepsilon(t) + b(t).$$

Thus

$$\frac{d}{dt}\Big(g_\varepsilon(t)\ \exp\Big(-\int_0^t a(s)ds\Big)\Big) \leq b(t) \exp\Big(-\int_0^t a(s)ds\Big).$$

So, for $t \in [0,T)$,

$$g_\varepsilon(t) \leq g_\varepsilon(0) \exp \int_0^t a(s)ds + \int_0^t b(s) \exp\Big(\int_s^t a(\tau)d\tau\Big) ds.$$

Thus, we obtain the conclusion of the lemma by tending ε to zero. $\square$

Proof. of Lemma 4.4. We prove this estimate by induction on s. We firstly assume that u belongs to $[C^1(I, L^2_{A_k}(\mathbf{R}^d))]^m \bigcap [C(I, H^1_k(\mathbf{R}^d))]^m$. We then have $f \in [C(I, L^2_{A_k}(\mathbf{R}^d))]^m$, and the function $t \mapsto \|u(t)\|^2_{0,k}$ is C^1 on the interval I. By definition of f we have

$$\begin{aligned}\frac{d}{dt}\|u(t)\|^2_{0,k} &= 2\langle \partial_t u, u\rangle_{L^2_{A_k}(\mathbf{R}^d)} \\ &= 2\langle f, u\rangle_{L^2_{A_k}(\mathbf{R}^d)} + 2\langle A_0 u, u\rangle_{L^2_{A_k}(\mathbf{R}^d)} + 2\sum_{p=1}^{d}\langle A_p T_p u, u\rangle_{L^2_{A_k}(\mathbf{R}^d)}.\end{aligned}$$

We will estimate the third term of the sum above by using the symmetric hypothesis of the matrix A_p. In fact from (2.4) and (2.5) we have

$$\begin{aligned}\langle A_p T_p u, u\rangle_{L^2_{A_k}(\mathbf{R}^d)} &= \sum_{1\le i,j\le m}\int_{\mathbf{R}^d} a_{p,i,j}(t,x)[(T_p)_x\, u_j(t,x)]u_i(t,x)A_k(x)dx \\ &= -\sum_{1\le i,j\le m}\int_{\mathbf{R}^d} a_{p,i,j}(t,x)[(T_p)_x\, u_i(t,x)]u_j(t,x)A_k(x)dx \\ &\qquad - \sum_{1\le i,j\le m}\int_{\mathbf{R}^d}[(T_p)_x\, a_{p,i,j}(t,x)]u_j(t,x)u_i(t,x)A_k(x)dx.\end{aligned}$$

The matrix A_p being symmetric, we have

$$-\sum_{1\le i,j\le m}\int_{\mathbf{R}^d} a_{p,i,j}(t,x)T_p u_i(t,x)u_j(t,x)A_k(x)dx = -\langle A_p T_p u, u\rangle_{L^2_{A_k}(\mathbf{R}^d)}.$$

Thus

$$\begin{aligned}&\langle A_p T_p u, u\rangle_{L^2_{A_k}(\mathbf{R}^d)} = \\ &\qquad -\frac{1}{2}\sum_{1\le i,j\le m}\int_{\mathbf{R}^d}(T_p\, a_{p,i,j}(t,x))u_i(t,x)u_j(t,x)A_k(x)dx.\end{aligned}$$

Since the coefficients of the matrix A_p, as well as their derivatives are bounded on $I \times \mathbf{R}^d$, there exists a positive constant λ_0 such that

$$\frac{d}{dt}\|u(t)\|^2_{0,k} \le 2\|f(t)\|_{0,k}\|u(t)\|_{0,k} + 2\lambda_0\|u(t)\|^2_{0,k}. \tag{4.3}$$

To complete the proof of Lemma 4.4 in the case $s = 0$ it suffices to apply Lemma 4.5. We assume now that Lemma 4.4 is proved for s.

Let u the function of $[C^1(I, H_k^{s+1}(\mathbf{R}^d))]^m \cap [C(I, H_k^{s+2}(\mathbf{R}^d))]^m$, we now introduce the function (with $m(d+1)$ components) U defined by

$$U = (u, T_1 u, \dots, T_d u).$$

Since

$$\partial_t u = f + \sum_{p=1}^{d} A_p T_p u + A_0 u,$$

for any $j \in \{1, \dots, d\}$, applying the operator T_j on the last equation and using (2.4), we obtain

$$\partial_t (T_j u) = \sum_{p=1}^{d} A_p T_p (T_j u) + \sum_{p=1}^{d} (T_j A_p) T_p u + T_j (A_0 u) + T_j f.$$

We can then write

$$\partial_t U = \sum_{p=1}^{d} B_p T_p U + B_0 U + F,$$

with

$$F = (f, T_1 f, \dots, T_d f),$$

and

$$B_p = \begin{pmatrix} A_p & 0 & . & . & 0 \\ 0 & A_p & 0 & . & . \\ . & 0 & . & . & . \\ . & . & . & . & 0 \\ 0 & . & . & 0 & A_p \end{pmatrix}, \quad p = 1, \dots, d,$$

and the coefficients of B_0 can be calculated from the coefficients of A_p and from $T_j A_p$ with $(p = 0, \dots, d)$ and $(j = 1, \dots, d)$. Using the induction hypothesis we then deduce the result, and the proof of Lemma 4.4 is finished. $\square$

B: Estimate about the approximated solution. We notice that the necessary hypothesis to the proof of the inequalities of Lemma 4.4 require exactly one more derivative than the regularity which appears in the statement of the theorem that we have to prove. We then have to regularize the system (S) by adapting the Friedrichs method. More precisely we consider the system (S_n) defined by

$$(S_n)\begin{cases}\partial_t u_n - \sum_{p=1}^{d} J_n(A_p T_p(J_n u_n)) - J_n(A_0 J_n u_n) = J_n f\\ u_{n\,|t=0} = J_n u_0,\end{cases}$$

with J_n is the cut off operator given by

$$J_n w := (J_n w_1, \dots, J_n w_m) \quad \text{and} \quad J_n w_j := \mathcal{F}_k^{-1}(1_{B(0,n)}(\xi)\mathcal{F}_k(w_j)), \quad (4.4)$$

for $j = 1, \dots, m$.

Now we state the following proposition (cf. [2] p. 389) which we need in the sequel of this step.

Proposition 4.6. *Let E be a Banach space, I an open interval of $\mathbf{R}$, $f \in C(I, E)$, $u_0 \in E$ and M be a continuous map from I into $\mathcal{L}(E)$, the set of linear continuous applications from E into itself. There exists a unique solution $u \in C^1(I, E)$ satisfying*

$$\begin{cases}\frac{du}{dt} & = M(t)u + f\\ u_{\,|t=0} & = u_0.\end{cases}$$

By taking $E = [L^2_{A_k}(\mathbf{R}^d)]^m$, and using the continuity of the operators $T_p J_n$ on $[L^2_{A_k}(\mathbf{R}^d)]^m$, the system (S_n) appears as an evolution equation

$$\begin{cases}\frac{du_n}{dt} & = M_n(t)u_n + J_n f\\ u_{n\,|t=0} & = J_n u_0\end{cases}$$

on $[L^2_{A_k}(\mathbf{R}^d)]^m$, where

$$t \mapsto M_n(t) := \sum_{p=1}^{d} J_n A_p(t, .) T_p J_n + J_n A_0(t, .) J_n,$$

is a continuous application from I into $\mathcal{L}([L^2_{A_k}(\mathbf{R}^d)]^m)$. Then from Proposition 4.6 there exists a unique function u_n continuous on I with values in $[L^2_{A_k}(\mathbf{R}^d)]^m$. Moreover, as the matrices A_p are C^∞ functions of t, $J_n f \in [C(I, L^2_{A_k}(\mathbf{R}^d))]^m$ and u_n verify

$$\frac{du_n}{dt} = M_n(t)u_n + J_n f.$$

Then $\frac{du_n}{dt} \in [C(I, L^2_{A_k}(\mathbf{R}^d))]^m$ which implies that $u_n \in [C^1(I, L^2_{A_k}(\mathbf{R}^d))]^m$. Moreover, as $J_n^2 = J_n$, it is obvious that $J_n u_n$ is also a solution of (S_n). We apply Proposition 4.6 we deduce that $J_n u_n = u_n$. The function u_n is then belongs to $[C^1(I, H^s_k(\mathbf{R}^d))]^m$ for any integer s and so (S_n) can be written as

$$(S_n) \begin{cases} \partial_t u_n - \sum_{p=1}^{d} J_n(A_p T_p u_n) - J_n(A_0 u_n) & = J_n f \\ u_{n\,|t=0} & = J_n u_0. \end{cases}$$

Now, let us estimate the evolution of $\|u_n(t)\|_{s,k}$.

Lemma 4.7. *For any positive integer s, there exists a positive constant λ_s such that for any integer n and any t in the interval I, we have*

$$\|u_n(t)\|_{s,k} \leq e^{\lambda_s t}\|J_n u(0)\|_{s,k} + \int_0^t e^{\lambda_s(t-t')}\|J_n f(t')\|_{s,k} dt'.$$

Proof. The proof uses the same ideas as in Lemma 4.4. $\square$

C: Construction of solution. This step consists on the proof of the following existence and uniqueness result:

Proposition 4.8. *For $s \geq 0$, we consider the symmetric system*

$$(S) \begin{cases} \partial_t u - \sum_{p=1}^{d} A_p T_p u - A_0 u & = f \\ u_{\,|t=0} & = v, \end{cases}$$

with f in $[C(I, H^{s+3}_k(\mathbf{R}^d))]^m$ and v in $[H^{s+3}_k(\mathbf{R}^d)]^m$. There exists a unique solution u belonging to the space $[C^1(I, H^s_k(\mathbf{R}^d))]^m \cap [C(I, H^{s+1}_k(\mathbf{R}^d))]^m$ and satisfying the energy estimate :

$$\|u(t)\|_{\sigma,k} \leq e^{\lambda_s t}\|v\|_{\sigma,k} + \int_0^t e^{\lambda_s(t-\tau)}\|f(\tau)\|_{\sigma,k} d\tau, \tag{4.5}$$

for all $\sigma \leq s+3$ and $t \in I$.

Proof. Us consider the sequence $(u_n)_n$ defined by the Friedrichs method and let us prove that this sequence is a Cauchy one in $[L^\infty(I, H^{s+1}_k(\mathbf{R}^d))]^m$.

We put $V_{n,p} = u_{n+p} - u_n$, we have

$$\begin{cases} \partial_t V_{n,p} - \sum_{j=1}^{d} J_{n+p}(A_j T_j V_{n,p}) - J_{n+p}(A_0 V_{n,p}) & = & f_{n,p} \\ V_{n,p\ |t=0} & = & (J_{n+p} - J_n)v, \end{cases}$$

with

$$f_{n,p} = -\sum_{j=1}^{d}(J_{n+p} - J_n)(A_j T_j V_{n,p}) \\ - (J_{n+p} - J_n)(A_0 V_{n,p}) + (J_{n+p} - J_n)f.$$

From Lemma 4.7 it follows that the sequence $(u_n)_{n\in\mathbf{N}}$ is bounded in $[L^\infty(I, H_k^{s+3}(\mathbf{R}^d))]^m$. Moreover, by a simple calculation we find

$$\|(J_{n+p} - J_n)(A_j T_j V_{n,p})\|_{s+1,k} \le \frac{C}{n}\|A_j T_j V_{n,p}\|_{s+2,k} \le \frac{C}{n}\|u_n(t)\|_{s+3,k}.$$

Similarly, we have

$$\|(J_{n+p}-J_n)(A_0 V_{n,p})+(J_{n+p}-J_n)f\|_{s+1,k} \le \frac{C}{n}\Big(\|u_n(t)\|_{s+3,k}+\|f(t)\|_{s+3,k}\Big).$$

By Lemma 4.7 we deduce that

$$\|V_{n,p}(t)\|_{s+1,k} \le \frac{C}{n}\, e^{\lambda_s t}.$$

Then $(u_n)_n$ is a Cauchy sequence in $[L^\infty(I, H_k^{s+1}(\mathbf{R}^d))]^m$. We then have the existence of a solution u of (S) in $[C(I, H_k^{s+1}(\mathbf{R}^d))]^m$. Moreover by the equation stated in (S) we deduce that $\partial_t u$ is in $[C(I, H_k^s(\mathbf{R}^d))]^m$, and so u is in $[C^1(I, H_k^s(\mathbf{R}^d))]^m$. The uniqueness follows immediately from Lemma 4.7.

Finally we will prove the inequality (4.5). From Lemma 4.7 we have

$$\|u_n(t)\|_{s+3,k} \le e^{\lambda_s t}\|J_n u(0)\|_{s+3,k} + \int_0^t e^{\lambda_s(t-\tau)}\|J_n f(\tau)\|_{s+3,k} d\tau.$$

Thus

$$\limsup_{n\to\infty} \|u_n(t)\|_{s+3,k} \le e^{\lambda_s t}\|v\|_{s+3,k} + \int_0^t e^{\lambda_s(t-\tau)}\|f(\tau)\|_{s+3,k} d\tau.$$

Since for any $t \in I$, the sequence $(u_n(t))_{n\in \mathbf{N}}$ tends to $u(t)$ in $[H_k^{s+1}(\mathbf{R}^d)]^m$, $(u_n(t))_{n\in \mathbf{N}}$ converge weakly to $u(t)$ in $[H_k^{s+3}(\mathbf{R}^d)]^m$, and then

$$u(t) \in [H_k^{s+3}(\mathbf{R}^d)]^m \text{ and } \|u(t)\|_{s+3,k} \leq \lim_{n\to\infty} \sup \|u_n(t)\|_{s+3,k}.$$

Now, combining the uniform bounds for$(u_n)_n$ in $[L^\infty(I; H_k^{s+3}(\mathbf{R}^d))]^m$ with the above result on convergence in $[L^\infty(I; H_k^{s+1}(\mathbf{R}^d))]^m$ and using the interpolation inequality (3.8), we obtain that for any $s' < s$, the sequence $(u_n)_n$ converges in $[C(I; H_k^{s'+3}(\mathbf{R}^d))]^m$. Thus, u belongs to the set $[C(I; H_k^{s'+3}(\mathbf{R}^d))]^m$. Using the fact that u is a solution of (S), we get that u belongs to $[C(I; H_k^{s'+3}(\mathbf{R}^d))]^m \bigcap [C^1(I; H_k^{s'+2}(\mathbf{R}^d))]^m$. Thus, by passing to the limit in Lemma 4.7 we obtain the inequality (4.5). The Proposition 4.8 is thus proved. □

Now we will prove the existence part of Theorem 4.3.

Proposition 4.9. *Let s be an integer. If v is in $[H_k^s(\mathbf{R}^d)]^m$ and f is in $[C(I, H_k^s(\mathbf{R}^d))]^m$, then there exists a solution of a symmetric system (S) in the space $[C(I, H_k^s(\mathbf{R}^d))]^m \cap [C^1(I, H_k^{s-1}(\mathbf{R}^d))]$.*

Proof. We consider the sequence $(\tilde{u}_n)_{n\in\mathbf{N}}$ of solutions of

$$\begin{cases} \partial_t \tilde{u}_n - \displaystyle\sum_{j=1}^d (A_j T_j \tilde{u}_n) - (A_0 \tilde{u}_n) & = J_n f \\ \tilde{u_n}_{|t=0} & = J_n v. \end{cases}$$

From Proposition 4.8 $(\tilde{u}_n)_n$ is in $[C^1(I, H_k^s(\mathbf{R}^d))]^m$. We will prove that $(\tilde{u}_n)_n$ is a Cauchy sequence in $[L^\infty(I, H_k^s(\mathbf{R}^d))]^m$. We put $\tilde{V}_{n,p} = \tilde{u}_{n+p} - \tilde{u}_n$. By difference, we find

$$\begin{cases} \partial_t \tilde{V}_{n,p} - \displaystyle\sum_{j=1}^d A_j T_j \tilde{V}_{n,p} - A_0 \tilde{V}_{n,p} & = (J_{n+p} - J_n) f \\ \tilde{V_{n,p}}_{|t=0} & = (J_{n+p} - J_n) v. \end{cases}$$

By Lemma 4.7 we deduce that

$$\|\tilde{V}_{n,p}\|_{s,k} \leq e^{\lambda_s t} \|(J_{n+p} - J_n)v\|_{s,k} + \int_0^t e^{\lambda_s (t-\tau)} \|(J_{n+p} - J_n) f(\tau)\|_{s,k} d\tau.$$

Since f is in $[C(I, H_k^s(\mathbf{R}^d))]^m$, the sequence $(J_n f)_n$ converges to f in $[L^\infty([0,T], H_k^s(\mathbf{R}^d))]^m$, and since v is in $[H_k^s(\mathbf{R}^d)]^m$, the sequence $(J_n v)_n$

converge to v in $[H^s_k(\mathbf{R}^d)]^m$ and so $(\tilde{u}_n)_n$ is a Cauchy sequence in

$$[L^\infty(I, H^s_k(\mathbf{R}^d))]^m.$$

Hence it converges to a function u of $[C(I, H^s_k(\mathbf{R}^d))]^m$, solution of the system (S). Thus $\partial_t u$ is in $[C(I, H^{s-1}_k(\mathbf{R}^d))]^m$ and the proposition is proved. $\square$

The existence in Theorem 4.3 is then proved as well as the uniqueness, when $s \geq 1$.

D: Uniqueness of solutions. In the following we give the result of uniqueness for $s = 0$ and hence Theorem 4.3 is proved.

Proposition 4.10. *Let u be a solution in $[C(I, L^2_{A_k}(\mathbf{R}^d))]^m$ of the symmetric system*

$$(S) \begin{cases} \partial_t u - \sum_{j=1}^d A_j T_j u - A_0 u & = 0 \\ u_{\,|t=0} & = 0. \end{cases}$$

Then $u \equiv 0$.

Proof. Let ψ a function in $[D((0,T), D(\mathbf{R}^d))]^m$; we consider the following system

$$({}^tS) \begin{cases} -\partial_t \varphi + \sum_{j=1}^d T_j(A_j \varphi) - {}^tA_0 \varphi & = \psi \\ \varphi_{\,|t=T} & = 0. \end{cases}$$

Since

$$T_j(A_j\varphi) = A_j T_j \varphi + (T_j A_j)\varphi,$$

the system $({}^tS)$ can be written

$$({}^tS) \begin{cases} -\partial_t \varphi + \sum_{j=1}^d A_j T_j \varphi - \tilde{A}_0 \varphi & = \psi \\ \varphi_{\,|t=T} & = 0, \end{cases}$$

with

$$\tilde{A}_0 = {}^tA_0 - \sum_{j=1}^d T_j A_j.$$

Due to Proposition 4.8, for any integer s there exists a solution φ of $({}^tS)$ in $[C^1([0,T],H^s_k(\mathbf{R}))]^m$. We then have

$$\begin{aligned}\langle u,\psi\rangle_k &= \langle u,-\partial_t\varphi+\sum_{j=1}^d A_jT_j\varphi-\tilde{A}_0\varphi\rangle_k\\ &= -\int_I\langle u(t,.),\partial_t\varphi(t,.)\rangle_k dt+\sum_{j=1}^d\int_{I\times\mathbf{R}^d}u(t,x)T_j(A_j\varphi)(t,x)A_k(x)dtdx\\ &\qquad\qquad -\int_{I\times\mathbf{R}^d}u(t,x)\ {}^tA_0\varphi(t,x)A_k(x)dtdx,\end{aligned}$$

with $\langle .,.\rangle_k$ defined by

$$\begin{aligned}\langle u,\chi\rangle_k &= \int_I\langle u(t,.),\chi(t,.)\rangle_k dt\\ &= \int_{I\times\mathbf{R}^d}u(t,x)\chi(t,x)A_k(x)dxdt,\ \chi\in[\mathcal{S}(\mathbf{R},\mathcal{S}_2(\mathbf{R}^d))]^m.\end{aligned}$$

By using that $u(t,.)$ is in $[L^2_{A_k}(\mathbf{R}^d)]^m$ for any t in I and the fact that A_j is symmetric we obtain

$$\int_{I\times\mathbf{R}^d}u(t,x)T_j(A_j\varphi)(t,x)A_k(x)dtdx=-\int_I\langle A_jT_ju(t,.),\varphi(t,.)\rangle_k dt.$$

So

$$\langle u,\psi\rangle_k=-\int_I\langle u(t,.),\partial_t\varphi(t,.)\rangle_k dt-\sum_{j=1}^d\langle A_jT_ju+A_0u,\varphi\rangle_k.$$

As u is not very regular, we have to justify the integration by parts in time on the quantity $\int_I\langle u(t,.),\partial_t\varphi(t,.)\rangle_k dt$. Since $J_nu(.,x), J_n\varphi(.,x)$ are C^1 functions on I, then by integration by parts, we obtain, for any $x\in\mathbf{R}^d$,

$$\begin{aligned}\int_I J_nu(t,x)\partial_t(J_n\varphi)(t,x)dt &= -J_nu(T,x)J_n\varphi(T,x)\\ &\qquad + J_nu(0,x)J_n\varphi(0,x)+\int_I\partial_tJ_nu(t,x)J_n\varphi(t,x)dt.\end{aligned}$$

Since $u(0,.)=\varphi(T,.)=0$, we have

$$-\int_I J_nu(t,x)\partial_t\varphi(t,x)dt=\int_I\partial_t(J_nu)(t,x)J_n\varphi(t,x)dt.$$

Integrating with respect to $A_k(x)dx$ we obtain

$$-\int_{I\times\mathbf{R}^d} J_n u(t,x)\partial_t J_n\varphi(t,x)A_k(x)dtdx = \int_I \langle \partial_t(J_n u)(t,.), J_n\varphi(t,.)\rangle_k dt. \tag{4.6}$$

Since u is in $[C(I, L^2_{A_k}(\mathbf{R}^d))]^m \cap [C^1(I, H_k^{-1}(\mathbf{R}^d))]^m$, we have

$$\lim_{n\to\infty} J_n u = u \text{ in } [L^\infty(I, L^2_{A_k}(\mathbf{R}^d))]^m$$

and

$$\lim_{n\to\infty} J_n \partial_t u = \partial_t u \text{ in } [L^\infty(I, H_k^{-1}(\mathbf{R}^d))]^m.$$

Similarly, we see that

$$\lim_{n\to\infty} J_n \varphi = \varphi \text{ in } [L^\infty(I, H^1_{A_k}(\mathbf{R}^d))]^m$$

and

$$\lim_{n\to\infty} J_n \partial_t\varphi = \partial_t\varphi \text{ in } [L^\infty(I, L^2_k(\mathbf{R}^d))]^m.$$

By passing to the limit in (4.6) we obtain

$$-\int_I \langle u(t,.), \partial_t\varphi(t,.)\rangle_k dt = \int_I \langle \partial_t u(t,.), \varphi(t,.)\rangle_k dt.$$

Hence

$$\langle u, \psi\rangle_k = \int_I \langle \partial_t u(t,.) - \sum_{j=1}^d \langle A_j T_j u(t,.) - A_0 u(t,.), \varphi(t,.)\rangle_k dt.$$

However since u is a solution of (S) with $f \equiv 0$, then $u \equiv 0$. This ends the proof. $\square$

5. **Applications**

5.1. **The Cherednik-wave equations with variable coefficients**

For $t \in \mathbf{R}$ and $x \in \mathbf{R}^d$, let $P(t,x,\partial_t,T_x)$ a differential-difference operator of degree 2 defined by:

$$P(u) = \partial_t^2 u - \mathrm{div}_k[A.\nabla_{k,x} u] + Q(t,x,\partial_t u, T_x u), \tag{5.1}$$

where

$$\nabla_{k,x} u := (T_1 \, u, \dots, T_d \, u)\,, \ \mathrm{div}_k\,(v_1, \dots, v_d) := \sum_{i=1}^{d} T_i v_i,$$

A is a real symmetric matrix such that there exists $m > 0$ satisfying

$$\langle A(t,x)\xi, \xi\rangle \geq m\|\xi\|^2, \quad \text{for all } (t,x) \in \mathbf{R} \times \mathbf{R}^d, \text{ and } \xi \in \mathbf{R}^d \tag{5.2}$$

and $Q(t, x, \partial_t u, T_x u)$ is differential-difference operator of degree 1, and we assume also that the matrix A is W-invariant with respect to x; the coefficients of A and Q are C^∞ and all derivatives are bounded. If we put $B = \sqrt{A}$ it is easy to see that the coefficients of B are C^∞ and all derivatives are bounded.

We introduce the vector U with $d + 2$ components

$$U = (u, \partial_t u, B\nabla_{k,x} u)\,. \tag{5.3}$$

Then, the equation $P(u) = f$ can be written as

$$\partial_t U = \left(\sum_{p=1}^{d} A_p T_p\right) U + A_0 U + (0, f, 0)\,, \tag{5.4}$$

with

$$A_p = \begin{pmatrix} 0 & . & . & . & . & . & 0 \\ . & 0 & b_{p\,1} & . & . & . & b_{p\,d} \\ . & b_{1\,p} & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & b_{d\,p} & 0 & . & . & . & 0 \end{pmatrix}$$

and $B = (b_{ij})$. Thus the system (5.4) is symmetric and from Theorem 4.3 we deduce the following.

Theorem 5.1. *For all $s \in \mathbf{N}$ and $u_0 \in H_k^{s+1}(\mathbf{R}^d)$, $u_1 \in H_k^s(\mathbf{R}^d)$ and f belongs to $C(\mathbf{R}, H_k^s(\mathbf{R}^d))$, there exists a unique solution*

$$u \in C^1(\mathbf{R}, H_k^s(\mathbf{R}^d)) \cap C(\mathbf{R}, H_k^{s+1}(\mathbf{R}^d))$$

such that

$$\begin{cases} \partial_t^2 u - \mathrm{div}_k[A.\nabla_{k,x} u] + Q(t, x, \partial_t u, T_x u) & = f \\ u_{\,|t=0} & = u_0 \\ \partial_t u_{\,|t=0} & = u_1. \end{cases}$$

5.2. Finite speed of propagation

Theorem 5.2. *Let (S) be a symmetric system. There exists a positive constant C_0 such that, for any positive real R, any function*

$$f \in [C(I, H^1_k(\mathbf{R}^d))]^m \text{ and any } v \in [H^1_k(\mathbf{R}^d)]^m$$

satisfying

$$f(t,x) \equiv 0 \quad \text{for} \quad \|x\| < R - C_0 t \tag{5.5}$$

$$v(x) \equiv 0 \quad \text{for} \quad \|x\| < R, \tag{5.6}$$

the unique solution u of system (S) belongs to $[C(I, H^1_k(\mathbf{R}^d))]^m$ with

$$u(t,x) \equiv 0 \quad \text{for} \quad \|x\| < R - C_0 t.$$

Proof. For $\tau \geq 1$, we put

$$u_\tau(t,x) = \exp\Big(\tau(-t + \psi(x))\Big)u(t,x),$$

where the function $\psi \in \mathcal{E}(\mathbf{R}^d)$ will be chosen later.
By a simple calculation we see that

$$\partial_t u_\tau - \sum_{j=1}^d A_j T_j u_\tau - B_\tau u_\tau = f_\tau,$$

with

$$f_\tau(t,x) = \exp\Big(\tau\Big[-t + \psi(x)\Big]\Big)f(t,x), \quad B_\tau = A_0 + \tau(-Id - \sum_{j=1}^d (T_j\psi)A_j).$$

There exists a positive constant K such that if $\|T_j\,\psi\|_{L^\infty_k(\mathbf{R}^d)} \leq K$ for any $j = 1, \ldots, d$, we have for any (t,x)

$$\langle \mathrm{Re}(B_\tau y), \bar{y}\rangle \leq \langle \mathrm{Re}(A_0 y), \bar{y}\rangle \quad \text{for all} \quad \tau \geq 1 \quad \text{and} \quad y \in \mathbf{C}^m.$$

We proceed as in the proof of energy estimate (4.1), we obtain the existence of positive constant δ_0, independent of τ, such that for any t in I, we have

$$\|u_\tau(t)\|_{0,k} \leq e^{\delta_0 t}\|u_\tau(0)\|_{0,k} + \int_0^t e^{\delta_0(t-t')}\|f_\tau(t')\|_{0,k}dt'. \tag{5.7}$$

We put $C_0 = \frac{1}{K}$ and choose $\psi = \psi(\|x\|)$ such that ψ is C^∞ and such that

$$-2\varepsilon + K(R - \|x\|) \leq \psi(x) \leq -\varepsilon + K(R - \|x\|).$$

There exists $\varepsilon > 0$ such that $\psi(x) \leq -\varepsilon + K(R - \|x\|)$. Hence

$$\|x\| \geq R - C_0 t, \quad \text{for all } (t,x) \quad \Longrightarrow -t + \psi(x) \leq -\varepsilon.$$

We tend τ to ∞ in (5.7), we deduce that

$$\lim_{\tau\to\infty} \int_{\mathbf{R}^d} \exp(2\tau(-t+\psi(x)))\|u(t,x)\|^2 A_k(x)dx = 0, \quad \text{for all } t \in I.$$

Then

$$u(t,x) = 0 \text{ on } \Big\{(t,x) \in I \times \mathbf{R}^d : \ t < \psi(x)\Big\}.$$

However if (t_0, x_0) verifies $\|x_0\| < R - C_0 t_0$, we can find a function ψ of precedent type such that $t_0 < \psi(x_0)$. Thus the theorem is proved. $\square$

Corollary 5.3. *Let (S) be a symmetric system. There exists a positive constant C_0 such that, for any positive real R, any function f belongs to $[C(I, L^2_{A_k}(\mathbf{R}^d))]^m$ and any $v \in [L^2_{A_k}(\mathbf{R}^d)]^m$ satisfying*

$$\begin{aligned} f(t,x) &\equiv 0 \quad \text{for} \quad \|x\| < R - C_0 t \qquad (5.8)\\ v(x) &\equiv 0 \quad \text{for} \quad \|x\| < R, \qquad (5.9)\end{aligned}$$

the unique solution u of system (S) belongs to $[C(I, L^2_{A_k}(\mathbf{R}^d))]^m$ with

$$u(t,x) \equiv 0 \quad \text{for} \quad \|x\| < R - C_0 t.$$

Proof. If $f_\varepsilon \in [C(I, H^1_k(\mathbf{R}^d))]^m$, $v_\varepsilon \in [H^1_k(\mathbf{R}^d))]^m$ are given such that $f_\varepsilon \to f$ in $[C(I, L^2_{A_k}(\mathbf{R}^d))]^m$ and $v_\varepsilon \to v$ in $[L^2_{A_k}(\mathbf{R}^d)]^m$, we know by Section 4 that the solution u_ε belongs to $[C(I, H^1_k(\mathbf{R}^d))]^m$ and verifies $u_\varepsilon \to u$ in $[C(I, L^2_{A_k}(\mathbf{R}^d))]^m$. Therefore, if we construct f_ε and v_ε satisfying (5.8) and (5.9) with R replaced by $R-\varepsilon$, we obtain the result by applying Theorem 5.2. To this end let us consider $\chi \in D(\mathbf{R}^d)$ and radial such that supp$\chi \subset B(0,1)$ and

$$\int_{\mathbf{R}^d} \chi(x) A_k(x) dx = 1.$$

For $\varepsilon > 0$, we put

$$\begin{array}{lllll} u_{0,\varepsilon} &=& \chi_\varepsilon *_k v &:=& (\chi_\varepsilon *_k v_1, \dots, \chi_\varepsilon *_k v_d),\\ f_\varepsilon(t,.) &=& \chi_\varepsilon *_k f(t,.) &:=& (\chi_\varepsilon *_k f_1(t,.), \dots, \chi_\varepsilon *_k f_d(t,.)),\end{array}$$

with

$$\chi_\varepsilon(x) = \frac{A_k(\frac{x}{\varepsilon})}{\varepsilon^d A_k(x)} \chi(\frac{x}{\varepsilon}).$$

The hypothesis (5.8) and (5.9) are then satisfied by f_ε and $u_{0,\varepsilon}$ if we replace R by $R-\varepsilon$. On the other hand the solution u_ε associated with f_ε and $u_{0,\varepsilon}$

is $[C^1(I, H_k^s(\mathbf{R}^d))]^m$ for any integer s. Finally applying Proposition 2.9 and Theorem 5.2 we obtain the result. □

Theorem 5.4. *Let (S) be a symmetric system. We assume that the functions $f \in [C(I, H_k^1(\mathbf{R}^d))]^m$ and $v \in [H_k^1(\mathbf{R}^d)]^m$ verify*

$$\begin{cases} f(t,x) & \equiv & 0 & \text{for} & \|x\| > R + C_0 t \\ v(x) & \equiv & 0 & \text{for} & \|x\| > R. \end{cases}$$

Then the unique solution u of system (S) belongs to $[C(I, H_k^1(\mathbf{R}^d))]^m$ with

$$u(t,x) \equiv 0 \quad \text{for } \|x\| > R + C_0 t.$$

Proof. The proof uses the same ideas as in Theorem 5.2. □

As above we obtain the following result.

Corollary 5.5. *Let (S) be a symmetric system. We assume that the functions $f \in [C(I, L^2_{A_k}(\mathbf{R}^d))]^m$ and $v \in [L^2_{A_k}(\mathbf{R}^d)]^m$ verify*

$$\begin{cases} f(t,x) & \equiv & 0 & \text{for} & \|x\| > R + C_0 t \\ v(x) & \equiv & 0 & \text{for} & \|x\| > R. \end{cases}$$

Then the unique solution u of system (S) belongs to $[C(I, L^2_{A_k}(\mathbf{R}^d))]^m$ with

$$u(t,x) \equiv 0 \quad \text{for } \|x\| > R + C_0 t.$$

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Hatem Mejjaoli
Department of Mathematics
College of Sciences
King Faisal University
Ahsaa, Kingdom of Saudi Arabia
hmejjaoli@kfu.edu.sa,
hatem.mejjaoli@ipest.rnu.tn