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The second Yamabe invariant with singularities


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Abstract

Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 3\). We suppose that \(g\) is a metric in the Sobolev space \(H^p_2(M, T^* M \otimes T^* M)\) with \(p > \frac{n}{2}\) and there exist a point \(P \in M\) and \(\delta > 0\) such that \(g\) is smooth in the ball \(B_p(\delta)\). We define the second Yamabe invariant with singularities as the infimum of the second eigenvalue of the singular Yamabe operator over a generalized class of conformal metrics to \(g\) and of volume 1. We show that this operator is attained by a generalized metric, we deduce nodal solutions to a Yamabe type equation with singularities.

Dedicated to the memory of T. Aubin.

1. Introduction

Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 3\). The problem of finding a metric conformal to the original one with constant scalar curvature was first formulated by Yamabe ([9]) and a variational method was initiated by this latter in an attempt to solve the problem. Unfortunately or fortunately a serious gap in the Yamabe problem was pointed out by Trudinger who addressed the question in the case of non positive scalar curvature ([9]). Aubin ([2]) solved the problem for arbitrary non locally conformally flat manifolds of dimension \(n \geq 6\). Finally Shoen ([8]) solved completely the problem using the positive-mass theorem found previously by Shoen himself and Yau. The method to solve the Yamabe problem could be described as follows: let \(u\) be a smooth positive function and let \(\bar{g} = u^{N-2}g\) be a conformal metric where \(N = 2n/(n-2)\). Up to a multiplying constant, the following equation is satisfied

\[ L_g(u) = S_{\bar{g}} |u|^{N-2} u \]

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where
\[ L_g = \frac{4(n-1)}{n-2} \Delta + S_g \]
and \( S_g \) denotes the scalar curvature of \( g \). \( L_g \) is conformally invariant called the conformal operator. Consequently, solving the Yamabe problem is equivalent to finding a smooth positive solution to the equation
\[ L_g(u) = ku^{N-1} \quad (1) \]
where \( k \) is a constant.

In order to obtain solutions to this equation, Yamabe defined the quantity
\[ \mu(M, g) = \inf_{u \in C^\infty(M), u > 0} Y(u) \]
where
\[ Y(u) = \frac{\int_M \left( \frac{4(n-1)}{n-2} |\nabla u|^2 + S_g u^2 \right) dv_g}{\left( \int_M |u|^N dv_g \right)^{2/N}}. \]

\( \mu(M, g) \) is called the Yamabe invariant, and \( Y \) the Yamabe functional. In the sequel we write \( \mu \) instead of \( \mu(M, g) \). Writing the Euler-Lagrange equation associated to \( Y \), we see that there exists a one to one correspondence between critical points of \( Y \) and solutions of equation (1). In particular, if \( u \) is a positive smooth function such that \( Y(u) = \mu \), then \( u \) is a solution of equation (1) and \( \bar{g} = u^{(N-2)} g \) is metric of constant scalar curvature.

The key point to solve the Yamabe problem is the following fundamental results due to Aubin ([2]). Let \( S_n \) be the unit euclidean sphere.

**Theorem 1.1.** Let \((M, g)\) be a compact Riemannian manifold of dimension \( n \geq 3 \). If \( \mu(M, g) < \mu(S_n) \), then there exists a positive smooth solution \( u \) such that \( Y(u) = \mu(M, g) \).

This strict inequality \( \mu(M, g) < \mu(S_n) \) avoids concentration phenomena. Explicitly \( \mu(S_n) = n(n-1)\omega_n^{2/n} \) where \( \omega_n \) stands for the volume of \( S_n \).

**Theorem 1.2.** Let \((M, g)\) be a compact Riemannian manifold of dimension \( n \geq 3 \). Then
\[ \mu(M, g) \leq \mu(S_n). \]
Moreover, the equality holds if and only if \((M,g)\) is conformally diffeomorphic to the sphere \(S^n\).

Amman and Humbert ([1]) defined the second Yamabe invariant as the infimum of the second eigenvalue of the Yamabe operator over the conformal class of the metric \(g\) with volume 1. Their method consists in considering the spectrum of the operator \(L_g\)

\[
\text{spec}(L_g) = \{\lambda_{1,g}, \lambda_{2,g}, \ldots\}
\]

where the eigenvalues \(\lambda_{1,g} < \lambda_{2,g} \ldots\) appear with their multiplicities. The variational characterization of \(\lambda_{1,g}\) is given by

\[
\lambda_{1,g} = \inf_{u \in C^\infty(M), u > 0} \frac{\int_M \left(\frac{4(n-1)}{n-2} |\nabla u|^2 + S_g u^2\right) dv_g}{\int_M u^2 dv_g}.
\]

Then they defined the \(k^{th}\) Yamabe invariant with \(k \in \mathbb{N}^*\), by

\[
\mu_k = \inf_{\tilde{g} \in [g]} \lambda_{k,\tilde{g}} \frac{Vol(M, \tilde{g})^{2/n}}{\text{Vol}(M, g)^{2/n}}
\]

where

\[
[g] = \{u^{N-2}g, \ u \in C^\infty(M), \ u > 0\}.
\]

With these notations \(\mu_1\) is the Yamabe invariant. They studied the second Yamabe invariant \(\mu_2\), they found that contrary to the Yamabe invariant, \(\mu_2\) cannot be attained by a regular metric. In other words, there does not exist \(\tilde{g} \in [g]\), such that

\[
\mu_2 = \lambda_{2,\tilde{g}} \frac{Vol(M, \tilde{g})^{2/n}}{\text{Vol}(M, g)^{2/n}}.
\]

In order to find minimizers, they enlarged the conformal class to a larger one. A generalized metric is the one of the form \(\tilde{g} = u^{N-2}g\), which is not necessarily positive and smooth, but only \(u \in L^N(M), u \geq 0, u \neq 0\) and where \(N = 2n/(n-2)\). The definitions of \(\lambda_{2,\tilde{g}}\) and of \(Vol(M, \tilde{g})^{2/n}\) can be extended to generalized metrics. The key points to solve this problem is the following theorems ([1]).

**Theorem 1.3.** Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 3\), then \(\mu_2\) is attained by a generalized metric in the following cases.

\[
\mu > 0, \ \mu_2 < \left[\left(\mu^{n/2} + (\mu(S_n))^n/2\right)^{2/n}\right]
\]
The assumptions of the last theorem are satisfied in the following cases

If \((M, g)\) is not locally conformally flat and, \(n \geq 11\) and \(\mu > 0\)

If \((M, g)\) is not locally conformally flat and, \(\mu = 0\) and \(n \geq 9\).

Theorem 1.5. Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 3\), assume that \(\mu_2\) is attained by a generalized metric \(\tilde{g} = u^{N-2}g\) then there exists a nodal solution \(w \in C^{2,\alpha}(M)\) of equation

\[
L_g(w) = \mu_2 |u|^{N-2}w
\]

such that

\[
|w| = u
\]

where \(\alpha \leq N - 2\).

Recently F. Madani studied (see [6]) the Yamabe problem with singularities when the metric \(g\) admits a finite number of points with singularities and is smooth outside these points. Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 3\), assume that \(g\) is a metric in the Sobolev space \(H^p_2(M, T^*M \otimes T^*M)\) with \(p > \frac{n}{2}\) and there exist a point \(P \in M\) and \(\delta > 0\) such that \(g\) is smooth in the ball \(B_p(\delta)\), and let \((H)\) be these assumptions. By Sobolev’s embedding, we have for \(p > \frac{n}{2}\), \(H^p_2(M, T^*M \otimes T^*M) \subset C^{1-\left[\frac{n}{p}\right],\beta}(M, T^*M \otimes T^*M)\), where \(\left[\frac{n}{p}\right]\) denotes the entire part of \(\frac{n}{p}\). Hence the metric satisfying assumption \((H)\) is of class \(C^{1-\left[\frac{n}{p}\right],\beta}\) with \(\beta \in (0,1)\) provided that \(p > n\). The Christoffel symbols belong to \(H^p_1(M)\) (to \(C^0(M)\) in case \(p > n\)), the Riemannian curvature tensor, the Ricci tensor and scalar curvature are in \(L^p(M)\).

F. Madani proved under the assumption \((H)\) the existence of a metric \(\tilde{g} = u^{N-2}g\) conformal to \(g\) such that \(u \in H^p_2(M), u > 0\) and the scalar curvature \(S_{\tilde{g}}\) of \(\tilde{g}\) is constant and \((M, g)\) is not conformal to the round sphere. Madani proceeded as follows: let \(u \in H^p_2(M), u > 0\) be a function and \(\tilde{g} = u^{N-2}g\) a particular conformal metric where \(N = 2n/(n - 2)\). Then, multiplying \(u\) by a constant, the following equation is satisfied

\[
L_g u = \frac{n - 2}{4(n - 1)} S_{\tilde{g}} |u|^{N-2}u
\]
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where

\[ L_g = \Delta_g + \frac{n-2}{4(n-1)} S_g \]

and the scalar curvature \( S_g \) is in \( L^p(M) \). Moreover \( L_g \) is weakly conformally invariant hence solving the singular Yamabe problem is equivalent to finding a positive solution \( u \in H^2(M) \) of

\[ L_g u = k|u|^{N-2} u \]  \hspace{1cm} (2)

where \( k \) is a constant. In order to obtain solutions of equation (2) we define the quantity

\[ \mu = \inf_{u \in H^2(M), u > 0} Y(u) \]

where

\[ Y(u) = \frac{\int_M \left( |\nabla u|^2 + \frac{(n-2)}{4(n-1)} S_g u^2 \right) \, dv_g}{\left( \int_M |u|^N \, dv_g \right)^{2/N}}. \]

\( \mu \) is called the Yamabe invariant with singularities. Writing the Euler-Lagrange equation associated to \( Y \), we see that there exists a one to one correspondence between critical points of \( Y \) and solutions of equation (2). In particular, if \( u \in H^2(M) \) is a positive function which minimizes \( Y \), then \( u \) is a solution of equation (2) and \( \bar{g} = u^{N-2} g \) is a metric of constant scalar curvature and \( \mu \) is attained by a particular conformal metric. The key points to solve the above problem are the following theorems ([6]).

**Theorem 1.6.** If \( p > n/2 \) and \( \mu < K^{-2} \) then equation 2 admits a positive solution \( u \in H^2(M) \subset C^{1,\beta}(M); [n/p] \) is the integer part of \( n/p \), \( \beta \in (0,1) \) which minimizes \( Y \), where \( K^2 = \frac{4}{n(n-1)} \omega_n^{-2/n} \) with \( \omega_n \) denotes the volume of \( S_n \). If \( p > n \) , then \( u \in H^2(M) \subset C^1(M) \).

**Theorem 1.7.** Let \((M,g)\) be a compact Riemannian manifold of dimension \( n \geq 3 \). \( g \) is a metric which satisfies the assumption (H). If \((M,g)\) is not conformal to the sphere \( S_n \) with the standard Riemannian structure then

\[ \mu < K^{-2} \]

**Theorem 1.8.** Let \((M,g)\) be a \( n \)-dimensional compact Riemannian manifold. If \( u \geq 0 \) is a non trivial weak solution in \( H^2_1(M) \) of equation
\[ \Delta u + hu = 0, \text{ with } h \in L^p(M) \text{ and } p > n/2, \] then \( u \in C^{1-[n/p],\beta} \) and \( u > 0; \) \( [n/p] \) is the integer part of \( n/p \) and \( \beta \in (0,1). \)

Denote by
\[ L^N_+(M) = \{ u \in L^N(M) : u \geq 0, u \neq 0 \}. \]

For regularity argument we need the following results

**Lemma 1.9.** Let \( u \in L^N_+(M) \) and \( v \in H^2_1(M) \) a weak solution to \( L_g(v) = u^{N-2}v, \) then
\[ v \in L^{N+\epsilon}(M) \]
for some \( \epsilon > 0. \)

The proof is the same as in ([6]) with some modifications. As a consequence of Lemma 7, \( v \in L^s(M), \forall s \geq 1. \)

**Proposition 1.10.** If \( g \in H^p_2(M, T^*M \otimes T^*M) \) is a Riemannian metric on \( M \) with \( p > n/2. \) If \( \bar{g} = u^{N-2}g \) is a conformal metric to \( g \) such that \( u \in H^2_1(M), u > 0 \) then \( L_g \) is weakly conformally invariant, which means that \( \forall v \in H^2_1(M), |u|^{N-1}L_{\bar{g}}(v) = L_g(uv) \) weakly. Moreover if \( \mu > 0, \) then \( L_g \) is coercive and invertible.

In this paper, let \((M,g)\) be a compact Riemannian manifold of dimension \( n \geq 3. \) We suppose that \( g \) is a metric in the Sobolev space \( H^p_2(M, T^*M \otimes T^*M) \) with \( p > n/2 \) and there exist a point \( P \in M \) and \( \delta > 0 \) such that \( g \) is smooth in the ball \( B_P(\delta) \) and we call these assumptions the condition \((H).\)

In the smooth case the operator \( L_g \) is an elliptic operator on \( M \) self-adjoint, and has a discrete spectrum \( \text{Spec}(L_g) = \{ \lambda_{1,g}, \lambda_{2,g}, \ldots \}, \) where the eigenvalues \( \lambda_{1,g} < \lambda_{2,g} \ldots \) appear with their multiplicities. These properties remain valid also in the case where \( S_g \in L^p(M). \) The variational characterization of \( \lambda_{1,g} \) is given by

\[
\lambda_{1,g} = \inf_{u \in H^2_1, u > 0} \frac{\int_M \left( |\nabla u|^2 + \frac{(n-2)}{4(n-1)} S_g u^2 \right) dv_g}{\int_M u^2 dv_g}
\]

Let \([g] = \{ u^{N-2}g : u \in H^2_1 \) and \( u > 0 \}),\) Let \( k \in \mathbb{N}^*, \) we define the \( k^{th} \) Yamabe invariant with singularities \( \mu_k \) as
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$$\mu_k = \inf_{\tilde{g}} \lambda_k \tilde{V}ol(M, \tilde{g})^{2/n}$$

with these notations, $\mu_1$ is the first Yamabe invariant with singularities.

In this work we are concerned with $\mu_2$. In order to find minimizers to $\mu_2$ we extend the conformal class to a larger one consisting of metrics of the form $\tilde{g} = u^{N-2}g$ where $u$ is no longer necessarily in $H^p(M)$ and positive but $u \in L^{N}_+(M) = \{L^N(M), u \geq 0, u \neq 0\}$ such metrics will be called for brevity generalized metrics. First we are going to show that if the singular Yamabe invariant $\mu \geq 0$ then $\mu_1$ it is exactly $\mu$. Next we consider $\mu_2$ and show that $\mu_2$ is attained by a conformal generalized metric.

Our main results state as follows:

**Theorem 1.11.** Let $(M, g)$ be a compact Riemannian manifold of dimension $n \geq 3$. We suppose that $g$ is a metric in the Sobolev space $H^p_2(M, T^*M \otimes T^*M)$ with $p > n/2$. If there exist a point $P \in M$ and $\delta > 0$ such that $g$ is smooth in the ball $B_P(\delta)$, then

$$\mu_1 = \mu.$$

**Theorem 1.12.** Let $(M, g)$ be a compact Riemannian manifold of dimension $n \geq 3$, we suppose that $g$ is a metric in the Sobolev space $H^p_2(M, T^*M \otimes T^*M)$ with $p > n/2$. There exist a point $P \in M$ and $\delta > 0$ such that $g$ is smooth in the ball $B_P(\delta)$. Assume that $\mu_2$ is attained by a metric $\tilde{g} = u^{N-2}g$ where $u \in L^{N}_+(M)$, then there exist a nodal solution $w \in C^{1-[n/p], \beta}(M)$, $\beta \in (0, 1)$, of equation

$$L_g w = \mu_2 u^{N-2}w.$$

Moreover there exist real numbers $a, b > 0$ such that

$$u = aw_+ + bw_-$$

with $w_+ = \sup(w, 0)$ and $w_- = \sup(-w, 0)$.

**Theorem 1.13.** Let $(M, g)$ be a compact Riemannian manifold of dimension $n \geq 3$, suppose that $g$ is a metric in the Sobolev space $H^p_2(M, T^*M \otimes T^*M)$ with $p > n/2$. There exist a point $P \in M$ and $\delta > 0$ such that $g$ is smooth in the ball $B_P(\delta)$ then $\mu_2$ is attained by a generalized metric in the following cases:
If \((M, g)\) is not locally conformally flat and, \(n \geq 11\) and \(\mu > 0\)
If \((M, g)\) is not locally conformally flat and, \(\mu = 0\) and \(n \geq 9\).

2. Generalized metrics and the Euler-Lagrange equation

Let
\[
L^N_+(M) = \left\{ u \in L^N_+(M) : u \geq 0, u \neq 0 \right\}
\]
where \(N = \frac{2n}{n-2}\).

As in ([1])

**Definition 2.1.** For all \(u \in L^N_+(M)\), we define \(Gr^u_k(H^2_1(M))\) to be the set of all \(k\)-dimensional subspaces of \(H^2_1(M)\) with \(\text{span}(v_1, v_2, \ldots, v_k) \in Gr^u_k(H^2_1(M))\) if and only if \(v_1, v_2, \ldots, v_k\) are linearly independent on \(M - u^{-1}(0)\).

Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 3\). For a generalized metric \(\overline{g}\) conformal to \(g\), we define
\[
\lambda_{k,\overline{g}} = \inf_{V \in Gr^u_k(H^2_1(M))} \sup_{v \in V} \frac{\int_M vL_g(v)dv_g}{\int_M |u|^{N-2}v^2dv_g}.
\]
We quote the following regularity theorem

**Theorem 2.2.** [7] On a \(n\)-dimensional compact Riemannian manifold \((M, g)\), if \(u \geq 0\) is a non trivial weak solution in \(H^2_1(M)\) of the equation
\[
\Delta u + hu = cu^{N-1}
\]
with \(h \in L^p(M)\) and \(p > n/2\), then
\[
u \in H^p_2(M) \subset C^{1-[n/p],\beta}(M)
\]
and \(u > 0\), where \([n/p]\) denotes the integer part of \(n/p\) and \(\beta \in (0, 1)\).

**Proposition 2.3.** Let \((v_m)\) be a sequence in \(H^2_1(M)\) such that \(v_m \to v\) strongly in \(L^2(M)\), then for all any \(u \in L^N_+(M)\)
\[
\int_M u^{N-2}(v^2 - v_m^2)dv_g \to 0.
\]

**Proof.** The proof is the same as in ([3]).
Proposition 2.4. If $\mu > 0$, then for all $u \in L^N_+(M)$, there exist two functions $v, w$ in $H^2_1(M)$ with $v \geq 0$ satisfying in the weak sense the equations

$$L_g v = \lambda_{1,\tilde{g}} u^{N-2} v$$

(7)

and

$$L_g w = \lambda_{2,\tilde{g}} u^{N-2} w$$

(8)

Moreover we can choose $v$ and $w$ such that

$$\int_M u^{N-2} w^2 dv_g = \int_M u^{N-2} v^2 dv_g = 1$$

and

$$\int_M u^{N-2} wvdv_g = 0.$$  

(9)

Proof. Let $(v_m)_m$ be a minimizing sequence for $\lambda_{1,\tilde{g}}$ i.e. a sequence $v_m \in H^2_1(M)$ such that

$$\lim_m \frac{\int_M v_m L_g(v_m) dv_g}{\int_M |u|^{N-2} v_m^2 dv_g} = \lambda_{1,\tilde{g}}$$

It is well known that $(|v_m|)_m$ is also minimizing sequence. Hence we can assume that $v_m \geq 0$. We normalize $(v_m)_m$ by

$$\int_M |u|^{N-2} v_m^2 dv_g = 1.$$  

Now by the fact that $L_g$ is coercive

$$c\|v_m\|_{H^2_1} \leq \int_M v_m L_g(v_m) dv_g \leq \lambda_{1,\tilde{g}} + 1.$$  

$(v_m)_m$ is bounded in $H^2_1(M)$ and after restriction to a subsequence we may assume that there exist $v \in H^2_1(M)$, $v \geq 0$ such that $v_m \to v$ weakly in $H^2_1(M)$, strongly in $L^2(M)$ and almost everywhere in $M$, then $v$ satisfies in the sense of distributions

$$L_g v = \lambda_{1,\tilde{g}} u^{N-2} v.$$  

If $u \in H^p_2(M) \subset C^{1-\left[\frac{n}{p}\right],\beta}(M)$ then

$$\int_M u^{N-2}(v^2 - v_m^2) dv_g \to 0$$

and

$$\int_M u^{N-2} v^2 dv_g = 1.$$  

Then $v$ is not trivial and is a nonnegative minimizer of $\lambda_{1,\tilde{g}}$, by Lemma 7

$$h = S_g - \lambda_{1,\tilde{g}} u^{N-2} \in L^p(M)$$

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and by Theorem 1.8
\[ v \in C^{1-[\frac{p}{2}],\beta}(M) \]
and
\[ v > 0. \]
If \( u \in L^N_+(M) \), by Proposition 2.3, we get
\[ \int_M u^{N-2}(v^2 - v_m^2)dv_g \to 0 \]
so
\[ \int_M u^{N-2}v^2dv_g = 1. \]
\( v \) is a non-negative minimizer in \( H^2_1(M) \) of \( \lambda_{1,g} \) such that
\[ \int_M u^{N-2}v^2dv_g = 1. \]
Now consider the set
\[ E = \{ w \in H^2_1(M) : \text{such that } u^{\frac{N-2}{2}}w \neq 0 \text{ and } \int_M u^{N-2}wvdv_g = 0 \}. \]
Obviously \( E \) is not empty and define
\[ \lambda'_{2,g} = \inf_{w \in E} \frac{\int_M wL_g(w)dv_g}{\int_M |u|^{N-2}w^2dv_g}. \]
Let \( (w_m) \) be a minimizing sequence for \( \lambda'_{2,g} \) i.e. a sequence \( w_m \in E \) such that
\[ \lim_m \frac{\int_M w_mL_g(w_m)dv_g}{\int_M |u|^{N-2}w_m^2dv_g} = \lambda'_{2,g}. \]
The same arguments lead to a minimizer \( w \) to \( \lambda'_{2,g} \) with \( \int_M u^{N-2}w^2 = 1. \)
Now writing
\[ \int_M u^{N-2}wvdv_g = \int_M u^{N-2}v(w - w_m)dv_g + \int_M u^{N-2}w_mvdv_g \]
and taking account of \( \int_M u^{N-2}w_mvdv_g = 0 \) and the fact that \( w_m \to w \) weakly in \( L^N(M) \) and since \( u^{N-2}v \in L^{\frac{N}{N-1}}(M) \), we infer that
\[ \int_M u^{N-2}wvdv_g = 0. \]
Hence (8) and (9) are satisfied with \( \lambda'_{2,g} \) instead of \( \lambda_{2,g} \). \( \square \)
Proposition 2.5. We have 
\[ \lambda'_{2,g} = \lambda_{2,\tilde{g}}. \]

Proof. The proof is the same as in ([3]) so we omit it. \qed

Remark 2.6. If \( p > n \) then \( u \in H^p_2(M) \subset C^1(M) \), by Theorem 9, \( v \) and \( w \in C^1(M) \) with \( v > 0 \).

Remark 2.7. If \( p > n \) then \( u \in H^p_2(M) \subset C^1(M) \) and \( \lambda_{2,\tilde{g}} = \lambda_{1,\tilde{g}} \), we see that \( |w| \) is a minimizer for the functional associated to \( \lambda_{1,\tilde{g}} \), then \( |w| \) satisfies the same equation as \( v \) and by Theorem 9 we get \( |w| > 0 \), this contradicts relation (9), necessarily
\[ \lambda_{2,\tilde{g}} > \lambda_{1,\tilde{g}}. \]

3. Variational characterization and existence of \( \mu_1 \)

In this section we need the following Sobolev’s inequality (see [5])

Theorem 3.1. Let \((M, g)\) be a compact \( n \)-dimensional Riemannian manifold. For any \( \varepsilon > 0 \), there exists \( A(\varepsilon) > 0 \) such that \( \forall u \in H^2_2(M) \),
\[ \|u\|_N^2 \leq (K^2 + \varepsilon)\|\nabla u\|_2^2 + A(\varepsilon)\|u\|_2^2 \]
where \( N = 2n/(n - 4) \) and \( K^2 = 4/(n(n - 2)) \omega_n^{-2} \). \( \omega_n \) is the volume of the round sphere \( S_n \).

Let \( [g] = \{u^{N-2}g : u \in H^p_2(M) \text{ and } u > 0\} \), we define the first singular Yamabe invariant \( \mu_1 \) as 
\[ \mu_1 = \inf_{\tilde{g} \in [g]} \lambda_{1,\tilde{g}} \text{Vol}(M, \tilde{g})^{2/n} \]
then we get
\[ \mu_1 = \inf_{u \in H^p_2, \tilde{V} \in \text{Gr}^u_1(H^p_2)} \sup_{v \in \tilde{V}} \frac{\int_M vL_g(v)dv_g}{\int_M |u|^{N-2}v^2dv_g} \left( \int_M u^Ndv_g \right)^{2/n}. \]

Lemma 3.2. We have 
\[ \mu_1 \leq \mu < K^{-2}. \]
Proof. If $p \geq 2n/(n+2)$, the embedding $H^p_2(M) \subset H^2_1(M)$ is true, so

$$\mu_1 = \inf_{u \in H^p_2, V \in Gr^1_1(H^p_2(M))} \sup_{v \in V} \frac{\int_M vL_g(v)dv_g}{\int_M |u|^{N-2}v^2dv_g} \left( \int_M u^N dv_g \right)^{2/n}$$

$$\leq \inf_{u \in H^p_2, V \in Gr^1_1(H^p_2(M))} \sup_{v \in V} \frac{\int_M vL_g(v)dv_g}{\int_M |v|^{N-2}v^2dv_g} \left( \int_M u^N dv_g \right)^{2/n}.$$ 

In particular for $p > \frac{n}{2}$ and $u = v$ we get

$$\mu_1 \leq \inf_{v \in H^p_2, V \in Gr^1_1(H^p_2(M))} \sup_{v \in V} \frac{\int_M vL_g(v)dv_g}{\int_M |v|^{N-2}v^2dv_g} \left( \int_M v^N dv_g \right)^{2/n} = \mu$$

i.e

$$\mu_1 \leq \mu < K^{-2}.$$

Theorem 3.3. If $\mu > 0$, there exists conform metric $\overline{g} = u^{N-2}g$ which minimizes $\mu_1$.

Proof. The proof will take several steps.

Step 1: We study a sequence of metrics $g_m = u_m^{N-2}g$ with $u_m \in H^p_2(M)$, $u_m > 0$ which minimize $\mu_1$ i.e. a sequence of metrics such that

$$\mu_1 = \lim_m \lambda_{1,m}(Vol(M, g_m)^{2/n}).$$

Without loss of generality, we may assume that $Vol(M, g_m) = 1$ i.e.

$$\int_M u_m^N dv_g = 1.$$ 

In particular, the sequence of functions $u_m$ is bounded in $L^N(M)$ and there exists $u \in L^N(M)$, $u \geq 0$ such that $u_m \to u$ weakly in $L^N(M)$. We are going to prove that the generalized metric $u^{N-2}g$ minimizes $\mu_1$. Proposition 2.4 implies the existence of a sequence $(v_m)$ in $H^2_1(M)$, $v_m > 0$ such that

$$L_g(v_m) = \lambda_{1,m}u_m^{N-2}v_m$$

and

$$\int_M u_m^{N-2}v_m^2 dv_g = 1.$$
now since \( \mu > 0 \), by Proposition 1.10, \( L_g \) is coercive and we infer that
\[
\|v_m\|_{H^2_1} \leq \int_M v_m L_g(v_m) \, dv_g = \lambda_{1,m} \leq \mu_1 + 1.
\]
The sequence \((v_m)_m\) is bounded in \( H^2_1(M) \), we can find \( v \in H^2_1(M) \), \( v \geq 0 \) such that \( v_m \to v \) weakly in \( H^2_1(M) \). Together with the weak convergence of \((u_m)_m\), we obtain in the sense of distributions
\[
L_g(v) = \mu_1 u^{N-2} v.
\]

**Step 2:** Now we are going to show that \( v_m \to v \) strongly in \( H^2_1(M) \).

We put
\[
z_m = v_m - v
\]
then \( z_m \to 0 \) weakly in \( H^2_1(M) \) and strongly in \( L^q(M) \) with \( q < N \), and writing
\[
\int_M |\nabla v_m|^2 \, dv_g = \int_M |\nabla z_m|^2 \, dv_g + \int_M |\nabla v|^2 \, dv_g + 2 \int_M \nabla z_m \nabla v \, dv_g
\]
we see that
\[
\int_M |\nabla v_m|^2 \, dv_g = \int_M |\nabla z_m|^2 \, dv_g + \int_M |\nabla v|^2 \, dv_g + o(1).
\]
Now because of \( 2p/(p-1) < N \), we have
\[
\int_M \frac{n-2}{4(n-1)} S_g (v_m - v)^2 \, dv_g \leq \frac{n-2}{4(n-1)} \|S_g\|_p \|v_m - v\|_{2p}^2 \to 0
\]
so
\[
\int_M \frac{n-2}{4(n-1)} S_g v_m^2 \, dv_g = \int_M \frac{n-2}{4(n-1)} S_g v^2 \, dv_g + o(1)
\]
and
\[
\int_M |\nabla v_m|^2 \, dv_g + \int_M \frac{n-2}{4(n-1)} S_g (v_m)^2 \, dv_g
\]
\[
= \int_M |\nabla z_m|^2 \, dv_g + \int_M |\nabla v|^2 \, dv_g + \int_M \frac{n-2}{4(n-1)} S_g(v)^2 \, dv_g + o(1).
\]
Then
\[ \int_M v_m L_g v_m \, dv_g = \int_M |\nabla z_m|^2 \, dv_g + \int_M |\nabla v|^2 \, dv_g + \int_M \frac{n - 2}{4(n - 1)} S_g v^2 \, dv_g + o(1) \]

And by the definition of \( \mu \) and Lemma 3.2 we get
\[ \int_M |\nabla v|^2 \, dv_g + \int_M \frac{n - 2}{4(n - 1)} S_g (v)^2 \, dv_g \geq \mu (\int_M v^N \, dv_g)^{\frac{2}{N}} \geq \mu_1 (\int_M v^N \, dv_g)^{\frac{2}{N}} \]

then
\[ \int_M v_m L_g (v_m) \, dv_g \geq \int_M |\nabla z_m|^2 \, dv_g + \mu_1 (\int_M v^N \, dv_g)^{\frac{2}{N}} + o(1). \]

And since
\[ \int_M v_m L_g (v_m) \, dv_g = \lambda_{1,m} \leq \mu_1 + o(1) \]

and
\[ \int_M |\nabla z_m|^2 \, dv_g + \mu_1 (\int_M v^N \, dv_g)^{\frac{2}{N}} \leq \mu_1 + o(1) \]
i.e.
\[ \mu_1 \|v\|_N^2 + \|\nabla z_m\|_2^2 \leq \mu_1 + o(1) \quad (10) \]

Now by Brézis-Lieb lemma ([4]), we get
\[ \lim_m \int_M \left( v_m^N + z_m^N \right) \, dv_g = \int_M v^N \, dv_g \]
i.e.
\[ \lim_m \|v_m\|_N^N - \|z_m\|_N^N = \|v\|_N^N. \]

Hence
\[ \|v_m\|_N^N + o(1) = \|z_m\|_N^N + \|v\|_N^N. \]

By Hölder’s inequality and \( \int_M u_m^{N-2} v_m^2 \, dv_g = 1 \), we get
\[ \|v_m\|_N^N \geq 1 \]
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i.e.

$$\int_M (v^N + z_m^N) dv = \int_M v^N_m dv + o(1) \geq 1 + o(1).$$

Then

$$\left(\int_M v^N dv\right)^{\frac{2}{N}} + \left(\int_M z_m^N dv\right)^{\frac{2}{N}} \geq 1 + o(1)$$

i.e.

$$\|z_m\|^2_N + \|v\|^2_N \geq 1 + o(1).$$

Now by Theorem 3.1 and the fact $z_m \to 0$ strongly in $L^2(M)$, we get

$$\|z_m\|^2_N \leq (K^2 + \varepsilon)\|\nabla z_m\|^2_2 + o(1)$$

$$1 + o(1) \leq \|z_m\|^2_N + \|v\|^2_N \leq \|v\|^2_N + (K^2 + \varepsilon)\|\nabla z_m\|^2_2 + o(1).$$

So we deduce

$$1 + o(1) \leq \|v\|^2_N + (K^2 + \varepsilon)\|\nabla z_m\|^2_2 + o(1)$$

and from inequality (10), we get

$$\|\nabla z_m\|^2_2 + \mu_1 \|v\|^2_N \leq \mu_1((K^2 + \varepsilon)\|\nabla z_m\|^2_2 + \|v\|^2_N) + o(1).$$

So if $\mu_1 K^2 < 1$, we get

$$(1 - \mu_1(K^2 + \varepsilon))\|\nabla z_m\|^2_2 \leq o(1)$$

i.e. $v_m \to v$ strongly in $H^2_1(M)$. 

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Step 3: We have

\[ \lim_m \int_M \left( u_m^{N-2} v_m^2 - u^{N-2} v^2 + u_m^{N-2} v^2 - u^{N-2} v^2 \right) dv_g = \lim_m \int_M \left( u_m^{N-2} (v_m^2 - v^2) + (u_m^{N-2} - u^{N-2}) v^2 \right) dv_g. \]

Now since \( u_m \to u \) a.e. so does \( u_m^{N-2} \to u^{N-2} \) and \( \int_M u_m^{N-2} dv_g \leq c \), hence \( u_m^{N-2} \) is bounded in \( L^{N/(N-2)}(M) \) and up to a subsequence \( u_m^{N-2} \to u^{N-2} \) weakly in \( L^{N/(N-2)}(M) \). Since \( v^2 \in L^\frac{N}{2}(M) \), we have

\[ \lim_m \int_M (u_m^{N-2} - u^{N-2}) v^2 dv_g = 0 \]

and by Hölder’s inequality

\[ \lim_m \int_M u_m^{N-2} (v_m - v)^2 dv_g \leq (\int_M u_m^N dv_g)^{N-2/N} (\int_M |v_m - v|^N dv_g)^{\frac{2}{N}} \leq 0. \]

By the strong convergence of \((v_m)\) in \( L^N(M)\), we get

\[ \int_M u^{N-2} v^2 dv_g = 1, \]

then \( v \) and \( u \) are non trivial functions.

Step 4: Let \( \bar{u} = av \in L^N_+(M) \) with \( a > 0 \) a constant such that \( \int_M \bar{u}^N dv_g = 1 \) with \( v \) a solution of

\[ L_g(v) = \mu_1 u^{N-2} v \]

with the constraint

\[ \int_M u^{N-2} v^2 dv_g = 1. \]

We claim that \( u = v \); indeed,
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\[ \mu_1 \leq \frac{\int_M vL_g(v)dv_g}{\int_M \bar{u}^{N-2}v^2dv_g} \]

\[ \leq \frac{\int_M vL_g(v)dv_g}{\int_M (av)^{N-2}v^2dv_g} = \frac{a^2\mu_1\int_M u^{N-2}v^2dv_g}{\int_M \bar{u}^{N-2}(av)^2dv_g} \]

and Hölder’s inequality lead

\[ \leq \mu_1 \int_M (u)^{N-2}(av)^2dv_g \]

\[ \leq \mu_1 (\int_M (u)^{N-2}\frac{N-2}{N}v^2dv_g)^\frac{N-2}{N} (\int_M (av)^{2\frac{N}{2}}dv_g)^\frac{2}{N} \leq \mu_1. \]

And since the equality in Hölder’s inequality holds if

\[ \bar{u} = u = av \]

then \( a = 1 \) and

\[ u = v. \]

Then \( v \) satisfies \( L_g v = \mu_1 v^{N-1} \), by Theorem 2.2 we get \( v = u \in H^p_2 (M) \subset C^{1-\frac{\beta}{p},\beta} (M) \) with \( \beta \in (0,1) \) and \( v = u > 0 \).

Resuming, we have

\[ L_g (v) = \mu_1 v^{N-1}, \int_M v^N dv_g = 1 \] and \( v = u \in H^p_2 (M) \subset C^{1-\frac{\beta}{p},\beta} (M) \)

so the metric \( \tilde{g} = u^{N-2}g \) minimizes \( \mu_1 \).

\[ \square \]

4. Yamabe conformal invariant with singularities

**Theorem 4.1.** If \( \mu \geq 0 \), then \( \mu_1 = \mu \)

**Proof.**  **Step 1:** If \( \mu > 0 \). Let \( v \) such that \( L_g (v) = \mu_1 v^{N-1} \) and \( \int_M v^N dv_g = 1 \) then

\[ \mu_1 = \int_M vL_g(v)dv_g \geq c \|v\|_{H^2_1} \]

and \( v \) in non trivial function then \( \mu_1 > 0 \). On the other hand

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\[ \mu = \inf \frac{\int_M vL_g(v)dv_g}{(\int_M v^N dv_g)^{\frac{2}{n}}} \]

\[ \leq \int_M vL_g(v)dv_g = \mu_1 \]

and by Lemma 3.2, we get

\[ \mu_1 = \mu \]

**Step 2:** If \( \mu = 0 \), Lemma 3.2 implies that \( \mu_1 \leq 0 \), hence

\[ \mu_1 = 0. \]

\[ \square \]

5. Variational characterization of \( \mu_2 \)

Let \([g] = \{u^{N-2}g, u \in H^p_2(M) \text{ and } u > 0\}\), we define the second Yamabe invariant \( \mu_2 \) as

\[ \mu_2 = \inf_{\bar{g} \in [g]} \lambda_{2, \bar{g}} Vol(M, \bar{g})^{2/n} \]

or more explicitly

\[ \mu_2 = \inf_{u \in H^p_2(M), V \in Gr(V^2(H_1^2(M)))} \sup_{v \in V} \frac{\int_M vL_g(v)dv_g}{\int_M |u|^{N-2}v^2dv_g} \left(\int_M u^N dv_g\right)^{\frac{2}{n}} \]

**Theorem 5.1** ([1]). On a compact Riemannian manifold \((M, g)\) of dimension \(n \geq 3\), we have for all \(v \in H^2_1(M)\) and for all \(u \in L^N_+(M)\)

\[ 2^{\frac{2}{n}} \int_M |u|^{N-2}v^2dv_g \leq (K^2 \int_M |\nabla v|^2dv_g + \int_M B_0 v^2dv_g)\left(\int_M u^N dv_g\right)^{\frac{2}{n}} \]

Or

\[ 2^{\frac{2}{n}} \int_M |u|^{N-2}v^2dv_g \leq \mu_1(S_n)\left(\int_M C_n |\nabla v|^2 + B_0 v^2dv_g\right)\left(\int_M u^N dv_g\right)^{\frac{2}{n}} \]
Theorem 5.2. ([1]) For any compact Riemannian manifold \((M, g)\) of dimension \(n \geq 3\), there exists \(B_0 > 0\) such that

\[ \mu_1(S_n) = n(n-1)\omega_n^{2/n} = \inf_{H^2_1} \frac{\int_M \frac{4(n-1)}{(n-2)}|\nabla u|^2 + B_0u^2dv_g}{(\int_M |u|^N dv_g)^{2/N}} \]

where \(\omega_n\) is the volume of the unit round sphere

or

\[ (\int_M |u|^N dv_g)^{2/N} \leq K^2 \int_M |\nabla u|^2 dv_g + \int_M B_0u^2 dv_g \]

\[ K^2 = \mu_1(S_n)^{-1}C_n \text{ and } C_n = (4(n-1))/(n-2) \]

6. Properties of \(\mu_2\)

We know that \(g\) is smooth in the ball \(B_p(\delta)\) by assumption \((H)\), this assumption is sufficient to prove that Aubin’s conjecture is valid. The case \((M, g)\) is not conformally flat in a neighborhood of the point \(P\) and \(n \geq 6\), let \(\eta\) is a cut-off function with support in the ball \(B_p(2\varepsilon)\) and \(\eta = 1\) in \(B_p(\varepsilon)\), where \(2\varepsilon \leq \delta\) and

\[ v_\varepsilon(q) = \left(\frac{\varepsilon}{r^2 + \varepsilon^2}\right)^{n-2} \]

with \(r = d(p, q)\). We let \(u_\varepsilon = \eta v_\varepsilon\) and define

\[ Y(u) = \frac{\int_M \left(|\nabla u|^2 + \frac{n-2}{4(n-1)}S_gu^2\right) dv_g}{(\int_M |u|^N dv_g)^{2/N}}. \]

We obtain the following lemma

Lemma 6.1. ([1])

\[ \mu = Y(v_\varepsilon) \leq \left\{ \begin{array}{ll} \{(K^{-2} - c|w(P)|^2\varepsilon^4 + 0(\varepsilon^4) \text{ if } n > 6 \\ K^{-2} - c|w(P)|^2\varepsilon^4 \log \frac{1}{\varepsilon} + 0(\varepsilon^4) \text{ if } n = 6 \end{array} \right. \]

where \(|w(P)|\) is the norm of the Weyl tensor at the point \(P\) and \(c > 0\).

Theorem 6.2. If \((M, g)\) is not locally conformally flat and \(n \geq 11\) and \(\mu > 0\), we find

\[ \mu_2 < (\mu_{\frac{n}{2}}^n + (K^{-2})^n)^{\frac{n}{2}} \]

and if \(\mu = 0\), \(n \geq 9\) then

\[ \mu_2 < K^{-2} \]
7. Existence of a minimizer to $\mu_2$

**Lemma 7.1.** Assume that $v_m \to v$ weakly in $H^2_1(M)$, $u_m \to u$ weakly in $L^N(M)$ and $\int_M u_m^{N-2} v_m^2 dv_g = 1$ then

$$\int_M u_m^{N-2} (v_m - v)^2 dv_g = 1 - \int_M u^{N-2} v^2 dv_g + o(1)$$

**Proof.** we have

$$\int_M u_m^{N-2} (v_m - v)^2 dv_g$$

$$= \int_M u_m^{N-2} v_m^2 dv_g + \int_M u_m^{N-2} v^2 dv_g - \int_M 2u_m^{N-2} v_m vdvg$$

$$= 1 + \int_M u_m^{N-2} v^2 dv_g - \int_M 2u_m^{N-2} v_m vdvg.$$ \hspace{1cm}(15)

Now $\left( u_m^{N-2} \right)_m$ is bounded in $L^{N-2}_M(M)$ and $u_m^{N-2} \to u^{N-2}$ a.e., then $u_m^{N-2} \to u^{N-2}$ weakly in $L^{N-2}_M(M)$ and $\forall \phi \in L^2(M)$

$$\int_M \phi u_m^{N-2} dv_g \to \int_M \phi u^{N-2} dv_g$$

in particular for $\phi = v^2$

$$\int_M v^2 u_m^{N-2} dv_g \to \int_M v^2 u^{N-2} dv_g.$$ 

$\int_M u_m^{N-2} v_m dv_g$ is bounded in $L^{N-1}_M(M)$, because of

$$\int_M u_m^{N-2} \frac{N}{N-1} v_m^{\frac{N}{N-1}} dv_g \leq \left( \int_M u_m^{N} dv_g \right)^{\frac{N-2}{N-1}} \left( \int_M v_m^{N} dv_g \right)^{\frac{1}{N-1}}$$

and $u_m^{N-2} v_m \to u^{N-2} v$ a.e., then $u_m^{N-2} v_m \to u^{N-2} v$ weakly in $L^{N-1}_M(M)$.

Hence
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\[ \int_M u_m^{N-2} v_m d\nu_g \to \int_M u^{N-2} v^2 d\nu_g \]

and

\[ \int_M u_m^{N-2} (v_m - v)^2 d\nu_g = 1 - \int_M u^{N-2} v^2 d\nu_g + o(1). \]

\[ \square \]

**Theorem 7.2.** If \( 1 - 2^{-\frac{2}{n}} K^2 \mu_2 > 0 \), then the generalized metric \( u^{N-2} g \) minimizes \( \mu_2 \).

**Proof.**  

**Step 1:** We study a sequence of metrics \( g_m = u_m^{N-2} g \) with \( u_m \in H^2_2(M) \), \( u_m > 0 \) which minimizes the infimum in the definition of \( \mu_2 \) i.e. a sequence of metrics such that

\[ \mu_2 = \lim \lambda_{2,m}(Vol(M, g_m)^{2/n}. \]

Without loss generality, we may assume that \( Vol(M, g_m) = 1 \) i.e. that \( \int_M u_m^N d\nu_g = 1 \). In particular, the sequence of functions \((u_m)_m\) is bounded in \( L^N(M) \) and there exists \( u \in L^N(M) \), \( u \geq 0 \) such that \( u_m \to u \) weakly in \( L^N \). We are going to prove that the generalized metric \( u^{N-2} g \) minimizes \( \mu_2 \). Proposition 2.4, implies the existence of \( v_m, w_m \in H^2(M), v_m > 0 \) such that

\[ L_g(v_m) = \lambda_{1,m} u_m^{N-2} v_m \]

\[ L_g(w_m) = \lambda_{2,m} u_m^{N-2} w_m \]

And such that

\[ \int_M u_m^{N-2} v_m^2 d\nu_g = \int_M u_m^{N-2} w_m^2 d\nu_g = 1, \int_M u_m^{N-2} v_m w_m d\nu_g = 0. \]

The sequence \( v_m, w_m \) is bounded in \( H^2_1(M) \), we can find \( v, w \in H^2_1(M), v \geq 0 \) such that \( v_m \to v \), \( w_m \to w \) weakly in \( H^2_1(M) \). Together with the weak convergence of \((u_m)\), we get in weak sense

\[ L_g(v) = \tilde{\mu}_1 u^{N-2} v \]

and

\[ L_g(w) = \mu_2 u^{N-2} w \]

where

\[ \tilde{\mu}_1 = \lim \lambda_{1,m} \leq \mu_2. \]

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Step 2: Now we show $v_m \rightarrow v$, $w_m \rightarrow w$ strongly in $H^2_1(M)$.

Applying Theorem 5.1 to the sequence $v_m - v$, we get

$$
\int_M |u_m|^{N-2} (v_m - v)^2 \, dv_g
\leq (2^{-\frac{2}{n}} K^2 \int_M |\nabla (v_m - v)|^2 \, dv_g + \int_M B_0 (v_m - v)^2 \, dv_g) (\int_M u^N \, dv_g)^{\frac{2}{n}}
$$

and since $v_m \rightarrow v$ strongly in $L^2(M)$,

$$
\int_M |u_m|^{N-2} (v_m - v)^2 \, dv_g \leq (2^{-\frac{2}{n}} K^2 \int_M |\nabla (v_m - v)|^2 \, dv_g + o(1)
\leq 2^{-\frac{2}{n}} K^2 \int_M (|\nabla (v_m)|^2 + |\nabla v|^2 - 2\nabla v_m \nabla v) \, dv_g + o(1).
$$

By the weak convergence of $(v_m)$, $\int_M \nabla v_m \nabla v \, dv_g = \int_M |\nabla v|^2 \, dv_g + o(1)$

$$
\int_M |u_m|^{N-2} (v_m - v)^2 \, dv_g \leq 2^{-\frac{2}{n}} K^2 \int_M (|\nabla (v_m)|^2 - |\nabla v|^2) \, dv_g + o(1)
$$

and since

$$
\int_M \frac{n-2}{4(n-1)} S_g v_m^2 \, dv_g = \int_M \frac{n-2}{4(n-1)} S_g v^2 \, dv_g + o(1)
$$

we get

$$
\int_M |u_m|^{N-2} (v_m - v)^2 \, dv_g
\leq 2^{-\frac{2}{n}} K^2 \int_M (|\nabla (v_m)|^2 - |\nabla v|^2) \, dv_g + \int_M \frac{n-2}{4(n-1)} S_g (v_m^2 - v^2) \, dv_g + o(1)
\leq 2^{-\frac{2}{n}} K^2 \int_M (v_m L_g(v_m) - v L_g(v)) \, dv_g + o(1)
\leq 2^{-\frac{2}{n}} K^2 \lambda_{1,m} - \hat{\mu}_1 \int_M u^{N-2} v^2 \, dv_g + o(1)
$$

By the fact $\hat{\mu}_1 = \lim \lambda_{1,m} \leq \mu_2$

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\[
\leq 2^{-\frac{2}{n}} K^2 \mu_2 \left(1 - \int_M u^{N-2} v^2 dv_g \right) + o(1)
\]

Then

\[
\int_M |u_m|^{N-2} (v_m - v)^2 dv_g \leq 2^{-\frac{2}{n}} K^2 \mu_2 \left(1 - \int_M u^{N-2} v^2 dv_g \right) + o(1)
\]

Now using the weak convergence of \((v_m)\) in \(H^2_1(M)\) and the weak convergence of \((u_m)\) in \(L^N(M)\), we infer by Lemma 7.1 that

\[
\int_M |u_m|^{N-2} (v_m - v)^2 dv_g = 1 - \int_M u^{N-2} v^2 dv_g + o(1)
\]

then

\[
1 - \int_M u^{N-2} v^2 dv_g \leq 2^{-\frac{2}{n}} K^2 \mu_2 \left(1 - \int_M u^{N-2} v^2 dv_g \right) + o(1)
\]

and

\[
1 - 2^{-\frac{2}{n}} K^2 \mu_2 \leq (1 - 2^{-\frac{2}{n}} K^2 \mu_2) \int_M u^{N-2} v^2 dv_g + o(1).
\]

So if \(1 - 2^{-\frac{2}{n}} K^2 \mu_2 > 0\) then

\[
\int_M u^{N-2} v^2 dv_g \geq 1.
\]

and by Fatou's lemma, we obtain

\[
\int_M u^{N-2} v^2 dv_g \leq \lim \int_M u_m^{N-2} v_m^2 dv_g = 1.
\]

We find that

\[
\int_M u^{N-2} v^2 dv_g = 1.
\] (16)

So \(u\) and \(v\) are not trivial.

Moreover

\[
\int_M |\nabla (v_m - v)|^2 dv_g = \int_M \left(|\nabla (v_m)|^2 + |\nabla v|^2 - 2\nabla v_m \nabla v \right) dv_g
\]

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\[ = \int_M \left( |\nabla (v_m)|^2 - |\nabla v|^2 \right) dv_g + o(1) \]

and since \( \int_M S_g (v_m^2 - v^2) dv_g = o(1) \), we get

\[ \int_M |\nabla (v_m - v)|^2 dv_g = \int_M (v_m L_g (v_m) - v L_g (v)) dv_g + o(1) \]

\[ \leq \mu_2 (1 - \int_M u^{N-2} v^2 dv_g) + o(1) \]

Then, by relation (16)

\[ \int_M |\nabla (v_m - v)|^2 dv_g = o(1) \]

and \( v_m \rightarrow v \) strongly in \( H^2_1 (M) \). The same argument holds with \( (w_m) \), hence \( w_m \rightarrow w \) strongly in \( H^2_1 (M) \) and \( \int_M u^{N-2} w^2 dv_g = 1 \).

To show that \( \int_M u^{N-2} v w dv_g = 0 \), first writing and using Hölder’s inequality, we get

\[ \int_M \left( u_m^{N-2} v_m w_m - u_m^{N-2} v w \right) dv_g = \int_M \left( u_m^{N-2} v_m w_m - u_m^{N-2} v w_m \right) dv_g \]

\[ + \int_M \left( u_m^{N-2} v w_m - u_m^{N-2} v w \right) dv_g \]

\[ = \int_M u_m^{N-2} (v_m - v) w_m dv_g + \int_M \left( u_m^{N-2} v w_m - u_m^{N-2} v w \right) dv_g \]

\[ = \int_M \frac{N-2}{u_m^2} w_m \left[ u_m^{N-2} (v_m - v) \right] dv_g + \int_M \left( u_m^{N-2} v w_m - u_m^{N-2} v w \right) dv_g \]

\[ \leq \left( \int_M u_m^{N-2} w_m^2 dv_g \right)^{\frac{1}{2}} \left( \int_M u_m^{N-2} (v_m - v)^2 dv_g \right)^{\frac{1}{2}} \]

\[ + \int_M \left( u_m^{N-2} v w_m - u_m^{N-2} v w \right) dv_g \]

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\[
\leq \left( \int_M u_m^{N-2}(v_m - v)^2 dv_g \right)^{\frac{1}{2}} + \int_M \left( u_m^{N-2}v w_m - u^{N-2}v w \right) dv_g
\]

\[
\leq \left[ \left( \int_M u_m^{-2N}dv_g \right)^{\frac{N-2}{N}} \left( \int_M |v_m - v|^N dv_g \right)^{\frac{2}{N}} \right]^\frac{1}{2} + \int_M \left( u_m^{N-2}v w_m - u^{N-2}v w \right) dv_g
\]

\[
\leq \left( \int_M |v_m - v|^N dv_g \right)^{\frac{1}{N}} + \int_M \left( u_m^{N-2}v w_m - u^{N-2}v w \right) dv_g
\]

\[
\leq \left( \int_M |v_m - v|^N dv_g \right)^{\frac{1}{N}} + \int_M \left( u_m^{N-2}v w_m - u_m^{N-2}v w + u_m^{N-2}v w - u^{N-2}v w \right) dv_g
\]

\[
\leq \left( \int_M |v_m - v|^N dv_g \right)^{\frac{1}{N}} + \int_M \left( u_m^{N-2}v (w_m - w) + (u_m^{N-2} - u^{N-2})v w \right) dv_g
\]

\[
\leq \left( \int_M |v_m - v|^N dv_g \right)^{\frac{1}{N}} + \int_M \left( u_m^{N-2}v (w_m - w) + (u_m^{N-2} - u^{N-2})v w \right) dv_g
\]

\[
\leq \left( \int_M |v_m - v|^N dv_g \right)^{\frac{1}{N}} + \left( \int_M u_m^{N-2}v^2 dv_g \right)^{\frac{1}{2}} \left( \int_M u_m^{N-2}(w_m - w)^2 dv_g \right)^{\frac{1}{2}} + \int_M \left( u_m^{N-2} - u^{N-2} \right)vdv_g
\]

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\[
\leq \left( \int_M |v_m - v|^N \, dv_g \right)^{\frac{1}{N}} + \left( \int_M u_m^{N-2} v^2 \, dv_g \right)^{\frac{1}{2}} \left( \int_M |v_m - w|^N \, dv_g \right)^{\frac{1}{N}} \]
\[
+ \int_M (u_m^{N-2} - u^{N-2}) v w \, dv_g.
\]

Now noting that
\[
\int_M u_m^{N-2} v^2 \, dv_g \leq \left( \int_M u_m^N \, dv_g \right)^{\frac{N-2}{N}} \left( \int_M v^N \, dv_g \right)^{\frac{2}{N}} < +\infty
\]
and taking account of \(u_m^{N-2} \to u^{N-2}\) weakly in \(L_{N-2}^N (M)\) and the fact that \(v w \in L_{N-2}^N (M)\), we deduce
\[
\int_M (u_m^{N-2} - u^{N-2}) v w \, dv_g \to 0
\]
hence
\[
\int_M u^{N-2} v w \, dv_g = 0.
\]
Consequently the generalized metric \(u^{N-2} g\) minimizes \(\mu_2\).

\[\square\]

**Theorem 7.3.** If \(\mu_2 < K^{-2}\), then generalized metric \(u^{N-2} g\) minimizes \(\mu_2\)

**Proof.**  **Step 1:** We study a sequence of metrics \(g_m = u_m^{N-2} g\) with \(u_m \in H^2_p (M)\), \(u_m > 0\) which attains \(\mu_2\) i.e. a sequence of metrics such that

\[
\mu_2 = \lim_{m} \lambda_{2,m}(Vol(M,g_m))^{2/n}.
\]
Without loss of generality, we may assume that \(Vol(M,g_m) = 1\) i.e. \(\int_M u_m^N \, dv_g = 1\). In particular, the sequence \((u_m)_m\) is bounded in \(L^N (M)\) and there exists \(u \in L^N (M)\), \(u \geq 0\) such that \(u_m \to u\) weakly in \(L^N (M)\). We are going to prove that the metric \(u^{N-2} g\) minimizes \(\mu_2\). Proposition 2.4 and Theorem 1.8 imply the existence of \(v_m, w_m \in C^{1-\left[\frac{n}{p}\right],\beta}\), with \(\beta \in (0,1) (M)\), \(v_m > 0\) such that

\[
L_g(v_m) = \lambda_{1,m} u_m^{N-2} v_m
\]
\[ L_g(w_m) = \lambda_{2,m} u_m^{N-2} w_m \]

and
\[ \int_M u_m^{N-2} v_m^2 dv_g = \int_M u_m^{N-2} w_m^2 dv_g = 1, \quad \int_M u_m^{N-2} v_m w_m dv_g = 0. \]

The sequences \((v_m)_m\) and \((w_m)_m\) are bounded in \(H^2_1(M)\), we can find \(v, w \in H^2_1(M)\) with \(v \geq 0\) such that \(v_m \to v, w_m \to w\) weakly in \(H^2_1(M)\). Together with the weak convergence of \((u_m)_m\), we get in the weak sense
\[ L_g(v) = \hat{\mu}_1 u^{N-2} v \]

and
\[ L_g(w) = \mu_2 u^{N-2} w \]

where
\[ \hat{\mu}_1 = \lim \lambda_{1,m} \leq \mu_2. \]

**Step 2:** Now we are going to show that \(v_m \to v, w_m \to w\) strongly in \(H^2_1(M)\).

By Hölder’s inequality, Theorem 3.1, the strong convergence of \((v_m)\) in \(L^2(M)\), we get
\[
\int_M |u_m|^{N-2}(v_m - v)^2 dv_g \leq \|v_m - v\|_N^2 \leq K^2 \|\nabla(v_m - v)\|_2 + o(1)
\]

\[
\leq K^2 \int_M \left( |\nabla(v_m)|^2 + |\nabla v|^2 - 2 \nabla v_m \nabla v \right) dv_g + o(1)
\]

\[
\leq K^2 \int_M \left( |\nabla(v_m)|^2 - |\nabla v|^2 \right) dv_g + o(1)
\]

\[
\leq K^2 \int_M (v_m L_g(v_m) - v L_g(v)) dv_g + o(1)
\]

\[
\leq K^2 \mu_2 (1 - \int_M u^{N-2} v^2 dv_g) + o(1)
\]

and with Lemma 7.1
\[
\int_M |u_m|^{N-2}(v_m - v)^2 dv_g = 1 - \int_M u^{N-2} v^2 dv_g + o(1)
\]
then
\[ 1 - \int_M u^{N-2}v^2 dv_g \leq K^2 \mu_2 (1 - \int_M u^{N-2}v^2 dv_g) + o(1) \]
i.e.
\[ 1 - K^2 \mu_2 \leq (1 - K^2 \mu_2) \int_M u^{N-2}v^2 dv_g \]
so if \( 1 - K^2 \mu_2 > 0 \),
\[ \int_M u^{N-2}v^2 dv_g \geq 1. \]
On the other hand since by Fatou’s lemma
\[ \int_M u^{N-2}v^2 dv_g \leq \lim \int_M u^{N-2}v_m^2 dv_g = 1. \]
Then
\[ \int_M u^{N-2}v^2 dv_g = 1. \]
and
\[ \int_M |\nabla (v_m - v)|^2 dv_g = o(1) \]
Hence \( v_m \to v \) strongly in \( H^2_1(M) \subset L^N(M) \).
The same conclusion also holds for \((w_m)_m\).

\[ \square \]

**Lemma 7.4.** Let \( u \in L^N(M) \) with \( \int_M u^N dv_g = 1 \) and \( z, w \) nonnegative functions in \( H^2_1(M) \) satisfying
\[ \int_M wL_g(w) dv_g \leq \mu_2 \int_M u^{N-2}w^2 dv_g \]  
(20)
and
\[ \int_M zL_g(z) dv_g \leq \mu_2 \int_M u^{N-2}z^2 dv_g \]  
(21)
And suppose that \((M - z^{-1}(0)) \cap (M - w^{-1}(0)) \) has measure zero. Then \( u \) is a linear combination of \( z \) and \( w \) and we have equality in (20) and (21).

**Proof.** The proof is the same as that of Aummann and Humbert in ([1]).

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**Theorem 7.5.** If a generalized metric \( u^{N-2}g \) minimizes \( \mu_2 \), then there exists a nodal solution \( w \in H^2_2(M) \subset C^{1-\lfloor n/p \rfloor, \beta}(M) \) of equation

\[
L_g(w) = \mu_2 u^{N-2}w
\]  

(22)

More over there exists \( b > 0 \) such that

\[
u = aw_+ + bw_-
\]

With \( w_+ = \sup(w, 0) \) and \( w_- = \sup(-w, 0) \).

**Proof.**  
**Step 1:** Applying Lemma 7.4 to \( w_+ = \sup(w, 0) \) and \( w_- = \sup(-w, 0) \), we get the existence of \( a, b > 0 \) such that

\[
u = aw_+ + bw_-
\]

Now by Lemma 1.9, \( w_+, w_- \in L^\infty(M) \) i.e. \( u \in L^\infty(M) \), \( u^{N-2} \in L^\infty(M) \), then

\[
h = S_g - \mu_2 u^{N-2} \in L^p(M)
\]

and from Theorem 2.2, we obtain

\[
w \in H^p_2(M) \subset C^{1-\lfloor n/p \rfloor, \beta}(M).
\]

**Step 2:** If \( \mu_2 = \mu_1 \), we see that \( |w| \) is a minimizer to the functional associated to \( \mu_1 \), then \( |w| \) satisfies the same equation as \( v \) and Theorem 2.2 shows that \( |w| = w \in H^p_2(M) \subset C^{1-\lfloor n/p \rfloor, \beta}(M) \) that is \( |w| > 0 \) everywhere, which contradicts the condition (9) in Proposition 2.4, then

\[
\mu_2 > \mu_1.
\]

**Step 3:** The solution \( w \) of the equation (22) changes sign. Since if it does not, we may assume that \( w \geq 0 \), by step2 the inequality in (20) is strict and by Lemma 7.4 we have the equality: a contradiction.

\[
\Box
\]

**Remark 7.6.** Step1 shows that \( u \) is not necessarily in \( H^p_2(M) \) and by the way the minimizing metric is not in \( H^p_2(M, T^*M \otimes T^*\hat{M}) \) contrary to the Yamabe invariant with singularities.

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