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Asymptotic behavior of weighted quadratic variation of bi-fractional Brownian motion


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Abstract

We prove, by means of Malliavin calculus, the convergence in $L^2$ of some properly renormalized weighted quadratic variations of bi-fractional Brownian motion (biFBM) with parameters $H$ and $K$, when $H < 1/4$ and $K \in (0, 1]$.

Comportement asymptotique de la variation quadratique à poids du mouvement brownien bifractionnaire

Résumé

Nous utilisons le calcul de Malliavin pour montrer la convergence dans $L^2$ de la variation quadratique à poids du mouvement brownien bifractionnaire (biFBM) d’indices $H$ et $K$ lorsque $H < 1/4$ et $K \in (0, 1]$.

1. Introduction

There has been recently a lot of interests in the literature to the study of weighted power variations. More precisely, for a given integer $p > 1$, a smooth enough function $h : \mathbb{R} \to \mathbb{R}$ and a process $X$, the analysis of the asymptotic behavior, as $n$ tends to infinity, of quantities such as

$$\sum_{l=0}^{n-1} h(X_{l/n})(\Delta X_{l/n})^p \quad (1.1)$$

(or some appropriate renormalized version of them) have been considered in [6, 5, 7]. Here $\Delta X_{l/n}$ stands for the increment $X_{l+1/n} - X_{l/n}$. Notice that (1.1) is called weighted power variations because of the presence of the factor $h(X_{l/n})$.

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This study originates in the work [5] by Nourdin, in the case where \( X \) is a fractional Brownian motion (f.B.m, in short). Then, the results of [5] have been improved in [6] by Nourdin, Nualart and Tudor. Let us also stress that the study in [6, 5] has been used in [2, 4] to deduce exact rate of convergence of some approximation schemes of scalar stochastic differential equations driven by a f.B.m. Moreover, for another motivation of this study, we can also mention that the analysis of the asymptotic behavior of (1.1), in the particular case \( p = 2 \) and \( X \) the fractional Brownian motion of Hurst parameter \( H \in (0, 1) \), allowed the authors of [7] to derive a new type of change of variable formula for \( X \), with a correction term that is an ordinary Itô integral with respect to a Wiener process that is independent of \( X \).

As we said, a complete description of the nature of the convergence of weighted \( p \)-power variation of the form (1.1) in the case where \( X \) is the fractional Brownian motion with Hurst parameter \( H \in (0, 1) \) has been given in [6, 5, 7]. More precisely, after adequate renormalization, central and non-central limit theorems have been derived there, depending on the value of \( p \) and \( H \). In particular, it is shown in [5] that, for weighted quadratic variations \( (p = 2) \), the following convergence holds for \( h \) regular enough and \( H \) strictly between 0 and 1/4:

\[
\sum_{l=0}^{n-1} h(X_{l/n})[(n^{2H}X_{l/n})^2 - 1] \xrightarrow{L^2}{\text{as}} \frac{1}{4} \int_0^1 h''(X_u) \, du. \tag{1.2}
\]

As pointed out by Nourdin in [5], (1.2) is somewhat surprising when it is compared to the situation where \( h \equiv 1 \). Indeed, since the seminal work of Breuer and Major [1], we know that, for any \( 0 < H < 3/4 \):

\[
\frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} [n^{2H}(\Delta X_{l/n})^2 - 1] \xrightarrow{\text{Law}} \mathcal{N}(0, C_H^2) \tag{1.3}
\]

where \( C_H \) denotes an explicit constant depending only on \( H \). So, instead of an \( L^2 \) convergence, we only have a convergence in law in (1.3). Observe that, since \( 2H - 1 < 1/2 \) if and only if \( H < 1/4 \), convergence (1.2) and (1.3) are, of course, not contradictory.

Motivated by this result, we shall show in the present note that the convergence (1.2) still holds in the case of a more general process, namely the bi-fractional Brownian motion (see below for a precise definition). As
in [5], our main tool for the proof is based on the integration by parts formula of Malliavin calculus.

The note is organized as follows. In Section 2 we recall the definition of the bi-f.B.m and present some preliminary results about its Malliavin calculus. In Section 3 we state and prove our result concerning the convergence similar to (1.2), but in the case where $X$ is a bi-f.B.m.

2. Preliminaries and notation

Here we recall the definition of the bi-fractional Brownian motion and present the elements of Malliavin calculus that will be needed in the sequel.

**Definition 2.1.** Let $H \in (0, 1)$ and $K \in (0, 1]$. A bi-fractional Brownian motion $(B_t^{H,K})_{t \geq 0}$ of indices $H$ and $K$ is a centered Gaussian process, starting from zero, with covariance function given by

$$R^{H,K}(s, t) := \frac{1}{2K} \left((t^{2H} + s^{2H})^K - |t - s|^{2HK}\right). \quad (2.1)$$

In particular, by choosing $K = 1$ and $H \in (0, 1)$ in (2.1), observe that we recover the covariance function of the fractional Brownian motion with Hurst parameter $H$.

The bi-fractional Brownian motion was introduced by Houdré and Villa in [3], and then further studied by Russo and Tudor in [9], and by Tudor and Xiao in [11]. It enjoys the self-similarity property, that is, for any constant $c > 0$, the processes \{c^{-HK}B_{ct}^{H,K}, t \geq 0\} and \{B_{t}^{H,K}, t \geq 0\} have the same distribution. Moreover, if $K \neq 1$, $B^{H,K}$ does not have stationary increments (see e.g. [10]). It is precisely the main difference with respect to f.B.m.

Let us introduce some basic facts on the Malliavin calculus with respect to $B^{H,K}$ on the time interval $[0, 1]$. For a more complete exposition, we refer to [8]. Let $\mathcal{H}$ be the Hilbert space defined as the closure of the linear space $\mathcal{E}$ generated by the indicator functions $(1_{[0,t]}, t \in [0, 1])$ with respect to the following inner product

$$\langle 1_{[0,s]}, 1_{[0,t]} \rangle_{\mathcal{H}} = R^{H,K}(s, t).$$
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The mapping \( 1_{[0,t]} \mapsto B^{H,K}_t \) can be extended to an isometry between \( \mathcal{H} \) and the Gaussian space generated by \( B^{H,K} \). We denote this isometry by \( \varphi \mapsto B^{H,K}(\varphi) \).

Let \( \mathcal{S} \) be the set of all smooth cylindrical random variables of the form
\[
F = f(B^{H,K}(\varphi_1), B^{H,K}(\varphi_2), \ldots, B^{H,K}(\varphi_n))
\]
where \( n \geq 0 \), \( f \in C^\infty \) has a compact support and \( \varphi_i \in \mathcal{H} \). The Malliavin derivative of \( F \) with respect to \( B^{H,K} \) is the element belonging to \( L^2(\Omega, \mathcal{H}) \) defined by
\[
D_s F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B^{H,K}(\varphi_1), B^{H,K}(\varphi_2), \ldots, B^{H,K}(\varphi_n))\varphi_i(s), \quad s \in [0,1].
\]

This operator can be extended to the closure \( \mathcal{D}^{1,2} \) of \( \mathcal{S} \) with respect to the norm
\[
\|F\|_{1,2}^2 := \mathbb{E}[F^2] + \mathbb{E}[\|D_F\|^2_{\mathcal{H}}].
\]
The Malliavin derivative satisfies the following chain rule. For every random vector \( F = (F_1, \ldots, F_n) \) with components in \( \mathcal{D}^{1,2} \) and for every continuously differentiable function \( \varphi : \mathbb{R}^n \to \mathbb{R} \) with bounded partial derivatives, we obtain \( \varphi(F_1, \ldots, F_1) \in \mathcal{D}^{1,2} \) and we have, for any \( s \in [0,1] \):
\[
D_s \varphi(F_1, \ldots, F_n) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(F_1, \ldots, F_n)D_s F_i.
\]

The divergence operator \( I \) is the adjoint of \( D \) in the following sense. A random process \( u \in L^2(\Omega, \mathcal{H}) \) belongs to the domain of \( I \) if and only if
\[
\mathbb{E}[\langle DF, u \rangle_{\mathcal{H}}] \leq C_u \|F\|_{L^2(\Omega)}, \quad \text{for every } F \in \mathcal{D}^{1,2},
\]
where \( C_u \) is a constant depending only on \( u \). In that case, \( I(u) \) verifies the integration by part formula:
\[
\mathbb{E}(FI(u)) = \mathbb{E}(\langle DF, u \rangle_{\mathcal{H}})
\]
for any \( F \in \mathcal{D}^{1,2} \).

3. Asymptotic behavior of weighted quadratic variations of bifractional Brownian motion.

We will make use of the following assumption on the weight function \( h \).

**Assumption (H_m):**
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$h : \mathbb{R} \to \mathbb{R}$ belongs to $C^m$ and, for any $p > 0$ and any $i = 1, \ldots, m$,
\[
\sup_{s \in [0,1]} \mathbb{E} \left[ |h^{(i)}(B_{s}^{H,K})|^p \right] < \infty.
\]

The main result of this section is the following:

**Theorem 3.1.** Let $B_{H,K}$ be a bifractional Brownian motion with parameters $H, K$ such that $0 < 4H < 1$, $K \in (0,1]$, and let $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying $(H_4)$. Then we have, as $n \to \infty$:

\[
n^{2HK-1} \sum_{l=0}^{n-1} h(B_{l/n}^{H,K}) [n^{2HK}(\Delta B_{l/n}^{H,K})^2 - 2^{1-K}] \xrightarrow{L^2} \frac{1}{22K} \int_0^1 h''(B_{u}^{H,K}) \, du.
\]  

(3.1)

**Remark 3.2.** When $K = 1$ (that is when $B_{H,K}$ is a fractional Brownian motion) we recover Theorem 1.1 in [5]. Our proof in the general case follows the same lines.

**Proof of the theorem.** Throughout the proof, we will denote for simplicity
\[
\delta_{k/n} = 1_{[k/n,(k+1)/n]} \quad \text{and} \quad \varepsilon_{k/n} = 1_{[0,k/n]}
\]
and we let $C$ stand for a positive generic constant independent of $k, l, n$ that can be different from line to line.

We will need several lemmas. The first one is immediate to check, so its proof is left to the reader.

**Lemma 3.3.**

1. If $2HK < 1$, then the sequence $\varphi$ defined by

\[
\varphi(l) := \left((l+1)^{2H} + l^{2H}\right)^K - 2^K l^{2HK}
\]

satisfies $\varphi(l) \sim \frac{2^K HK}{l^{2HK}}$ as $l$ goes to infinity. In particular, $\varphi$ is bounded.

2. If $2HK < 1$, then the sequence defined by

\[
\phi(l) := l^{2HK} + (l+1)^{2HK} - 2^{1-K} \left(l^{2H} + (l+1)^{2H}\right)^K
\]

satisfies $\phi(l) \sim C/l^{2-2HK}$ as $l$ goes to infinity. In particular, $\sum_{l \geq 0} \phi(l) < \infty$. 

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Lemma 3.4.  

(1) Assume that $2HK < 1$. Then, as $n \to \infty$,
\[
\sum_{k,l=0}^{n-1} |\langle \varepsilon_{l/n}, \delta_{k/n} \rangle_\mathcal{H}| = o(n^{2-2HK}). \tag{3.4}
\]

(2) Assume that $H < 1/4$. For $k,l = 0,1, \ldots, n-1$, set
\[
D_{k,l} := \left( (k+1)^{2H} + (l+1)^{2H} \right)^K - \left( (k+1)^{2H} + l^{2H} \right)^K + \left( k^{2H} + (l+1)^{2H} \right)^K \nonumber
\]
\[
- \left( k^{2H} + l^{2H} \right)^K.
\tag{3.5}
\]
Then, as $n \to \infty$,
\[
\sum_{k,l=0}^{n-1} |D_{k,l}| = o(n^{2-2HK}). \tag{3.6}
\]

(3) Assume that $H < 1/4$. Then, as $n \to \infty$,
\[
\sum_{k,l=0}^{n-1} |\langle \delta_{k/n}, \delta_{l/n} \rangle_\mathcal{H}| = o(n^{2-4HK}). \tag{3.7}
\]

Proof of Lemma 3.4. We prove the first point. For $0 \leq k,l \leq n-1$, we have
\[
\langle \varepsilon_{l/n}, \delta_{k/n} \rangle_\mathcal{H} = \frac{1}{2K} n^{-2HK} \left( (k+1)^{2H} + l^{2H} \right)^K \nonumber
\]
\[
- \left( k^{2H} + l^{2H} \right)^K + |l-k|^{2HK} - |l-k-1|^{2HK}
\]
and therefore
\[
\sum_{k,l=0}^{n-1} |\langle \varepsilon_{l/n}, \delta_{k/n} \rangle_\mathcal{H}| \leq \frac{1}{2K} n^{-2HK} \sum_{l=0}^{n-1} \left( (n^{2H} + l^{2H})^K - l^{2HK} \right) \nonumber
\]
\[
+ \frac{1}{2K} n^{-2HK} \sum_{k,l=0}^{n-1} |l-k|^{2HK} - |l-k-1|^{2HK} \nonumber
\]
\[
\sim 2^{-K} n \cdot \left( \int_0^1 ((1+x^{2H})^K - x^{2HK}) \, dx \right) + 2^{-K} n \nonumber
\]
\[
= Cn = o(n^{2-2HK}), \quad \text{since} \quad 2HK < 1.
\]
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Concerning the second point, we use the elementary inequality $|x|^K - |y|^K \leq |x-y|^K$, valid for any $x, y \in \mathbb{R}$ because $K \leq 1$, to see that

$$|D_{k,l}| \leq 2 \left( (l+1)^{2H} - l^{2H} \right)^K.$$  

Consequently, since $\left( (l+1)^{2H} - l^{2H} \right)^K$ behaves as $l^{2HK-K}$ for large $l$, we get

$$\sum_{k,l=0}^{n-1} |D_{k,l}| \leq C n \sum_{l=1}^{n} \frac{1}{l^{2HK}},$$  \hspace{1cm} (3.8)

which is $o(n^{-2HK})$. Indeed, let $\gamma$ such that $2HK+1-K < \gamma < -2HK+1$. Then, the series $\sum_{l=1}^{\infty} 1/l^{2HK+\gamma}$ is convergent and

$$n^{-1+2HK} \sum_{l=1}^{n} \frac{1}{l^{2HK}} = n^{\gamma-1+2HK} \frac{1}{n^{\gamma}} \sum_{l=1}^{n} \frac{1}{l^{2HK}} \leq n^{\gamma-1+2HK} \sum_{l \geq 1} \frac{1}{l^{2HK+\gamma}} \to 0.$$  

For the third point, we have

$$\langle \delta_{k/n}, \delta_{l/n} \rangle_H = \frac{1}{2K} n^{-2HK} \left( D_{k,l} + |k-l-1|^{2HK} + |k-l+1|^{2HK} - 2|k-l|^{2HK} \right),$$

with $D_{k,l}$ defined by (3.5). Then, we obtain as previously

$$\sum_{k,l=0}^{n-1} |\langle \delta_{k/n}, \delta_{l/n} \rangle_H| \leq \frac{1}{2K} n^{-2HK} \sum_{k,l=0}^{n-1} |D_{k,l}| + 2^{1-K} n.$$  

Thus, using (3.6) of Lemma 3.4 and the fact that $H < 1/4$, equality (3.7) follows since $n = o(n^{2-4HK})$, which completes the proof.  

□
Lemma 3.5. If $2H < 1$, $0 < K \leq 1$ and $g, h$ are two functions satisfying the condition $(H_2)$, then

$$
\sum_{k,l=0}^{n-1} \mathbb{E}\{h(B_{k/n}^H)g(B_{l/n}^H)[n^{2HK} (\Delta B_{k/n}^H)^2 - 2^{1-K}]\} = \frac{1}{2^2K} \frac{1}{n^{2HK}} \sum_{k,l=0}^{n-1} \mathbb{E}[h''(B_{k/n}^H)g(B_{l/n}^H)] + o(n^{2-2HK})
$$

Proof of Lemma 3.5. For $k, l = 0, 1, \ldots, n - 1$, we use the integration by parts formula to write

$$
\mathbb{E}\{h(B_{k/n}^H)g(B_{l/n}^H)n^{2HK} (\Delta B_{k/n}^H)^2\} = \mathbb{E}\{h'(B_{k/n}^H)g(B_{l/n}^H)n^{2HK} (\Delta B_{k/n}^H) I(\delta_{k/n})\} \langle \varepsilon_{k/n}, \delta_{k/n} \rangle + \mathbb{E}\{h''(B_{k/n}^H)g'(B_{l/n}^H)n^{2HK} (\Delta B_{k/n}^H)^2\} \langle \delta_{k/n}, \delta_{k/n} \rangle.
$$

But,

$$
n^{2HK} \langle \delta_{k/n}, \delta_{k/n} \rangle = 2^{1-K} + \phi(k)
$$

with $\phi$ defined as in (3.3). Thus,

$$
\mathbb{E}\{h(B_{k/n}^H)g(B_{l/n}^H)[n^{2HK} (\Delta B_{k/n}^H)^2 - 2^{1-K}]\} = \mathbb{E}\{h'(B_{k/n}^H)g(B_{l/n}^H)n^{2HK} (\Delta B_{k/n}^H) I(\delta_{k/n})\} \langle \varepsilon_{k/n}, \delta_{k/n} \rangle + \mathbb{E}\{h''(B_{k/n}^H)g'(B_{l/n}^H)n^{2HK} (\Delta B_{k/n}^H)^2\} \langle \varepsilon_{k/n}, \varepsilon_{l/n}, \delta_{k/n}, \delta_{k/n} \rangle + \mathbb{E}\{h''(B_{k/n}^H)g''(B_{l/n}^H)n^{2HK} (\Delta B_{k/n}^H)^2\} \langle \delta_{k/n}, \delta_{k/n} \rangle \langle \delta_{k/n}, \delta_{k/n} \rangle + \mathbb{E}\{h''(B_{k/n}^H)g''(B_{l/n}^H)n^{2HK} (\Delta B_{k/n}^H)^2\} \langle \delta_{k/n}, \delta_{k/n} \rangle \langle \delta_{k/n}, \delta_{k/n} \rangle + \mathbb{E}\{h''(B_{k/n}^H)g''(B_{l/n}^H)n^{2HK} (\Delta B_{k/n}^H)^2\} \langle \delta_{k/n}, \delta_{k/n} \rangle \langle \delta_{k/n}, \delta_{k/n} \rangle + \mathbb{E}\{h''(B_{k/n}^H)g''(B_{l/n}^H)n^{2HK} (\Delta B_{k/n}^H)^2\} \langle \delta_{k/n}, \delta_{k/n} \rangle \langle \delta_{k/n}, \delta_{k/n} \rangle + \mathbb{E}\{h''(B_{k/n}^H)g''(B_{l/n}^H)n^{2HK} (\Delta B_{k/n}^H)^2\} \langle \delta_{k/n}, \delta_{k/n} \rangle \langle \delta_{k/n}, \delta_{k/n} \rangle (3.10)
$$

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Now, we have
\[
\langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{H}^{2} = \frac{1}{2n^{2K}} \frac{1}{n^{4HK}} (\varphi(k) - 1)^{2}
\] (3.11)

with \( \varphi \) defined by (3.2).

Therefore, using Lemma 3.3, we get
\[
\left| \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{H}^{2} - \frac{1}{2n^{2K}} \frac{1}{n^{4HK}} \right| = \frac{1}{2n^{2K}} \frac{1}{n^{4HK}} |\varphi(k)(\varphi(k) - 2)| \leq C \frac{1}{n^{4HK}} \varphi(k).
\]

Since \( 2HK < 1 \), we can choose \( \beta > 0 \) such that \( 2HK < \beta < 1 \) and we set \( \gamma = 1 - \beta \). Then
\[
\sum_{l \geq 1} \varphi(l)/l^{\beta} < \infty \quad \text{and consequently,}
\]
\[
\sum_{k,l=0}^{n-1} \left| \langle \varepsilon_{l/n}, \delta_{l/n} \rangle_{H}^{2} - \frac{1}{2n^{2K}} \frac{1}{n^{4HK}} \right| \leq Cn^{1-4HK-\gamma}.
\]

This implies that, under condition \((H_2)\)
\[
n^{2HK} \sum_{k,l=0}^{n-1} \left| \mathbb{E}\{ h''(B_{k/n}^{H,K})g(B_{l/n}^{H,K})\} \left( \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{H}^{2} - \frac{1}{2n^{2K}} \frac{1}{n^{4HK}} \right) \right| \leq Cn^{2-2HK-\gamma} = o(n^{2-2HK}).
\]

Furthermore, using the fact that \( 2HK \leq 2H \leq 1 \), we see that
\[
n^{2HK} \left| \langle \varepsilon_{l/n}, \delta_{k/n} \rangle_{H} \right| := 2^{-K} \left( (k + 1)^{2H} + l^{2H} \right)^{K} - \left( (k^{2H} + l^{2H})^{K} + |l - k|^{2HK} - |l - 1 - k|^{2HK} \right)
\]
\[
\leq 2^{-K} \left\{ \left| (k + 1)^{2H} - k^{2H} \right|^{K} + \left| |l - k|^{2HK} - |l - 1 - k|^{2HK} \right| \right\} \leq 2^{1-K} \quad (3.12)
\]

is bounded independently of \( k \) and \( l \). Now, since
\[
\sum_{k,l=0}^{n-1} \left\{ \left( (k + 1)^{2H} + l^{2H} \right)^{K} - \left( k^{2H} + l^{2H} \right)^{K} \right.
\]
\[
\left. + \left| |l - k|^{2HK} - |l - 1 - k|^{2HK} \right| \right\} \leq Cn^{1+2HK}
\]

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by telescoping sum, we deduce that
\[
\begin{align*}
& n^{2HK} \sum_{k,l=0}^{n-1} \left\{ |\langle \xi_{k/n}, \delta_{k/n} \rangle \mathcal{H} \langle \xi_{l/n}, \delta_{k/n} \rangle \mathcal{H} | + |\langle \xi_{l/n}, \delta_{k/n} \rangle \mathcal{H} |^2 \right\} \\
& \leq C \sum_{k,l=0}^{n-1} |\langle \xi_{l/n}, \delta_{k/n} \rangle \mathcal{H} | \\
& \leq C n^{-2HK} \sum_{k,l=0}^{n-1} \left\{ (k+1)^{2H} + l^{2H} \right\}^K \\
& \quad - \left( k^{2H} + l^{2H} \right)^K + |(l - k)^{2HK} - (l - 1 - k)^{2HK} | \\
& \leq C n^{-2HK} n^{1+2HK} = Cn = o(n^{2-2HK}) \quad (\text{since } 2HK < 1).
\end{align*}
\]
Thus, under condition (\(H_2\)), we obtain
\[
\sum_{k,l=0}^{n-1} \mathbb{E} \left\{ |h'(B_{k/n}^H)g'(B_{l/n}^H)\right\} n^{2HK} |\langle \xi_{k/n}, \delta_{k/n} \rangle \mathcal{H} \langle \xi_{l/n}, \delta_{k/n} \rangle \mathcal{H}| \\
+ \mathbb{E} \left\{ |h(B_{k/n}^H)g''(B_{l/n}^H)\right\} n^{2HK} |\langle \xi_{l/n}, \delta_{k/n} \rangle \mathcal{H}|^2 \\
= o(n^{2-2HK}).
\]
On the other hand, by Lemma 3.3 and once again using condition (\(H_2\))
\[
\sum_{k,l=0}^{n-1} \mathbb{E} \{ |h(B_{k/n}^H)g(B_{k/n}^H)\phi(k)| \} \leq C \left( \sum_{k=0}^{\infty} |\phi(k)| \right) \cdot n = o(n^{2-2HK})
\]
(since \(2HK < 1\)).

Finally, by combining all the previous estimates with (3.10), the proof of Lemma 3.5 is done. \(\square\)

**Lemma 3.6.** If \(H < 1/4, 0 < K \leq 1\) and \(g, h\) are two functions satisfying the condition \((H_4)\), then
\[
\sum_{k,l=0}^{n-1} \mathbb{E} \left\{ h(B_{k/n}^H)g(B_{l/n}^H) \right\} \\
\left[ n^{2HK} (\Delta B_{k/n}^H)^2 - 2^{1-K} \right] \left[ n^{2HK} (\Delta B_{l/n}^H)^2 - 2^{1-K} \right] \\
= \frac{1}{2^{4K}} \frac{1}{n^{4HK}} \sum_{k,l=0}^{n-1} \mathbb{E} \left[ h''(B_{k/n}^H)g''(B_{l/n}^H) \right] + o(n^{2-4HK}).
\]

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Proof of Lemma 3.6. Using the integration by part formula we have

\[
\mathbb{E}\{h(B_{k/n}^H,K)g(B_{l/n}^H,K)[n^{2HK}(\Delta B_{k/n}^{H,K})^2 - 2^{1-K}]\phi(l/n)\}
\]

It follows from (3.9), that

\[
\mathbb{E}\{h(B_{k/n}^H,K)g(B_{l/n}^H,K)[n^{2HK}(\Delta B_{k/n}^{H,K})^2 - 2^{1-K}]\}
\]

and, once again by an integration by part formula, it leads to

\[
\mathbb{E}\{h(B_{k/n}^H,K)g(B_{l/n}^H,K)[n^{2HK}(\Delta B_{k/n}^{H,K})^2 - 2^{1-K}]\}
\]

\[
= \sum_{i=1}^{7} A_{k,l,n}^i.
\]
Consequently, the proof of the lemma will be deduced after the study of the asymptotic behavior of $\sum_{k,l=0}^{n} A_{k,l,n}^i$, as $n \to \infty$, for each $i \in \{1, \ldots, 7\}$.

**Claim 3.1.** We have, as $n$ goes to infinity,

1. \[ \sum_{k,l=0}^{n-1} |A_{k,l,n}^i| = o(n^{2-4HK}) \quad \text{for every } i \neq 6. \]

2. \[ \sum_{k,l=0}^{n-1} A_{k,l,n}^6 = \frac{1}{24K n^{2HK}} \sum_{k,l=0}^{n-1} \mathbb{E}\{h''(B_{k/n}^H)g''(B_{l/n}^H)\} + o(n^{2-4HK}). \]

**Proof of Claim 3.1.** We first consider the term $A_{k,l,n}^1$, the study of $A_{k,l,n}^2$ being similar. Using $h''$ instead of $h$ in (3.10), we can write

\[
A_{k,l,n}^1 := n^{2HK} \mathbb{E}\{h''(B_{k/n}^H)g(B_{l/n}^H)[n^{2HK}(\Delta B_{k/n}^H)^2 - 2^{1-K}]\} \langle \varepsilon_{k/n}, \delta_{l/n} \rangle_H^2
\]

\[
= n^{AHK} \mathbb{E}\{h^{(4)}(B_{k/n}^H)g(B_{l/n}^H)\} \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_H^2 \langle \varepsilon_{k/n}, \delta_{l/n} \rangle_H^2 + 2n^{AHK} \mathbb{E}\{h^{(3)}(B_{k/n}^H)g'(B_{l/n}^H)\} \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_H \langle \varepsilon_{l/n}, \delta_{k/n} \rangle_H \langle \varepsilon_{k/n}, \delta_{l/n} \rangle_H^2
\]

\[
+ n^{AHK} \mathbb{E}\{h''(B_{k/n}^H)g''(B_{l/n}^H)\} \langle \varepsilon_{l/n}, \delta_{k/n} \rangle_H^2 \langle \varepsilon_{k/n}, \delta_{l/n} \rangle_H^2 + n^{2HK} \mathbb{E}\{h''(B_{k/n}^H)g(B_{l/n}^H)\} \langle \varepsilon_{k/n}, \delta_{l/n} \rangle_H^2 \phi(k).
\]

Using that $n^{2HK}\langle \varepsilon_{k/n}, \delta_{l/n} \rangle_H$ and $\phi(k)$ are bounded with respect to $k$, $l$, $n$, see (3.12), and using condition (H4), we have

\[ |A_{k,l,n}^1| \leq C n^{-2HK} |\langle \varepsilon_{k/n}, \delta_{l/n} \rangle_H|. \]

According to Lemma 3.4, we deduce

\[ \sum_{k,l=0}^{n-1} |A_{k,l,n}^1| = o(n^{2-4HK}). \]

Now, let us consider the term $A_{k,l,n}^3$, the study of the cases $A_{k,l,n}^i$ where $i = 4, 5$ being similar, since each of this terms contains the factor $\langle \delta_{k/n}, \delta_{l/n} \rangle_H$. 

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As previously, by the Malliavin integration by parts formula, we can write

\[
A_{k,l,n}^3 := 4n^{4HK}E\left\{h'(B_{k/n}^H)g(B_{l/n}^H)I(k/n)\right\}\langle\varepsilon_{k/n},\delta_{l/n}\rangle_H\langle\delta_{k/n},\delta_{l/n}\rangle_H
\]

\[
= 4n^{4HK}E\left\{h''(B_{k/n}^H)g(B_{l/n}^H)\right\}\langle\varepsilon_{k/n},\delta_{k/n}\rangle_H\langle\delta_{k/n},\delta_{l/n}\rangle_H
\]

\[
+ 4n^{4HK}E\left\{h'(B_{k/n}^H)g'(B_{l/n}^H)\right\}\langle\varepsilon_{l/n},\delta_{k/n}\rangle_H\langle\delta_{k/n},\delta_{l/n}\rangle_H.
\]

Hence, using again that \(n^{2HK}\langle\varepsilon_{k/n},\delta_{l/n}\rangle_H\) is bounded and the condition (H2), we obtain

\[
\sum_{k,l=0}^{n-1} |A_{k,l,n}^3| \leq C \sum_{k,l=0}^{n-1} |\langle\delta_{k/n},\delta_{l/n}\rangle_H|
\]

which is \(o(n^{2-4HK})\) by using point (3) of Lemma 3.4.

For the term \(A_{k,l,n}^6\), we use (3.11) and point (1) of Lemma 3.3 to write

\[
\langle\varepsilon_{k/n},\delta_{k/n}\rangle_H^2 = \frac{1}{22K} \frac{1}{n^{4HK}} + O\left(\frac{1}{n^{4HK}}\right).
\]  \hspace{1cm} (3.13)

Substituting (3.13) into the expression of \(A_{k,l,n}^6\), yields

\[
A_{k,l,n}^6 := E\left\{h(B_{k/n}^H)g''(B_{l/n}^H)\right\}\left[n^{2HK}(\Delta B_{k/n}^H)^2 - 2^{1-K}\right]n^{2HK}\langle\varepsilon_{l/n},\delta_{l/n}\rangle_H^2
\]

\[
= \frac{1}{22K} \frac{1}{n^{2HK}} E\left\{h(B_{k/n}^H)g''(B_{l/n}^H)\right\}\left[n^{2HK}(\Delta B_{k/n}^H)^2 - 2^{1-K}\right] + O\left(\frac{1}{n^{2HK}}\right).
\]

Therefore, using Lemma 3.5, with \(g''\) instead of \(g\), we obtain

\[
\sum_{k,l=0}^{n-1} A_{k,l,n}^6 = \frac{1}{24K} \frac{1}{n^{4HK}} \sum_{k,l=0}^{n-1} E\left\{h''(B_{k/n}^H)g''(B_{l/n}^H)\right\} + o(n^{2-4HK}).
\]
Finally, we consider the term $A^7_{k,l,n}$. Still using Malliavin integration by parts formula, we write

$$A^7_{k,l,n} := \phi(l) \mathbb{E}\{h(B^H,K_k)g(B^H,K_l)[n^{2HK}(\Delta B^H,K_k)^2 - 2^{1-K}]\}$$

$$= \mathbb{E}\{h''(B^H,K_k)g(B^H,K_l)\langle \varepsilon_{k/n}, \delta_{k/n} \rangle \} \mathbb{E}\{\varepsilon_{l/n}, \delta_{k/n} \rangle \}$$

$$+ 2n^{2HK} \phi(l) \mathbb{E}\{h'(B^H,K_k)g'(B^H,K_l)\langle \varepsilon_{k/n}, \delta_{k/n} \rangle \} \langle \varepsilon_{l/n}, \delta_{l/n} \rangle \mathbb{E}\{h''(B^H,K_k)g''(B^H,K_l)\langle \varepsilon_{l/n}, \delta_{l/n} \rangle \}$$

$$+ \phi(k) \phi(l) \mathbb{E}\{h(B^H,K_k)g(B^H,K_l)\}$$

$$=: (a), (b), (c), (d), (k,l,n).$$

We claim that $\sum_{k,l=0}^{n-1} |A^7_{k,l,n}| = o(n^{2-4HK})$. Indeed, we have by condition (H$_2$)

$$|a| + |b| \leq C |\varepsilon_{k/n}, \delta_{k/n} \rangle |\phi(l)|.$$

But,

$$n^{2HK} \langle \varepsilon_{k/n}, \delta_{k/n} \rangle \mathbb{E} = 2^{-K} ((k+1)^{2H} + k^{2H})^K - 2^{K} k^{2HK} - 1$$

$$= 2^{-K} \{\varphi(k) - 1\}.$$

Then, thanks to the first point of Lemma 3.3, we can write

$$\sum_{k,l=0}^{n-1} \sum_{k,l=0}^{n-1} |a| + |b| \leq C n^{-2HK} \left( \sum_{l=0}^{n-1} |\phi(l)| \right) \left( \sum_{k=0}^{n-1} |\varphi(k) - 1| \right)$$

$$\leq C n^{-2HK} \left( \sum_{l=0}^{\infty} |\phi(l)| \right) \left( n + \sum_{k=1}^{n} |\varphi(k)| \right).$$

Since $\varphi(k) \sim 1/k^{K-2HK}$ as $k$ goes to infinity and $4H < 1$, we have

$$n^{-2HK} \sum_{k=0}^{n-1} |\varphi(k)| = o(n^{2-4HK}).$$

Combining with $n^{1-2HK} = o(n^{2-4HK})$, since $2HK < 1$, it follows that

$$\sum_{k,l=0}^{n-1} |a| + |b| = o(n^{2-4HK}).$$
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For the term $\sum_{k,l=0}^{n-1} \left| (c)_{k,l,n} \right|$, we have similarly

$$\sum_{k,l=0}^{n-1} \left| (c)_{k,l,n} \right| \leq Cn \sum_{k=0}^{n-1} \left| \phi(k) \right| \left| \langle \varepsilon_{k/n}, \delta_{k/n} \rangle \right|$$

$$\leq Cn^{1-2HK} \sum_{k=0}^{n-1} \left| \phi(k) \right| \left| ((k+1)^{2H} - k^{2H})^K + 1 \right|$$

so that (recall that $H < 1/4 < 1/2$)

$$\sum_{k,l=0}^{n-1} \left| (c)_{k,l,n} \right| \leq Cn^{1-2HK} \sum_{k=0}^{\infty} \left| \phi(k) \right| = o(n^{2-4HK}), \quad \text{since} \quad 2HK < 1.$$

For the last term $\sum_{k,l=0}^{n} \left| (d)_{k,l,n} \right|$, we have

$$\sum_{k,l=0}^{n} \left| (d)_{k,l,n} \right| \leq \left( \sum_{k=0}^{\infty} \left| \phi(k) \right| \right)^2 = o(n^{2-4HK}), \quad \text{since} \quad 2HK < 1.$$

This finishes the proof of Claim 3.1, and thus the proof of Lemma 3.6. □

Combining these two lemmas, the proof of the theorem can be completed along the same lines as in [5]. Indeed, by Lemma 3.6, we have

$$\mathbb{E} \left\{ n^{2HK-1} \sum_{k=0}^{n-1} h(B_{k/n}^H) [n^{2HK} (\Delta B_{k/n}^H)^2 - 2^{1-K}] \right\}^2$$

$$= n^{4HK-2} \sum_{k,l=0}^{n-1} \mathbb{E} \left\{ h(B_{k/n}^H) h(B_{l/n}^H) \right\} \left[ n^{2HK} (\Delta B_{k/n}^H)^2 - 2^{1-K} \right] \left[ n^{2HK} (\Delta B_{l/n}^H)^2 - 2^{1-K} \right]$$

$$= \frac{1}{2^{2K} n^2} \sum_{k,l=0}^{n-1} \mathbb{E} \left[ h^{''}(B_{k/n}^H) h^{''}(B_{l/n}^H) \right] + o(1)$$

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and, using Lemma 3.5, we have

\[
\mathbb{E} \left\{ n^{2HK-1} \sum_{k=0}^{n-1} h(B_{k/n}^H) \right\} \\
= \sum_{k,l=0}^{n-1} \mathbb{E} \left\{ h(B_{k/n}^H) h''(B_{l/n}^H) \left[ n^{2HK} (\Delta B_{k/n}^H)^2 - 2^{1-K} \right] \right\} \\
= \frac{1}{24K} \frac{1}{n^2} \sum_{k,l=0}^{n-1} \mathbb{E} \left[ h''(B_{k/n}^H) h''(B_{l/n}^H) \right] + o(1).
\]

As a consequence, we obtain the convergence

\[
\mathbb{E} \left\{ n^{2HK-1} \sum_{k=0}^{n-1} h(B_{k/n}^H) \left[ n^{2HK} (\Delta B_{k/n}^H)^2 - 2^{1-K} \right] \right\} \\
- \frac{1}{24K} \frac{1}{n} \sum_{k=0}^{n-1} h''(B_{k/n}^H) \xrightarrow[n \to \infty]{} 0
\]

which implies (3.1), since

\[
\frac{1}{n} \sum_{k=0}^{n-1} h''(B_{k/n}^H) \xrightarrow{L^2(\Omega)} \frac{1}{n} \int_0^1 h''(B_u^H) \, du.
\]

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References

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