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Necessary condition for measures which are \((L^q, L^p)\) multipliers

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Abstract

Let \(G\) be a locally compact group and \(\rho\) the left Haar measure on \(G\). Given a non-negative Radon measure \(\mu\), we establish a necessary condition on the pairs \((q, p)\) for which \(\mu\) is a multiplier from \(L^q(G, \rho)\) to \(L^p(G, \rho)\). Applied to \(\mathbb{R}^n\), our result is stronger than the necessary condition established by Oberlin in [14] and is closely related to a class of measures defined by Fofana in [7].

When \(G\) is the circle group, we obtain a generalization of a condition stated by Oberlin [15] and improve on it in some cases.

Résumé

Soit \(G\) un groupe localement compact et \(\rho\) la mesure de Haar à gauche sur \(G\). Etant donné une mesure de Radon positive \(\mu\), nous établissons une condition nécessaire sur les couples \((q, p)\) pour lesquels \(\mu\) est un multiplicateur de \(L^q(G, \rho)\) dans \(L^p(G, \rho)\). Appliqué à \(\mathbb{R}^n\), notre résultat est plus fort que la condition nécessaire établie par Oberlin dans [14] et est très lié à une classe de mesures définie par Fofana dans [7].

Lorsque \(G\) est le tore, nous obtenons une généralisation d’une condition énoncée par Oberlin [15] et l’améliorons dans certains cas.

1. Introduction

We suppose that \(G\) is a locally compact group and \(\rho\) is the left Haar measure on \(G\).

For \(1 \leq q < \infty\), a Radon measure \(\mu\) on \(G\) is said to be \(L^q\)-improving if there exists a real number \(p > q\) such that

\[
\mu \ast f \in L^p(G, \rho) \quad \text{and} \quad \|\mu \ast f\|_{L^p(G, \rho)} \leq c \|f\|_{L^q(G, \rho)}
\]

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**Math. classification:** 43A05, 43A15.
for all \( f \in L^q(G, \rho) \), where \( c \) is a real number not depending on \( f \).

Of course absolutely continuous measures with Radon-Nikodym derivatives with respect to \( \rho \) in \( L^r(G, \rho) \) with \( \frac{1}{q} + \frac{1}{r} - 1 > 0 \) are \( L^q \)-improving. But \( L^q \)-improving singular measures also exist.

Bonami [2] showed that all tame Riesz products on the Walsh group are \( L^q \)-improving, and that was extended to all compact abelian groups by Ritter [16]. Moreover it is well known that on the circle group \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \), the Cantor-Lebesgue measure \( \mu_3^2 \) associated with the Cantor set of constant ratio of dissection \( \delta > 2 \) is \( L^q \)-improving for \( 1 < q < \infty \).(See Section 4 for a precise definition of this measure.) This result was proved by Oberlin [12] for \( \delta = 3 \). Ritter [17], Beckner, Janson and Jerison [1] proved the same for \( \delta \) rational and Christ [3] for \( \delta \) irrational.

In fact, Christ has extended the result to Cantor-Lebesgue measures with variable but bounded ratios \( 2 < \delta_t \leq c \) of dissection.

In this note, we are interested in the following problem: given a non-negative Radon measure \( \mu \) on \( G \), determine the indices \( 1 \leq q < p < \infty \) for which there exists a non-negative constant \( c(\mu, q, p) \) such that

\[
\| \mu \ast f \|_{L^p(G, \rho)} \leq c(\mu, q, p) \| f \|_{L^q(G, \rho)}, \quad f \in L^q(G, \rho).
\]

In [15] Oberlin stated the following

**Proposition 1.1.** If the Cantor-Lebesgue measure \( \mu_3^2 \) associated to the middle third Cantor set satisfies (1.1), then

\[
\frac{1}{q} + \left( 1 - \frac{\log 2}{\log 3} \right) \left( 1 - \frac{1}{p} \right) \leq 1.
\]

Graham, Hare and Ritter obtained in [9] the following

**Proposition 1.2.** Let \( \mu \) be a measure on the circle group \( \mathbb{T} \) and \( 1 \leq q < 2 \). If there exists a non-negative constant \( c(\mu, q) \) such that

\[
\| \mu \ast f \|_{L^2(\mathbb{T})} \leq c(\mu, q) \| f \|_{L^q(\mathbb{T})}, \quad f \in L^q(\mathbb{T}),
\]

then there exists a positive real number \( K \) such that for any interval \( I \) whose endpoints are \( x \) and \( x + h \), we have

\[
| \mu(I) | \leq K |h|^{\frac{1}{2} - \frac{1}{q}}.
\]
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Inequality (1.3) means that \(\mu\) satisfies a Lipschitz condition of order 
\(\frac{1}{q} - \frac{1}{2}\).

Replacing \(T\) by \(\mathbb{R}^n\), Oberlin proved a similar necessary condition (see the proof of Proposition 2 in [14]).

**Proposition 1.3.** If a non-negative Radon measure on \(\mathbb{R}^n\) satisfies (1.1), then there exists a positive real number \(K\) such that

\[
\mu(R) \leq K |R|^\frac{1}{q} - \frac{1}{p}
\]

(1.4)

for all rectangles \(R\) in \(\mathbb{R}^n\).

In the present paper, we establish the following necessary condition:

**Proposition 1.4.** Suppose that \(\mu\) is a non-negative Radon measure on \(G\) satisfying (1.1). Then for any subsets \(V\) and \(\{x_i / i \in I\}\) of \(G\) such that

i) \(V\) is relatively compact,

ii) \(I\) is countable and \((x_i V) \cap (x_j V) = \emptyset\) for \(i \neq j\),

we have

\[
\rho(V)^\frac{1}{p} \left( \sum_{i \in I} \mu(x_i V)^p \right)^\frac{1}{p} \leq c(\mu, q, p) \rho(V^{-1} V)^\frac{1}{q}.
\]

(1.5)

We show that all the necessary conditions stated in Proposition 1.1, Proposition 1.2 and Proposition 1.3 follow from Proposition 1.4.

Moreover any non-negative Radon measure \(\mu\) on \(\mathbb{T}\) or \(\mathbb{R}^n\) satisfying the conclusion of Proposition 1.4 belongs to the space \(M^{p, \alpha}\), \(\frac{1}{\alpha} = 1 - \frac{1}{q} + \frac{1}{p}\) (see Notation 3.4 and Section 4 for the definition of \(M^{p, \alpha}\)). In [7], Fofana used these spaces of measures and their subspaces \((L^q, l^p)^\alpha\) to express a necessary condition for Fourier multipliers. He also obtained a generalization of Hausdorff-Young inequality. For other results related to these spaces see [6], [8] and [11].

Inequality (1.2) means exactly that \(\mu_3^2\) belongs to \(M^{p, \alpha}\) where \(\frac{1}{\alpha} = 1 - \frac{1}{q} + \frac{1}{p}\) (see the comment after the proof of Proposition 4.2).

Applied to the Cantor-Lebesgue measure associated to the Cantor set of constant ratio of dissection \(\delta > 3\), Proposition 1.4 yields the following

**Proposition 1.5.** Let \(\delta > 3\) and \(1 < q < p < \infty\). Assume that

\[
\left\| \mu_3^2 * f \right\|_{L^p(\mathbb{T})} \leq c \left( \mu_3^2, p, q \right) \|f\|_{L^q(\mathbb{T})}, \quad f \in L^q(\mathbb{T}).
\]
Then
\[ p \leq \frac{\log \left( \frac{\delta}{2} \right)}{\log \left( \frac{\delta}{3} \right)} q \]  
(1.6)
and
\[ \frac{1}{q} + \left( 1 - \frac{\log 2}{\log \delta} \right) \left( 1 - \frac{1}{p} \right) \leq 1. \]  
(1.7)

Notice that (1.6) is stronger than (1.7) if \( q > \frac{\log 3}{\log 2} \).

The remainder of this paper is organized as follows: in Section 2 we prove Proposition 1.4 and apply it to \( G = \mathbb{R}^n \) in Section 3. In Section 4 we examine the case \( G = \mathbb{T} \).

2. Proof of Proposition 1.4

Proof. Let \( V \) be a relatively compact subset of \( G \). Then \( f = \chi_{V^{-1}V} \) belongs to \( L^q (G, \rho) \). We have, for all \( i \in I \) and all \( x \in x_iV \),
\[ \mu \ast f (x) = \int_G f \left( y^{-1}x \right) d\mu (y) \geq \int_{x_iV} f \left( y^{-1}x \right) d\mu (y), \]
\[ y \in x_iV \implies y^{-1}x \in V^{-1}V \quad \text{and} \quad f \left( y^{-1}x \right) = 1 \]
and therefore \( \mu \ast f (x) \geq \mu (x_iV) \). It follows that
\[ \int_G (\mu \ast f (x))^p d\rho (x) \geq \sum_{i \in I} \int_{x_iV} (\mu \ast f (x))^p d\rho (x) \geq \sum_{i \in I} \mu (x_iV)^p \rho (x_iV). \]

Therefore
\[ \rho \left( V \right)^{\frac{1}{p}} \left( \sum_{i \in I} \mu (x_iV)^p \right)^{\frac{1}{p}} \leq \| \mu \ast f \|_{L^p(G, \rho)} \]
\[ \leq c(\mu, q, p) \| f \|_{L^q(G, \rho)} \]
\[ = c(\mu, q, p) \rho \left( V^{-1}V \right)^{\frac{1}{q}}. \]

This completes the proof. \( \square \)
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3. Case \(G = \mathbb{R}^n\)

**Notation 3.1.** Let \(R\) be a rectangle in \(\mathbb{R}^n\) with sides \(a_i v_i, i = 1, \ldots, n\), where \((v_i)_{1 \leq i \leq n}\) is a direct orthonormal basis in \(\mathbb{R}^n\) and \(a_i > 0, i = 1, \ldots, n\).

For any \(r > 0\) and \(k = (k_1, \ldots, k_n) \in \mathbb{Z}^n\), set

\[
R^r_k = \left\{ \sum_{i=1}^n (k_i r a_i + x_i) v_i / 0 \leq x_i < r a_i, i = 1, \ldots, n \right\}.
\]

In other words, \(R^r_k\) is a rectangle which \(i\)-th edge is parallel to the vector \(v_i\) and of length \(r a_i\). Notice that for \(r > 0\), the family \(\{R^r_k / k \in \mathbb{Z}^n\}\) is a partition of \(\mathbb{R}^n\).

**Proposition 3.2.** Let \(1 \leq q \leq p < \infty\). If a non-negative Radon measure \(\mu\) on \(\mathbb{R}^n\) satisfies (1.1), then for all rectangles \(R\) in \(\mathbb{R}^n\)

\[
\sup_{r > 0} (r^n |R|)^{\frac{1}{\alpha} - 1} \left( \sum_{k \in \mathbb{Z}^n} \mu(R^r_k)^p \right)^{\frac{1}{p}} \leq c(\mu, q, p) 2^n
\]

where \(\frac{1}{\alpha} = 1 - \frac{1}{q} + \frac{1}{p}\).

**Proof.** Let \(r > 0\). Notice that for every \(k \in \mathbb{Z}^n\) we have that \(R^r_k = R_0 + u_k\), where

\[
R_0 = \left\{ \sum_{i=1}^n x_i v_i / 0 \leq x_i < r a_i, i = 1, \ldots, n \right\} \quad \text{and} \quad u_k = \sum_{i=1}^n k_i r a_i v_i.
\]

It follows from Proposition 1.4 that

\[
|R_0 - R_0|^{-\frac{1}{q}} |R_0|^\frac{1}{p} \left( \sum_{k \in \mathbb{Z}^n} \mu(R^r_k)^p \right)^{\frac{1}{p}} \leq c(\mu, q, p).
\]

Since \(|R_0| = r^n |R|\), we have

\[
2^{-n} (r^n |R|)^{-\frac{1}{q}} (r^n |R|)^\frac{1}{p} \left( \sum_{k \in \mathbb{Z}^n} \mu(R^r_k)^p \right)^{\frac{1}{p}} \leq c(\mu, q, p).
\]

Hence

\[
(r^n |R|)^{\frac{1}{\alpha} - 1} \left( \sum_{k \in \mathbb{Z}^n} \mu(R^r_k)^p \right)^{\frac{1}{p}} \leq c(\mu, q, p) 2^n.
\]

The assertion follows. \(\square\)
Remark 3.3. Proposition 1.3 is a direct consequence of Proposition 3.2. In fact, suppose that \( \mu \) satisfies (1.1) and let \( R \) be any rectangle. As \( \{ R_k \mid k \in \mathbb{Z}^n \} \) is a partition of \( \mathbb{R}^n \), \( R \subset \bigcup_{k \in M} R_k \), where \( M \) is a subset of \( \mathbb{Z}^n \) which number of elements does not exceed \( 2^n \). So \( \mu(R) \leq \sum_{k \in M} \mu(R_k) \) and by Hölder inequality we have

\[
|R|^{\frac{1}{p} - \frac{1}{q}} \mu(R) \leq 2^{\frac{n(p-1)}{p}} |R|^{\frac{1}{p} - \frac{1}{q}} \left( \sum_{k \in M} \mu(R_k)^p \right)^{\frac{1}{p}}
\]

\[
\leq 2^{\frac{n(p-1)}{p}} \left( \sum_{k \in \mathbb{Z}^n} \mu(R_k)^p \right)^{\frac{1}{p}}
\]

\[
\leq 2^{\frac{n(p-1)}{p}} \sup_{r > 0} (r^n |R|)^{\frac{1}{\alpha} - 1} \left( \sum_{k \in \mathbb{Z}^n} \mu(R_k)^p \right)^{\frac{1}{p}}
\]

where \( \frac{1}{\alpha} = 1 - \frac{1}{q} + \frac{1}{p} \). Thus, by Proposition 3.2 we obtain

\[
|R|^{\frac{1}{p} - \frac{1}{q}} \mu(R) \leq 2^{n \left(1 - \frac{1}{p} + \frac{1}{q} \right)} c(\mu, q, p).
\]

Notation 3.4. For any \( k \in \mathbb{Z}^n \), \( x \in \mathbb{R}^n \) and \( r > 0 \), set

\[
I^r_k = \prod_{i=1}^n [k_i r, (k_i + 1) r) \quad \text{and} \quad J^r_x = \prod_{i=1}^n \left( x_i - \frac{r}{2}, x_i + \frac{r}{2} \right).
\]

Let \( M^0 \) denote the space of Radon measures (not necessarily non-negative) on \( \mathbb{R}^n \). For \( \mu \in M^0 \), \( |\mu| \) stands for its total variation. Let \( 1 \leq \alpha, p \leq \infty \). For \( \mu \in M^0 \) and \( r > 0 \), we set

\[
r \| \mu \|_p = \begin{cases} \left( \sum_{k \in \mathbb{Z}^n} |\mu(I^r_k)|^p \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\
\sup_{x \in \mathbb{R}^n} |\mu(J^r_x)| & \text{if } p = \infty \end{cases}
\]

and \( \| \mu \|_{p, \alpha} = \sup_{r > 0} r^{\alpha \left(\frac{1}{\alpha} - 1 \right)} r \| \mu \|_p \).

We define \( M^{p, \alpha}(\mathbb{R}^n) = \{ \mu \in M^0 / \| \mu \|_{p, \alpha} < \infty \} \).

Another consequence of Proposition 3.2 is the following
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**Corollary 3.5.** Assume that \(1 \leq q \leq p < \infty\) and \(\mu\) satisfies (1.1). Then \(\mu\) belongs to \(M^{p, \alpha}(\mathbb{R}^n)\) where \(\frac{1}{\alpha} = 1 - \frac{1}{q} + \frac{1}{p}\).

**Proof.** It follows by choosing \(a_i = 1\) for \(i \in \{1, \ldots, n\}\) and \((v_i)_{1 \leq i \leq n} = (e_i)_{1 \leq i \leq n}\) the usual basis of \(\mathbb{R}^n\) in the definition of \(R_k^n\) in Proposition 3.2. \(\square\)

4. Case \(G = \mathbb{T}\)

In this section we suppose that \(m \geq 2\) is an integer. Let us describe the construction of the Cantor set with variable ratios of dissection and its associated Cantor-Lebesgue measure. We take the interval \([0, 1)\) as a model for \(\mathbb{T}\). Let \(\delta_t > m\) for \(t = 1, 2, \ldots\). Delete from \([0, 1)\), \((m-1)\) left closed intervals of equal length \(\frac{1}{m-1} \left(1 - \frac{m}{\delta_1}\right)\) so that the \(m\) remaining left closed intervals denoted by \(E^1_l\), \(1 \leq l \leq m\), are equally spaced and have the same length \(\frac{1}{\delta_1}\). From each interval \(E^1_l\), \(1 \leq l \leq m\), delete \((m-1)\) left closed intervals of equal length \(\frac{1}{(m-1)\delta_1} \left(1 - \frac{m}{\delta_1 \delta_2}\right)\) so that the \(m\) remaining left closed subintervals \(E^2_l\), \(1 \leq l \leq m^2\), are equally spaced and have the same length \(\frac{1}{\delta_1 \delta_2}\). At this stage, the remaining subset of \([0, 1)\) is \(C^m_{\delta_1, \delta_2} = \bigcup_{l=1}^{m^2} E^2_l\). By iteration, we obtain a sequence of subsets \(C^m_{\delta_1, \delta_2, \ldots, \delta_j} = \bigcup_{l=1}^{m^j} E^j_l\)

where each \(E^j_l\) is a left closed interval of length \(r_j = \prod_{t=1}^{j} \delta_t^{-1}\). \(C^m_{\delta_1} = \bigcap_{j=1}^{\infty} C^m_{\delta_1, \delta_2, \ldots, \delta_j}\) is the \((m, (\delta_t))\)-Cantor set and the \(\delta_t\)'s are called its ratios of dissection. Associated to \(C^m_{\delta_1}\) in a natural way is a probability measure \(\mu^m_{(\delta_1)}\) satisfying \(\mu^m_{(\delta_1)}(E^j_l) = \frac{1}{m^j}\) for \(j = 1, 2, \ldots\) and for \(l = 1, 2, \ldots, m^j\). This measure is the Cantor-Lebesgue measure associated to the \((m, (\delta_t))\)-Cantor set. When \(\delta_t = \delta, t = 1, 2, \ldots\), we write \(\mu^m_{(\delta_1)} = \mu^m_{\delta}\). It follows that \(\mu^3_{\delta}\) is the usual Cantor-Lebesgue measure associated to the middle third Cantor set. For a detailed exposition on Cantor sets see Zygmund [19].

Notice that if \(\mu\) is a non-negative Radon measure on \(\mathbb{T}\), then in a natural way, we may identify \(\mu\) with a non-negative Radon measure \(\nu\) on \(\mathbb{R}\) having support in the interval \([0, 1)\). In addition, we have the following result established by Ritter in [17].
Proposition 4.1. Let \( 1 \leq q \leq p < \infty \), and suppose there is a constant \( K > 0 \) such that
\[
\| \mu * f \|_{L^p(\mathbb{T})} \leq K \| f \|_{L^q(\mathbb{T})}, \quad f \in L^q(\mathbb{T}).
\]
Then there is a constant \( K_0 > 0 \) such that
\[
\| \nu * f \|_{L^p(\mathbb{R})} \leq K_0 \| f \|_{L^q(\mathbb{R})}, \quad f \in L^q(\mathbb{R}).
\]
Defining, for \( 1 \leq \alpha, p \leq \infty \),
\[
M_{p, \alpha}(\mathbb{T}) = \{ \mu \in M_{p, \alpha}(\mathbb{R}) / \text{supp}(\mu) \subset [0, 1) \}
\]
where \( \text{supp}(\mu) \) denotes the support of \( \mu \), it is easy to see that Corollary 3.5 holds in this setting.

The following result gives a characterization of measures \( \mu_m^\alpha(\delta_t) \) which belong to \( M_{p, \alpha}(\mathbb{T}) \).

Proposition 4.2. Let \( \delta_t > m, t = 1, 2, \ldots \). Assume that \( 1 < \alpha \leq p < \infty \).
Then \( \mu_m^\alpha(\delta_t) \) belongs to \( M_{p, \alpha}(\mathbb{T}) \) if and only if there exists a constant \( c > 0 \) such that
\[
\prod_{t=1}^{j} \delta_t \leq cm^{\alpha(p-1)/p}, \quad j = 1, 2, \ldots.
\]
In particular, the Cantor-Lebesgue measure \( \mu_m^\alpha(\delta) \) of constant ratio of dissection \( \delta \) belongs to \( M_{p, \alpha}(\mathbb{T}) \) if and only if
\[
1 - \frac{1}{\alpha} - \frac{\log m}{\log \delta} \left( 1 - \frac{1}{p} \right) \leq 0.
\]
Proof. a) For all \( r \geq 1 \)
\[
r^{\frac{1}{\alpha}-1} r \left\| \mu_m^\alpha(\delta_t) \right\|_p = r^{\frac{1}{\alpha}-1} \leq 1.
\]
b) Let \( j \) be a positive integer and \( r_j = \prod_{t=1}^{j} \delta_t^{-1} \). Recall that for \( l = 1, 2, \ldots, m^j \), \( |E_i^j| = r_j \) and \( \mu_m^\alpha(\delta_t) \left(E_i^j\right) = \frac{1}{m^j} \). For each fixed \( l \), put \( K_l = \{ k \in \mathbb{N} / E_i^j \cap I_k^j \neq \emptyset \} \). Then \( K_l \) has at most 2 elements. In the same way, for each fixed \( k \) in \( \mathbb{N} \) set \( L_k = \{ l \in \{ 1, 2, \ldots, m^j \} / E_i^j \cap I_k^j \neq \emptyset \} \).
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Then the number of elements of \(L_k\) is at most 2. We have

\[
m^j m^{-jp} = \sum_{l=1}^{m^j} \mu^m_{(\delta_t)} \left( E^j_l \right)^p
= \sum_{l=1}^{m^j} \left( \sum_{k \in K_l} \mu^m_{(\delta_t)} \left( E^j_l \cap I^r_k \right) \right)^p
\leq 2^{p-1} \sum_{l \in L_k} \sum_{k \in K_l} \mu^m_{(\delta_t)} \left( E^j_l \cap I^r_k \right)^p
= 2^{p-1} \sum_{k \in \mathbb{N}} \sum_{l \in L_k} \mu^m_{(\delta_t)} \left( E^j_l \cap I^r_k \right)^p
\leq 2^p \sum_{k \in \mathbb{N}} \mu^m_{(\delta_t)} \left( I^r_k \right)^p.
\]

Then

\[
\left( r^j \left( \frac{1}{\alpha} - 1 \right) m^{-1} \right)^j \left( \prod_{t=1}^{\delta_t} \right)^{1-\frac{1}{\alpha}} m^{-j \left( 1 - \frac{1}{p} \right)} \leq 2r^{\frac{1}{\alpha}-1} \| \mu^m_{(\delta_t)} \|_p.
\]

c) Let \(r \in (0, 1)\). There exists an integer \(j \geq 1\) such that \(r_j \leq r < r_{j-1}\) where \(r_0 = 1\) and \(r_n = \prod_{t=1}^{\delta_t} \delta_t^{-1}\) for \(n \geq 1\). Furthermore, each \(I^r_k\) intersects at most \(m\) intervals \(E^j_l\). So \(\mu^m_{(\delta_t)} \left( I^r_k \right) \leq m^{-j} m\). The number of \(I^r_k\) which intersect the intervals \(E^j_l\) is at most \(2m^j\). It follows that

\[
\sum_{k \in \mathbb{N}} \mu^m_{(\delta_t)} \left( I^r_k \right)^p \leq 2m^j (1-p) m^p.
\]

Hence

\[
r^{\frac{1}{\alpha}-1} r \| \mu^m_{(\delta_t)} \|_p \leq 2r^\frac{1}{\alpha}-1 m^j \left( \frac{1}{p} - 1 \right) m
\leq 2r^\frac{1}{\alpha}-1 m^j \left( \frac{1}{p} - 1 \right) m
= 2^\frac{1}{\alpha} m \left( \prod_{t=1}^{\delta_t} \delta_t \right)^{\frac{1}{\alpha}} \left( \frac{1}{\alpha} - 1 \right) m^{\frac{1}{\alpha}-1} \right)^j.
\]
Finally,
\[ \mu_{\delta_t}^m \in M^{p, \alpha}(\mathbb{T}) \iff \sup_j \left( \prod_{t=1}^j \delta_t \right)^{\frac{1}{j} \left( 1 - \frac{1}{\alpha} \right) \frac{1}{m^{\frac{1}{p}} - 1}} < \infty \]

where \( c \) is a positive constant not depending on \( j \).

d) Now, let \( \delta_t = \delta \) for all \( t \geq 1 \). From c) we know that:
\[ \mu_{\delta_t}^m \in M^{p, \alpha}(\mathbb{T}) \iff \prod_{t=1}^j \delta_t \leq cm^{\alpha(p-1)/p(\alpha-1)}, \quad j = 1, 2, ... \]

Notice that for \( 1 - \frac{1}{\alpha} = \frac{1}{q} - \frac{1}{p} \), (4.1) reduces to (1.2) when \( m = 2 \) and \( \delta = 3 \).

**Proposition 4.3.** Let \( \mu_{\delta_t}^m \) be the Cantor-Lebesgue measure with variable ratios \( \delta_t > m \) of dissection. Let \( 1 < q < p < \infty \). Assume that
\[ \left\| \mu_{\delta_t}^m * f \right\|_{L^p(\mathbb{T})} \leq c \left( \mu_{\delta_t}^m, p, q \right) \left\| f \right\|_{L^q(\mathbb{T})}, \quad f \in L^q(\mathbb{T}). \]

Then there exists a constant \( c > 0 \) such that
\[ \prod_{t=1}^j \delta_t \leq cm^{\alpha(p-1)/pq}, \quad j = 1, 2, ... . \]

In particular, if \( \delta_t = \delta \) for all \( t \geq 1 \), then
\[ \frac{1}{q} + \left( 1 - \frac{\log m}{\log \delta} \right) \left( 1 - \frac{1}{p} \right) \leq 1. \]

**Proof.** Let \( 1 - \frac{1}{\alpha} = \frac{1}{q} - \frac{1}{p} \). Then the desired result follows from Corollary 3.5 and Proposition 4.2. \( \square \)

Proposition 1.1 is obtained from Proposition 4.3 by taking \( m = 2 \) and \( \delta_t = 3 \) for all \( t \geq 1 \).
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**Proof of Proposition 1.5.** We are in the case \(m = 2\) and \(\delta_t = \delta > 3\) for all \(t \geq 1\). Let \(j\) be a positive integer. Observe that for any non-negative integer \(k\), any \(l \in \{1, 2, \ldots, 2^j + k\}\), \(E^{j+k}_l = x_l^{j+k} + \left[0, \delta^{-j-k}\right]\) and \(\mu_\delta^2 (E^{j+k}_l) = 2^{-j-k}\). Set \(A_0 = [0, \delta^{-j}]\) and \(B_0 = A_0 - A_0 = (-\delta^{-j}, \delta^{-j})\). From Proposition 1.4 we obtain

\[
|B_0|^{-\frac{1}{q}} |A_0|^{-\frac{1}{p}} \left(\sum_{l=1}^{2^j} \mu_\delta^2 (E^j_l)^p\right)^{\frac{1}{p}} \leq c(\mu_\delta^2, p, q).
\]

Observe that for fixed \(l\) in \(\{1, 2, \ldots, 2^j\}\), \(E^j_l\) contains two intervals \(E^{j+1}_{l,1}\) and \(E^{j+1}_{l,2}\) satisfying

\[
\mu_\delta^2 (E^j_l) = \mu_\delta^2 (E^{j+1}_{l,1} \cup E^{j+1}_{l,2})
\]

and

\[
E^{j+1}_{l,1} \cup E^{j+1}_{l,2} = x_l^{j} + \left([0, \delta^{-j-1}] \cup [\delta^{-j} - \delta^{-j-1}, \delta^{-j}]\right).
\]

Setting \(A_1 = [0, \delta^{-j-1}] \cup [\delta^{-j} - \delta^{-j-1}, \delta^{-j}]\) and applying Proposition 1.4 we obtain

\[
|A_1 - A_1|^{-\frac{1}{q}} |A_1|^{-\frac{1}{p}} \left(\sum_{l=1}^{2^j} \mu_\delta^2 (E^j_l)^p\right)^{\frac{1}{p}} \leq c(\mu_\delta^2, p, q).
\]

But each preceding interval \(E^{j+1}_{l,1}, i \in \{1, 2\}\), contains two intervals \(E^{j+2}_{l,1}\) and \(E^{j+2}_{l,2}\) such that

\[
\mu_\delta^2 (E^{j+1}_{l,1}) = \mu_\delta^2 (E^{j+2}_{l,1} \cup E^{j+2}_{l,2}) = \frac{1}{2^{j+1}}.
\]

Moreover

\[
\bigcup_{l=1}^{2} (E^{j+2}_{l,1} \cup E^{j+2}_{l,2}) = x_l^{j} + A_2
\]

where

\[
A_2 = \left[0, \delta^{-j-2}\right] \cup [\delta^{-j-1} - \delta^{-j-2}, \delta^{-j-1}] \cup [\delta^{-j} - \delta^{-j-1} + \delta^{-j-2}, \delta^{-j}].
\]

This remark enables us to apply again Proposition 1.4. Thus we obtain

\[
|A_2 - A_2|^{-\frac{1}{q}} |A_2|^{-\frac{1}{p}} \left(\sum_{l=1}^{2^j} \mu_\delta^2 (E^j_l)^p\right)^{\frac{1}{p}} \leq c(\mu_\delta^2, p, q).
\]

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The iteration of the process leads us to two sequences of sets \((A_k)_{k \geq 0}\) and \((\widetilde{A}_k)_{k \geq 0}\) defined by:

\[
A_{k+1} = \frac{1}{\delta} A_k \cup \left( \delta^{-j} - \frac{1}{\delta} \widetilde{A}_k \right), \quad \widetilde{A}_{k+1} = \frac{1}{\delta} \widetilde{A}_k \cup \left( \delta^{-j} - \frac{1}{\delta} A_k \right)
\]

with \(A_0 = [0, \delta^{-j}], \widetilde{A}_0 = (0, \delta^{-j}]\) and satisfying

\[
|B_k|^{-\frac{1}{q}} |A_k|^{\frac{1}{p}} \left( \sum_{l=1}^{2^j} \mu^2_k \left( E^j_l \right)^p \right)^{\frac{1}{p}} \leq c \left( \mu^2_k, p, q \right),
\]

where \(B_k = A_k - \widetilde{A}_k\) for all \(k \geq 0\).

Notice that \(A_0 - A_0 = \widetilde{A}_0 - \widetilde{A}_0\) and \(|A_0| = |\widetilde{A}_0|\). Furthermore, for any \(k \geq 0\), clearly \(A_{k+1} - A_{k+1} = \widetilde{A}_{k+1} - \widetilde{A}_{k+1}\) and since \(\frac{1}{\delta} A_k \cap \left( \delta^{-j} - \frac{1}{\delta} \widetilde{A}_k \right) = \emptyset = \frac{1}{\delta} \widetilde{A}_k \cap \left( \delta^{-j} - \frac{1}{\delta} A_k \right)\)

we have \(|A_{k+1}| = |\widetilde{A}_{k+1}|\). Thus

\[
A_k - A_k = \widetilde{A}_k - \widetilde{A}_k \quad \text{and} \quad |A_k| = |\widetilde{A}_k|, \quad k \geq 0.
\]

Observe that: \(|A_0| = \delta^{-j}, \ |A_1| = 2\delta^{-j-1}\) and \(|A_2| = 2^2\delta^{-j-2}\). Suppose that for some integer \(k \geq 0, |A_k| = 2^k\delta^{-j-k}\). By the preceding remarks we get \(|A_{k+1}| = \frac{1}{\delta} |A_k| + \frac{1}{\delta} |\widetilde{A}_k| = \frac{2}{\delta} |A_k| = 2^{k+1}\delta^{-j-(k+1)}\). We conclude that

\[
|A_k| = 2^k\delta^{-j-k}, \quad k \geq 0.
\]

Notice that \(A_0 + \widetilde{A}_0 = (0, 2\delta^{-j}) = \delta^{-j} - (\delta^{-j}, \delta^{-j}) = \delta^{-j} - (A_0 - A_0) = \delta^{-j} - B_0\). Furthermore, for any \(k \geq 0\), on the one hand

\[
B_{k+1} = \left[ \frac{1}{\delta} A_k \cup \left( \delta^{-j} - \frac{1}{\delta} \widetilde{A}_k \right) \right] - \left[ \frac{1}{\delta} A_k \cup \left( \delta^{-j} - \frac{1}{\delta} \widetilde{A}_k \right) \right]
\]

\[
= \frac{1}{\delta} (A_k - A_k) \cup \left( \frac{1}{\delta} (A_k + \widetilde{A}_k) - \delta^{-j} \right) \cup \left( \delta^{-j} - \frac{1}{\delta} (\widetilde{A}_k + A_k) \right) \cup \frac{1}{\delta} (\widetilde{A}_k - \widetilde{A}_k)
\]

\[
= \frac{1}{\delta} (A_k - A_k) \cup \left( \frac{1}{\delta} (A_k + \widetilde{A}_k) - \delta^{-j} \right) \cup \left( \delta^{-j} - \frac{1}{\delta} (\widetilde{A}_k + A_k) \right)
\]

(because of (4.3))

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and on the other hand
\[
A_{k+1} + \tilde{A}_{k+1} = \frac{1}{\delta} (A_k + \tilde{A}_k) \cup \left( \delta^{-j} + \frac{1}{\delta} (A_k - A_k) \right) \cup \\
\cup \left( \delta^{-j} + \frac{1}{\delta} (\tilde{A}_k - \tilde{A}_k) \right) \cup \left( 2\delta^{-j} - \frac{1}{\delta} (\tilde{A}_k + A_k) \right) \\
= \frac{1}{\delta} (A_k + \tilde{A}_k) \cup \left( \delta^{-j} + \frac{1}{\delta} (A_k - A_k) \right) \cup \\
\cup \left( 2\delta^{-j} - \frac{1}{\delta} (\tilde{A}_k + A_k) \right) \quad \text{(because of (4.3))}
\]
and so \( A_{k+1} + \tilde{A}_{k+1} = \delta^{-j} - B_{k+1} \). Thus
\[
|A_k - A_k| = |A_k + \tilde{A}_k|, \quad k \geq 0. \tag{4.4}
\]
Notice that for all \( k \geq 0 \), the sets \( \frac{1}{\delta} (A_k - A_k), \frac{1}{\delta} (A_k + \tilde{A}_k) - \delta^{-j} \) and \( \delta^{-j} - \frac{1}{\delta} (\tilde{A}_k + A_k) \) form a partition of \( B_{k+1} \). Thus, by (4.4) we have \( |B_{k+1}| = \frac{3}{\delta} |A_k - A_k| = \frac{3}{\delta} |B_k|, \ k \geq 0 \). As \( |B_0| = 2\delta^{-j} \), we conclude that for all \( k \geq 0 \), \( |B_k| = \left( \frac{3}{\delta} \right)^k 2\delta^{-j} \).

Finally, using inequality (4.2) we get:
\[
2 \left( \frac{3}{\delta} \right)^k \delta^{-j} \left( 2^{k \delta^{-j-k}} \right)^{\frac{1}{p}} 2^{j \left( \frac{1}{p} - 1 \right)} \leq c \left( \mu_2, p, q \right), \quad k \geq 0, \ j \geq 1
\]
\[
2^{-\frac{1}{q}} \left( 3^{-\frac{1}{q}} \delta^{-\frac{1}{q}} - \frac{1}{\delta} \frac{1}{2} 2^\frac{1}{p} \right)^k \left( \delta^{-\frac{1}{q}} - \frac{1}{\delta} \frac{1}{p} \frac{1}{2} - 1 \right)^j \leq c \left( \mu_2, p, q \right), \quad k \geq 0, \ j \geq 1
\]
\[
3^{-\frac{1}{q}} \delta^{-\frac{1}{q}} - \frac{1}{\delta} \frac{1}{2} 2^\frac{1}{p} \leq 1 \quad \text{and} \quad \delta^{-\frac{1}{q}} - \frac{1}{\delta} \frac{1}{p} \frac{1}{2} - 1 \leq 1
\]
\[
p \leq \frac{\log \left( \frac{\delta}{2} \right)}{\log \left( \frac{\delta}{3} \right)} q \quad \text{and} \quad \frac{1}{q} + \left( 1 - \frac{\log 2}{\log \delta} \right) \left( 1 - \frac{1}{p} \right) \leq 1.
\]
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