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$L_{p,q}$-cohomology of warped cylinders


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Abstract

We extend some results by Gol’dshtein, Kuz’minov, and Shvedov about the $L_p$-cohomology of warped cylinders to $L_{p,q}$-cohomology for $p \neq q$. As an application, we establish some sufficient conditions for the nontriviality of the $L_{p,q}$-torsion of a surface of revolution.

Cohomologie $L_{p,q}$ des cylindres tordus

Résumé

On généralise quelques résultats par Gol’dshitein, Kuz’minov et Shvedov sur la cohomologie $L_p$ des cylindres tordus à cohomologie $L_{p,q}$ pour $p \neq q$. Comme application, on établit des conditions suffisantes pour la non-nullité de la torsion $L_{p,q}$ d’une surface de révolution.

1. Introduction

Let $M$ be a Riemannian manifold. For $1 \leq p \leq \infty$ and a positive continuous function $\sigma : M \to \mathbb{R}$, denote by $L^j_p(M, \sigma)$ the Banach space of measurable forms of degree $j$ on $M$ with the finite norm

$$
\|\omega\|_{L^j_p(M, \sigma)} = \left\{ \int_M |\omega(x)|^p \sigma^p(x) \, dx \right\}^{1/p} \quad \text{if } 1 \leq p < \infty,
$$

$$
\text{ess sup}_{x \in M} |\omega(x)| \sigma(x) \quad \text{if } p = \infty.
$$

Here $dx$ stands for the volume element of $M$ and $|\omega(x)|$ is the modulus of the exterior form $\omega(x)$. In the usual way, we also define the spaces $L_{p,\text{loc}}(M)$.

Denote by $D^j(M) = C_0^\infty(M)$ the space of smooth forms of degree $j$ on $M$ having compact support included in $\text{Int} M$. A form $\psi \in L^{j+1}_{1,\text{loc}}(M)$

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is called the (weak) differential $d\omega$ of $\omega \in L^2_{1,\text{loc}}(M)$ if

$$\int_U \omega \wedge du = (-1)^{j+1} \int_U \psi \wedge u$$

for every orientable domain $U \subset \text{Int } M$ and every form $u \in D^{\text{dim } M - j - 1}(M)$ having support in $U$.

For two weights $\sigma_j, \sigma_{j+1}$ on $M$, put

$$W^j_{p,q}(M, \sigma_j, \sigma_{j+1}) = \{ \omega \in L^j_p(M, \sigma_j) \mid d\omega \in L^{j+1}_q(M, \sigma_{j+1}) \}.$$

The space $W^j_{p,q}(M, \sigma_j, \sigma_{j+1})$ is endowed with the norm

$$\|\omega\|_{W^j_{p,q}(M, \sigma_j, \sigma_{j+1})} = \|\omega\|_{L^p_j(M, \sigma_j)} + \|d\omega\|_{L^{j+1}_q(M, \sigma_{j+1})}.$$

If $p = q$ then it is often more convenient to consider the equivalent norm

$$\|\omega\|_{W^j_p(M, \sigma_j, \sigma_{j+1})} = \left(\|\omega\|_{L^p_j(M, \sigma_j)}^p + \|d\omega\|_{L^{j+1}_p(M, \sigma_{j+1})}^p\right)^{1/p}.$$

In the sequel we let $V^j_{p,q}(M, \sigma_j, \sigma_{j+1})$ denote the closure of $D^j(M)$ in the norm of $W^j_{p,q}(M, \sigma_j, \sigma_{j+1})$.

Given an arbitrary subset $A \subset M$, let $W^j_{p,q}(M, A, \sigma_j, \sigma_{j+1})$ be the closure in $W^j_{p,q}(M, \sigma_j, \sigma_{j+1})$ of the subspace spanned by all forms $\omega \in W^j_{p,q}(M, \sigma_j, \sigma_{j+1})$ which vanish on some neighborhood of $A$ (depending on $\omega$).

Let $Z^j_q(M, \sigma_j)$ be the subspace in $W^j_{q,q}(M, \sigma_j, \sigma_j)$ that consists of all forms $\omega$ such that $d\omega = 0$ and let

$$B^j_{p,q}(M, \sigma_{j-1}, \sigma_j) = \{ \theta \in W^j_{q,q}(M, \sigma_j, \sigma_j) \mid \theta = d\psi \text{ for some } \psi \in W^{j-1}_{p,q}(M, \sigma_{j-1}, \sigma_j) \}.$$

The spaces

$$H^j_{p,q}(M, \sigma_{j-1}, \sigma_j) = Z^j_q(M, \sigma_j)/B^j_{p,q}(M, \sigma_{j-1}, \sigma_j)$$

and

$$\overline{H}^j_{p,q}(M, \sigma_{j-1}, \sigma_j) = Z^j_q(M, \sigma_j)/\overline{B}^j_{p,q}(M, \sigma_{j-1}, \sigma_j),$$

where $\overline{B}^j_{p,q}(M, \sigma_{j-1}, \sigma_j)$ is the closure of $B^j_{p,q}(M, \sigma_{j-1}, \sigma_j)$ in $L^j_q(M, \sigma_j)$ (equivalently, in $W^j_{q,q}(M, \sigma_j, \sigma_j)$) are called the $j$th $L^{p,q}$-cohomology and
the $j$th reduced $L_{p,q}$-cohomology of the Riemannian manifold $M$ with weights $\sigma_{j-1}$ and $\sigma_j$. The quotient space

\[ T_{p,q}^j(M, \sigma_{j-1}, \sigma_j) = \overline{B}_{p,q}^j(M, \sigma_{j-1}, \sigma_j) / B_{p,q}^j(M, \sigma_{j-1}, \sigma_j) \]

will be referred to as the $L_{p,q}$-torsion of $M$ with the given weights. Clearly, the space $T_{p,q}^j(M, \sigma_{j-1}, \sigma_j)$ is isomorphic to the closure of the zero in $H_{p,q}^j(M, \sigma_{j-1}, \sigma_j)$.

Given a subset $A \subset M$, the relative nonreduced and reduced $L_{p,q}$-cohomology spaces $H_{p,q}^j(M, A, \sigma_{j-1}, \sigma_j)$ and $\overline{H}_{p,q}^j(M, A, \sigma_{j-1}, \sigma_j)$ are defined as

\[ H_{p,q}^j(M, A, \sigma_{j-1}, \sigma_j) = Z_q^j(M, A, \sigma_j) / B_{p,q}^j(M, A, \sigma_{j-1}, \sigma_j) \]
\[ \overline{H}_{p,q}^j(M, A, \sigma_{j-1}, \sigma_j) = Z_q^j(M, A, \sigma_j) / \overline{B}_{p,q}^j(M, A, \sigma_{j-1}, \sigma_j), \]

where the relative spaces $Z_q^j(M, A, \sigma_j)$ and $B_{p,q}^j(M, A, \sigma_{j-1}, \sigma_j)$ are defined as their absolute analogs above with the spaces $W_{p,q}^j(M, \sigma_j, \sigma_j)$ and $W_{p,q}^{j-1}(M, \sigma_{j-1}, \sigma_j)$ replaced by the spaces

\[ W_{p,q}^j(M, A, \sigma_j, \sigma_j) \text{ and } W_{p,q}^{j-1}(M, A, \sigma_{j-1}, \sigma_j). \]

For $p = q$, we write the subscript $p$ instead of $p, p$ throughout. If the weights involved in the definition of the corresponding space are equal to 1 then they will be omitted.

The spaces $W_{p,q}$ and $L_{p,q}$-cohomology were introduced at the beginning of the 1980’s by Gol’dshein, Kuz’minov, and Shvedov [3, 4, 5, 6, 7, 8], who obtained many results concerning $W_{p,q}$-forms and especially $L_p$-cohomology. Later $L_{p,q}$-cohomology was considered in [11, 12, 13, 14, 15, 17, 22].

In this paper, we, following [9, 10], look for conditions of the nontriviality of the $L_{p,q}$-cohomology and $L_{p,q}$-torsion on warped cylinders, a class of warped products of Riemannian manifolds. By the warped product $X \times_f Y$ of two Riemannian manifolds $(X, g_X)$ and $(Y, g_Y)$ with the warping function $f : X \to \mathbb{R}_+$ we mean the product manifold $X \times Y$ endowed with the metric $g_X + f^2(x)g_Y$. If $X = [a, b]$ is a half-interval on the real line then $X \times_f Y$ is referred to as the warped cylinder. The study of the $L_2$-cohomology of warped cylinders was initiated by Cheeger [2].

The structure of the article is as follows. In Section 2, we adapt the results of [9] about the $L_p$-cohomology of a half-interval to the case $p \neq q$. 

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After that, using these $L_{p,q}$-results, in Section 3, we prove a partial $L_{p,q}$-generalization of Theorem 1 of [9] about the $L_p$-cohomology of a warped cylinder $[a, b] \times f\gamma$ depending on the analytic properties of the function $f$. As an application, we obtain an extension of the necessary condition for the triviality of the $L_{p,q}$-torsion of a surface of revolution in $\mathbb{R}^{n+2}$ [16] from the case $p = q$ to arbitrary $p,q$ such that $\frac{1}{q} - \frac{1}{p} < \frac{1}{n+1}$.

2. Weighted $L_{p,q}$-cohomology of a half-interval

Consider a half-interval $[a, b]$, $-\infty < a < b \leq \infty$ and positive continuous functions $v_0, v_1 : [a, b] \rightarrow \mathbb{R}$. For $1 < p, q < \infty$, the space $W^0_{p,q}([a, b], v_0, v_1)$ can be identified with the space of the functions $g \in L_p([a, b], v_0)$ whose weak derivative $g' \in L_q([a, b], v_1)$. As above, endow $W^0_{p,q}([a, b], v_0, v_1)$ with the norm

$$
\|g\|_{W^0_{p,q}([a,b],v_0,v_1)} = \left(\int_a^b |g(t)|^pv_0^pdt\right)^{1/p} + \left(\int_a^b |g'(t)|^qv_1^qdt\right)^{1/q}.
$$

From the classical Sobolev Embedding Theorem it follows that the functions of the class $W^0_{p,q}([a, b], v_0, v_1)$ are continuous on $[a, b]$. Consider also the space

$$W^0_{p,q}([a, b], \{a\}, v_0, v_1) = \{f \in W^0_{p,q}([a, b], \{a\}, v_0, v_1) \mid f(a) = 0\}.
$$

We have

$$H^1_{p,q}([a, b], v_0, v_1) = W^1_{q}([a, b], v_1, v_1)/dW^0_{p,q}([a, b], v_0, v_1);$$

$$H^1_{p,q}([a, b], \{a\}, v_0, v_1) = W^1_{q}([a, b], \{a\}, v_1, v_1)/dW^0_{p,q}([a, b], \{a\}, v_0, v_1).$$

The spaces $\overline{H}^1_{p,q}([a, b], v_0, v_1)$ and $\overline{H}^1_{p,q}([a, b], \{a\}, v_0, v_1)$ are described similarly.

We call the following assertion the lemma about the Hardy inequality [1, 10, 21]:

**Lemma 2.1.** Suppose that $1 \leq p, q \leq \infty$, $\frac{1}{q} + \frac{1}{p} = 1$, $\alpha, \beta \in [-\infty, \infty]$, $I_{\alpha, \beta}$ is the interval with endpoints $\alpha$ and $\beta$, $v_0$ and $v_1$ are continuous positive functions on $I_{\alpha, \beta}$. Then for the existence of a global constant $C$ such that

$$\left|\int_\alpha^\beta v_0(t) \int_\alpha^\tau g(t)dt \right|^p d\tau \leq C \left|\int_\alpha^\beta |v_1(t)|^q dt \right|^q$$

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for every \( g \in L_q(I_{\alpha,\beta}, v_1) \), it is necessary and sufficient that
\[
\chi_{p,q}(\alpha, \beta, v_0, v_1) < \infty.
\]
Here
\[
\chi_{p,q}(\alpha, \beta, v_0, v_1) = \sup_{\tau \in I_{\alpha,\beta}} \left\{ \left| \int_{\alpha}^{\beta} |v_0(t)|^p dt \right|^{1/p} \left| \int_{\alpha}^{\tau} |v_1(t)|^{-q'} dt \right|^{1/q'} \right\}
\]
if \( p \geq q \);
\[
\chi_{p,q}(\alpha, \beta, v_0, v_1) = \left| \int_{\alpha}^{\beta} \left( \int_{\alpha}^{\tau} |v_1(t)|^{-q'} dt \right)^{p-1} \left| \int_{\alpha}^{\beta} |v_0(t)|^p dt \right| \right|^{\frac{q}{q-p}} |v_1(\tau)|^{-q'} d\tau \right|^{\frac{q-p}{pq}}
\]
if \( p < q \).

If \( p = 1 \) (\( q' = \infty \)) then the corresponding integral must be replaced by \( \text{ess sup} \).

The constant \( \chi_{p,q}(\alpha, \beta, v_0, v_1) \) will be referred to as the Hardy constant.

The following lemma was proved in [9] for \( p = q \) and \( v_0 = v_1 \). The proof given in [9] holds for different \( p \) and \( q \) and different \( v_0 \) and \( v_1 \).

**Lemma 2.2.** Suppose that \( \alpha, \beta \in [-\infty, \infty] \), \( v_0, v_1 : I_{\alpha,\beta} \to \mathbb{R} \) are positive continuous functions, and \( \chi_{p,q}(\alpha, \beta, v_0, v_1) = \infty \). Then there exists a nonnegative function \( h \) such that
\[
\left| \int_{\alpha}^{\beta} v_1^q(t)h^q(t) dt \right| < \infty, \quad \left| \int_{\alpha}^{\beta} v_0^p(\tau) \left[ \int_{\alpha}^{\tau} h(t) dt \right]^p d\tau \right| = \infty.
\]

As in [9], Lemma 2 yields the following assertion.

**Theorem 2.3.** If \( v_0, v_1 \) are positive continuous functions on \( [a, b] \) and
\( 1 < p, q < \infty \) then
\[\begin{array}{l}
(1) \quad H^1_{p,q}([a, b], \{a\}, v_0, v_1) = 0 \iff \chi_{p,q}(a, b, v_0, v_1) < \infty; \\
(2) \quad H^1_{p,q}([a, b], v_0, v_1) = 0 \iff \chi_{p,q}(a, b, v_0, v_1) < \infty \quad \text{or} \quad \chi_{p,q}(b, a, v_0, v_1) < \infty.
\end{array}\]

Let
\[
0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0
\]
be an exact sequence of Banach complexes, i.e., complexes in the category of Banach spaces and bounded linear operators. Sequence (2.1) yields an exact sequence of the cohomology spaces
\[
\cdots \to H^{k-1}(C) \xrightarrow{\partial} H^k(A) \xrightarrow{\phi^*} H^k(B) \xrightarrow{\psi^*} H^k(C) \to \cdots
\]

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with continuous operators $\partial^*$, $\varphi^*$, $\psi^*$ and a semi-exact sequence of the reduced cohomology spaces
\[
\cdots \rightarrow \overline{H}^{k-1}(C) \xrightarrow{\partial^*} \overline{H}^k(A) \xrightarrow{\varphi^*} \overline{H}^k(B) \xrightarrow{\psi^*} \overline{H}^k(C) \rightarrow \ldots 
\]
(2.2)

Under certain conditions, sequence (2.2) is exact at some terms (see [10, 18, 20]). In particular, Gol’dshtein, Kuz’minov, and Shvedov proved the following assertion in [10, Theorem 1(1)]:

**Lemma 2.4.** If $H^k(C)$ is separated and $\dim \partial(H^{k-1}(C)) < \infty$ then the sequence $\overline{H}^{k-1}(C) \xrightarrow{\partial^*} \overline{H}^k(A) \xrightarrow{\varphi^*} \overline{H}^k(B) \xrightarrow{\psi^*} \overline{H}^k(C)$ is exact.

As was explained in [12], we can describe the $j$th weighted $L_{p,q}$-cohomology of an $n$-dimensional Riemannian manifold $M$ with given weights $\sigma_{j-1}$ and $\sigma_j$ in terms of Banach complexes. To this end, consider an arbitrary sequence $\pi = \{p_0, p_1, \ldots, p_n\} \subset [1, \infty]$ with $p_{j-1} = p$ and $p_j = q$ and a sequence of positive continuous weights $\sigma = \{\sigma_k\}_{k=0}^n$ with the given $\sigma_{j-1}$ and $\sigma_j$. Given a subset $A \subset M$, put
\[
W^k_\pi(M, A, \sigma) = W_{p_k,p_{k+1}}(M, A, \sigma_k, \sigma_{k+1}).
\]
Here we have assumed that $p_{n+1} = p_n$ and $\sigma_{n+1} = \sigma_n$.

Since the exterior differential is a bounded operator
\[
d^{k-1} : W^{k-1}_\pi(M, A, \sigma) \rightarrow W^k_\pi(M, A, \sigma),
\]
we obtain a Banach complex
\[
0 \rightarrow W^0_\pi(M, A, \sigma) \xrightarrow{d^0} W^1_\pi(M, A, \sigma) \rightarrow \ldots \rightarrow W^{n-1}_\pi(M, A, \sigma) \rightarrow W^n_\pi(M, A, \sigma) \rightarrow 0. \tag{2.3}
\]
By the $k$-th $L_\pi$-cohomology $H^k_\pi(M, A, \sigma)$ (reduced $k$-th $L_\pi$-cohomology $\overline{H}^k_\pi(M, A, \sigma)$) of the Riemannian manifold $M$ with respect to $A$ with weight $\sigma$ we mean the cohomology (reduced cohomology) of (2.3). Thus,
\[
H^k_\pi(M, A, \sigma) = H^k_{p_{k-1},p_k}(M, A, \sigma_{k-1}, \sigma_k)
\]
and
\[
\overline{H}^k_\pi(M, A, \sigma) = \overline{H}^k_{p_{k-1},p_k}(M, A, \sigma_{k-1}, \sigma_k)
\]
for all $k$. In particular,
\[
H^j_\pi(M, A, \sigma) = H^j_{p,q}(M, A, \sigma_{j-1}, \sigma_j),
\]
\[
\overline{H}^j_\pi(M, A, \sigma) = \overline{H}^j_{p,q}(M, A, \sigma_{j-1}, \sigma_j).
\]
Take $M = [a, b[, A = \{a\}$, $1 < p, q < \infty$, $\pi = \{p, q\}$, and a pair of weights $v = \{v_0, v_1\}$. We have the following exact sequence of Banach complexes:

$$0 \rightarrow W^*_\pi([a, b[, \{a\}, v]) \overset{i}{\rightarrow} W^*_\pi([a, b[, v]) \overset{i}{\rightarrow} H^*(\{a\}) \rightarrow 0,$$

where $H^*(\{a\})$ is the complex with the only nontrivial term $H^0(\{a\}) = \mathbb{R}$. Here the mappings $i$ and $j$ are defined as follows: $j$ is the inclusion mapping; if $g \in W^0_\pi([a, b[, v)$ then $i(g) = g(a)$ (recall that $g$ is continuous) and in dimension one $j$ is zero. Lemma 2.4 yields the exact sequence

$$\mathbb{R} = H^0(\{a\}) \overset{j}{\rightarrow} \overline{H}^1_{p, q}([a, b[, \{a\}, v_0, v_1]) \overset{j}{\rightarrow} \overline{H}^1_{p, q}([a, b[, v_0, v_1]).$$

Thus, we infer the following assertion, proved for $p = q$ in [9]. With what has been said above, the proof of [9] extends to the case of $p \neq q$ without change.

**Theorem 2.5.** If $v_0$, $v_1$ are positive continuous functions on $[a, b[, 1 < p < \infty$, $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ then

1. $\overline{H}^1_{p, q}([a, b[, v_0, v_1]) = 0$;
2. $\overline{H}^1_{p, q}([a, b[, \{a\}, v_0, v_1]) = 0$ if and only if $\int_a^b v_1^{1/q'}(t)dt = \infty$ or $\int_a^b v_0^p(t)dt < \infty$;
3. If $\overline{H}_{p, q}([a, b[, \{a\}, v_0, v_1]) \neq 0$ then

$$\overline{\partial} : \mathbb{R} = H^0(\{a\}) \rightarrow \overline{H}^1_{p, q}([a, b[, \{a\}, v_0, v_1)$$

is an isomorphism.

3. $L_{p, q}$-cohomology of the warped cylinder $C_{a, b}^f$

Let $Y$ be an orientable manifold of dimension $n$, $C_{a, b}^f Y = [a, b[ \times fY$. Put $Y_a = \{a\} \times Y$. Generally speaking, $C_{a, b}^f$ is a Lipschitz Riemannian manifold in the sense of [3] but we will assume throughout for simplicity that $\partial Y = \emptyset$ to make $C_{a, b}^f$ smooth, which will be enough for our purposes.

Suppose that $1 < p < \infty$ and $1 < q < \infty$. 

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Suppose that continuous mappings \( \nu : L_p^{j-1}(Y) \times L_p^{1}([a, b], f^{\frac{n}{p}}) \to L_p^{j}(C_{a,b}^f Y) \),
\( \nu(\varphi, gdt) = gdt \land \varphi. \) In [9] it was proved that \( \nu \) is continuous and if \( \varphi \in Z_p^{j-1}(Y) \) then \( \nu_\varphi = \nu(\varphi, \cdot) : L_p^{1}([a, b], f^{\frac{n}{p}}) \to L_p^{j}(C_{a,b}^f Y) \) induces continuous mappings
\[
\nu^*_\varphi : H_p^1([a, b], f^{\frac{n}{p}}) \to H_p^j(C_{a,b}^f Y);
\]
\[
\tilde{\nu}^*_\varphi : H_p^1([a, b], [a], f^{\frac{n}{q}}) \to H_p^j(C_{a,b}^f Y, Y_a).
\]

Supposing that \( \varphi \in Z_p^{j-1}(Y) \cap Z_q^{j-1}(Y) \), we similarly become convinced that the mapping \( \nu_\varphi = \nu(\varphi, \cdot) \) induces continuous mappings
\[
\nu^*_\varphi : H_{p,q}^1([a, b], f^{\frac{n}{p}}) \to H_{p,q}^j(C_{a,b}^f Y);
\]
\[
\tilde{\nu}^*_\varphi : H_{p,q}^1([a, b], [a], f^{\frac{n}{q}}) \to H_{p,q}^j(C_{a,b}^f Y, Y_a).
\]

Now, assume that \( \psi \in L_p^{n-1+j}(Y) (p' = \frac{p}{p-1}) \) and \( \omega \in L_p^1(C_{a,b}^f Y) \). Write \( \omega \) in the form \( \omega = \omega_A + dt \land \omega_B \), where \( \omega_A, \omega_B \) do not contain \( dt \) [10]. Following [9], introduce the continuous operator
\[
\mu_\psi : L_p^{j}(C_{a,b}^f Y) \to L_p^{1}([a, b], f^{\frac{n}{p}})
\]
by the formula
\[
\mu_\psi \omega = \left( \int_Y \omega_B(t) \land \psi \right) dt.
\]

The following lemma was proved in [9] for \( p = q \) and \( \psi \in V_{p'}^{n-1+j}(Y) \). The proof in [9] easily extends to \( p \neq q \):

**Lemma 3.1.** If \( \psi \in D^{n-1+j}(Y) \) and \( d\psi = 0 \) then \( \mu_\psi \) induces continuous mappings
\[
\mu^*_\psi : H_{p,q}^1(C_{a,b}^f Y) \to H_{p,q}^1([a, b], f^{\frac{n}{p}});
\]
\[
\tilde{\mu}^*_\psi : H_{p,q}^1(C_{a,b}^f Y, Y_a) \to H_{p,q}^1([a, b], [a], f^{\frac{n}{q}}).
\]

We have the following theorem partially generalizing item 7 of Theorem 1 in [9]:

**Theorem 3.2.** Suppose that \( Y \) is an orientable \( n \)-dimensional Riemannian manifold, \( \infty < a < b \leq \infty, f : [a,b] \to \mathbb{R} \) is a positive continuous function, \( 1 < p \leq \infty, 1 < q < \infty \). Assume that there exists \( \varphi \in \)
Let\( d\gamma = 0 \).

Thus, \( \int_{\gamma} \varphi \wedge \gamma \neq 0 \) for some form \( \gamma \in D^{n-j+1}(Y) \).

The following hold:

1. if \( \chi_{p,q}(a,b, f_p^n-j+1, f_q^n-j+1) = \infty \) then \( H^j_{p,q}(C_{a,b}^f, Y_a) \neq 0 \);
2. if \( \chi_{p,q}(a,b, f_p^n-j+1, f_q^n-j+1) = \infty \) and \( \chi_{p,q}(b,a, f_p^n-j+1, f_q^n-j+1) = \infty \) then \( T^j_{p,q}(C_{a,b}^f, Y) \neq 0 \) and, hence, \( \dim H^j_{p,q}(C_{a,b}^f, Y) = \infty \).

**Proof.** Let \( \varphi \in Z^j_{p-1}(Y) \cap Z^j_{q-1}(Y) \) be a cocycle having the property mentioned in the theorem and let \( \gamma \in D^{n-j+1}(M) \) be a form such that \( \int_Y \varphi \wedge \gamma = 1 \). Then \( \mu^*_\gamma \circ \nu^*_\varphi = \text{id} \), \( \tilde{\mu}^*_\gamma \circ \tilde{\nu}^*_\varphi = \text{id} \) \cite{9}. Consequently, the mappings

\[
\nu^*_\varphi : H^1_{p,q}([a,b], f_p^n-j+1, f_q^n-j+1) \to H^j_{p,q}(C_{a,b}^f, Y)
\]

and

\[
\tilde{\nu}^*_\varphi : H^1_{p,q}([a,b], \{a\}, f_p^n-j+1, f_q^n-j+1) \to H^j_{p,q}(C_{a,b}^f, Y, Y_a)
\]

are injective.

Suppose that \( \chi_{p,q}(a,b, f_p^n-j+1, f_q^n-j+1) = \infty \). Then, by Theorem 2.3, \( H^1_{p,q}([a,b], \{a\}, f_p^n-j+1, f_q^n-j+1) \neq 0 \). Therefore, \( H^j_{p,q}(C_{a,b}^f, Y, Y_a) \neq 0 \).

Assume now that

\[
\chi_{p,q}(a,b, f_p^n-j+1, f_q^n-j+1) = \infty
\]

and

\[
\chi_{p,q}(b,a, f_p^n-j+1, f_q^n-j+1) = \infty.
\]

Then, by Theorem 2.3, \( H^1_{p,q}([a,b], f_p^n-j+1, f_q^n-j+1) \neq 0 \). Since, by Theorem 2.5, \( \overline{H}^1_{p,q}([a,b], f_p^n-j+1, f_q^n-j+1) = 0 \), we have

\[
T^1_{p,q}([a,b], f_p^n-j+1, f_q^n-j+1) \neq 0.
\]

Now, if we had \( T^j_{p,q}(C_{a,b}^f, Y) = 0 \), \( \nu^*_\varphi \) would be a continuous injective mapping with values in the Hausdorff space \( H^j_{p,q}(C_{a,b}^f, Y) \), and so the cohomology space \( H^1_{p,q}([a,b], f_p^n-j+1, f_q^n-j+1) \) would also be Hausdorff, i.e., without torsion. Thus, \( T^j_{p,q}(C_{a,b}^f, Y) \neq 0 \). The theorem is proved. \( \square \)
\( L_{p,q}\)-torsion of a surface of revolution

Let \( M \) be a surface of revolution in \( \mathbb{R}^{n+2} \), i.e., the \((n+1)\)-dimensional surface defined by the equation
\[
f^2(x_1) = x_2^2 + \cdots + x_{n+2}^2, \quad (x_1, \ldots, x_{n+2}) \in \mathbb{R}^{n+2}, \quad x_1 \geq 0, \tag{3.1}
\]
where \( f : [0, \infty[ \rightarrow \mathbb{R} \) is a positive smooth function. The manifold \( M \) is the product \([0, \infty[ \times S^n\) endowed with the metric
\[
g_M = (1 + f'^2(x_1))dx_1^2 + f^2(x_1)dy^2
\]
induced from \( \mathbb{R}^{n+2} \), where \( dx_1^2 \) and \( dy^2 \) are the conventional Riemannian metrics on \([0, \infty[ \) and the sphere \( S^n \). In other words, \( M \) may be considered as the warped product \([0, \infty[ \times F S^n\), where \( F = f \circ G^{-1}, \ G(x) = \int_0^x \sqrt{1 + f'^2(t)}dt \).

In [17], we have proved the following fact:

**Theorem 3.3.** Suppose that \( f \) is unbounded, \( p, q \in [1, \infty[, \ \frac{1}{q} - \frac{1}{p} < \frac{1}{n+1}, \ 1 \leq j \leq n+1 \). Then \( T_{j,p,q}(M) \neq 0 \).

Kuz’minov and Shvedov [19] established that when \( f \) is bounded from above, \( T_{j,p}(M) \) is zero for all \( j, \ 2 \leq j \leq n \) and that, for \( j = 1, n+1 \), the triviality of \( T_{j,p}(M) \) depends on the finiteness of some Hardy constants. This is due to the connection between the \( L_p \)-cohomology of the warped product \( C_{a,b}^f Y \) and the weighted \( L_p \)-cohomology of \([a,b[\) given in the mentioned papers [9, 10]. Above we have shown that there is a connection of this type for \( L_{p,q} \)-cohomology. Namely, by Theorem 3.2, since \( S^n \) is compact and the de Rham cohomology \( H^{j-1}(S^n) \) of \( S^n \) is nontrivial if \( j = 1, n+1 \), for \( T_{j,p,q}(M) (j = 1, n+1) \) to be zero, it is necessary that \( \chi_{p,q}(0, \infty, F^{n-j+1}_p, F^{n-j+1}_q) < \infty \) or \( \chi_{p,q}(\infty, 0, F^{n-j+1}_p, F^{n-j+1}_q) < \infty \).

The main result of this section is a generalization of Theorems 2 and 2’ of [16] and is formulated as follows:

**Theorem 3.4.** Let \( M \) be the surface of revolution (3.1). Suppose that \( 1 < p < \infty, \ 1 < q < \infty, \ \frac{1}{q} - \frac{1}{p} < \frac{1}{n+1}, \ j \in \{1, n+1\}. \) If \( T_{j,p,q}(M) = 0 \) then \( \lim_{x \to \infty} f(x) = 0 \) and \( \text{vol} \ M \ < \infty \).

**Proof.** Put \( k = j - 1, \ q' = \frac{q}{q-1} \).
We have the following equalities:
\[ \chi_{p,q}^0 \equiv \chi_{p,q}(0, \infty, F_{\frac{n}{p}}^{n-k}, F_{\frac{n}{q}}^{n-k}) \]
\[ = \sup_{\tau > 0} \left\{ \left( \int_0^{\tau} f^{n-kp}(t) \sqrt{1 + f'^2(t)} dt \right)^{1/p} \left( \int_0^{\tau} f^{-(\frac{n}{q}-k)q'}(t) \sqrt{1 + f'^2(t)} dt \right)^{1/q'} \right\}; \]
\[ \chi_{p,q}^\infty \equiv \chi_{p,q}(\infty, 0, F_{\frac{n}{p}}^{n-k}, F_{\frac{n}{q}}^{n-k}) \]
\[ = \sup_{\tau > 0} \left\{ \left( \int_0^{\tau} f^{n-kp}(t) \sqrt{1 + f'^2(t)} dt \right)^{1/p} \left( \int_0^{\infty} f^{-(\frac{n}{q}-k)q'}(t) \sqrt{1 + f'^2(t)} dt \right)^{1/q'} \right\} \]
if \( p \geq q; \)
\[ \chi_{p,q}^0 \equiv \chi_{p,q}(0, \infty, F_{\frac{n}{p}}^{n-k}, F_{\frac{n}{q}}^{n-k}) \]
\[ = \left( \int_0^{\infty} \left[ \left( \int_0^{H(x)} f^{-(\frac{n}{q}-k)q'}(t) \sqrt{1 + f'^2(t)} dt \right)^{1/p} \int_{H(x)}^{\infty} f^{n-kp}(t) \sqrt{1 + f'^2(t)} dt \right]^{q/q-p} \right. \]
\[ \times \left. f^{-(\frac{n}{q}-k)q'}(x) \sqrt{1 + f'^2(x)} dx \right)^{q/q-p}; \quad (3.2) \]
\[ \chi_{p,q}^\infty \equiv \chi_{p,q}(\infty, 0, F_{\frac{n}{p}}^{n-k}, F_{\frac{n}{q}}^{n-k}) \]
\[ = \left( \int_0^{\infty} \left[ \left( \int_0^{H(x)} f^{-(\frac{n}{q}-k)q'}(t) \sqrt{1 + f'^2(t)} dt \right)^{1/p} \int_{H(x)}^{\infty} f^{n-kp}(t) \sqrt{1 + f'^2(t)} dt \right]^{q/q-p} \right. \]
\[ \times \left. f^{-(\frac{n}{q}-k)q'}(x) \sqrt{1 + f'^2(x)} dx \right)^{q/q-p} \]
if \( p < q. \) Here \( H(x) \) is the function inverse to the arc length function \( G(x) = \int_0^x \sqrt{1 + f'^2(t)} dt. \)

The main element in the proof of Theorem 3.4 is the following lemma which has some independent interest.

**Lemma 3.5.** If \( \frac{1}{q} - \frac{1}{p} < \frac{1}{n+1}, 1 < p < \infty, 1 < q < \infty, 0 \leq k \leq n, \) then the following hold:

1. if \( \chi_{p,q}^0 \not< \infty \) or \( \chi_{p,q}^\infty \not< \infty \) then \( \lim_{t \to \infty} f(t) = 0; \)
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(2) if $\frac{n}{p} - k \leq 0$ then $\chi_{p,q}^0 = \infty$;
(3) if $\frac{n}{q} - k \geq 0$ then $\chi_{p,q}^\infty = \infty$.

Proof. Suppose first that $p \geq q$.
Assume that $\chi_{p,q}^0 < \infty$. Then

$$\int_0^\infty f^{n-kp}(t)\sqrt{1 + f'^2(t)}dt < \infty \quad (3.3)$$

Since

$$f^{n-kp}(t)\sqrt{1 + f'^2(t)} \geq f^{n-kp}(t)|f'(t)|,$$

it follows that the integral

$$\int_0^\infty f^{n-kp}(t)f'(t)dt$$

$$= \begin{cases} \frac{1}{n-4k+1} \lim_{t \to \infty} (f^{n-4k+1}(t) - f^{n-4k+1}(0)) & \text{if } n - kp \neq -1, \\ \lim_{t \to \infty} \log \frac{f(t)}{f(0)} & \text{if } n - kp = -1 \end{cases} \quad (3.4)$$

is finite.
There appear several possibilities:
(a) $\frac{n}{p} - k > 0$. The above implies that there exists a finite limit $\lim_{t \to \infty} f(t)$, which is zero by (3.3).
(b) $\frac{n}{p} - k = 0$. This is impossible in view of (3.3).
(c) $-\frac{1}{p} < \frac{n}{p} - k < 0$. Then $n - kp + 1 > 0$ and $f(t)$ has a finite limit as $t \to \infty$, which contradicts (3.3).
(d) $\frac{n}{p} - k = -\frac{1}{p}$. A contradiction to (3.3).
(e) $\frac{n}{p} - k < -\frac{1}{p}$. In this case, $n - kp < -1$. Hence, $\lim_{t \to \infty} f(t) = \infty$. 

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Note that, since $\frac{1}{q} - \frac{1}{p} < \frac{1}{n+1}$, we have $k + 1 > \frac{n+1}{p} + 1 > \frac{n+1}{q}$, whence $-(\frac{n}{q} - k)q' + 1 > 0$. We infer

$$
\left( \int_{\tau}^{\infty} f^{n-\frac{kp}{q}}(t) \sqrt{1 + f'^2(t)} \, dt \right)^{1/p} \left( \int_{0}^{\tau} f^{-\left(\frac{n}{q} - k\right)q'}(t) \sqrt{1 + f'^2(t)} \, dt \right)^{1/q'}
$$

$$
\geq \left( \int_{\tau}^{\infty} f^{n-\frac{kp}{q}}(t) |f'(t)| \, dt \right)^{1/p} \left( \int_{0}^{\tau} f^{-\left(\frac{n}{q} - k\right)q'}(t) |f'(t)| \, dt \right)^{1/q'}
$$

$$
\geq \left( \int_{\tau}^{\infty} f^{n-\frac{kpq+1}{n-qp+1}}(t) \left| f^{-\left(\frac{n}{q} - k\right)q'+1}(\tau) - f^{-\left(\frac{n}{q} - k\right)q'+1}(0) \right|^{1/q'} \right.
$$

$$
\left. \frac{-(\frac{n}{q} - k)q'+1}{-(\frac{n}{q} - k)q'+1} \right|\tau
$$

$$
= C \cdot f^{\frac{n+1}{p} - \frac{n+1}{q} + 1}(\tau)|1 - f^{-\left(\frac{n}{q} - k\right)q'+1}(0) f^{\left(\frac{n}{q} - k\right)q'-1}(\tau)|^{1/q'}. \quad (3.5)
$$

The last quantity in (3.5) is equivalent to $C f^{\frac{n+1}{p} - \frac{n+1}{q} + 1}(\tau)$ as $\tau \to \infty$ and, hence, tends to infinity. Therefore, $\chi_{p,q}^0 = \infty$, and we obtain a contradiction.

Thus, if $\chi_{p,q} < \infty$ then $\lim_{t \to 0} f(t) = 0$ and $\frac{n}{p} - k > 0$.

Suppose now that $\chi_{p,q}^\infty < \infty$. Then

$$
\int_{0}^{\infty} f^{-\left(\frac{n}{q} - k\right)q'}(t) \sqrt{1 + f'^2(t)} \, dt < \infty \quad (3.6)
$$

and, hence, there exists a finite integral

$$
\int_{0}^{\infty} f^{-\left(\frac{n}{q} - k\right)q'}(t) f'(t) \, dt
$$

$$
= \begin{cases} 
\lim_{t \to \infty} f^{-\left(\frac{n}{q} - k\right)q'+1}(t) - f^{-\left(\frac{n}{q} - k\right)q'+1}(0) & \text{if } -(\frac{n}{q} - k)q' \neq -1, \\
\lim_{t \to \infty} \log \frac{f(t)}{f(0)} & \text{if } -(\frac{n}{q} - k)q' = -1.
\end{cases} \quad (3.7)
$$
As in the case $\chi_{p,q}^0 < \infty$, we infer that either $\frac{n}{q} - k < 0$ and $\lim_{t \to \infty} f(t) = 0$ or $(\frac{n}{q} - k)q' > 1$ and $\lim_{t \to \infty} f(t) = \infty$. In the latter case we have:

$$\chi_{p,q}^\infty \geq \sup_{\tau > 0} \left\{ \left( \frac{f^{-(\frac{n}{q} - k)q' + 1}((\tau))}{1 - (\frac{n}{q} - k)q'} \right)^{1/q'} \left| \frac{f^{n-kp+1}(\tau) - f^{n-kp+1}(0)}{n - kp + 1} \right|^{1/p} \right\}$$

$$= C \sup_{\tau > 0} \left\{ f^{\frac{n+1}{p} - \frac{n+1}{q} + 1}(\tau) \left| 1 - f^{n-kp+1}(0) f^{-(n-kp+1)}(\tau) \right|^{1/p} \right\}, \quad (3.8)$$

where $C = \text{const} > 0$. Since $k < \frac{n+1}{q} - 1 < \frac{n+1}{p}$, we have $n - kp + 1 > 0$, and, hence, the last quantity in $(3.8)$ behaves like $C f^{\frac{n+1}{p} - \frac{n+1}{q} + 1}(\tau)$ and, consequently, tends to infinity as $\tau \to \infty$. Hence, $\chi_{p,q}^\infty = \infty$; a contradiction.

Thus, if $\chi_{p,q}^\infty < \infty$ then $\lim_{t \to 0} f(t) = 0$ and $\frac{n}{q} - k < 0$.

We now pass to the case $p < q$.

Suppose that $\chi_{p,q}^0 < \infty$. Then, as above, we have $(3.3)$ and $(3.4)$ and conclude that either $\frac{n}{p} - k > 0$ and $\lim_{t \to \infty} f(t) = 0$ or $\frac{n}{p} - k < -\frac{1}{p}$ and $\lim_{t \to \infty} f(t) = \infty$. Show that the latter case is impossible. By $(3.2)$, we infer

$$\left(\chi_{p,q}^0\right)^{\frac{pq}{q-p}} \geq \int_0^\infty \int_0^{H(x)} \left| f^{-(\frac{n}{q} - k)q'}(t) f'(t) dt \right|^{p-1} \int_{H(x)}^{\infty} \left| f^{n-kp}(t) f'(t) dt \right|^{q/(q-p)} \times f^{-(\frac{n}{q} - k)q'}(x) \sqrt{1 + f'^2(x)} dx$$

$$= \int_0^\infty \left[ \left| f^{-(\frac{n}{q} - k)q' + 1}(H(x)) - f^{-(\frac{n}{q} - k)q' + 1}(0) \right|^{p-1} \left| \frac{f^{n-kp+1}(H(x))}{n - kp + 1} \right|^{q/(q-p)} \times f^{-(\frac{n}{q} - k)q'}(x) \sqrt{1 + f'^2(x)} dx \right.$$

$$= \int_0^\infty \left[ \left| f^{-(\frac{n}{q} - k)q' + 1}(s) - f^{-(\frac{n}{q} - k)q' + 1}(0) \right|^{p-1} \left| \frac{f^{n-kp+1}(s)}{n - kp + 1} \right|^{q/(q-p)} F^{-(\frac{n}{q} - k)q'}(s) ds \right.$$

$$= C \int_0^\infty F^N(s) \left| 1 - F^{-(\frac{n}{q} - k)q' + 1}(0) F^{\frac{n}{q} - k} q' - 1(s) \right| ds. \quad (3.9)$$
Here \( C = \text{const} > 0 \) and
\[
N = \left( -\left( \frac{n}{q} - k \right)q' + 1 \right) (p - 1) + n - kp + 1 \frac{q}{q - p} - \left( \frac{n}{q} - k \right)q' \\
= \left[ \left( 1 - \frac{p - 1}{q - 1} \right) \frac{q}{q - p} - \frac{1}{q - 1} \right] n - \left[ \left( \frac{q(p - 1)}{q - 1} - p \right) \frac{q}{q - p} + \frac{1}{q - 1} \right] k + \frac{pq}{q - p} \\
= n - k + \frac{pq}{q - p} > 0.
\]

Moreover, \( (\frac{n}{q} - k)q' - 1 < 0 \), since \( \frac{n}{q} < \frac{n + 1}{q} < \frac{n + 1}{p} < k \). Consequently, the expression under the last integral in (3.9) is equivalent to \( CF^{n-k+\frac{pq}{q-p}}(s) \), i.e., tends to \( \infty \) as \( s \to \infty \) and thus, the integral does not exist. A contradiction.

Suppose now that \( \chi_{p,q}^\infty < \infty \). Then we have (3.6) and (3.7) and infer that, in this case, either \( \frac{n}{q} - k < 0 \) and \( \lim_{t \to \infty} f(t) = 0 \) or \( q' (\frac{n}{q} - k) > 1 \) and \( \lim_{t \to \infty} f(t) = \infty \). In the latter case, we infer
\[
(\chi_{p,q}^\infty)^\frac{pq}{q-p} \geq \int_0^\infty \left[ \int_0^{H(x)} f^{n-kp}(t) f'(t) dt \right] \left[ \int_0^{H(x)} f^{-(\frac{n}{q} - k)q'}(t) f'(t) dt \right] \frac{q}{q-p} \\
\times \frac{f^{-(\frac{n}{q} - k)q'}(x)}{\sqrt{1 + f'^2(x)}} dx \\
= \int_0^\infty \left[ \left( \frac{f^{n-kp+1}(H(x)) - f^{n-kp+1}(0)}{n - kp + 1} \right) \left( \frac{f^{-(\frac{n}{q} - k)q'+1}(H(x))}{-(\frac{n}{q} - k)q' + 1} \right) \frac{q}{q-p} \right] \\
\times \frac{f^{-(\frac{n}{q} - k)q'}(x)}{\sqrt{1 + f'^2(x)}} dx \\
= \int_0^\infty \left[ \frac{F^{n-kp+1}(s) - F^{n-kp+1}(0)}{n - kp + 1} \left( \frac{F^{-(\frac{n}{q} - k)q'+1}(s)}{-(\frac{n}{q} - k)q' + 1} \right) \frac{q}{q-p} \right] F^{-(\frac{n}{q} - k)q'}(s) ds \\
= C \int_0^\infty F^N(s) |1 - F^{n-kp+1}(0)F^{-(n-kp+1)}(s)| ds. \quad (3.10)
\]

Here, as above, \( C = \text{const} > 0 \), \( N = n - k + \frac{pq}{q-p} > 0 \), and \( -(n - kp + 1) < 0 \). Thus, the expression under the integral is equivalent to \( CF^{n-k+\frac{pq}{q-p}}(s) \), i.e., tends to infinity as \( s \to \infty \).

Lemma 3.5 is completely proved. \( \square \)

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Now, return to the proof of Theorem 3.4. Suppose that \( T^j_{p,q}(M) = 0 \) for \( j = 1 \) or \( j = n + 1 \). Then, by Theorem 3.2,
\[
\chi_{p,q}(0, \infty, F_{p}^{n-j+1}, F_{q}^{n-j+1}) < \infty
\]
(and, hence, \( \int_{0}^{\infty} f^{(n-j+1)p} \sqrt{1 + f'^2} dt < \infty \)) or
\[
\chi_{p,q}(\infty, 0, F_{p}^{n-j+1}, F_{q}^{n-j+1}) < \infty
\]
(and, hence, \( \int_{0}^{\infty} f^{-(n-j+1)q'} \sqrt{1 + f'^2} dt < \infty \)). By Lemma 3.5, this implies that \( \lim_{t \to \infty} f(t) = 0 \) and, in both cases,
\[
\text{vol } M = s_n \int_{0}^{\infty} f^n(t) \sqrt{1 + f'^2} dt < \infty.
\]
Here \( s_n \) stands for the volume of the \( n \)-dimensional unit sphere in \( \mathbb{R}^{n+1} \).

The theorem is proved. \( \square \)

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References

$L_{p,q}$-COHOMOLOGY OF WARPED CYLINDERS


