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# On minimal non-*PC*-groups

FRANCESCO RUSSO  
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## Abstract

A group  $G$  is said to be a *PC*-group, if  $G/C_G(x^G)$  is a polycyclic-by-finite group for all  $x \in G$ . A minimal non-*PC*-group is a group which is not a *PC*-group but all of whose proper subgroups are *PC*-groups. Our main result is that a minimal non-*PC*-group having a non-trivial finite factor group is a finite cyclic extension of a divisible abelian group of finite rank.

## *Sur les non-PC-groupes minimaux*

### Résumé

On dit qu'un groupe  $G$  est un *PC*-groupe, si pour tout  $x \in G$ ,  $G/C_G(x^G)$  est une extension d'un groupe polycyclique par un groupe fini. Un non-*PC*-groupe minimal est un groupe qui n'est pas un *PC*-groupe mais dont tous les sous-groupes propres sont des *PC*-groupes. Notre principal résultat est qu'un non-*PC*-groupe minimal ayant un groupe quotient fini non-trivial est une extension cyclique finie d'un groupe abélien divisible de rang fini.

## 1. Introduction

A group  $G$  is said to have *polycyclic-by-finite* (resp., *Černikov*) *conjugacy classes*, or to be a *PC-group* (resp., *CC-group*), if  $G/C_G(x^G)$  is a polycyclic-by-finite (resp., *Černikov*) group for all  $x \in G$ . These groups generalize the *FC*-groups and were introduced in [5] (resp., [10]).

If  $\mathfrak{X}$  is a class of groups, a group  $G$  is said to be a *minimal non- $\mathfrak{X}$ -group* if it is not an  $\mathfrak{X}$ -group but all of whose proper subgroups are  $\mathfrak{X}$ -groups. Many results have been obtained on minimal non- $\mathfrak{X}$ -groups, for various choices of  $\mathfrak{X}$ . In particular, in [2] and [3] (see also [12, Section 8]) V. V. Belyaev and N. F. Sesekin characterized minimal non-*FC*-groups when

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they have a non-trivial finite or abelian factor group. They proved that a minimal non- $FC$ -group is a finite cyclic extension of a divisible  $p$ -group of finite rank ( $p$  a prime). Then J. Otál and J. M. Peña proved in [9] that there are no minimal non- $CC$ -groups which have a non-trivial finite or abelian factor group. In this paper we prove that a minimal non- $PC$ -group having a non-trivial finite factor group is a finite cyclic extension of a divisible group of finite rank. Contrary to the  $FC$ -case, these groups are not necessarily periodic. Note that the imposition of the condition of having a non-trivial finite factor group is to avoid Tarski groups, that is, infinite non-abelian groups whose proper subgroups are finite.

In Section 2 we consider finitely generated minimal non- $PC$ -groups. Among these there are Tarski groups [8], for this reason we cannot obtain a good description in such a case. However in Theorem 1.1 we give some conditions which have to be satisfied.

We can find in [1] the definitions of the  $PC$ -centre and the  $PC$ -hypercentre which are introduced in the same way as the  $FC$ -centre and the  $FC$ -hypercentre (see [11, Section 4.3]). More precisely, an element  $x$  of a group  $G$  is said to be a  $PC$ -element, if  $G/C_G(x^G)$  is a polycyclic-by-finite group. The set  $P(G)$  of all  $PC$ -elements of  $G$  forms a characteristic subgroup of  $G$ . This was already noted by R. Baer in the finite case, that is, in the case of  $FC$ -groups (see [11, Lemma 4.31]). Then we may consider the series

$$\{1\} = P_0 \triangleleft P_1 \triangleleft \dots \triangleleft P_\alpha \triangleleft P_{\alpha+1} \triangleleft \dots,$$

where  $P_1 = P(G)$ ,  $P_{\alpha+1}/P_\alpha = P(G/P_\alpha)$  and  $P_\lambda = \bigcup_{\alpha < \lambda} P_\alpha$ , with  $\alpha$  ordinal and  $\lambda$  limit ordinal. This is a characteristic ascending series of  $G$  and is called *upper  $PC$ -central series* of  $G$ . Its last term is called  *$PC$ -hypercentre* of  $G$ . If  $G = P_\beta$ , for some ordinal  $\beta$ , we say that  $G$  is a  *$PC$ -hypercentral* group (of length at most  $\beta$ ). In the finite case we have the corresponding notions of *upper  $FC$ -central series* of  $G$ ,  *$FC$ -hypercentre* of  $G$  and  *$FC$ -hypercentral* group.

Our first result is the following.

**Theorem 1.1.** *Let  $G$  be a finitely generated minimal non- $PC$ -group. Then:*

- (i)  *$G$  has no non-trivial locally graded factor groups. In particular,  $G$  is a perfect group and has no proper subgroups of finite index.*
- (ii) *The centre, the hypercentre, the  $PC$ -centre and the  $PC$ -hypercentre of  $G$  coincide.*

- (iii)  $G/\text{Frat}(G)$  is an infinite simple group, where  $\text{Frat}(G)$  stands for the Frattini subgroup of  $G$ , and for all  $x$  in  $G \setminus \text{Frat}(G)$  we have that  $G = x^G$ .

Note that the simplicity of  $G/\text{Frat}(G)$  has been proved for minimal non- $\mathfrak{X}$ -groups when  $\mathfrak{X}$  is the class of nilpotent groups [7], finite-by-nilpotent groups [14], locally finite-by-nilpotent groups [4] and [13] and torsion-by-nilpotent groups [13].

In Section 3 we characterize minimal non- $PC$ -groups having a non-trivial finite factor group.

Recall that a group is said to have *finite (Prüfer) rank*  $n$  if every finitely generated subgroup can be generated by  $n$  elements and  $n$  is the least such an integer. This is our second result.

**Theorem 1.2.** *Let  $G$  be a group and suppose that  $G^*$ , the finite residual of  $G$ , is a proper subgroup of  $G$ . Then  $G$  is a minimal non- $PC$ -group if, and only if, the following conditions hold:*

- (i) *There exists  $x \in G$  such that  $G = \langle G^*, x \rangle$ . Moreover,  $G^*$  is non-trivial and there is a prime  $p$  and a positive integer  $n$  such that  $x^{p^n} \in G^*$ ;*
- (ii)  *$G^*$  is either a  $q$ -group for a suitable prime  $q$ , or a torsion-free group. Furthermore,  $G^*$  is a divisible abelian group of finite rank;*
- (iii)  *$G' = G^*$ ;*
- (iv) *If  $N$  is a proper  $G$ -admissible subgroup of  $G^*$ , then  $N$  is a finitely generated group;*
- (v) *If  $H$  is a proper normal subgroup of  $G$ , then  $HG^*$  is a proper abelian subgroup of  $G$ . In particular,  $H$  is an abelian group.*

Clearly, every periodic minimal non- $PC$ -group is a minimal non- $FC$ -group. But in view of the results of [12, Theorem 8.11], one can deduce from Theorem 1.2 that the converse holds for groups having a non-trivial finite factor group. So we have the following consequence.

**Corollary 1.3.** *Let  $G$  be a group having a non-trivial finite factor group. Then  $G$  is a periodic minimal non- $PC$ -group if and only if  $G$  is a minimal non- $FC$ -group.*

Since abelian  $q$ -groups of finite rank are the direct product of finitely many quasicyclic groups or finite cyclic groups, they satisfy the minimal condition on subgroups. So one can deduce the following result.

**Corollary 1.4.** *Let  $G$  be a group having a non-trivial finite factor group. If  $G$  is a periodic minimal non-PC-group, then  $G$  is a Černikov group.*

## 2. Finitely generated minimal non-PC-groups

We will use the next lemma which is referred to [5, Theorem 2.2] and [5, Lemma 2.4].

**Lemma 2.1.**

- (i) *A group  $G$  is a PC-group if and only if each finite set of elements of  $G$  is contained in a polycyclic-by-finite normal subgroup of  $G$ .*
- (ii) *The class of PC-groups is closed with respect to subgroups, quotients and direct products of its members.*

**Lemma 2.2.** *Let  $G$  be a finitely generated minimal non-PC-group. Then  $G$  has no proper subgroups of finite index.*

*Proof.* Suppose that  $H$  is a proper subgroup of  $G$  of finite index. Then  $H$  is a finitely generated PC-group. So by Lemma 2.1(i)  $H$ , and therefore  $G$ , is a polycyclic-by-finite group. Hence  $G$  is a PC-group, which is a contradiction.  $\square$

Since finitely generated locally graded groups and in particular polycyclic-by-finite groups, have proper subgroups of finite index, we can deduce the following result.

**Corollary 2.3.** *Let  $G$  be a minimal non-PC-group. Then  $G$  is a locally graded group if and only if  $G$  is not a finitely generated group.*

**Lemma 2.4.** *Let  $G$  be a finitely generated minimal non-PC-group. Then  $G$  has no non-trivial locally graded factor groups. In particular,  $G$  is a perfect group.*

*Proof.* Let  $N$  be a normal subgroup of  $G$  such that  $G/N$  is a non-trivial locally graded group. Since  $G/N$  is finitely generated, it has a proper normal subgroup  $H/N$  such that  $(G/N)/(H/N)$  is finite. Thus  $H$  is a proper subgroup of  $G$  of finite index, which is a contradiction to Lemma 2.2.  $\square$

Since a finitely generated  $PC$ -group is a polycyclic-by-finite group, we can deduce the following result.

**Corollary 2.5.** *A finitely generated minimal non- $PC$ -group has no non-trivial factor groups which are  $PC$ -groups.*

*Proof of Theorem 1.1.* (i). Follows from Lemma 2.2 and Lemma 2.4.

(ii). Since  $G$  is a perfect group, the centre of  $G/Z(G)$  is trivial. So that  $Z(G)$  is the hypercentre of  $G$ . Now let  $x$  be a  $PC$ -element of  $G$ , then  $G/C_G(x^G)$  is a polycyclic-by-finite group. We deduce from Lemma 2.4 that  $G = C_G(x^G)$ . Thus  $x \in Z(G)$ , hence  $Z(G)$  is the  $PC$ -centre of  $G$ . Clearly Corollary 2.5 gives that  $\bar{G} = G/P(G)$  is a minimal non- $PC$ -group. Thus as before, if  $\bar{x}$  is a  $PC$ -element of  $\bar{G}$  then  $\bar{x}$  belongs to the centre of  $\bar{G}$ . Now  $\bar{G} = G/Z(G)$ , so  $Z(\bar{G})$  is trivial. Thus the set of  $PC$ -elements of  $\bar{G}$  is trivial and therefore the  $PC$ -centre of  $G$  is the  $PC$ -hypercentre of  $G$ .

(iii). Since  $G$  is a finitely generated group,  $Frat(G)$  is a proper subgroup, so  $G/Frat(G)$  is an infinite group. Suppose that  $G/Frat(G)$  is not a simple group and let  $N$  be a normal subgroup of  $G$  such that  $Frat(G) \leq N \leq G$ . Therefore there is a maximal subgroup  $M$  of  $G$  such that  $N \not\leq M$ . It follows that  $G = MN$  and therefore  $G/N \simeq M/M \cap N$ . Since  $M$  is a proper subgroup of  $G$ , it is a  $PC$ -group. We deduce that  $G/N$  is a  $PC$ -group, which is a contradiction by Corollary 2.5. Therefore  $G/Frat(G)$  is a simple group. Let  $x$  be an element of  $G \setminus Frat(G)$ . Since  $G/Frat(G)$  is a simple group,  $x^G Frat(G) = G$  and this gives that  $G = x^G$ .  $\square$

### 3. The presence of a non-trivial finite factor group

To prove Theorem 1.2 we adapt [12, Proof of Theorem 8.11].

**Lemma 3.1.** *Let  $G = HK$  be the product of a normal polycyclic-by-finite subgroup  $H$  by a  $PC$ -subgroup  $K$ . Then  $G$  is a  $PC$ -group.*

*Proof.* Let  $x \in G$ ; since  $G/H$  is a  $PC$ -group,  $(xH)^{G/H}$  is a polycyclic-by-finite group. So that  $x^K H/H$  is a polycyclic-by-finite group and therefore  $x^K$  is a polycyclic-by-finite group. Now  $x^G \leq Hx^K$ , hence  $x^G$  is a polycyclic-by-finite group, which gives that  $G$  is a  $PC$ -group.  $\square$

**Lemma 3.2.** *Let  $H$  be a subgroup of a  $PC$ -group  $G$ . If  $H$  has no proper subgroups of finite index, then it is contained in  $Z(G)$ .*

*Proof.* Since  $G$  is a  $PC$ -group,  $G/Z(G)$  is a residually polycyclic-by-finite group and therefore it is a residually finite group. So  $HZ(G)/Z(G)$  is a residually finite group which has no proper subgroups of finite index, hence it is trivial and therefore  $H \leq Z(G)$ .  $\square$

**Lemma 3.3.** *Let  $H$  be a normal subgroup of finite index in a minimal non- $PC$ -group  $G$ . Then  $G/H$  is a cyclic group of  $p$ -power order, where  $p$  is a prime.*

*Proof.* First, note that in view of Theorem 1.1,  $G$  is not a finitely generated group. So that every finitely generated subgroup of  $G$  is a proper subgroup and therefore it is a polycyclic-by-finite group by Lemma 2.1 (i). Since  $G$  is not a  $PC$ -group, there exists a non-trivial element  $x$  of  $G$  such that  $x^G$  is not a polycyclic-by-finite group. Assume that  $\langle H, x \rangle$  is a proper subgroup of  $G$ . Then  $\langle H, x \rangle$  is a  $PC$ -group so that  $x^H$  is a polycyclic-by-finite group. Since  $G/H$  is a finite group,  $G = HF$ , where  $F$  is a finitely generated subgroup of  $G$ . So that

$$x^G = x^{HF} = (x^H)^F = K^F$$

where  $K = x^H$ . But both  $K$  and  $F$  are finitely generated groups, so  $\langle K, F \rangle$  is also a finitely generated group and therefore it is a polycyclic-by-finite group. It follows that  $K^F$  is a polycyclic-by-finite group and this gives the contradiction that  $x^G$  is a polycyclic-by-finite group. Thus  $G = \langle H, x \rangle$  and  $G/H$  is a cyclic group, as claimed.

It remains to prove that  $G/H$  has  $p$ -power order, where  $p$  is a prime. Let  $|G : H| = mn$  for suitable integers  $m, n > 1$  such that  $(m, n) = 1$ . Then  $\langle H, x^m \rangle$  and  $\langle H, x^n \rangle$  are proper subgroups of  $G$  so that they are  $PC$ -groups. So both  $(x^m)^H$  and  $(x^n)^H$  are polycyclic-by-finite groups. But for every positive integer  $i$  we have that

$$(x^i)^G = (x^i)^{\langle x \rangle H} = (x^i)^H$$

so that both  $(x^m)^G$  and  $(x^n)^G$  are polycyclic-by-finite groups and therefore  $(x^m)^G (x^n)^G$  is a polycyclic-by-finite group too. But  $(m, n) = 1$  so that

$$x^G = (x^m)^G (x^n)^G$$

and therefore  $x^G$  is a polycyclic-by-finite group, which is a contradiction. Then  $G/H$  has  $p$ -power order, where  $p$  is a suitable prime, as claimed.  $\square$

*Proof of Theorem 1.2.* (i). Let  $H_1$  and  $H_2$  be two normal subgroups of finite index in  $G$ . Then  $|G : H_1 \cap H_2|$  is also a finite number which is divisible by both  $|G : H_1|$  and  $|G : H_2|$ . We deduce that the prime  $p$  of Lemma 3.3 is the same for all normal subgroups of finite index in  $G$ . On the other hand, if  $|G : H_1| = p^n$  and  $|G : H_2| = p^m$ , where  $n \leq m$ , then  $H_2 \leq H_1$  because the finite cyclic group  $G / (H_1 \cap H_2)$  has a unique quotient of a given order. Thus the set of normal subgroups of  $G$  of finite index is a chain and therefore  $G/G^*$  is a finite group. It follows, by Lemma 3.3, that there is  $x$  in  $G$  and a prime  $p$  and a positive integer  $n$  such that  $G = \langle G^*, x \rangle$  and  $x^{p^n} \in G^*$ . Clearly  $G^*$  is a non-trivial group.

(ii). By Lemma 3.2,  $G^*$  is an abelian group and  $G^*$  is divisible because it has no proper subgroups of finite index.

From (i) we have that  $G = \langle G^*, x \rangle$ . We claim that  $\langle H, x \rangle$  is a proper subgroup of  $G$ , whenever  $H$  is a  $G$ -admissible proper subgroup of  $G^*$ . If  $G = \langle H, x \rangle$ , then  $G/H$ , and therefore  $G^*/H$ , is a non-trivial cyclic group. So  $G^*/H$ , and therefore  $G^*$ , has a proper subgroup of finite index, which is a contradiction.

We know from [6, Section 19] that  $G^*$  is the direct product of quasicyclic groups and copies of the additive group  $\mathbb{Q}$  of the rational numbers. Since  $G$  is not abelian,  $G^* \not\leq Z(G)$ . So there exists a subgroup  $K$  of  $G^*$  such that  $K$  is isomorphic to a quasicyclic group  $C_{q^\infty}$  for some prime  $q$  or  $K$  is isomorphic to  $\mathbb{Q}$  and  $K \not\leq Z(G)$ . By Lemma 3.2 we deduce that  $G = \langle K, x \rangle$ . Write  $N = K^G$ . Then  $N = K^{\langle x \rangle} = KK^x \dots K^{x^{p^n-1}}$ . This implies that  $N$  is either an abelian  $q$ -group or a torsion-free abelian group. Since  $K, K^x, \dots, K^{x^{p^n-1}}$  have rank 1,  $N$  has finite rank, because it is a product of finitely many subgroups of finite rank [11, Lemma 1.44].

Now, since  $G = \langle K, x \rangle$ , we have  $G = \langle N, x \rangle$  and  $N$  is a  $G$ -admissible subgroup of  $G^*$ . We deduce by the previous argument that  $N = G^*$ . Therefore, the result follows.

(iii). Suppose that  $G'$  is a proper subgroup of  $G^*$ . Then as before  $H = \langle G', x \rangle$  is proper in  $G$  so that  $x^H$  is a polycyclic-by-finite group. But  $H' = [G', \langle x \rangle]$  since  $G'$  is an abelian group, so  $H'$  is a polycyclic-by-finite group. By Lemma 3.1, we deduce that  $G/H'$  is a minimal non-PC-group, and since a factor group of a divisible abelian group is also divisible, there is no loss of generality if we suppose that  $H' = 1$ . It follows that  $G'$  is a central subgroup of  $G$ , from which we deduce that

$$[G^*, x] = [(G^*)^{p^n}, x] = [G^*, x]^{p^n} = [G^*, x^{p^n}] = 1.$$



This gives that  $G$  is an abelian group, which is a contradiction. Therefore  $G' = G^*$ , as claimed.

(iv). Let  $N$  be a proper  $G$ -admissible subgroup of  $G^*$ . Then, as before,  $\langle N, x \rangle$  is a proper subgroup of  $G$ , so that  $x^N$ , and therefore  $[N, \langle x \rangle]$ , is a polycyclic-by-finite group. Hence it suffices to prove that  $N/[N, \langle x \rangle]$  is a finitely generated group. Since a section of a group of finite rank is of finite rank [11, Lemma 1.44], there is no loss of generality if we assume that  $[N, \langle x \rangle] = 1$ , so that  $N$  is a central subgroup of  $G$ . Now,

$$G^* = G' = [G^*, \langle x \rangle] = \langle [a, x^r] : a \in G^*, r \in \mathbb{Z} \rangle.$$

Since  $N$  is central we deduce that each element of  $N$  is of the form  $[a, x]$ , where  $a \in G^*$ . We have  $[a, x]^{p^n} = [a, x^{p^n}] = 1$ , so that  $N$  is of finite exponent. Therefore  $N$  is an abelian  $p$ -group of finite rank and of finite exponent, hence  $N$  is a finite group, as required.

(v). Let  $H$  be a proper normal subgroup of  $G$ . Assume that  $G = HG^*$ . Then  $H/H \cap G^*$  is finite. Note that since  $H$  is a proper subgroup of  $G$ ,  $H \cap G^*$  is a proper subgroup of  $G^*$ . We deduce by (iv) that  $H \cap G^*$  is a polycyclic group and this gives that  $H$  is a polycyclic-by-finite group. It follows that  $G$  is an extension of a polycyclic-by-finite group by an abelian group. Therefore  $G$  is a  $PC$ -group by Lemma 3.1, which is a contradiction, so that  $HG^*$  is a proper subgroup of  $G$ . Since  $G/G^*$  is a cyclic group of order  $p^n$ , there is a positive integer  $i \leq n$  such that  $HG^* = \langle G^*, x^{p^i} \rangle$ . By Lemma 3.2, we deduce that  $HG^*$ , and therefore  $H$ , is an abelian group.

Conversely, suppose that  $G$  is a group which satisfies conditions (i)-(v). If  $G$  is a  $PC$ -group, then Lemma 3.2 gives that  $G$  is an abelian group and therefore the condition (iii) gives that  $G^*$  is a trivial group which contradicts (i). Let  $H$  be a proper subgroup of  $G$ . If  $HG^*$  is a proper subgroup of  $G$ , then by the condition (v) we have that  $HG^*$ , and therefore  $H$ , is an abelian group. So that  $H$  is a  $PC$ -group. Now assume that  $G = HG^*$ . Since  $G/G^*$  is a group of order  $p^n$ ,  $H^{p^n}$  is a subgroup of  $G^*$ . Clearly  $H^{p^n}$  is a proper  $G$ -admissible subgroup of  $G^*$ . We deduce by (iv) that  $H^{p^n}$  is a polycyclic group. Now  $G$  is of finite rank because  $G^*$  is of finite rank and  $G/G^*$  is a cyclic group. It follows that  $H/H^{p^n}$  is a metabelian group of finite rank and of finite exponent, hence it is a finite group. So  $H$  is a polycyclic-by-finite group and therefore it is a  $PC$ -group. We conclude that  $G$  is a minimal non- $PC$ -group, as claimed.  $\square$

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## References

- [1] J. C. Beidleman, A. Galoppo, and M. Manfredino, *On  $PC$ -hypercentral and  $CC$ -hypercentral groups*, *Comm. Alg.* **26** (1998), 3045–3055.
- [2] V. V. Belyaev, *Minimal non- $FC$ -groups*, VI All Union Symposium Group Theory (Čerkassy, 1978), Naukova Dumka, 1980, pp. 97–102.
- [3] V. V. Belyaev and N. F. Sesekin, *Infinite groups of Miller-Moreno type*, *Acta Math. Hungar.* **26** (1975), 369–376.
- [4] A. Dilmi, *Groups whose proper subgroups are locally finite-by-nilpotent*, *Ann. Math. Blaise Pascal* **14** (2007), 29–35.
- [5] S. Franciosi, F. de Giovanni, and M. J. Tomkinson, *Groups with polycyclic-by-finite conjugacy classes*, *Boll. U. M. I.* **7** (1990), 35–55.
- [6] L. Fuchs, *Abelian groups*, Pergamon Press, London, 1967.
- [7] M. F. Newman and J. Wiegold, *Arch. math.*, *Soviet Math. Dokl.* **15** (1964), 241–250.
- [8] A. Yu. Ol’shanskii, *Infinite groups with cyclic subgroups*, *Soviet Math. Dokl.* **20** (1979), 343–346.
- [9] J. Otál and J. M. Peña, *Minimal non- $CC$ -groups*, *Comm. Algebra* **16** (1988), 1231–1242.
- [10] Ya. D. Polovickii, *Groups with extremal classes of conjugated elements*, *Sibirski Math. Z.* **5** (1964), 891–895.
- [11] D. J. Robinson, *Finiteness conditions and generalized soluble groups*, Springer Verlag, Berlin, 1972.
- [12] M. J. Tomkinson,  *$Fc$ -groups*, Pitman, Boston, 1984.
- [13] N. Trabelsi, *On minimal non-(torsion-by-nilpotent) and non-((locally finite)-by-nilpotent) groups*, *C. R. Acad. Sci. Paris Ser. I* **344** (2007), 353–356.

- [14] M. Xu, *Groups whose proper subgroups are finite-by-nilpotent*, Arch. Math. **66** (1996), 353–359.

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