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Abstract

In this paper, we give a generalization of Baer Theorem on the injective property of divisible abelian groups. As consequences of the obtained result we find a sufficient condition for a group $G$ to express as semi-direct product of a divisible subgroup $D$ and some subgroup $H$. We also apply the main Theorem to the $p$-groups with center of index $p^2$, for some prime $p$. For these groups we compute $N_c(G)$ the number of conjugacy classes and $N_a$ the number of abelian maximal subgroups and $N_{na}$ the number of nonabelian maximal subgroups.

1. Introduction

We shall recall some definitions:

If $H$ is a subgroup of a group $G$, a subgroup $K$ is called a complement of $H$ in $G$ if $G = HK$ and $H \cap K = \{1\}$. Therefore if $H \triangleleft G$ and $K \triangleleft G$, then $G$ is said to be the direct product of $H$ and $K$, in symbols, $G = H \odot K$.

If $H \triangleleft G$, then $G$ is said to be the semi-direct product of $H$ and $K$, in symbols, $G = H \rtimes K$.

An abelian group $D$ is called divisible if for every $x \in D$ and every positive integer $n$ there is a $y \in D$ so that $x = ny$, i.e., each element of $D$ is divisible by every positive integer. The main property of divisible groups is that they satisfy the following “injectivity” condition:

**Theorem 1.1** (Baer Theorem [3]). *If $D$ is a divisible group, then any homomorphism $f : A \to D$ from any abelian group $A$ into $D$ extends to any abelian group $G$ which contains $A$ i.e., there exists a homomorphism $ar{f} : G \to D$ so that $\bar{f}|_A = f$.***

The purpose of this paper is to generalize this result to the nonabelian groups. To this end, we introduce the property “$N$” in subgroups: Let $H$ be a subgroup of an arbitrary group $G$.

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$H$ satisfies the condition "N" if and only if $\exists g \in G - H \ [g, H] = 1$.

This is equivalent to saying that $C_G(H) \nsubseteq H$ where $C_G(H)$ is the centralizer of $H$ in $G$ which is defined to be the set of all $g$ in $G$ such that $hg = gh$ for all $h$ in $H$, it is clearly a subgroup of $G$.

By the definition of the condition “N” we deduce that

1) If $G$ is abelian, then every proper subgroup $H$ satisfies the condition "N".

2) If $G$ is a nonabelian nilpotent group, then every maximal normal abelian subgroup $H$ of $G$ does not satisfy the condition “N” because $C_G(H) = H$, [3].

3) There exist a nonabelian groups $G$ whose a subgroup $H$ satisfies the condition "N", for example let $G = Q_8 \times \mathbb{Z}/2\mathbb{Z}$ where $H = Q_8$ is the quaternion group of order 8, [2].

2. Main results and proofs

**Theorem 2.1.** Let $G$ be a group and let $H$ be a subgroup of $G$ such that each proper subgroup $H'$ of $G$ which contains $H$, satisfies the condition “N”. Then any homomorphism $f : H \rightarrow D$ from $H$ into divisible group $D$ extends to the group $G$.

**Proof.** Let us consider the set $S$ of all pairs $(H_i, f_i)$ where $H_i$ is a subgroup of $G$ containing $H$ and $f_i : H_i \rightarrow D$ is an extension of $f$, i.e., $f_i|_{H_i} = f$. Let $(H_i, f_i) \leq (H_j, f_j)$ if $H_i \subset H_j$ and $f_j|_{H_i} = f_i$. The set $S$ is partially ordered by the relation $\leq$. We aim to apply Zorn’s Lemma to $S$ and to this end we consider a chain $(H_i, f_i)_{i \in I}$. It has an upper bound $(\bigcup_{i \in I} H_i, f')$ where $f' : \bigcup_{i \in I} H_i \rightarrow D$ is defined by $f'(x_i) = f_i(x_i)$ for every $x_i \in H_i$. This is unambiguous since $x_i \in H_i \subset H_m \Rightarrow f_m(x_i) = f_i(x_i)$. Consequently, by Zorn’s Lemma, $S$ has a maximal element, say $(\overline{H}, \overline{f})$. We claim that $\overline{H} = G$ and $\overline{f}$ is the desired extension of $f$ to $G$. To see this suppose $\overline{H} \neq G$. By hypotheses $\overline{H}$ satisfies the condition “N”, consequently, there is an $g \in G - \overline{H}$ such that $[g, \overline{H}] = 1$, therefore $\langle \overline{H}, g \rangle = \overline{H} \cdot \langle g \rangle$. There are two cases:

**Case 1.**

$\overline{H} \cap \langle g \rangle \neq \{0\}$. Let $n$ the smallest positive integer so that $g^n = u \in \overline{H}$. Since $D$ is divisible there is a $x \in D$ so that $\overline{f}(u) = nx$. Let $\overline{f}^* : \overline{H} \cdot \langle g \rangle \rightarrow D$ be defined by $\overline{f}^*(ag^t) = \overline{f}(a) + tx$ where $a \in \overline{H}$ and $0 \leq t < n$, if
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Any divisible subgroup \( ag^t = a'g'^t \) then \( a^{t-1}a = g^{t-t} \in H \), so \( t' - t = 0 \) and \( a = a' \). Hence \( x = f(x) = f(y) \). We consider the identity map: \( f \) to abelian groups, also we deduce the known result, 

\[
\begin{align*}
H \cap (g) &= \{0\}. \\
H \cdot (g) &= H \circ (g). \\
D &\text{ splits}.
\end{align*}
\]

Let \( x_0 \) be an element of \( D \), in this case we can define \( f^{-1} : H \cdot (g) \to D \) by writing \( f^{-1} (ag^t) = f(a) + kx_0 \), it is easy to verify that \( f^{-1} \) is a homomorphism so that \( f^{-1} = f \) contradicting the maximality of \( (H, f) \). Thus \( H = G \) and \( f \) is the desired extension of \( f \) to \( G \). 

\text{Case 2.}

Since every subgroup of an abelian group satisfies the condition “\( N \)”, we can apply Theorem 2.1 to abelian groups, also we deduce the known result, [3]:

**Corollary 2.2.** Any divisible subgroup \( D \) of an abelian group \( G \) splits, i.e., \( D \) has a complement \( H \) so that \( G = H \oplus D \).

If \( G \) is a group (not necessarily abelian), we write:

**Corollary 2.3.** Let \( G \) be a group and let \( D \) be a divisible subgroup of \( G \) such that every subgroup \( H \) of \( G \) which contains \( D \), satisfies the condition “\( N \)”. Then \( D \) has a complement \( H \) so that \( G = H \ltimes D \).

\text{Proof.} We consider the identity map: \( id_D : D \to D \), by Theorem 2.1, \( id_D \) extends to the group \( G \), i.e., there exists a homomorphism \( f : G \to D \) so that \( f|_D = id_D \). Let \( i : D \to G \) be the inclusion map, then \( f \circ i = id_D \) implies that \( f(G) = D \). Let \( H = Ker \) \( f \), if \( x \in G \), then \( x = x f \). Since \( f \) \( x f \) \( x^{-1} f \) \( x \), we have \( f \) \( x f \) \( x^{-1} f \) \( x = 1 \), so \( G = H.D \). If \( x \) belongs to \( H \cap D \), then \( x = f \) \( x' \) and \( x = f \) \( x' \), that is, \( H \cap D = \{1\} \). Hence \( G = H \ltimes D \). 

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Corollary 2.4. Let $G$ be a finite $p$-group with center of index $p^2$. If $H$ is a nonabelian maximal subgroup of $G$ then any homomorphism $f : H \to D$ from $H$ into divisible group $D$ extends to the group $G$.

To prove Corollary 2.4, we need the following.

Lemma 2.5. Let $G$ be a finite $p$-group such that its center $Z(G)$ has index $p^2$. If $H$ is a maximal subgroup of $G$, then the following properties are equivalent.

i) $H$ is abelian

ii) $Z(G) \subseteq H$

iii) $H$ does not satisfy the condition “N”.

Proof of Lemma 2.5. “i) $\Rightarrow$ ii)” Let us assume that $H$ is abelian. If $Z(G) \not\subseteq H$, there exists $g \in Z(G) - H$ and $G = H \cdot \langle g \rangle$. Then $G$ is abelian, this contradicts $|G : Z(G)| = p^2$.

“ii) $\Rightarrow$ iii)” Assume that $Z(G) \subseteq H$. Then $Z(G) \subseteq Z(H) \subseteq H \subseteq G$. By hypothesis $|G : Z(G)| = p^2$. Since the center does not have a prime index and $|G : H| = p$, $Z(H) = H$, consequently $H$ is abelian. Hence $H$ is a maximal normal abelian subgroup of the nilpotent group $G$, so $C_G(H) = H$, [3], and $H$ does not satisfy the condition “N”.

“iii) $\Rightarrow$ i)” If $C_G(H) \subseteq H$, then $Z(G) \subseteq Z(H) \subseteq H$. By the same way we deduce that $H$ is abelian. $\square$

Proof of Corollary 2.4. Let $H$ be a subgroup of $G$ so that $H \subseteq H'$, since $H$ is nonabelian, by Lemma 2.5, there is $g \in G - H$ so that $[g, H] = 1$. Then $[g, H'] = 1$ by maximality of $H$. Thus the conditions of Theorem 2.1 are satisfied, so we obtain Corollary 2.4. $\square$

3. Subgroups satisfying the condition “N”

If $A$ is finitely generated abelian group, the rank of $A$ is defined by $rk(A)$ the minimum number of generators of $A$.

We denote us by $x^G$ the conjugacy class of $x$ in an arbitrary group $G$ and $C_G(x)$ the centralizer of $x$ in $G$ and $N_c(G)$ the number of the conjugacy classes.

If $G$ is a finite $p$-group of class $c$, then from [4], we know that

$$N_c(G) \geq c|G|^{1/c} - c + 1.$$
Let $G$ be a finite $p$-group of order $p^n$ such that its center has index $p^2$. In this section, we compute the number $N_c(G)$ and $N_0$ the number of maximal subgroups in $G$ satisfying the condition “N”.

**Theorem 3.1.** Let $G$ be a finite $p$-group of order $p^n$ such that its center has index $p^2$, then

1) $G$ has precisely $p + 1$ abelian maximal subgroups.

2) The number of maximal subgroups satisfying the condition “N” equals

$$N_0 = (p^r - p^2)/(p - 1)$$

where $r$ is the rank of $G/G' = \mathbb{Z}/p^{n_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{n_r}\mathbb{Z}$.

3) $N_c(G) = p^{n-1} + p^{n-2} - p^{n-3}$ and each nontrivial conjugacy class has $p$ elements.

The proof of Theorem 3.1 results from the following Lemmas.

**Lemma 3.2.** Let $G$ be an abelian finite $p$-group, then the number of subgroups of order $p$ equals

$$\left(\frac{p^r - 1}{p - 1}\right) = p^r - 1.$$ 

Proof of Lemma 3.2. Since $rk(G) = r$, G is isomorphic with the group $\mathbb{Z}/p^{n_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{n_r}\mathbb{Z}$. If $g$ is an element of order $p$ in $G$, then $g = g_1g_2\cdots g_r$ such that $g_i$ has order $p$ or 1 and $g \neq 1$. The number of such elements $g$ equals

$$(p - 1) \cdot C_1^1 + (p - 1)^2 C_2^2 + \cdots + (p - 1)^r C_r^r = p^r - 1.$$ 

Since a group of order $p$ has $p - 1$ elements of order $p$, the number of subgroups of order $p$ is $(p^r - 1)/(p - 1)$.

**Lemma 3.3.** Let $G$ be a $p$-group satisfying $|G : Z(G)| = p^m$, then $|G'| \leq p^{m(m - 1)/2}$.

Proof. By induction on $m$.

Proof of Theorem 3.1. 1) Let $H$ be an abelian maximal subgroup of $G$, then $H$ does not satisfy the condition “N”, so $Z(G) \subset H$. Consequently $H/Z(G)$ is a subgroup of order $p$ of the elementary $p$-group $G/Z(G) \cong \mathbb{Z}/p\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z}$ by Lemma 3.2, there is $(p^2 - 1)(p^2 - 1) = p + 1$ such subgroups.

2) By Lemma 3.3, $|G : Z(G)| = p^2$ implies that $|G'| = p$. Let $H$ be a maximal subgroup of $G$, then $G' \subset H \subset G$ implies that $H/G'$ is a maximal subgroup of $G/G'$. By using the known result of Steinitz [5]: The number of subgroups of order $p^k$ equals the number of subgroups of order $p^{n-k}$ in a finite abelian group of order $p^n$, we conclude that the number of the maximal subgroups $H$ is equal to the number of subgroups of order $p$ of $G/G'$. If $rk(G/G') = r$, then the number of maximal subgroups satisfying
the condition “N” is \((p^r - 1)(p - 1) - (p + 1) = (p^r - p^2)(p - 1)\) by lemma 3.2.

3) If \(x \in Z(G)\), the conjugacy class of \(x\) is trivial, i.e., \(x^G = \{x\}\). If \(x \notin Z(G)\), then \(Z(G) \subsetneq C_G(x) \subsetneq G\) therefore \(|x^G| = |G : C_G(x)| = p\). Let \(k\) the number of nontrivial conjugacy classes, then \(|G| = p^n = p^{n-2} + kp\). Consequently \(k = p^{n-1} - p^{n-3}\) and \(N_c(G) = p^{n-1} + p^{n-2} - p^{n-3}\), so the proof is complete. □

**Corollary 3.4.** Let \(G\) be a \(p\)-group of order \(p^n\) such that \(|G : Z(G)| = p^2\). If \(G/G'\) is elementary \(p\)-group, then \(G\) has exactly \((p^{n-1} - 1)(p - 1)\) maximal subgroups.

**Proof.** This follows easily from Theorem 3.1. □

### 4. Examples

The following examples illustrate some applications of the previous results.

**Example 4.1.** Let \(G\) be a \(p\)-group of order \(p^3\), then

1) The number \(N\) of maximal subgroups is given in the following table

<table>
<thead>
<tr>
<th>(G)</th>
<th>(\frac{\mathbb{Z}}{p^3\mathbb{Z}})</th>
<th>(\frac{\mathbb{Z}}{p^2\mathbb{Z}} \times \frac{\mathbb{Z}}{p^2\mathbb{Z}})</th>
<th>(\frac{\mathbb{Z}}{p\mathbb{Z}} \times \frac{\mathbb{Z}}{p^2\mathbb{Z}} \times \frac{\mathbb{Z}}{p^2\mathbb{Z}})</th>
<th>(G) is nonabelian</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N)</td>
<td>1</td>
<td>1 + (p)</td>
<td>1 + (p + p^2)</td>
<td>1 + (p)</td>
</tr>
</tbody>
</table>

2) The number of conjugacy classes is \(N_c(G) = p^2 + p - 1\) (if \(G\) is nonabelian).

To prove the result 1) we consider the two cases

a) If \(G\) is abelian, we apply Lemma 2.5.

b) If \(G\) is nonabelian, \(|G : Z(G)| = p^2\) because the index of center does not equal to a prime, so \(G' = Z(G)\) and \(G\) is extra-special, [3]. Now \(G' \subset \text{Frat}(G) \subset G\) implies that \(|\text{Frat}(G)| = p^2\) or \(p\), the first case is impossible because, by Burnside Basis Theorem, [3], \(|G : \text{Frat}(G)| = p\) implies that \(G\) is generated by one element, that is, \(G\) is cyclic. Hence \(G' = Z(G) = \text{Frat}(G)\). If \(G/G'\) is not elementary \(p\)-group, then by Theorem 3.1 \(G\) has one maximal subgroup, so \(|\text{Frat}(G)| = p^2\), a contradiction. Thus \(G/G'\) is elementary \(p\)-group and the result is an immediate consequence of Corollary 3.4.
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2) Since \(|G : Z(G)| = p^2\), to calculate \(N_c(G)\) it is enough to apply Theorem 3.1 for \(n = 3\).

Example 4.2. Let \(G\) be a \(p\)-group of order \(p^4\) such that \(|G : Z(G)| = p^2\).

1) If \(G/G'\) is an elementary \(p\)-group, then \(G\) has exactly \(p + 1\) abelian maximal subgroups and \(p^2\) nonabelian maximal subgroups.

2) If \(G/G'\) is not an elementary \(p\)-group, \(G\) has \(p + 1\) maximal subgroups, all abelian.

In order to prove this, we consider two cases.

1) In the first place, if \(G/G'\) is an elementary \(p\)-group, then \(G\) has \(p^3 - 1\) maximal subgroups by Corollary 3.4. To calculate the number of nonabelian maximal subgroups we can apply the second assertion of Theorem 3.1.

2) Now assume that \(G/G'\) is not an elementary \(p\)-group, then the rank \(rk(G/G') = 1\) or \(2\), the first case implies that \(|G : Frat(G)| = p^1\) and \(G\) is cyclic. Hence \(rk(G/G') = 2\), by Theorem 3.1, \(G\) has not a nonabelian maximal subgroup and it has exactly \(p + 1\) maximal subgroups, all abelian, as required.

Example 4.3. Let \(G\) be a \(p\)-group of order \(p^4\) such that \(|G : Z(G)| = p^3\). Then

1) \(G\) has one maximal abelian subgroup and \(p\) nonabelian maximal subgroups.

2) \(G\) has exactly \(2p^2 - 1\) conjugacy classes.

In order to prove this result, we first note that every element \(x\) of \(G\) belongs to a maximal subgroup. Second, we establish two Lemmas.

Lemma 4.4. Let \(G\) be a \(p\)-group of order \(p^4\) such that \(|G : Z(G)| = p^3\). Then

\[ |x^G| = p \iff x \in M - Z(G), \]

where \(M\) is an abelian maximal subgroup of \(G\).

Proof of Lemma 4.4. If \(|x^G| = p\), then \(|C_G(x)| = p^3\), let \(M = C_G(x)\). Since \(G\) is a finite nilpotent group, \(M \triangleleft G\) and \(G/M\) is abelian, consequently \(G' \subset M\). If \(G/Z(G)\) is abelian, then \(G' \subset Z(G) \Rightarrow G' = Z(G)\), so \(G\) is extra-special and \(|G| = p^{2k+1}\), [3], a contradiction. Hence \(G/Z(G)\) is not abelian, since \(G\) is nilpotent, \(G' \cap Z(G) \neq \{1\}\), so \(Z(G) \nsubseteq G' \nsubseteq M\) and \(|G'| = p^2\). Assume that \(M\) is not abelian, then \(Z(G) \subset Z(M) \subset M\).
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Since the index of center does not equal to a prime, $Z(G) = Z(M)$. If $y \in M$, $yx = xy$ so $x \in Z(M)$ and we reach the contradiction $x \in Z(G)$. Hence $M$ is abelian. Conversely, let $M$ be an abelian maximal subgroup of $G$ and $x \in M - Z(G)$. If $|x^G| = p^2$, then $|C_G(x)| = p^2$. Since $M$ is abelian, $M \subset C_G(x) \subset G$, so $M = C_G(x)$ and $|C_G(x)| = p^3$, by this contradiction we obtain $|x^G| = p$. □

Lemma 4.5. Let $G$ be a $p$–group of order $p^4$ such that $|G : Z(G)| = p^3$ and let $M$ be a maximal subgroup of $G$. Then

1) If $M$ is abelian, $M$ contains exactly $p^2 - 1$ nontrivial conjugacy classes which has $p$ elements.

2) If $M$ is not abelian, $M - G'$ contains exactly $p - 1$ nontrivial conjugacy classes which has $p^2$ elements.

Proof of Lemma 4.5. 1) Assume that $M$ is abelian. Let $x \in M - Z(G)$, since $M \triangleleft G$, $x^G \subset M$. By Lemma 13, $|x^G| = p$, consequently $G$ has

$$\frac{p^3 - p}{p} = p^2 - 1$$

nontrivial conjugacy classes which has $p$ elements.

2) If $M$ is not abelian, let $x \in M - G'$. From Lemma 13 it follows that $|x^G| = p^2$, so $M - G'$ has exactly $\frac{p^3 - p^2}{p^2} = p - 1$ nontrivial conjugacy classes which has $p^2$ elements. □

We will prove the last result as following. If $M_1$ and $M_2$ are two maximal subgroups in $G$, it is clear that $M_1 \cap M_2 = G'$. We denote by $k_a$ (respectively $k_{na}$) the number of abelian (respectively nonabelian) maximal subgroups in $G$. If $x \in G' - Z(G)$, $x^G \subset G'$, so $|x^G| = p$ and $|C_G(x)| = p^3$, we have shown in the proof of Lemma 4.4 that $C_G(x)$ is abelian, consequently $k_a \neq 0$.

Let $M_1, M_2$ be two abelian maximal subgroups of $G$. Let $x \in G' - Z(G)$, then $G' \subset M_1 \subset G$ and by Lemma 4.4, $|x^G| = p$.

Let $x \in M_1$. Since $M_1$ is abelian, $M_1 \subset C_G(x) \subset G$, so $M_1 = C_G(x)$. By the same way we obtain $M_2 = C_G(x)$. Hence $M_1 = M_2$ and $k_a = 1$. Each maximal subgroup $M$ satisfy $G' \subset M \subset G$, so $M/G'$ is a subgroup of order $p$ of the group $G/G' \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. By Lemma 3.2, $G$ has $p + 1$ maximal subgroups, so $k_{na} = p$.

2) By using Lemma 4.4 and Lemma 4.5 and the first assertion of Example 4.2, we obtain $N_c(G) = p + (p^2 - 1) + p(p - 1) = 2p^2 - 1$. 274
Remark 4.6. In [1], M. Reid proved that if $G$ is a finite group whose order is not divisible by 3, and $G$ has $m$ conjugacy classes, then the congruence $|G| \equiv m \mod 3$ holds. With the hypotheses of Theorem 3.1, we have the congruence $|G| \equiv N_c(G) \mod 6$ because
\[ |G| - N_c(G) = p^n - p^{n-1} - p^{n-2} + p^{n-3} = p^{n-3}(p - 1)^2(p + 1). \]

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References


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