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Properties of subgroups not containing their centralizers

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Abstract

In this paper, we give a generalization of Baer Theorem on the injective property of divisible abelian groups. As consequences of the obtained result we find a sufficient condition for a group G to express as semi-direct product of a divisible subgroup D and some subgroup H. We also apply the main Theorem to the p-groups with center of index p^2, for some prime p. For these groups we compute $N_c(G)$ the number of conjugacy classes and N_a the number of abelian maximal subgroups and N_{na} the number of nonabelian maximal subgroups.

1. Introduction

We shall recall some definitions:

If H is a subgroup of a group G, a subgroup K is called a complement of H in G if $G = HK$ and $H \cap K = \{1\}$. Therefore if $H \lhd G$ and $K \lhd G$, then G is said to be the direct product of H and K, in symbols, $G = H \odot K$.

If $H \lhd G$, then G is said to be the semi-direct product of H and K, in symbols, $G = H \rtimes K$.

An abelian group D is called divisible if for every $x \in D$ and every positive integer n there is a $y \in D$ so that $x = ny$, i.e., each element of D is divisible by every positive integer. The main property of divisible groups is that they satisfy the following “injectivity" condition:

Theorem 1.1 (Baer Theorem [3])**.** *If D is a divisible group, then any homomorphism $f : A \to D =$ from any abelian group A into D extends to any abelian group G which contains A i.e., there exists a homomorphism $\overline{f} : G \to D$ so that $\overline{f}_{|A} = f$.*

The purpose of this paper is to generalize this result to the nonabelian groups. To this end, we introduce the property “N” in subgroups: Let H be a subgroup of an arbitrary group G.

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H satisfies the condition “N" if and only if $\exists g \in G - H \quad [g,\ H] = 1$.

This is equivalent to saying that $C_G(H) \nsubseteq H$ where $C_G(H)$ is the centralizer of H in G which is defined to be the set of all g in G such that $hg = gh$ for all h in H, it is clearly a subgroup of G.

By the definition of the condition “N” we deduce that

1) If G is abelian, then every proper subgroup H satisfies the condition "N".

2) If G is a nonabelian nilpotent group, then every maximal normal abelian subgroup H of G does not satisfy the condition “N” because $C_G(H) = H$, [3].

3) There exist a nonabelian groups G whose a subgroup H satisfies the condition“N”, for example let $G = Q_8 \times \mathbb{Z}/2\mathbb{Z}$ where $H = Q_8$ is the quaternion group of order 8, [2].

2. Main results and proofs

Theorem 2.1. *Let G be a group and let H be a subgroup of G such that each proper subgroup H' of G which contains H, satisfies the condition “N”. Then any homomorphism $f : H \to D$ from H into divisible group D extends to the group G.*

Proof. Let us consider the set S of all pairs $(H_i, f_i\ )$ where H_i is a subgroup of G containing H and $f_i : H_i \to D$ is an extension of f, i.e., $f_{|H_i} = f_i$. Let $(H_i, f_i) \leq (H_j, f_j)$ if $H_i \subset H_j$ and $f_{j|H_i} = f_i$. The set S is partially ordered by the relation $\leq$. We aim to apply Zorn’s Lemma to S and to this end we consider a chain $(H_i, f_i)_{i\in I}$. It has an upper bound $\left(\cup_{i\ \in I} H_i, f\ '\right)$ where $f\ ' : \cup_{i\ \in I} H_i \to D$ is defined by $f'(x_i) = f_i(x_i)$ for every $x_i \in H_i$. This is unambiguous since $x_i \in H_i \subset H_m \Rightarrow f_m\ (x_i) = f_i\ (x_i)$. Consequently, by Zorn’s Lemma, S has a maximal element, say $(\overline{H},\ \overline{f}\ )$. We claim that $\overline{H} = G$ and $\overline{f}$ is the desired extension of f to G. To see this suppose $\overline{H} \neq G$. By hypotheses $\overline{H}$ satisfies the condition “N”, consequently, there is an $g \in G - \overline{H}$ such that $\left[g,\ \overline{H}\right] = 1$, therefore $\left\langle \overline{H}, g\right\rangle = \overline{H} \cdot \langle\ g\ \rangle$. There are two cases:

Case 1.

$\overline{H} \cap \langle\ g\ \rangle \neq \{0\}$. Let n the smallest positive integer so that $g^n = u \in \overline{H}$. Since D is divisible there is a $x \in D$ so that $\overline{f}(u) = nx$. Let $\overline{f}^* : \overline{H} \cdot \langle\ g\ \rangle \to D$ be defined by $\overline{f}\ ^*(ag^t) = \overline{f}(a) + tx$ where $a \in \overline{H}$ and $0 \leq t \prec n$, if

$ag^t = a'g^{t'}$ then $a'^{-1}a = g^{t'-t} \in H$, so $t' - t = 0$ and $a = a'$. Hence $\overline{f}^{\,*}$ is well-defined mapping. Let $z_1 = a_1 g^{t_1}$, $z_2 = a_2 g^{t_2}$ be two elements of the group $\overline{H} \cdot \langle\, g\,\rangle$, then $\overline{f}^*(z_1) + \overline{f}^{\,*}(z_2) = \overline{f}\,(a_1a_2) + (\,t_1 + t_2\,)x$ and $t_1 + t_2 = kn + t_0$ where $0 \leq t_0 \prec n$. In the other hand $\overline{f}^{\,*}(z_1z_2) = \overline{f}^{\,*}(a_1g^{t_1}a_2g^{t_2})$, since $\left[g,\ \overline{H}\right] = 1$, $\overline{f}^{\,*}(z_1z_2) = \overline{f}^{\,*}(a_1a_2g^{t_1+t_2}) = \overline{f}^{\,*}(a_1a_2u^kg^{t_0})$, so $\overline{f}^{\,*}(z_1z_2) = \overline{f}\,(a_1a_2) + (\,kn + t_0\,)x$. Finally $\overline{f}^{\,*}(z_1z_2) = \overline{f}^{\,*}(z_1) + \overline{f}^{\,*}(z_2)$, so $\overline{f}^{\,*}$ is a homomorphism so that $\overline{f}^{\,*}_{|\overline{H}} = \overline{f}$. This contradicts the maximality of $(\overline{H},\ \overline{f})$.

Case 2.

$\overline{H} \cap \langle\, g\,\rangle = \{0\}$. Then $\overline{H} \cdot \langle\, g\,\rangle = \overline{H} \odot \langle\, g\,\rangle$. Let x_0 be an element of D, in this case we can define $\overline{f}^{\,*} : \overline{H} \odot \langle\, g\,\rangle \to D$ by writing $\overline{f}^{\,*}(ag^t) = \overline{f}(a) + kx_0$, it is easy to verify that $\overline{f}^*$ is a homomorphism so that $\overline{f}^{\,*}_{|\overline{H}} = \overline{f}$ contradicting the maximality of $(\overline{H}, \overline{f})$. Thus $\overline{H} = G$ and $\overline{f}$ is the desired extension of f to G. □

Since every subgroup of an abelian group satisfies the condition"N", we can apply Theorem 2.1 to abelian groups, also we deduce the known result, [3]:

Corollary 2.2. *Any divisible subgroup D of an abelian group G splits, i.e., D has a complement H so that $G = H \oplus D$.*

If G is a group (not necessarily abelian), we write:

Corollary 2.3. *Let G be a group and let D be a divisible subgroup of G such that every subgroup H of G which contains D, satisfies the condition "N". Then D has a complement H so that $G = H \rtimes D$.*

Proof. We consider the identity map: $id_D : D \to D$, by Theorem 2.1, id_D extends to the group G, i.e., there exists a homomorphism $\overline{f}$: $G \to D$ so that $\overline{f}_{|D} = id_D$. Let $i : D \to G$ be the inclusion map, then $\overline{f} \circ i = id_D$ implies that $\overline{f}(G) = D$. Let $H = Ker\overline{f}$, if $x \in G$, then $x = x\overline{f}\,(x^{-1})\overline{f}\,(x)$. Since $\overline{f}\,\left[x\overline{f}\,(x^{-1})\right] = \overline{f}\,(x).\overline{f}\,\circ \overline{f}\,(x^{-1})$ and $\overline{f}_{|D} = id_D$, we have $\overline{f}\,\left[x\overline{f}\,(x^{-1})\right] = 1$, so $G = H.D$. If x belongs to $H \cap D$, then $x = \overline{f}\,(x')$ and $\overline{f}\,(x) = 1$, thus $\overline{f}\,(x) = \overline{f}\,\circ \overline{f}\,(x') = \overline{f}\,(x')$, that is, $H \cap D = \{1\}$. Hence $G = H \rtimes D$. □

Corollary 2.4. *Let G be a finite p-group with center of index p^2. If H is a nonabelian maximal subgroup of G then any homomorphism $f : H \to D$ from H into divisible group D extends to the group G.*

To prove Corollary 2.4, we need the following.

Lemma 2.5. *Let G be a finite p-group such that its center $Z(G)$ has index p^2. If H is a maximal subgroup of G, then the following properties are equivalent.*
i)H is abelian
ii) $Z(G) \subset H$
iii) H does not satisfy the condition "N".

Proof of Lemma 2.5. "*i*) $\Rightarrow$ *ii*)". Let us assume that H is abelian. If $Z(G) \not\subseteq H$, there exists $g \in Z(G) - H$ and $G = H \cdot \langle\ g\ \rangle$. Then G is abelian, this contradicts $|G : Z(G)| = p^2$.

"*ii*) $\Rightarrow$ *iii*)". Assume that $Z(G) \subset H$. Then $Z(G) \subset Z(H) \subset H \subset G$. By hypothesis $|G : Z(G)| = p^2$. Since the center does not have a prime index and $|G : H| = p$, $Z(H) = H$, consequently H is abelian. Hence H is a maximal normal abelian subgroup of the nilpotent group G, so $C_G(H) = H$, [3], and H does not satisfy the condition "N".

"*iii*) $\Rightarrow$ *i*)". If $C_G(H) \subset H$, then $Z(G) \subset Z(H) \subset H$. By the same way we deduce that H is abelian. □

Proof of Corollary 2.4. Let H be a subgroup of G so that $H \subset H'$, since H is nonabelian, by Lemma 2.5, there is $g \in G - H$ so that $[g,\ H] = 1$. Then $[g,\ H'] = 1$ by maximality of H. Thus the conditions of Theorem 2.1 are satisfied, so we obtain Corollary 2.4. □

3. Subgroups satisfying the condition "N"

If A is finitely generated abelian group, the rank of A is defined by $rk(A)$ the minimum number of generators of A.

We denote us by x^G the conjugacy class of x in an arbitrary group G and $C_G(x)$ the centralizer of x in G and $N_c(G)$ the number of the conjugacy classes.

If G is a finite p-group of class c, then from [4], we know that

$$N_c(G) \geq c\,|G\,|^{1/c} - c + 1.$$

Let G be a finite p-group of order p^n such that its center has index p^2. In this section, we compute the number $N_c(G)$ and N_0 the number of maximal subgroups in G satisfying the condition "N".

Theorem 3.1. *Let G be a finite p-group of order p^n such that its center has index p^2, then*

1) G has precisely $p+1$ abelian maximal subgroups.

2) The number of maximal subgroups satisfying the condition "N" equals $N_0 = (p^r - p^2)/(p-1)$ where r is the rank of $G/G' = \mathbb{Z}/p^{n_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{n_r}\mathbb{Z}$.

3) $N_c(G) = p^{n-1} + p^{n-2} - p^{n-3}$ and each nontrivial conjugacy class has p elements.

The proof of Theorem 3.1 results from the following Lemmas.

Lemma 3.2. *Let G be an abelian finite $p-$ group, then the number of subgroups of order p equals $(p^r - 1)/(p-1)$ where r is the rank of G.*

Proof of Lemma 3.2. Since $rk(G) = r$, G is isomorphic with the group $\mathbb{Z}/p^{n_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{n_r}\mathbb{Z}$. If g is an element of order p in G, then $g = g_1 g_2 \ldots g_r$ such that g_i has order p or 1 and $g \neq 1$. The number of such elements g equals

$$(p-1) \cdot C_r^1 + (p-1)^2 C_r^2 + \cdots + (p-1)^r C_r^r = p^r - 1.$$

Since a group of order p has $p-1$ elements of order p, the number of subgroups of order p is $(p^r - 1)(p-1)$. ☐

Lemma 3.3. *Let G be a $p-$group satisfying $|G : Z(G)| = p^m$, then $|G'| \leq p^{m(m-1)/2}$.*

Proof. By induction on m. ☐

Proof of Theorem 3.1. 1) Let H be an abelian maximal subgroup of G, then H does not satisfy the condition "N", so $Z(G) \subset H$. Consequently $H/Z(G)$ is a subgroup of order p of the elementary $p-$ group $G/Z(G) \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ by Lemma 3.2, there is $(p^2-1)(p-1) = p+1$ such subgroups.
2) By Lemma 3.3, $|G : Z(G)| = p^2$ implies that $|G'| = p$. Let H be a maximal subgroup of G, then $G' \subset H \subset G$ implies that H/G' is a maximal subgroup of G/G'. By using the known result of Steinitz [5]: The number of subgroups of order p^k equals the number of subgroups of order p^{n-k} in a finite abelian group of order p^n, we conclude that the number of the maximal subgroups H is equal to the number of subgroups of order p of G/G'. If $rk(G/G') = r$, then the number of maximal subgroups satisfying

the condition "N" is $(p^r-1)(p-1)-(p+1) = (p^r-p^2)(p-1)$ by lemma 3.2.
3) If $x \in Z(G)$, the conjugacy class of x is trivial, i.e, $x^G = \{x\}$. If $x \notin Z(G)$, then $Z(G) \subsetneq C_G(x) \subsetneq G$ therefore $\left|x^G\right| = |G : C_G(x)| = p$. Let k the number of nontrivial conjugacy classes, then $|G| = p^n = p^{n-2} + kp$. Consequently $k = p^{n-1} - p^{n-3}$ and $N_c(G) = p^{n-1} + p^{n-2} - p^{n-3}$, so the proof is complete. □

Corollary 3.4. *Let G be a $p-$group of order p^n such that $|G : Z(G)| = p^2$. If G/G' is elementary $p-$group, then G has exactly $(p^{n-1}-1)(p-1)$ maximal subgroups.*

Proof. This follows easily from Theorem 3.1. □

4. **Examples**

The following examples illustrate some applications of the previous results.

Example 4.1. Let G be a $p-$group of order p^3, then
1) The number N of maximal subgroups is given in the following table

G	$\frac{\mathbb{Z}}{p^3\mathbb{Z}}$	$\frac{\mathbb{Z}}{p\mathbb{Z}} \times \frac{\mathbb{Z}}{p^2\mathbb{Z}}$	$\frac{\mathbb{Z}}{p\mathbb{Z}} \times \frac{\mathbb{Z}}{p\mathbb{Z}} \times \frac{\mathbb{Z}}{p\mathbb{Z}}$	G is nonabelian
N	1	$1+p$	$1+p+p^2$	$1+p$

2) The number of conjugacy classes is $N_c(G) = p^2 + p - 1$ (if G is nonabelian).

To prove the result 1) we consider the two cases
a) If G is abelian, we apply Lemma 2.5.
b) If G is nonabelian, $|G : Z(G)| = p^2$ because the index of center does not equal to a prime, so $G' = Z(G)$ and G is extra-special, [3]. Now $G' \subset$ Frat(G) $\subset G$ implies that $|\text{Frat}(G)| = p^2$ or p, the first case is impossible because, by Burnside Basis Theorem, [3], $|G : \text{Frat}(G)| = p$ implies that G is generated by one element, that is, G is cyclic. Hence $G' = Z(G) = \mathrm{F}rat(G)$. If G/G' is not elementary $p-$group, then by Theorem 3.1 G has one maximal subgroup, so $|\mathrm{F}rat(G)| = p^2$, a contradiction. Thus G/G' is elementary $p-$group and the result is an immediate consequence of Corollary 3.4.

2) Since $|G : Z(G)| = p^2$, to calculate $N_c(G)$ it is enough to apply Theorem 3.1 for $n = 3$.

Example 4.2. Let G be a $p-$group of order p^4 such that $|G : Z(G)| = p^2$.

1) If G/G' is an elementary $p-$group, then G has exactly $p+1$ abelian maximal subgroups and p^2 nonabelian maximal subgroups.

2) If G/G' is not an elementary $p-$ group, G has $p+1$ maximal subgroups, all abelian.

In order to prove this, we consider two cases.

1) In the first place, if G/G' is an elementary $p-$ group, then G has $\frac{p^3-1}{p-1} = p^2+p+1$ maximal subgroups by Corollary 3.4. To calculate the number of nonabelian maximal subgroups we can apply the second assertion of Theorem 3.1.

2) Now assume that G/G' is not an elementary $p-$group, then the rank $rk(G/G') = 1$ or 2, the first case implies that $|G : Frat(G)| = p^1$ and G is cyclic. Hence $rk(G/G') = 2$, by Theorem 3.1, G has not a nonabelian maximal subgroup and it has exactly $p+1$ maximal subgroups, all abelian, as required.

Example 4.3. Let G be a $p-$group of order p^4 such that $|G : Z(G)| = p^3$. Then

1) G has one maximal abelian subgroup and p nonabelian maximal subgroups.

2) G has exactly $2p^2 - 1$ conjugacy classes.

In order to prove this result, we first note that every element x of G belongs to a maximal subgroup. Second, we establish two Lemmas.

Lemma 4.4. *Let G be a $p-$group of order p^4 such that $|G : Z(G)| = p^3$. Then*

$$\left|x^G\right| = p \Longleftrightarrow x \in M - Z(G),$$

where M is an abelian maximal subgroup of G.

Proof of Lemma 4.4. If $\left|x^G\right| = p$, then $|C_G(x)| = p^3$, let $M = C_G(x)$. Since G is a finite nilpotent group, $M \lhd G$ and G/M is abelian, consequently $G' \subset M$. If $G/Z(G)$ is abelian, then $G' \subset Z(G) \Rightarrow G' = Z(G)$, so G is extra-special and $|G| = p^{2k+1}$, [3], a contradiction. Hence $G/Z(G)$ is not abelian, since G is nilpotent, $G' \cap Z(G) \neq \{1\}$, so $Z(G) \subsetneq G' \subsetneq M$ and $|G'| = p^2$. Assume that M is not abelian, then $Z(G) \subset Z(M) \subset M$.

Since the index of center does not equal to a prime, $Z(G) = Z(M)$. If $y \in M$, $yx = xy$ so $x \in Z(M)$ and we reach the contradiction $x \in Z(G)$. Hence M is abelian. Conversely, let M be an abelian maximal subgroup of G and $x \in M - Z(G)$. If $\left|x^G\right| = p^2$, then $|C_G(x)| = p^2$. Since M is abelian, $M \subset C_G(x) \subsetneq G$, so $M = C_G(x)$ and $|C_G(x)| = p^3$, by this contradiction we obtain $\left|x^G\right| = p$. □

Lemma 4.5. *Let G be a $p-$group of order p^4 such that $|G : Z(G)| = p^3$ and let M be a maximal subgroup of G. Then*

1) If M is abelian, M contains exactly p^2-1 nontrivial conjugacy classes which has p elements.

2) If M is not abelian, $M-G'$ contains exactly $p-1$ nontrivial conjugacy classes which has p^2 elements.

Proof of Lemma 4.5. 1) Assume that M is abelian. Let $x \in M - Z(G)$, since $M \lhd G$, $x^G \subset M$. By Lemma 13, $\left|x^G\right| = p$, consequently G has $\frac{p^3-p}{p} = p^2 - 1$ nontrivial conjugacy classes which has p elements.

2) If M is not abelian, let $x \in M - G'$. From Lemma 13 it follows that $\left|x^G\right| = p^2$, so $M-G'$ has exactly $\frac{p^3-p^2}{p^2} = p-1$ nontrivial conjugacy classes which has p^2 elements. □

We will prove the last result as following. If M_1 and M_2 are two maximal subgroups in G, it is clear that $M_1 \cap M_2 = G'$. We denote by k_a (respectively k_{na}) the number of abelian (respectively nonabelian) maximal subgroups in G. If $x \in G' - Z(G)$, $x^G \subset G'$, so $\left|x^G\right| = p$ and $|C_G(x)| = p^3$, we have shown in the proof of Lemma 4.4 that $C_G(x)$ is abelian, consequently $k_a \neq 0$.

Let M_1, M_2 be two abelian maximal subgroups of G. Let $x \in G' - Z(G)$, then $G' \subset M_1 \subsetneq G$ and by Lemma 4.4, $\left|x^G\right| = p$.

Let $x \in M_1$. Since M_1 is abelian, $M_1 \subset C_G(x) \subsetneq G$, so $M_1 = C_G(x)$. By the same way we obtain $M_2 = C_G(x)$. Hence $M_1 = M_2$ and $k_a = 1$. Each maximal subgroup M satisfy $G' \subset M \subset G$, so $\frac{M}{G'}$ is a subgroup of order p of the group $G/G' \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. By Lemma 3.2, G has $p+1$ maximal subgroups, so $k_{na} = p$.

2) By using Lemma 4.4 and Lemma 4.5 and the first assertion of Example 4.2, we obtain $N_c(G) = p + (p^2 - 1) + p(p - 1) = 2p^2 - 1$.

Remark 4.6. In [1], M. Reid proved that if G is a finite group whose order is not divisible by 3, and G has m conjugacy classes, then the congruence $|G| \equiv m \bmod 3$ holds. With the hypotheses of Theorem 3.1, we have the congruence $|G| \equiv N_c(G) \bmod 6$ because

$$|G| - N_c(G) = p^n - p^{n-1} - p^{n-2} + p^{n-3} = p^{n-3}(p-1)^2(p+1).$$

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References

[1] M. Reid – The number of conjugacy classes, *Amer. Math. Monthly* **105** (1998), no. 4, p. 359–361.

[2] D. J. S. Robinson – *Finiteness conditions and generalized soluble groups. Part 1*, Springer-Verlag, New York, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 62.

[3] ———, *A course in the theory of groups*, second éd., Graduate Texts in Mathematics, vol. 80, Springer-Verlag, New York, 1996.

[4] G. Sherman – A lower bound for the number of conjugacy classes in a finite nilpotent group, *Pacific J. Math.* **80** (1979), no. 1, p. 253–254.

[5] N. Watson – Subgroups of finite abelian groups, 1995, Summer research paper, Haverford College.

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