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Bounds For Étale Capitulation Kernels II


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Abstract

Let $p$ be an odd prime and $E/F$ a cyclic $p$-extension of number fields. We give a lower bound for the order of the kernel and cokernel of the natural extension map between the even étale $K$-groups of the ring of $S$-integers of $E/F$, where $S$ is a finite set of primes containing those which are $p$-adic.

Bornes pour les noyaux de capitulations II

Résumé

Soit $p$ un nombre premier impair et $E/F$ une $p$-extension cyclique de corps de nombres. Nous donnons une minoration pour l’ordre du noyau et conoyau de l’application naturelle d’extension entre les $K$-groupes étalés des anneaux de $S$-entiers de $E/F$ où $S$ est un ensemble fini de places contenant les places $p$-adiques.

1. Introduction

Let $F$ be an algebraic number field and let $p$ be an odd prime number. For a finite set $S$ of primes of $F$ containing the primes above $p$, let $\mathcal{O}_F^S$ denote the ring of $S$-integers of $F$. For a Galois $p$-extension $E$ of $F$ with Galois group $G$ which is unramified outside $S$, the kernel and the cokernel of the natural functorial map between the even étale $K$-groups $f_i : K_{2i}^{\text{ét}}(\mathcal{O}_F^S) \longrightarrow (K_{2i-2}^{\text{ét}}(\mathcal{O}_F^S))^G$ are described by the cohomology of odd étale $K$-groups $K_{2i-1}^{\text{ét}}(\mathcal{O}_E^S)$. So using Borel’s results on the abelian group structure of odd $K$-groups, one can give an upper bound for the rank of the finite $p$-groups $\text{ker}(f_i)$ and $\text{coker}(f_i)$, as explained by B. KAHN [8, section 4], by means of the number of real and complex embeddings of the number field $F$. In [1], partially answering a question asked by B. KAHN loc.cit., we gave a lower bound for the order of $\text{ker}(f_i)$ and $\text{coker}(f_i)$, in

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the case where the extension \( E/F \) is cyclic of degree \( p \) in terms of tamely ramified primes. Our purpose in the present paper is to similarly treat the case where \( E/F \) is cyclic of degree \( p^n, n \geq 1 \).

When the number field \( F \) contains a primitive \( p \)-th root of unity \( \zeta_p \), the classical Tate kernel \( D_F \) consists of the non-zero elements \( a \) of \( F \), such that the symbol \( \{ a, \zeta_p \} \) is trivial in \( K_2 F \). Obviously, \( D_F \) lies between \( F^\times \), the multiplicative group of non zero elements of \( F \) and \( F^\times p \).

Answering a question raised by J. COATES [2], R. GREENBERG showed that even though in general \( A_F \neq D_F \), they coincide when the base field \( F \) contains enough \( p \)-primary roots of unity [4].

More generally, when \( F \) contains the \( p^n \)-th roots of unity, for each integer \( i \geq 2 \), there exists a subgroup \( D_F^{(i,n)} \) of \( F^\times \) containing \( F^\times p^n \), such that \( K_{2i-1} F/p^n \cong D_F^{(i,n)}/F^\times p^n \), and the order of coker(\( f_i \)) is minorized by the norm index in the generalized Tate kernel \( D_F^{(i,n)} \) (Proposition 2.1). Following Greenberg’s method, one can show that, once again under Leopoldt’s conjecture, \( D_F^{(i,n)} \) turns out to be the Kummer radical \( A_F^{(n)} \) of the compositum of the \( n \)-th layers of \( \mathbb{Z}_p \)-extensions of \( F \), provided \( F \) contains enough \( p \)-primary roots of unity. We then obtain our lower bound by minorizing the norm index \( [A_F^{(n)} : A_F^{(n)} \cap N_E/F(E^\times)] \) in terms of the ramification indices in \( E/F \) of non-\( p \)-adic primes belonging to the same "primitive" set for \( (F,p) \) (Proposition 4.3).

At the end of the paper, we treat the case where the base field \( F \) is "\( p \)-regular" and all the tamely ramified primes in \( E/F \) belong to the same primitive set. In particular, we show that there are infinitely many cyclic extensions \( E/F \) of degree \( p^n \), such that the order of the kernel (or the cokernel) takes any prescribed value between 1 and the trivial upper bound \( p^n(1+r_2) \).

2. A lower bound via the Tate kernel

Suppose that \( E/F \) is a cyclic extension of degree \( p^n \) with Galois group \( G \), and that \( F \) contains the \( p^n \)-th roots of unity \( \mu_{p^n} \). Denote by \( S \) the set
of p-adic primes, as well as those which ramify in $E/F$. Throughout this paper $i$ is an integer $\geq 2$. The exact sequence

$$0 \to \mathbb{Z}_p(i) \to \mathbb{Z}_p(i) \to \mathbb{Z}/p^n \mathbb{Z}(i) \to 0$$

induces an injection

$$K_{2i-1}^{\text{ét}} F/p^n \cong H^1(F, \mathbb{Z}_p(i))/p^n$$

$$\hookrightarrow H^1(F, \mathbb{Z}/p^n \mathbb{Z}(i))$$

$$= H^1(F, \mu_{p^n})(i - 1)$$

$$\cong F^\bullet/F^{\bullet}p^n(i - 1),$$

where $H^1(F, \cdot)$ denotes the first continuous cochain cohomology group of the absolute Galois group $G_F$ of $F$ and, for any $G_F$-module $M$, the notation $M(i)$ is the $i$-fold Tate twisted module $M[14]$.

Thus there exists a subgroup $D_{F}^{(i,n)}$ of $F^\bullet$ containing $F^\bullet p^n$ - the analogue of the Tate-kernel in the case of $i = 2$ and $n = 1$ -, such that

$$K_{2i-1}^{\text{ét}} F/p^n \cong (D_{F}^{(i,n)}/F^\bullet p^n)(i - 1).$$

Since the odd étale $K$-groups satisfy Galois descent, we have [1, Section 1]:

$$\text{coker}(f_i) \cong (K_{2i-1}^{\text{ét}} F/p^n)/N_{E/F}(K_{2i-1}^{\text{ét}} E/p^n)$$

$$\cong D_{F}^{(i,n)}/F^\bullet p^n N_{E/F}(D_{E}^{(i,n)})(i - 1).$$

Since $F^\bullet p^n N_{E/F}(D_{E}^{(i,n)}) \subset D_{F}^{(i,n)} \cap N_{E/F}(E^\bullet)$, we have the following lower bound for the order of the kernel or the cokernel of the natural natural functorial map between the even étale $K$-groups

$$f_i : K_{2i-2}^{\text{ét}}(\sigma_F^S) \longrightarrow (K_{2i-2}^{\text{ét}}(\sigma_E^S))^G$$

(with $G$ is cyclic, the Herbrand quotient $h(G, K_{2i-1}^{\text{ét}} (\sigma_E^S))$ is trivial, so that $\ker(f_i)$ and $\text{coker}(f_i)$ have the same order):

**Proposition 2.1.** Let $E/F$ be a cyclic extension of degree $p^n$ of algebraic number fields containing $\mu_{p^n}$. Then

$$|\text{coker}(f_i)| = |\ker(f_i)| \geq [D_{F}^{(i,n)} : D_{F}^{(i,n)} \cap N_{E/F}(E^\bullet)].$$

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A detailed account of these generalized Tate kernels $D_{F}^{(i,n)}$ can be found in [6, 15], see also [9] for the case $n = 1$.

3. Tate kernel and Kummer radical

In this section, we fix a positive integer $n$ and assume that our base number field $F$ contains the $p^n$-th roots of unity $\mu_{p^n}$. Let $\mu_{p^\infty} := \cup_{m \geq 1} \mu_{p^m}$ be the group of all $p$-primary roots of unity and $F_\infty := F(\mu_{p^\infty})$ be the cyclotomic $\mathbb{Z}_p$-extension of $F$. Denote by $F_n$ the $n$-th layer in $F_\infty$ and by $\Gamma$ the Galois group $\text{Gal}(F_\infty/F)$. Fix a topological generator $\gamma$ of $\Gamma$ in order to identify the Iwasawa algebra $\mathbb{Z}_p[[\Gamma]]$ with the power series algebra $\Lambda := \mathbb{Z}_p[[T]]$.

Let $K \colonequals F_\infty^\bullet \otimes \mathbb{Q}_p/\mathbb{Z}_p$, considered as a discrete group on which $\Gamma$ acts through the first factor. Let $\tilde{F}$ be the compositum of all $\mathbb{Z}_p$-extensions of $F$ and $A_{\tilde{F}}^{(n)} = \{a \in F^\bullet/F(\sqrt[p^n]{a}) \subset \tilde{F}\}$ be the Kummer radical of the compositum of the $n$-th layers of the $\mathbb{Z}_p$-extensions of $F$.

Following Greenberg [4],

$$A_{F}^{(n)} = \{a \in F^\bullet/a \otimes (p^{-n} \mod \mathbb{Z}_p) \in \text{Div}(K(-1)^\Gamma)\}$$

and one can establish as in [1, page 204] that for all $i \geq 2$

$$D_{F}^{(i,n)} = \{a \in F^\bullet/a \otimes (p^{-n} \mod \mathbb{Z}_p) \in \text{Div}(K(i - 1)^\Gamma)\}.$$ 

Here $\text{Div}$ stands for the maximal divisible subgroup.

Let $K_\infty$ be the maximal abelian pro-$p$-extension of $F_\infty$. Kummer theory yields a perfect pairing [7, Section 7]

$$\text{Gal}(K_\infty/F_\infty) \times K \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p, \quad (\sigma, a \otimes (p^{-m} \mod \mathbb{Z}_p)) \longmapsto \sigma(\mu_{p^\infty}/(\sqrt[p^m]{a})/\sqrt[p^m]{a}).$$

Now let $M_\infty$ be, as usual, the maximal abelian pro-$p$-extension of $F_\infty$ unramified outside $p$ and $\mathcal{X}_\infty := \text{Gal}(M_\infty/F_\infty)$. Let $N_\infty$ be the subfield of $M_\infty$ fixed by the torsion submodule $\text{Tor}_\Lambda(\mathcal{X}_\infty)$. Denote by $\mathcal{N}$ the subgroup of $K$ corresponding to the field $N_\infty$ by the above pairing. For every integer $i$, we then have a perfect pairing

$$\mathcal{X}(-i) \times \mathcal{N}(i - 1) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p,$$

where $X := \text{Fr}_\Lambda \mathcal{X}_\infty = \text{Gal}(N_\infty/F_\infty)$ is the maximal torsion-free quotient of $\mathcal{X}_\infty$.  

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It is well known that $X$ is a submodule of $\Lambda^{r_2}$ of finite index. The quotient module $H_F := \Lambda^{r_2}/X$ is isomorphic as an abelian group to the kernel of the natural map $K_2 F_n \to K_2 F_\infty$, for $n$ large [2]. The exponent of the finite group $H_F$ will play an important role in what follows and will be henceforth denoted by $p^e$.

From the above pairing we see that for all $i \in \mathbb{Z}$, $p^n \text{Div}(\mathcal{N}(i - 1)\Gamma)$ is the Pontryagin dual of $\text{Fr}_{\mathbb{Z}_p}(X(i)\Gamma)/p^n$.

The following lemma generalizes [1, Lemma 2.1] to the case of cyclic extensions of degree $p^n$ with which we are dealing:

**Lemma 3.1.** ([4, page 1242]) Let $j \equiv i \pmod{p^r}$ for an integer $r \leq n + e$. Then

$$\text{Fr}_{\mathbb{Z}_p}(X(i)\Gamma)/p^n \cong \text{Fr}_{\mathbb{Z}_p}(X(j)\Gamma)/p^n \ (i - j)$$

provided $\mu_{p^n+e-r} \subset F$.

**Proof.** As in the proof of [1, Lemma 2.1], we have, for each integer $i$,

$$\text{Fr}_{\mathbb{Z}_p}(X(i)\Gamma)/p^n \cong X(i)/(X(i) \cap T(\Lambda^{r_2}(i)) + p^n X(i)).$$

Let $Y_i := X(i) \cap T(\Lambda^{r_2}(i)) + p^n X(i)$. We have to show that the two submodules $Y_i$ and $Y_j$ are the same for any two integers $i$ and $j$ such that $j \equiv i \pmod{p^r}$.

Let $\kappa$ be the cyclotomic character and recall that $\gamma$, which we have already fixed, is a topological generator of $\Gamma$. Denote the action of $T$ on $\Lambda^{r_2}(i)$ by $T^{(i)} := \kappa(\gamma)^i \gamma - 1$. Each element $y \in Y_i$ can be written as $y = T^{(i)} \lambda + p^n x$, with $T^{(i)} \lambda \in X$, for a $\lambda \in \Lambda^{r_2}$ and an $x \in X$. Write $y = (T^{(i)} - T^{(j)}) \lambda + T^{(j)} \lambda + p^n x$. Since, by hypothesis $\mu_{p^n+e-r} \subset F$, we have

$$\kappa(\gamma) \equiv 1 \pmod{p^{n+e-r}}.$$ 

Moreover $p^r$ dividing $i - j$, we obtain from the preceding congruence

$$\kappa(\gamma)^{i-j} \equiv 1 \pmod{p^{n+e}}.$$ 

Thus $(T^{(i)} - T^{(j)}) \Lambda^{r_2}$ is contained in $p^{n+e} \Lambda^{r_2}$. On the other hand, as an abelian group $X/Y_j \simeq (\mathbb{Z}/p^n \mathbb{Z})^{r_2}$ is of exponent $p^n$, so the exponent of $\Lambda^{r_2}/Y_j$ is at most $p^{n+e}$. Thus $(T^{(i)} - T^{(j)}) \Lambda^{r_2} \subset Y_j$. The element $T^{(j)} \lambda$ of $T(\Lambda^{r_2}(j))$ is also in $X$ because $y$, $(T^{(i)} - T^{(j)}) \lambda$ and $p^n x$ are in $X$. We conclude that $y$ is in $Y_j$. The lemma follows. \qed
By duality, the previous lemma then shows that under the same conditions
\[ p^n \text{Div}(N(i)\Gamma) = p^n \text{Div}(N(j)\Gamma)(i - j). \]
In particular, putting \( j = 0 \):
\[ p^n \text{Div}(N(i)\Gamma) = p^n \text{Div}(N\Gamma)(i). \]
Recall now that for any rational integer \( i \geq 2 \) [13]
\[ \text{Div}(N(i - 1)\Gamma) = \text{Div}(\mathcal{K}(i - 1)\Gamma) \]
and for any \( i \neq 1 \) the above equality is conjectured to be true (Greenberg, Schneider). The case \( i = 0 \) corresponds to the Leopoldt conjecture for the base number field \( F \) at the prime \( p \). Thus we have the following corollaries:

**Corollary 3.2.** For two integers \( i \geq 2 \) and \( j \geq 2 \), if \( j \equiv i \pmod{p^r} \) for an integer \( r \leq n + e \), then
\[ D_F^{(i,n)} = D_F^{(j,n)}(i - j) \]
provided \( \mu_{p^{n+e-r}} \subset F \). Recall our assumption that \( F \) always contains at least \( \mu_p \).

In the following corollaries, we put \( j = 0 \) and \( i \geq 2 \).

**Corollary 3.3.** Assume the number field \( F \) contains \( \mu_p \) and satisfies Leopoldt’s conjecture at the prime \( p \). Then
\[ D_F^{(i,n)} = D_F^{(0,n)}(i) = A_F^{(n)}(i) \]
provided \( \mu_{p^{n+e-r}} \subset F \) for an integer \( r \leq n + e \) such that \( p^r \mid i \).

Since \( \mu_p \subset F \), for \( m \) large, the \( m \)-th layer \( F_m \) of the cyclotomic \( \mathbb{Z}_p \)-extension of \( F \) contains enough \( p \)-primary roots of unity and the condition \( \mu_{p^{n+e-r}} \subset F_m \) is automatically satisfied:

**Corollary 3.4.** Assume that the layers \( F_m \) of the cyclotomic \( \mathbb{Z}_p \)-extension of \( F \) satisfy Leopoldt’s conjecture at the prime \( p \). Then, we have
\[ D_{F_m}^{(i,n)} = D_{F_m}^{(0,n)}(i) = A_{F_m}^{(n)}(i) \]
for \( m \) large enough.

The preceding corollaries generalize those of [1, Section 2] where the case of cyclic extensions of degree \( p \) is treated.
4. Bounds For The Higher étale capitulation Kernels

Let $E/F$ be a cyclic extension of algebraic number fields of degree $p^n$, containing $\mu_{p^n}$, with Galois group $G$. The set $S$ consists of a finite set of primes containing $S_p$ and those primes which ramify in $E/F$. Since the étale $K$-groups $K^{\text{ét}}_{2i-2}F$ are finitely generated $\mathbb{Z}_p$-modules of rank $r_2$ and have cyclic torsion subgroup, we have the following upper bound for the kernel or the cokernel of the natural extension map $f_i : K^{\text{ét}}_{2i-2}(\sigma_F^S) \rightarrow (K^{\text{ét}}_{2i-2}(\sigma_E^S))^G$:

$$|\ker(f_i)| = |\text{coker}(f_i)| \leq p^{n(1+r_2)},$$

where $i \geq 2$ and $r_2$ is the number of complex places of $F$.

We also recall that the maps $f_i$ are not injective once a non-$p$-adic prime ramifies in $E/F$ [1, Proposition 4.2].

Assume that the number field $F$ contains $\mu_{p^n}$. Let $\tilde{F}_n$ be the compositum of the $n$-th layers of the $\mathbb{Z}_p$-extensions of $F$. By the definition of the Kummer radical $A_F^{(n)}$, we have a perfect pairing

$$\text{Gal}(\tilde{F}_n/F) \times A_F^{(n)}/F^{\mu_{p^n}} \rightarrow \mu_{p^n},$$

$$(\sigma, a) \mapsto \sigma(\sqrt[n]{a})^{\nu^n}.$$

**Definition 4.1.** ([3, 10, 11, 12]) A set $S$ of finite primes of $F$ containing $S_p$ is called primitive for $(F, p)$ if the Frobenius "attached" to the primes $v$ in $S - S_p$ generate a direct summand in $\text{Gal}(\tilde{F}_n/F)$ of $\mathbb{Z}_p$-rank the cardinality of $S - S_p$, where $\tilde{F}$ is the compositum of all the $\mathbb{Z}_p$-extensions of $F$.

Let $S - S_p = \{v_1, v_2, \ldots, v_s\}$ be the set of non-$p$-adic primes which ramify in $E/F$. We extract from this a set $S_p \cup \{v_1, v_2, \ldots, v_t\}$ primitive for $(F, p)$. Denote by $\sigma_j := \sigma_j(\tilde{F}_n/F)$ the Frobenius "attached" to the prime $v_j$ in the extension $\tilde{F}_n/F$. We consider $\text{Gal}(\tilde{F}_n/F)$ as a naturally free $\mathbb{Z}/p^n\mathbb{Z}$-module. By the definition of primitivity, the set $\{\sigma_1, \ldots, \sigma_t\}$ is $\mathbb{Z}/p^n\mathbb{Z}$-free and could be extended to a basis $\{\sigma_1, \ldots, \sigma_t, \sigma_{t+1}, \ldots, \sigma_{1+r_2+\delta_F}\}$ of $\text{Gal}(\tilde{F}_n/F)$. Here $\delta_F$ denotes the default of Leopoldt’s conjecture for $(F, p)$. Introduce the dual basis $\{a_1, \ldots, a_{1+r_2+\delta_F}\}$ with respect to the above pairing:

$$\begin{align*}
\sigma_j(\sqrt[n]{a_j}) &= \zeta_{p^n}^{\nu^n}\sqrt[n]{a_j} & \text{for all} & \quad j = 1, \ldots, 1 + r_2 + \delta_F \\
\sigma_j(\sqrt[n]{a_k}) &= \sqrt[n]{a_k} & \text{whenever} & \quad k \neq j.
\end{align*}$$
Here $ζ_{p^n}$ is a fixed primitive $p^n$-th root of unity. In particular, for each $j$, the prime $v_j$ remains inert in $F(\sqrt[n]{a_j})$ and splits in

$$F(\sqrt[n]{a_1}, \ldots, \sqrt[n]{a_{i-1}}, \sqrt[n]{a_{i+1}}, \ldots, \sqrt[n]{a_{1+r_2+δ_F}}).$$

Let $v$ be any of the primes in $\{v_1, v_2, \ldots, v_t\}$. Denote by $w$ a prime of $E$ above $v$. Let $F_v, E_w$ be the completion of $F$ and $E$ at $v$ and $w$ respectively. The natural composite map $A_F(n) \hookrightarrow F^e \hookrightarrow F_v^e$ induces the following injection

$$A_F(n) / A_F \cap N_{E_w/F_v}(E_w^e) \hookrightarrow F_v^e / N_{E_w/F_v}(E_w^e) \cong Gal(E_w/F_v)$$

showing that $A_F(n) / A_F \cap N_{E_w/F_v}(E_w^e)$ is cyclic. The following lemma gives the order of this cyclic group:

**Lemma 4.2.** Let $v = v_j$ for a $j = 1, 2, \ldots, t$ and $w$ a prime of $E$ dividing $v$. Denote by $p^e \geq p$ the ramification index of $v$ in $E/F$. The factor group $A_F(n) / A_F \cap N_{E_w/F_v}(E_w^e)$ is cyclic of order $p^e$.

**Proof.** By construction, all the $a_k$ for $k \neq j$ belong to $N_{E_w/F_v}(E_w^e)$ (since $\sqrt[n]{a_k} \in F_v$), so that $A_F(n) / A_F \cap N_{E_w/F_v}(E_w^e)$ is generated by the class of $a = a_j$.

Let $E = F(\sqrt[n]{b})$. Let $(\ , \ )_v$ be the Hilbert symbol in the local field $F_v$ with values in $μ_{p^n}$. For any integer $α$, we have the following equivalences:

$$a^{p^n} \in N_{E_w/F_v}(E_w^e) \iff (a^{p^n}, b)_v = 1 \iff (a, b^{p^n})_v = 1 \iff b^{p^n} \in N_{F_v(\sqrt[n]{b})/F_v}(F_v(\sqrt[n]{a})).$$

Since the extension $F_v(\sqrt[n]{a})/F_v$ is unramified of degree $p^n$, this last norm group consists of all elements whose valuation is exactly $p^n$. Accordingly, $a^{p^n} \in N_{E_w/F_v}(E_w^e)$ precisely when $p^{n-α}$ divides the valuation of $b$ in $F_v$. Finally, we have: $a^{p^n}$ is a norm in $E_w/F_v$ precisely when the local extension $F_v(\sqrt[n-α]{b})/F_v$ is unramified.

Now, by definition of $e$, $F_v(\sqrt[n-e]{b})$ being the maximal unramified extension of $F_v$ contained in $E_w = F_v(\sqrt[n]{b})$, we conclude that the order of the class of $a$ in $A_F(n) / A_F \cap N_{E_w/F_v}(E_w^e)$ is exactly $p^e$, as was to be shown. □
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Now consider the canonical map

$$A_F^{(n)} / A_F^{(n)} \cap v \in T \setminus S_p \ N_{E/F_v} (E_w^\bullet) \xrightarrow{\varphi} \prod_{v \in T \setminus S_p} A_F^{(n)} / A_F^{(n)} \cap N_{E/F_v} (E_w^\bullet)$$

where the set $T := S_p \cup \{ v_1, v_2, \cdots, v_t \}$ consists of a primitive set for $(F, p)$ inside $S$. The map $\varphi$ is obviously injective. On the other hand, by the construction of the dual basis $a_j$, we have

$$\begin{align*}
\varphi (a_1) &= (a_1, 0, \cdots, 0) \\
\varphi (a_2) &= (0, a_2, 0, \cdots, 0) \\
&\quad \cdots \\
\varphi (a_t) &= (0, \cdots, 0, a_t).
\end{align*}$$

Therefore, the map $\varphi$ is in fact an isomorphism. Now by the previous lemma, the target group is of order $p^{e_1 + \cdots + e_t}$ where $p^{e_j} \geq p$ is the ramification index of the non-$p$-adic prime $v_j$ in the cyclic $p$-extension $E/F$.

Accordingly

**Proposition 4.3.** Let $E/F$ be a cyclic extension of degree $p^n$ containing $\mu_{p^n}$. Let $\{ v_1, \cdots, v_t \}$ consist of a set of tamely ramified primes in $E/F$ belonging to a primitive set for $(F, p)$. We then have the following lower bound for the norm index in the Kummer radical $A_F^{(n)}$ of the $n$-th layers of the $\mathbb{Z}_p$-extensions of $F$:

$$[A_F^{(n)} : A_F^{(n)} \cap N_{E/F} (E^\bullet)] \geq p^{e_1 + \cdots + e_t},$$

where $p^{e_j}$ is the ramification index of $v_j$ in $E/F$.

Combining this proposition with the results of the previous sections we get the following lower bound for the kernel or the cokernel of the natural map $f_i : K^{\text{ét}}_{2i-2} (\mathcal{O}_E^S) \longrightarrow K^{\text{ét}}_{2i-2} (\mathcal{O}_E^S)^G$, $i \geq 2$, which we are interested in.

**Theorem 4.4.** Let $F$ be a number field satisfying Leopoldt’s conjecture at the prime $p$. Let $E/F$ be a cyclic extension of degree $p^n$. Let $\{ v_1, \cdots, v_t \}$ consist of a set of tamely ramified primes in $E/F$ belonging to a primitive set for $(F, p)$. Denote by $p^{e_j} \geq p$ the ramification index of $v_j$ in $E/F$ and by $p^e$ the exponent of $H_F$. Then

$$| \ker (f_i) | = | \text{coker} (f_i) | \geq p^{e_1 + \cdots + e_t},$$

provided $\mu_{p^{n+e-r}} \subset F$ for an integer $r \leq n + e$ such that $p^r | i$.

**Proof.** We successively have
\(|\ker(f_i)| = |\coker(f_i)| \geq \left[D_F^{i,n} : D_F^{i,n} \cap N_{E/F}(E^\bullet)\right] = \left[A_F^{(n)} : A_F^{(n)} \cap N_{E/F}(E^\bullet)\right] \geq p^{e_1 + \cdots + e_t}.

In the classical case of \(i = 2\), we necessarily have \(r = 0\) and obtain:

**Corollary 4.5.** Let \(F\) be a number field satisfying Leopoldt’s conjecture at the prime \(p\) and let \(\mu_{p^n} \subset F\). Let \(E/F\) be a cyclic extension of degree \(p^n\). Let \(\{v_1, \cdots, v_t\}\) consist of a maximal set of tamely ramified primes in \(E/F\) belonging to a primitive set for \((F, p)\). Denote by \(p^{e_j} \geq p^n\) the ramification index of \(v_j\) in \(E/F\). If \(\mu_{p^n+e} \subset F\), then we have the following lower bound

\[|\ker(f_i)| = |\coker(f_i)| \geq p^{e_1 + \cdots + e_t},\]

for the kernel and the cokernel of the natural extension map of the tame kernels \(f : K_2(o_F^S) \rightarrow K_2(o_F^S)^G\).

A set \(T\) primitive for \((F, p)\) is said to be maximal when \(T - S_p\) is as large as possible. When \(F\) satisfies Leopoldt’s conjecture, this is the case where \(T - S_p\) contains exactly \(1 + r_2\) primes, \(r_2\) being the number of non-conjugate complex embeddings of \(F\). When amongst totally and tamely ramified primes in \(E/F\) one can extract a set \(\{v_1, \cdots, v_1+r_2\}\) sitting in a primitive set, then the method developed here gives the exact size of \(|\ker(f_i)| = |\coker(f_i)|:\)

**Corollary 4.6.** Let \(F\) be a number field satisfying Leopoldt’s conjecture at the prime \(p\) and let \(\mu_{p^n} \subset F\). Let \(E/F\) be a cyclic extension of degree \(p^n\). Assume there exists a primitive set \(T\) for \((F, p)\) which is maximal, and such that each \(v \in T - S_p\) is totally ramified in \(E/F\). Then

\[|\ker(f_i)| = |\coker(f_i)| = p^{n(1+r_2)},\]

provided \(\mu_{p^n+e-r} \subset F\) for an integer \(r \leq n + e\) such that \(p^r \mid i\).

To finish, we establish that for each non-negative integer \(t \leq 1 + r_2\), there exist cyclic extensions \(E/F\) of degree \(p^n\) where the order of \(\ker(f_i)\) is exactly \(p^{nt}\). Start with the following short exact sequence

\[0 \rightarrow K_{2i-2}(o_F) \rightarrow K_{2i-2}(o_F^S) \rightarrow \bigoplus_{v \in S - S_p} H^2(F_v, \mathbb{Z}_p(i)) \rightarrow 0.\]
We choose the ground number field $F$ to be $p$-regular (that is to say $K^{\text{ét}}_{2i-2}(o_F) = 0$). This is for example the case of any cyclotomic field $\mathbb{Q}(\mu_{p^n})$, provided the prime $p$ is regular. Furthermore, we suppose that the set $S$ is primitive for $(F, p)$ so that the number field $E$ is also $p$-regular. In this way, we get the following commutative diagramme

$$
K^{\text{ét}}_{2i-2}(o_E^S)^G \xrightarrow{f_i} (\oplus_{v \in S-S_p} (\oplus_{w|v} H^2(E_w, \mathbb{Z}_p(i))))^G
$$

and all that remains to do is to estimate the order of the kernel of the right vertical map. For each prime $v$, by local duality, the kernel of $f_v$ has the same order as the cokernel of the canonical map

$$
(\oplus_{w|v} H^0(E_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i)))^G \longrightarrow H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))
$$

induced by the norm. Let $E'_w$ be the inertia field in $E_w/F_v$. Then $E'_w$ is obtained from $F_v$ by adjoining $p$-primary roots of unity (it is in fact a layer of the cyclotomic $\mathbb{Z}_p$-extension of $F_v$, namely $E'_w = F_{v,\infty} \cap E_w$). From this follows that the map

$$
\oplus_{w|v} H^0(E'_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \longrightarrow H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))
$$

is in fact surjective, whereas in the totally ramified extension $E_w/E'_w$ the cokernel of the map

$$
\oplus_{w|v} H^0(E_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \longrightarrow H^0(E'_w, \mathbb{Q}_p/\mathbb{Z}_p(1-i))
$$

is of order $p^{e_v} = [E_w : E'_w]$, the ramification index of $v$ in $E/F$ (for details see [5, Lemma 4.2.1]).

Thus we have the following:

**Proposition 4.7.** Let $F$ be a $p$-regular number field containing the $p^n$-th roots of unity and let $E/F$ be a cyclic extension of degree $p^n$. Then

$$
|\ker(f_i)| = |\coker(f_i)| = p^{\sum_{v \in S-S_p} e_v},
$$

provided the set $S$ of the $p$-adic prime of $F$ and those which ramify in $E$ is primitive for $(F, p)$.

Čebotarev’s density theorem guarantees that for each number field $F$ there exist infinitely many cyclic extensions $E$ of $F$ of degree $p^n$, such that the set $S$ of the $p$-adic primes of $F$ and the tamely ramified primes in $E/F$ is primitive for $(F, p)$, and such that each $v \in S - S_p$ has the
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prescribed ramified index $p^{e_v}$ in $E/F$. Thus, according to the preceding proposition, for each $p$-regular number field $F$ with $r_2$ non-conjugate complex embeddings, and for each $p$-power (given in advance) $p^m \leq p^{n(1+r_2)}$, we can find infinitely many cyclic extensions $E$ of $F$ of degree $p^n$, such that $|\ker(f_i)| = |\coker(f_i)| = p^m$.

References

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