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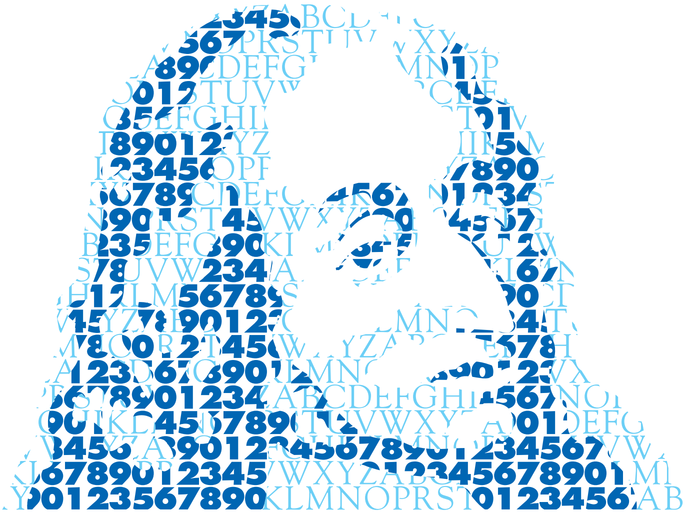

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Generalized Kummer theory and its applications

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Abstract

In this report we study the arithmetic of Rikuna's generic polynomial for the cyclic group of order n and obtain a generalized Kummer theory. It is useful under the condition that $\zeta \notin k$ and $\omega \in k$ where ζ is a primitive n-th root of unity and $\omega = \zeta + \zeta^{-1}$. In particular, this result with $\zeta \in k$ implies the classical Kummer theory. We also present a method for calculating not only the conductor but also the Artin symbols of the cyclic extension which is defined by the Rikuna polynomial.

1. Introduction

In this report we study the arithmetic of Rikuna's generic polynomial for the cyclic group of order n and obtain a generalized Kummer theory. It is useful under the condition that $\zeta \notin k$ and $\omega \in k$ where ζ is a primitive n-th root of unity and $\omega = \zeta + \zeta^{-1}$. In particular, this result with $\zeta \in k$ implies the classical Kummer theory. We also present a method for calculating not only the conductor but also the Artin symbols of the cyclic extension which is defined by the Rikuna polynomial. By an arithmetic argument we show that a certain cubic polynomial is not generic (cf. Corollary 3.6).

We first recall notion on the genericity of a polynomial (cf. Jensen-Ledet-Yui [3]). Let k be a field and G a finite group. The rational function field $k(t_1, t_2, \ldots, t_m)$ over k with m variables $t_1, t_2, \ldots, t_m$ is denoted by $k(\mathfrak{t})$ where $\mathfrak{t} = (t_1, t_2, \ldots, t_m)$. For a polynomial $F(X) \in K[X]$ over a field K let us denote by $\mathrm{Spl}_K F(X)$ the minimal splitting field of $F(X)$ over K. We say that a polynomial $F(\mathfrak{t}, X) \in k(\mathfrak{t})[X]$ is a k-regular G-polynomial or a regular polynomial over k for G if the field $\mathrm{Spl}_{k(\mathfrak{t})} F(\mathfrak{t}, X)$ is a Galois extension L of $k(\mathfrak{t})$ with two conditions $\mathrm{Gal}(L/k(\mathfrak{t})) \simeq G$ and $L \cap \overline{k} = k$ where $\overline{k}$ is an algebraic closure field of k. For example, if n is a positive

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integer greater than 2, then the Kummer polynomial $X^n - t \in \mathbf{Q}(t)[X]$ is a regular polynomial for the cyclic group $\mathcal{C}_n$ of order n not over $\mathbf{Q}$ but over $\mathbf{Q}(\zeta_n)$ where ζ_n is a primitive n-th root of unity in $\overline{\mathbf{Q}}$. A k-regular G-polynomial $F(\mathfrak{t}, X) \in k(\mathfrak{t})[X]$ is called to be generic over k if $F(\mathfrak{t}, X)$ yields all the Galois G-extensions containing k, that is, for every Galois extension L/K with $\mathrm{Gal}(L/K) \simeq G$ and $K \supseteq k$ there exists a K-specialization $\mathfrak{s} = (s_1, s_2, \ldots, s_m)$, $s_i \in K$ so that $L = \mathrm{Spl}_K F(\mathfrak{s}, X)$.

Let n be an odd number greater than 1 and $\zeta = \zeta_n$ a primitive n-th root of unity in $\overline{\mathbf{Q}}$. We put $\omega = \zeta + \zeta^{-1}$ and $k = \mathbf{Q}(\omega)$. We define a polynomial $R_n(t, X)$ by

$$R_n(t, X) = \frac{\zeta^{-1}(X-\zeta)^n - \zeta(X-\zeta^{-1})^n}{\zeta^{-1} - \zeta} - t\frac{(X-\zeta)^n - (X-\zeta^{-1})^n}{\zeta^{-1} - \zeta}.$$

Note that $R_n(t, X)$ is a polynomial in $k(t)[X]$.

Proposition 1.1 (Rikuna [11]). *The polynomial $R_n(t, X)$ is generic over the field k for the group $\mathcal{C}_n$.*

Remark 1.2. When n is even and K does not contain ζ, the polynomial $R_n(t, X)$ is not generic over K for $\mathcal{C}_n$ in general (cf. Komatsu [6]). For the case that n is even, Hashimoto and Rikuna [2] constructed a k-generic $\mathcal{C}_n$-polynomial with two parameters.

In a previous paper [6] we study the arithmetic of the polynomial $R_n(t, X)$. Let k be a field whose characteristic is equal to 0 or prime to n. Let ζ be a primitive n-th root of unity in $\overline{k}$ and put $\omega = \zeta + \zeta^{-1}$. For a field K containing $k(\omega)$ let $T(K) = \mathbf{P}^1(K) - \{\zeta, \zeta^{-1}\} = K \cup \{\infty\} - \{\zeta, \zeta^{-1}\}$ be a set with composition $\underset{T}{+}$ such that $s_1 \underset{T}{+} s_2 = (s_1 s_2 - 1)/(s_1 + s_2 - \omega)$. Then $T(K)$ is an algebraic torus of dimension 1 which has a group isomorphism $\varphi : T \to \mathbf{G}_m$, $t \mapsto (t-\zeta)/(t-\zeta^{-1})$ over $K(\zeta)$. In fact, the composition $\underset{T}{+}$ is defined as $s_1 \underset{T}{+} s_2 = \varphi^{-1}(\varphi(s_1)\varphi(s_2))$. The identity 0_T on T is equal to $\infty = \varphi^{-1}(1)$. The inverse $\underset{T}{-} s$ of an $s \in T(K)$ is $-s + \omega$. For a positive integer $m \in \mathbf{Z}$ let $[m]$ be the multiplication map by m with respect to $\underset{T}{+}$, that is, $[m]s = s \underset{T}{+} \cdots \underset{T}{+} s$ with m terms. We denote $[m]T(K) = \{[m]s | s \in T(K)\}$ and $T[m] = T(\overline{K})[m] = \{x \in T(\overline{K}) | [m]x = \infty\}$. Note that $-1 = \varphi^{-1}(\zeta)$ and $T[n] = \langle -1 \rangle_T = \{-1, 0, \ldots, \omega, \omega + 1, \infty\} \subset T(k(\omega))$. Let Γ_K be the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ of K where K^{sep} is the separable closure field of K. Then we have a descent Kummer theory.

Proposition 1.3 (Ogawa [10], Komatsu [6])**.** *There exists a group isomorphism*

$$\delta : T(K)/[n]T(K) \to \mathrm{Hom}_{\mathrm{cont}}(\Gamma_K, \mathcal{C}_n).$$

We have a relation between the polynomial $R_n(t, X)$ and the algebraic group T as follows. For an $s \in T(K)$ let L_s be the field $\mathrm{Spl}_K R_n(s, X)$ and $[n]^{-1}(s)$ the set $\{x \in T(\overline{K}) | [n]x = s\}$.

Lemma 1.4. *We have $L_s = K([n]^{-1}(s))$. In particular, the field L_s is equal to the fixed field $(K^{\mathrm{sep}})^{\mathrm{Ker}\delta(s)}$ of K^{sep} by the subgroup* $\mathrm{Ker}\delta(s)$ *of Γ_K.*

Corollary 1.5. *For elements s_1 and $s_2 \in K$ the equation $L_{s_1} = L_{s_2}$ holds if and only if $\langle s_1\rangle_T = \langle s_2\rangle_T$ in $T(K)/[n]T(K)$.*

Remark 1.6. Morton [9] and Chapman [1] essentially gave the composition $\underset{T}{+}$ for the case $n = 3$. Here $R_3(t/3, X) = X^3 - tX^2 - (t+3)X - 1$ is known as the simplest cubic polynomial of Shanks type.

2. Ramifications and Artin symbols

In this section we recall some results in [6] and [7]. Let l be an odd prime number and ζ a primitive l-th root of unity in $\overline{\mathbf{Q}}$. Let K be a finite algebraic number field containing $\mathbf{Q}(\omega)$ where $\omega = \zeta + \zeta^{-1}$. We assume that the extension $K/\mathbf{Q}(\omega)$ is unramified at all the prime ideals of K above l. For an $s \in K$ we denote by L_s the minimal splitting field $\mathrm{Spl}_K R_l(s, X)$ of the polynomial $R_l(s, X)$ over the field K. For a prime ideal $\mathfrak{p}$ of K let $v_\mathfrak{p}$ be a $\mathfrak{p}$-adic additive valuation which is normalized so that $v_\mathfrak{p}(K^\times) = \mathbf{Z}$. For a prime ideal $\mathfrak{l}$ of K above l we define a set $U_{K,\mathfrak{l}}$ by

$$U_{K,\mathfrak{l}} = \{s \in T(K) | v_\mathfrak{l}(s - \omega/2) \le -(l-1)/2 \text{ or } v_\mathfrak{l}(s - \omega/2) \ge (l+1)/2\}.$$

For a prime ideal $\mathfrak{q}$ of K with $\mathfrak{q} \nmid l$ the set $U_{K,\mathfrak{q}}$ is defined to be

$$U_{K,\mathfrak{q}} = \{s \in T(K) | v_\mathfrak{q}(s^2 - \omega s + 1) \le 0 \text{ or } v_\mathfrak{q}(s^2 - \omega s + 1) \equiv 0 \pmod{l}\}.$$

Lemma 2.1 (Komatsu [6]). *For an* $s \in K$ *the conductor* $\mathrm{cond}(L_s/K)$ *of the extension* L_s/K *is equal to* $\prod_{\mathfrak{p}} \mathfrak{p}^{\lambda_{\mathfrak{p}}}$ *where*

$$\lambda_{\mathfrak{p}} = \begin{cases} 1 & \text{if } \mathfrak{p} \nmid l \text{ and } s \notin U_{K,\mathfrak{p}}, \\ \mathfrak{c}_{\mathfrak{l}}(s) & \text{if } \mathfrak{p} = \mathfrak{l} \mid l \text{ and } s \notin U_{K,\mathfrak{l}}, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\mathfrak{c}_{\mathfrak{l}}(s)$ *is equal to a positive integer* $(l+2)/2 - |v_{\mathfrak{l}}(s - \omega/2) - 1/2|$ *for* $s \notin U_{K,\mathfrak{l}}$.

We denote by U_K the intersection $\cap_{\mathfrak{p}} U_{K,\mathfrak{p}}$ of the sets $U_{K,\mathfrak{p}}$ where $\mathfrak{p}$ runs through all of the prime ideals of K. In general, one has that $[l]T(K) \subseteq U_K$.

Corollary 2.2. *Vandiver conjecture for* $\mathbf{Q}(\omega)$ *is true, that is, the class number of* $\mathbf{Q}(\omega)$ *is not divisible by* l *if and only if it satisfies* $[l]T(\mathbf{Q}(\omega)) = U_{\mathbf{Q}(\omega)}$. *In particular, an unramified cyclic extension of* $\mathbf{Q}(\omega)$ *with degree* l *is obtained as* $\mathrm{Spl}_{\mathbf{Q}(\omega)} R_l(s, X)$ *for an* $s \in U_{\mathbf{Q}(\omega)} - [l]T(\mathbf{Q}(\omega))$.

Let us assume that $s \notin [l]T(K)$, that is, L_s/K is a cyclic extension of degree l. Then L_s is generated over K by a solution x of $R_l(s, X) = 0$. The Galois group $\mathrm{Gal}(L_s/K)$ is generated by an element σ such that $\sigma(x) = x \underset{T}{+} (-1)$. Note that $\langle -1 \rangle_T = T[l] \subset T(K)$. Let $\mathfrak{p}$ be a prime ideal of K which is unramified in the extension L_s/K. We denote by $\mathbf{F}_{\mathfrak{p}}$ the residue class field $\mathcal{O}_K/\mathfrak{p}$ and by q the cardinal number $\sharp \mathbf{F}_{\mathfrak{p}}$ of the finite field $\mathbf{F}_{\mathfrak{p}}$. Note that $q \equiv 0$ or $\pm 1 \pmod{l}$ since K contains ω. We fix a prime ideal $\mathfrak{P}$ of L_s above $\mathfrak{p}$. Then there exists an element $\tau \in \mathrm{Gal}(L_s/K)$ such that $v_{\mathfrak{P}}(\tau(\alpha) - \alpha^q) \geq 1$ for every algebraic integer $\alpha \in \mathcal{O}_{L_s}$ in L_s. The element τ depends not on the choice of the prime ideal $\mathfrak{P}$ but only on the prime ideal $\mathfrak{p}$. We call τ the Artin symbol of $\mathfrak{p}$ in L_s/K and denote it by $\mathrm{Art}_{\mathfrak{p}}(L_s/K)$. We put $\mu_{\mathfrak{p}}(s) = v_{\mathfrak{p}}(s^2 - \omega s + 1)$.

Theorem 2.3 (Komatsu [7]). *We assume that* $\mathfrak{p} \nmid l$. *If* $\mu_{\mathfrak{p}}(s) < 0$, *then* $\mathrm{Art}_{\mathfrak{p}}(L_s/K) = \mathrm{id}$, *that is,* $\mathfrak{p}$ *splits completely in* L_s/K. *For the case* $\mu_{\mathfrak{p}}(s) = 0$, *we have* $\mathrm{Art}_{\mathfrak{p}}(L_s/K) = \sigma^i$ *where* $i \in \mathbf{Z}$ *is an integer such that* $[i](-1) = [(\pm q - 1)/l]s$ *in* $T(\mathbf{F}_{\mathfrak{p}})$ *provided* $q \equiv \pm 1 \pmod{l}$, *respectively. When* $\mu_{\mathfrak{p}}(s) > 0$ *and* $\mu_{\mathfrak{p}}(s) \not\equiv 0 \pmod{l}$, *the extension* L_s/K *is totally ramified at* $\mathfrak{p}$.

Theorem 2.3 does not deal with an exceptional case that $\mu_{\mathfrak{p}}(s) > 0$ and $\mu_{\mathfrak{p}}(s) \equiv 0 \pmod{l}$, that is, $\mu_{\mathfrak{p}}(s) = jl$ for a positive integer $j \in \mathbf{Z}$. In the following we may reduce the exceptional case to the case $\mu_{\mathfrak{p}}(s) \leq 0$. For a number $s_0 \in K$ with $v_{\mathfrak{p}}(s - s_0) = j$ we put $s_1 = s \underset{T}{-} [l]s_0 \in K$.

Lemma 2.4. *We have $L_s = L_{s_1}$ and $\mu_{\mathfrak{p}}(s_1) \leq 0$.*

Proof. Corollary 1.5 shows that $L_s = L_{s_1}$. Let $\widetilde{\mathfrak{p}}$ be a prime ideal of $K(\zeta)$ above $\mathfrak{p}$. Then one has that $(v_{\widetilde{\mathfrak{p}}}(s - \zeta), v_{\widetilde{\mathfrak{p}}}(s - \zeta^{-1})) = (jl, 0)$ or $(0, jl)$ since $\widetilde{\mathfrak{p}} \nmid l$. When $(v_{\widetilde{\mathfrak{p}}}(s - \zeta^{\pm 1}), v_{\widetilde{\mathfrak{p}}}(s - \zeta^{\mp 1})) = (jl, 0)$, we have $(v_{\widetilde{\mathfrak{p}}}(s_0 - \zeta^{\pm 1}), v_{\widetilde{\mathfrak{p}}}(s_0 - \zeta^{\mp 1})) = (j, 0)$, respectively. It follows from $s_1 = s \underset{T}{-} [l]s_0$ that

$$\frac{s_1 - \zeta}{s_1 - \zeta^{-1}} = \frac{s - \zeta}{s - \zeta^{-1}} \left(\frac{s_0 - \zeta}{s_0 - \zeta^{-1}} \right)^{-l}.$$

This implies that $v_{\widetilde{\mathfrak{p}}}((s_1 - \zeta)/(s_1 - \zeta^{-1})) = 0$ and $v_{\widetilde{\mathfrak{p}}}(s_1 - \zeta^{\pm 1}) \leq 0$. Thus we have $\mu_{\mathfrak{p}}(s_1) = v_{\widetilde{\mathfrak{p}}}((s_1 - \zeta)(s_1 - \zeta^{-1})) \leq 0$. $\square$

Proposition 2.5 (Komatsu [7]). *We assume $(l, K, \mathfrak{p}) = (3, \mathbf{Q}, 3\mathbf{Z})$. For an $s \in \mathbf{Q}$ the decomposition of the prime ideal $3\mathbf{Z}$ in the extension $L_s/\mathbf{Q}$ is as follows:*
(i) *the prime $3\mathbf{Z}$ ramifies in $L_s/\mathbf{Q}$ if and only if $v_3(s + 1/2) \in \{0, 1\}$.*
(ii) *the prime $3\mathbf{Z}$ splits completely in $L_s/\mathbf{Q}$ if and only if $v_3(s + 1/2) \notin \{-1, 0, 1, 2\}$.*
(iii) *the ideal $3\mathbf{Z}$ remains prime in $L_s/\mathbf{Q}$ if and only if $v_3(s + 1/2) \in \{-1, 2\}$. When $v_3(s + 1/2) = -1$ and $3s \equiv \mp 1 \pmod 3$, we have $\mathrm{Art}_{3\mathbf{Z}}(L_s/\mathbf{Q}) = \sigma^{\pm 1}$, respectively. For the case $v_3(s + 1/2) = 2$ and $(s + 1/2)/9 \equiv \pm 1 \pmod 3$, it satisfies $\mathrm{Art}_{3\mathbf{Z}}(L_s/\mathbf{Q}) = \sigma^{\pm 1}$, respectively.*

Let $f_0(t, X)$ be the cubic polynomial $R_3(t, X) = X^3 - 3tX^2 - (3t + 3)X - 1$. For a rational number $s \in \mathbf{Q}$ let L_s denote the minimal splitting field $\mathrm{Spl}_{\mathbf{Q}} f_0(s, X)$ of $f_0(s, X)$ over $\mathbf{Q}$. Now assume that $s \notin [3]T(\mathbf{Q})$, that is, L_s is a cyclic cubic extension of $\mathbf{Q}$. Then it holds that $L_s = \mathbf{Q}(x)$ for a solution $x \in \overline{\mathbf{Q}}$ of $R_3(s, X) = 0$. Let σ be a generator of $\mathrm{Gal}(L_s/\mathbf{Q})$ such that $\sigma(x) = x \underset{T}{+} (-1) = (-x - 1)/x$. The following table shows the Artin symbols $\mathrm{Art}_p(L_s/\mathbf{Q})$ for prime numbers p with $2 \leq p \leq 19$ and $p \neq 3$.

p	σ^0 split	σ^1 inert	σ^2 inert	ram. or bl.up
2	∞	0	1	$-$
5	$\infty, 2$	$1, 4$	$0, 3$	$-$
7	$\infty, 3$	$0, 5$	$1, 6$	$2, 4$
11	$\infty, 2, 5, 8$	$0, 6, 7, 9$	$1, 3, 4, 10$	$-$
13	$\infty, 4, 6, 8$	$1, 2, 7, 12$	$0, 5, 10, 11$	$3, 9$
17	$\infty, 0, 1, 8, 15, 16$	$2, 6, 7, 11, 12, 13$	$3, 4, 5, 9, 10, 14$	$-$
19	$\infty, 0, 1, 9, 17, 18$	$4, 10, 12, 13, 15, 16$	$2, 3, 5, 6, 8, 14$	$7, 11$

The number m at p-row in the table above means that s is a p-adic integer with $s \equiv m \pmod{p}$. For example, if $s \in \mathbf{Q}$ satisfies that $v_5(s) \geq 0$ and $s \equiv 1 \pmod{5}$, then the ideal $5\mathbf{Z}$ remains prime in $L_s/\mathbf{Q}$ and the Artin symbol $\mathrm{Art}_{5\mathbf{Z}}(L_s/\mathbf{Q})$ is equal to $\sigma^1 = \sigma$. The symbol ∞ represents that $v_p(s)$ is negative, i.e., the image of s by the reduction map $T(\mathbf{Q}) \to T(\mathbf{F}_p)$, $s \mapsto s \pmod{p}$ is equal to ∞. On the column of "ram. or bl.up", it holds that $\mu_p(s) = v_p(s^2+s+1) \geq 1$. If $\mu_p(s)$ is not divisible by 3, then p ramifies in $L_s/\mathbf{Q}$. When $\mu_p(s) \equiv 0 \pmod{3}$, one can blow-up s to a new $s_1 \in \mathbf{Q}$ such that $L_s = L_{s_1}$ and $\mu_p(s_1) \leq 0$. In fact, for a number $s_0 \in \mathbf{Q}$ with $v_{\mathfrak{p}}(s - s_0) = \mu_p(s)/3$ we put $s_1 = s \underset{T}{-} [3]s_0 \in \mathbf{Q}$. Then we have $L_s = L_{s_1}$ and $\mu_{\mathfrak{p}}(s_1) \leq 0$. The decomposition type of p in $L_s/\mathbf{Q}$ coincides with that in $L_{s_1}/\mathbf{Q}$, which is determined completely by the data that s_1 belongs to the columns of "split" or "inert". In particular, p is unramified in $L_s/\mathbf{Q}$. The symbol $-$ at the column of ram. or bl.-up is denoted for the fact that $p \equiv 2 \pmod{3}$ cannot ramify in any cyclic cubic fields due to class field theory. Indeed, it satisfies $\mu_p(s) \leq 0$ provided $p \equiv 2 \pmod{3}$. The table for $p = 3$ is as follows.

$v_3(s)$	σ^0 split	σ^1 inert	σ^2 inert	ram.
≥ 0	$s \equiv 13 \pmod{27}$	$s \equiv 22 \pmod{27}$	$s \equiv 4 \pmod{27}$	$s \not\equiv 4 \pmod{9}$
-1	$\emptyset$	$3s \equiv 2 \pmod{3}$	$3s \equiv 1 \pmod{3}$	$\emptyset$
≤ -2	all cases	$\emptyset$	$\emptyset$	$\emptyset$

For example, if s is a 3-adic integer with $s \not\equiv 4 \pmod{9}$, then $3\mathbf{Z}$ ramifies in $L_s/\mathbf{Q}$. When $v_3(s) \leq -2$, the prime ideal $3\mathbf{Z}$ splits completely in $L_s/\mathbf{Q}$.

3. Numerical examples of cubic polynomials

In this section we study the Artin symbols in the cyclic cubic fields obtained by some cubic polynomials. Let ζ be a primitive 3rd root of unity in $\overline{\mathbf{Q}}$. Let K be a field containing $\mathbf{Q}$. Let $f(X)$ be a cubic polynomial over K of the form $f(X) = X^3 + a_1X^2 + a_2X + a_3$ whose discriminant is equal to a non-zero square δ^2 for $\delta \in K^\times$. Let b_2 and b_3 be elements in K such that $g(X) = f(X - a_1/3) = X^3 + b_2X + b_3$. One has that $b_2 = -a_1^2/3 + a_2$ and $b_3 = 2a_1^3/27 - a_1a_2/3 + a_3$. Then it holds that

$$\delta^2 = \mathrm{disc}_X f(X) = a_1^2a_2^2 - 4a_1^3a_3 + 18a_1a_2a_3 - 4a_2^3 - 27a_3^2 = -4b_2^3 - 27b_3^2.$$

When $b_2 \neq 0$, we define the invariant $c \in K$ of the polynomial $f(X)$ by $c = -(9b_3+\delta)/(2\delta)$. The invariant is determined up to $\underset{T}{-}$, that is, c or $-c-1$ due to the choice of the signature of the square root δ of the discriminant $\mathrm{disc}_X f(X)$. If $b_2 = 0$ and $b_3 \neq -1$, then the invariant c is defined to be $\varphi^{-1}(-b_3) = (\zeta^{-1}b_3 + \zeta)/(b_3 + 1)$. For the case $(b_2, b_3) = (0, -1)$ we set $c = \zeta$. Let $f_0(t, X)$ be the cubic polynomial $R_3(t, X) = X^3 - 3tX^2 - (3t + 3)X - 1$.

Lemma 3.1. *We have* $\mathrm{Spl}_K f(X) = \mathrm{Spl}_K f_0(c, X)$.

Proof. When $b_2 \neq 0$, it is seen that

$$\begin{aligned} f_0(c, X + c) &= X^3 - \frac{9(27b_3^2 + d^2)}{4\delta^2}X + \frac{27b_3(27b_3^2 + \delta^2)}{4\delta^3} \\ &= X^3 + (9b_2^3/\delta^2)X - 27b_2^3b_3/\delta^3 \\ &= g(X/\gamma)\gamma^3 \end{aligned}$$

where $\gamma = -3b_2/\delta \in K^\times$. If $b_2 = 0$, then $\delta^2 = -27b_3^2$ and $\zeta \in K$. This implies that $\varphi^{-1}(-b_3) \in K$ and $\mathrm{Spl}_K(X^3 + b_3) = \mathrm{Spl}_K f_0(\varphi^{-1}(-b_3), X)$ provided $b_3 \neq -1$. For the case of $(b_2, b_3) = (0, -1)$ it holds that $\mathrm{Spl}_K f(X) = K = \mathrm{Spl}_K f_0(\zeta, X)$ since $f(X) = (X - 1)(X - \zeta)(X - \zeta^2)$ and $f_0(\zeta, X) = (X - \zeta)^3$. $\square$

Let us start with $f(X) = X^3 - 3tX^2 - (3t + 3)X - 1$. Here it satisfies that $(a_1, a_2, a_3) = (-3t, -(3t + 3), -1)$ and $(b_2, b_3) = (-3(t^2 + t + 1), -(2t + 1)(t^2+t+1))$. One has that $\mathrm{disc}_X f(X) = 3^4(t^2+t+1)^2$. If $\delta = 3^2(t^2+t+1)$, then $c = t$ and $f_0(t, X) = X^3 - 3tX^2 - (3t + 3)X - 1$, which is the same as the starting one. Lecacheux [8] gave a cubic polynomial

$$f_1(t, X) = X^3 - (t^3 - 2t^2 + 3t - 3)X^2 - t^2X - 1$$

and Kishi [4] constructed cubic polynomials

$$\begin{aligned}
f_2(t,X) &= X^3 + 3(3t^2-3t+2)X^2 + 3X - 1,\\
f_3(t,X) &= X^3 - t(t^2+t+3)(t^2+2)X^2 - (t^3+2t^2+3t+3)X - 1,\\
f_4(t,X) &= X^3 + (t^8+2t^6-3t^5+3t^4-4t^3+5t^2-3t+3)X^2\\
&\quad -t^2(t^3-2)X - 1.
\end{aligned}$$

It is calculated that the discriminants $\mathrm{disc} f_i(t,X)$ of the polynomials $f_i(t,X)$ are

$$\begin{aligned}
\mathrm{disc}_X f_1(t,X) &= (t-1)^2(t^2+3)^2(t^2-3t+3)^2,\\
\mathrm{disc}_X f_2(t,X) &= 3^6(2t-1)^2(t^2-t+1)^2,\\
\mathrm{disc}_X f_3(t,X) &= (t^2+1)^2(t^2+3)^2(t^4+t^3+4t^2+3)^2,\\
\mathrm{disc}_X f_4(t,X) &= (t^2-t+1)^2(t^3+t-1)^2(t^4-t^3+t^2-3t+3)^2\\
&\quad \times(t^4+2t^3+4t^2+3t+3)^2.
\end{aligned}$$

Let $c_i(t)$ be rational functions in $\mathbf{Q}(t)$ such that

$$\begin{aligned}
c_1(t) &= \frac{t(t^4-3t^3+6t^2-8t+6)}{3(t-1)},\\
c_2(t) &= -\frac{9t^4-18t^3+18t^2-8t+1}{2t-1},\\
c_3(t) &= \frac{t(t^8+2t^7+9t^6+11t^5+25t^4+18t^3+25t^2+8t+9)}{3(t^2+1)},\\
c_4(t) &= -t(t^{13}+3t^{11}-5t^{10}+6t^9-12t^8+17t^7-18t^6+24t^5\\
&\quad -23t^4+21t^3-15t^2+11t-6)/(3(t^3+t-1)).
\end{aligned}$$

Lemma 3.2. *We have* $\mathrm{Spl}_{\mathbf{Q}(t)} f_i(t,X) = \mathrm{Spl}_{\mathbf{Q}(t)} f_0(c_i(t),X)$ *for* $i = 1,2,3$ *and* 4.

Proof. The equations of the assertion follow from Lemma 3.1 and the algorithm for computing the invariants $c = c_i(t)$ of $f_i(t,X)$, respectively. Indeed, the square roots $\delta_i(t)$ of the discriminants $\mathrm{disc}_X f_i(t,X)$ for the computations are

$$\begin{aligned}
\delta_1(t) &= (t-1)(t^2+3)(t^2-3t+3),\\
\delta_2(t) &= 3^3(2t-1)(t^2-t+1),\\
\delta_3(t) &= (t^2+1)(t^2+3)(t^4+t^3+4t^2+3),\\
\delta_4(t) &= (t^2-t+1)(t^3+t-1)(t^4-t^3+t^2-3t+3)\\
&\quad \times(t^4+2t^3+4t^2+3t+3).
\end{aligned}$$

□

It is seen that $c_i(t)^2+c_i(t)+1$ have the cubes of polynomials $\eta_i(t)$ as factors where $\eta_1(t)=t^2-t+1$, $\eta_2(t)=3t^2-3t+1$, $\eta_3(t)=t^4+t^3+3t^2+t+1$ and $\eta_4(t)=t^6+t^4-2t^3+t^2-t+1$, respectively. As the blow-up argument before Lemma 2.4 one may think that there exist rational functions $\widetilde{c_i}(t)$ more "suitable" than $c_i(t)$ such that $\mathrm{Spl}_{\mathbf{Q}(t)} f_0(\widetilde{c_i}(t),X)=\mathrm{Spl}_{\mathbf{Q}(t)} f_0(c_i(t),X)$. We define $\varepsilon_1(t)=-t, \varepsilon_2(t)=-3t+1, \varepsilon_3(t)=-(t^2+t+1)/t$ and $\varepsilon_4(t)=-(t^3+t-1)/t$. Indeed, it holds that $\eta_i(t)||(c_i(t)-\varepsilon_i(t))$ for $i=1,2,3$ and 4. Now put $\widetilde{c_i}(t)=c_i(t)\underset{T}{-}[3]\varepsilon_i(t)$, respectively. The direct computation implies

Lemma 3.3. *We have*

$$
\begin{aligned}
&\widetilde{c_1}(t)=\frac{t(t-3)}{3(t-1)}, \quad \widetilde{c_1}(t)^2+\widetilde{c_1}(t)+1=\frac{(t^2+3)(t^2-3t+3)}{3^2(t-1)^2},\\
&\widetilde{c_2}(t)=t-1, \qquad\quad \widetilde{c_2}(t)^2+\widetilde{c_2}(t)+1=t^2-t+1,\\
&\widetilde{c_3}(t)=\frac{t^2(t-1)}{3(t^2+1)}, \quad \widetilde{c_3}(t)^2+\widetilde{c_3}(t)+1=\frac{(t^2+3)(t^4+t^3+4t^2+3)}{3^2(t^2+1)^2},\\
&\widetilde{c_4}(t)=\frac{t(t+1)(t^3-t^2+t-3)}{3(t^3+t-1)},\\
&\widetilde{c_4}(t)^2+\widetilde{c_4}(t)+1=(t^2-t+1)(t^4-t^3+t^2-3t+3)\\
&\qquad\qquad\qquad\qquad\quad \times(t^4+2t^3+4t^2+3t+3)/(3^2(t^3+t-1)^2).
\end{aligned}
$$

For the equation

$$\mathrm{Spl}_{\mathbf{Q}(t)} f_2(t,X)=\mathrm{Spl}_{\mathbf{Q}(t)} f_0(\widetilde{c_2}(t),x)=\mathrm{Spl}_{\mathbf{Q}(t)} f_0(t-1,X),$$

we omit the following argument for the case of $f_2(t,X)$. Let us fix $i=1$, 3 and 4. For a rational number $s\in\mathbf{Q}$ we denote by M_s the field $L_{\widetilde{c_i}(s)}=\mathrm{Spl}_{\mathbf{Q}} f_0(\widetilde{c_i}(s),X)=\mathrm{Spl}_{\mathbf{Q}} f_i(s,X)$. Assume that $\widetilde{c_i}(s)\notin[3]T(\mathbf{Q})$. Let x be a solution of $f_0(\widetilde{c_i}(s),X)=0$ and σ a generator of $\mathrm{Gal}(M_s/\mathbf{Q})$ such that $\sigma(x)=x\underset{T}{+}(-1)=(-x-1)/x$. The decomposition types and the Artin symbols $\mathrm{Art}_p(M_s/\mathbf{Q})$ in $M_s/\mathbf{Q}$ of prime numbers $p\le 19$ are as follows.

For the polynomial $f_0(\widetilde{c_1}(t), X)$ we have

p	σ^0 split	σ^1 inert	σ^2 inert	ram. or bl.up
2	$\infty, 1(4)$	$0(2), 3(4)$	$\emptyset$	–
3	$\infty, 1$	$\emptyset$	2	$0 \Rightarrow$ ram.
5	$\infty, 1$	$2, 4$	$0, 3$	–
7	$\infty, 1$	$0, 3$	$\emptyset$	$2, 4, 5, 6$
11	$\infty, 1, 9$	$0, 3, 4$	$2, 5, 6, 7, 8, 10$	–
13	$\infty, 1, 2, 12$	$4, 9$	$0, 3, 8, 10$	$5, 6, 7, 11$
17	$\infty, 0, 1, 3, 4, 5, 6, 9$	$7, 10, 11, 12, 14$	$2, 8, 13, 15, 16$	–
19	$\infty, 0, 1, 3, 8, 17$	$2, 5, 7, 10, 18$	$6, 11, 12, 14, 16$	$4, 9, 13, 15$

The integer m at the p-row in the table above implies that s is a p-adic integer with $s \equiv m \pmod{p}$. The symbol ∞ at the p-row means that $v_p(s)$ is negative. The notation $m(p^j)$ represents that s is a p-adic integer with $s \equiv m \pmod{p^j}$. For the polynomial $f_0(\widetilde{c_3}(t), X)$ we have

p	σ^0 split	σ^1 inert	σ^2 inert	ram. or bl.up
2	∞	$0(2), 1(4)$	$3(4)$	–
3	$\infty, 43(81)$	$2(3), 16(81)$	$70(81)$	o.w.$^1 \Rightarrow$ ram.
5	$\infty, 2, 3$	$\emptyset$	$0, 1, 4$	–
7	$\infty, 4$	$0, 1$	$\emptyset$	$2, 3, 5, 6$
11	$\infty, 3, 9$	$0, 1, 7, 10$	$2, 4, 5, 6, 8$	–
13	$\infty, 4, 5, 8, 10, 12$	$2, 9, 11$	$0, 1, 3$	$6, 7$
17	$\infty, 0, 1, 2, 4, 13$	$9, 10, 11, 12, 15, 16$	$3, 5, 6, 7, 8, 14$	–
19	$\infty, 0, 1, 2, 9, 14$	$3, 5, 6, 10$	$\left\{\begin{array}{l} 7, 8, 11, 12, \\ 13, 16, 17, 18 \end{array}\right.$	$4, 15$

Here the "o.w.^{1}" in the table means the otherwise case, which is equivalent to the condition that 0(3), 1(9), 4(9), 7(27) and 25(27). In such a case, the extension $M_s/\mathbf{Q}$ is ramified at 3. For the polynomial $f_0(\widetilde{c_4}(t), X)$ we have

p	σ^0 split	σ^1 inert	σ^2 inert	ram. or bl.up
2	∞	$0, 1$	$\emptyset$	–
3	$\infty, 20(27), 14(81)$	$1(3), 2(27), 41(81)$	$11(27), 68(81)$	o.w.^{2} $\Rightarrow$ ram.
5	$\infty, 1$	2	$0, 3, 4$	–
7	$\infty, 2$	$0, 4, 6$	1	$3, 5$
11	$\infty, 2, 8, 9$	$0, 1, 7, 10$	$3, 4, 5, 6$	–
13	$\infty, 6$	5	$0, 2, 3, 7, 12$	$1, 4, 8, 9, 10, 11$
17	$\begin{cases} \infty, 0, 5, 6, 8, \\ 12, 13, 15, 16 \end{cases}$	$2, 3, 4, 7, 9, 11, 14$	$1, 10$	–
19	$\infty, 0, 4, 14, 18$	$9, 10, 13, 15, 17$	$1, 6, 7, 11$	$2, 3, 5, 8, 12, 16$

The "o.w.^{2}" in the table means the otherwise case, which is equivalent to the condition that $0(3)$, $8(9)$, $5(27)$ and $23(27)$. In such a case, $M_s/\mathbf{Q}$ is ramified at 3.

Theorem 3.4. *The family* $\{\mathrm{Spl}_{\mathbf{Q}} f_1(s, X) | s \in \mathbf{Q}\}$ *does not contain any cyclic cubic fields* E *which are unramified at two prime numbers* 2 *and* 3 *with* $\mathrm{Art}_2(E/\mathbf{Q}) = \mathrm{Art}_3(E/\mathbf{Q}) \neq \mathrm{id}$.

Let E_{13} and E_{19} be cyclic cubic fields with conductor 13 and 19, respectively.

Lemma 3.5. *For* $i = 13$ *and* 19 *we have* $\mathrm{Art}_2(E_i/\mathbf{Q}) = \mathrm{Art}_3(E_i/\mathbf{Q}) \neq \mathrm{id}$, *respectively.*

Corollary 3.6. *The polynomials* $f_1(t, X)$ *is not generic over* $\mathbf{Q}$ *for* $\mathcal{C}_3$.

Remark 3.7. By a geometric approach it is already shown that the polynomials $f_1(t, X)$, $f_3(t, X)$ and $f_4(t, X)$ are not generic for $\mathcal{C}_3$ over any finite algebraic number fields (cf. [5]).

Remark 3.8. There are symbols $\emptyset$ at 7-rows in the tables for $f_0(\widetilde{c_1}(t), X)$ and $f_0(\widetilde{c_3}(t), X)$, respectively. However, the case of $\mathrm{Art}_7(M_s/\mathbf{Q}) = \sigma^2$ occurs because of some blowing-up cases.

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